This is the March 2011 version of the High School Mathematics Model Curriculum for the conceptual category Functions. (Note: The conceptual categories Number and Quantity, Algebra, Functions, Modeling, Geometry, and Statistics and Probability do not directly align with high school mathematics courses.) The current focus of this document is to provide instructional strategies and resources, and identify misconceptions and connections related to the clusters and standards. The Ohio Department of Education is working in collaboration with assessment consortia, national professional organizations and other multi-state initiatives to develop common content elaborations and learning expectations

Functions	
Domain	Cluster
Interpreting Functions	Understand the concept of a function and use function notation. Interpret functions that arise in applications in terms of the context. Analyze functions using different representations.
Building Functions	Build a function that models a relationship between two quantities. Build new functions from existing functions.
Linear, Quadratic, and Exponential Models	Construct and compare linear, quadratic, and exponential models and solve problems. Interpret expressions for functions in terms of the situation they model
Trigonometric Functions	(+) Extend the domain of trigonometric functions using the unit circle. Model periodic phenomena with trigonometric functions. (+) Prove and apply trigonometric identities.

High School Conceptual Category: Functions

Domain	Interpreting Functions
Cluster	***Understand the concept of a function and use function notation***
Standards	1. Understand that a function from one set (called the domain) to another set (called the range) assigns to each element of the domain exactly one element of the range. If *f* is a function and *x* is an element of its domain, the *f(x)* denotes the output of *f* corresponding to the input *x*. The graph of *f* is the graph of the equation *y = f(x)*. 2. Use function notation, evaluate functions for inputs in their domains, and interpret statements that use function notation in terms of a context. 3. Recognize that sequences are functions, sometimes defined recursively, whose domain is a subset of the integers. *For example, the Fiboonacci sequence is defined recursively by F(0) = F(1) = 1, f(n + 1) = f(n) + f (n – 1) for n ≥ 1.*

Content Elaborations (in development)

This section will provide additional clarification and examples to aid in the understanding of the standards. To support shared interpretations across states, content elaborations are being developed through multi-state partnerships organized by CCSSO and other national organizations. This information will be included as it is developed.

Expectations for Learning (in development)

As the framework for the assessments, this section will be developed by the CCSS assessment consortia (SBAC and PARCC). Ohio is currently participating in both consortia and has input into the development of the frameworks. This information will be included as it is developed.

Instructional Strategies and Resources

Instructional Strategies

Provide applied contexts in which to explore functions. For example, examine the amount of money earned when given the number of hours worked on a job, and contrast this with a situation in which a single fee is paid by the “carload” of people, regardless of whether 1, 2, or more people are in the car.

Use diagrams to help students visualize the idea of a function machine. Students can examine several pairs of input and output values and try to determine a simple rule for the function.

Rewrite sequences of numbers in tabular form, where the input represents the term number (the position or index) in the sequence, and the output represents the number in the sequence.

Help students to understand that the word “domain” implies the set of all possible input values and that the integers are a set of numbers made up of {…-2, -1, 0, 1, 2, …}.

Distinguish between relationships that are not functions and those that are functions (e.g., present a table in which one of the input values results in multiple outputs to contrast with a functional relationship). Examine graphs of functions and non-functions, recognizing that if a vertical line passes through at least two points in the graph, then y (or the quantity on the vertical axis) is not a function of x (or the quantity on the horizontal axis).

Instructional Resources/Tools

Diagrams or drawings of function machines, as well as tables and graphs.

Function Machine virtual manipulatives, such as available at nlvm.usu.edu.

Common Misconceptions

Students may believe that all relationships having an input and an output are functions, and therefore, misuse the function terminology.

Students may also believe that the notation *f(x)* means to multiply some value *f* times another value *x*. The notation alone can be confusing and needs careful development. For example, *f*(2) means the output value of the function *f* when the input value is 2.

Diverse Learners

Information and instructional strategies for gifted students, English Language Learners (ELL), and students with

disabilities is available in the Introduction to Universal Design for Learning document located on the Revised Academic Content Standards and Model Curriculum Development Web page. Additional strategies and resources based on the Universal Design for Learning principles can be found at www.cast.ora.
Connections: Understanding a function as a rule that assigns exactly one output to each input is developed in Grade 8. In high school, the idea of a function is expanded to include use of function notation and evaluating functions for given input values.

Domain	**Interpreting Functions**
Cluster	***Interpret functions that arise in applications in terms of the context***
Standards	*4.* For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship. *Key features include: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; and periodicity.* 5. Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes. *For example, if the function h(n) gives the number of person-hours it takes to assemble n engines in a factory, then the positive integers would be an appropriate domain for the function.* 6. Calculate and interpret the average rate of change of a function (presented symbolically or as a table) over a specified interval. Estimate the rate of change from a graph.

Content Elaborations (in development)

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Expectations for Learning (in development)

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Instructional Strategies and Resources

Instructional Strategies

Flexibly move from examining a graph and describing its characteristics (e.g., intercepts, relative maximums, etc.) to using a set of given characteristics to sketch the graph of a function.

Examine a table of related quantities and identify features in the table, such as intervals on which the function increases, decreases, or exhibits periodic behavior.

Recognize appropriate domains of functions in real-world settings. For example, when determining a weekly salary based on hours worked, the hours (input) could be a rational number, such as 25.5. However, if a function relates the number of cans of soda sold in a machine to the money generated, the domain must consist of whole numbers.

Given a table of values, such as the height of a plant over time, students can estimate the rate of plant growth. Also, if the relationship between time and height is expressed as a linear equation, students should explain the meaning of the slope of the line. Finally, if the relationship is illustrated as a linear or non-linear graph, the student should select points on the graph and use them to estimate the growth rate over a given interval.

Instructional Resources/Tools

Tables, graphs, and equations of real-world functional relationships.
Graphing calculators to generate graphical, tabular, and symbolic representations of the same function for comparison.

Common Misconceptions

Students may believe that it is reasonable to input any *x*-value into a function, so they will need to examine multiple situations in which there are various limitations to the domains.

Students may also believe that the slope of a linear function is merely a number used to sketch the graph of the line. In reality, slopes have real-world meaning, and the idea of a rate of change is fundamental to understanding major concepts from geometry to calculus.

Diverse Learners

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Universal Design for Learning principles can be found at www.cast.org.

Specific strategies for mathematics may include:

Begin with simple, linear functions to describe features and representations, and then move to more advanced functions, including non-linear situations.

Provide students with many examples of functional relationships, both linear and non-linear. Use real-world examples, such as the growth of an investment fund over time, so that students can not only describe what they see in a table, equation, or graph, but also can relate the features to the real-life meanings.

Allow students to collect their own data sets, such as the falling temperature of a glass of hot water when removed from a flame versus the amount of time, to generate tables and graphs for discussion and interpretation.

Connections:

A basic study of functions takes place in Grade 8, as students examine simple linear and non-linear functions, including the idea that some functions have greater rates of change than others. In high school, more complex functions with inconsistent rates of change and their graphs are examined, including analysis of the domain of a function.

Domain	Interpreting Functions
Cluster	***Analyze functions using different representations***
Standards	7. Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases. a. Graph linear and quadratic functions and show intercepts, maxima, and minima. b. Graph square root, cube root, and piecewise-defined functions, including step functions and absolute value functions. c. Graph polynomial functions, identifying zeros when suitable factorizations are available, and showing end behavior. d. (+) Graph rational functions, identifying zeros and asymptotes when suitable factorizations are available, and showing end behavior. e. Graph exponential and logarithmic functions, showing intercepts and end behavior, and trigonometric functions, showing period, midline, and amplitude. 8. Write a function defined by an expression in different but equivalent forms to reveal and explain different properties of the function. a. Use the process of factoring and completing the square in a quadratic function to show zeros, extreme values, and symmetry of the graph, and interpret these in terms of a context. *b.* Use the properties of exponents to interpret expressions for exponential functions. *For example, identify percent rate of change in functions such as* $y = (1.02)^5$, $y = (0.97)^5$, $y = (1.01)^{12t}$, $y = (1.2)^{t/10}$, *and classify them as representing exponential growth or decay.* 9. Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions). *For example, given a graph of one quadratic function and an algebraic expression for another, say which has the larger maximum*.

Content Elaborations (in development)

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Expectations for Learning (in development)

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Instructional Strategies and Resources

Instructional Strategies

Explore various families of functions and help students to make connections in terms of general features. For example, just as the function $y = (x + 3)^2 – 5$ represents a translation of the function $y = x$ by 3 units to the left and 5 units down, the same is true for the function $y = | x + 3 | - 5$ as a translation of the absolute value function $y = | x |$.

Discover that the factored form of a quadratic or polynomial equation can be used to determine the zeros, which in turn can be used to identify maxima, minima and end behaviors.

Use various representations of the same function to emphasize different characteristics of that function. For example, the *y*-intercept of the function $y = x^2 -4x – 12$ is easy to recognize as (0, -12). However, rewriting the function as $y = (x – 6)(x + 2)$ reveals zeros at (6, 0) and at (-2, 0). Furthermore, completing the square allows the equation to be written as $y = (x – 2)^2 – 16$, which shows that the vertex (and minimum point) of the parabola is at (2, -16).

Examine multiple real-world examples of exponential functions so that students recognize that a base between 0 and 1 (such as an equation describing depreciation of an automobile [$f(x) = 15,000(0.8)^x$ represents the value of a $15,000 automobile that depreciates 20% per year over the course of *x* years]) results in an exponential decay, while a base greater than 1 (such as the value of an investment over time [$f(x) = 5,000(1.07)^x$ represents the value of an investment of $5,000 when increasing in value by 7% per year for *x* years]) illustrates growth.

Instructional Resources/Tools

Graphing utilities on a calculator and/or computer can be used to demonstrate the changes in behavior of a function as various parameters are varied.

Real-world problems, such as maximizing the area of a region bound by a fixed perimeter fence, can help to illustrate applied uses of families of functions.

Common Misconceptions

Students may believe that each family of functions (e.g., quadratic, square root, etc.) is independent of the others, so they may not recognize commonalities among all functions and their graphs.

Students may also believe that skills such as factoring a trinomial or completing the square are isolated within a unit on polynomials, and that they will come to understand the usefulness of these skills in the context of examining characteristics of functions.

Additionally, student may believe that the process of rewriting equations into various forms is simply an algebra symbol manipulation exercise, rather than serving a purpose of allowing different features of the function to be exhibited.

Diverse Learners

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Some information below includes additional mathematics that students should learn in order to take advanced courses such as calculus, advanced statistics, or discrete mathematics, and goes beyond the mathematics that all students should study in order to be college- and career-ready:

Add families of functions, one at a time, to the students' knowledge base so they can see connections among behaviors of the various functions.

Provide numerous examples of real-world contexts, such as exponential growth and decay situations (e.g., a population that declines by 10% per year) to help students apply an understanding of functions in context.

Examine rational functions on a graphing calculator and discuss why, for example, the tabular representation shows an "Error" message for some values of *y*. Students need to be able to verbalize why a function has asymptotes and distinguish between asymptotes and holes. For example, the function $f(x) = \frac{(x+1)(x-3)}{(2x+5)(x-3)}$ has an asymptote at $x = -2.5$ but a hole at $x = 3$.

Involve students in activities that include collection and analysis of data to generalize function behaviors. For example, they can take a cup filled with pennies, spill them onto a table, count how many came up "heads," put only those pennies back in the cup, and repeat this process several times. In the end, they will generate a table of values that will model an exponential decay function with a base of ½.

Connections:

In Grade 7, students are exposed to the idea that rewriting an expression can shed light on the meaning of the expression. This idea is expanded upon as students explore functions in high school and recognize how the form of the equation can provide clues about zeros, asymptotes, etc.

Domain	Building Functions
Cluster	***Build a function that models a relationship between two quantities***
Standards	1. Write a function that describes a relationship between two quantities. (a) Determine an explicit expression, a recursive process, or steps for calculation from a context. (b) Combine standard function types using arithmetic operations. *For example, build a function that models the temperature of a cooling body by adding a constant function to a decaying exponential and relate these functions to the model*. (c) (+) Compose functions. *For example, if T(y) is the temperature in the atmosphere as a function of height, and h(t) is the height of a weather balloon as a function of time, then T(h(t)) is the temperature at the location of the weather balloon as a function of time*. 2. Write arithmetic and geometric sequences both recursively and with an explicit formula; use them to model situations, and translate between the two forms.

Content Elaborations (in development)

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Expectations for Learning (in development)

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Instructional Strategies and Resources

Instructional Strategies

Provide a real-world example (e.g., a table showing how far a car has driven after a given number of minutes, traveling at a uniform speed), and examine the table by looking "down" the table to describe a recursive relationship, as well as "across" the table to determine an explicit formula to find the distance traveled if the number of minutes is known.

Write out terms in a table in an expanded form to help students see what is happening. For example, if the *y*-values are 2, 4, 8, 16, they could be written as 2, 2(2), 2(2)(2), 2(2)(2)(2), etc., so that students recognize that 2 is being used multiple times as a factor.

Focus on one representation and its related language – recursive or explicit – at a time so that students are not confusing the formats.

Provide examples of when functions can be combined, such as determining a function describing the monthly cost for owning two vehicles when a function for the cost of each (given the number of miles driven) is known.

Using visual approaches (e.g., folding a piece of paper in half multiple times), use the visual models to generate sequences of numbers that can be explored and described with both recursive and explicit formulas. Emphasize that there are times when one form to describe the function is preferred over the other.

Instructional Resources/Tools

Hands-on materials (e.g., paper folding, building progressively larger shapes using pattern blocks, etc.) can be used as a visual source to build numerical tables for examination.

Visuals available to assist students in seeing relationships are featured at the National Library of Virtual Manipulatives as well as The National Council of Teachers of Mathematics, Illuminations
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Common Misconceptions

Students may believe that the best (or only) way to generalize a table of data is by using a recursive formula. Students naturally tend to look "down" a table to find the pattern but need to realize that finding the 100th term requires knowing the 99th term unless an explicit formula is developed.

Students may also believe that arithmetic and geometric sequences are the same. Students need experiences with both types of sequences to be able to recognize the difference and more readily develop formulas to describe them.

Additionally, advanced students who study composition of functions may misunderstand function notation to represent multiplication (e.g., $f(g(x))$ means to multiply the f and g function values).

Diverse Learners

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Connections: In Grade 8, students compare functions by looking at equations, tables and graphs, and focus primarily on linear relationships. In high school, examination of functions is extended to include recursive and explicit representations and sequences of numbers that may not have a linear relationship.

Domain	Building Functions
Cluster	***Build new functions from existing functions***
Standards	3. Identify the effect on the graph of replacing *f(x)* by *f(x) + k, k f(x), f(kx),* and *f(x + k)* for specific values of *k* (both positive and negative); find the value of *k* given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. *Include recognizing even and odd functions from their graphs and algebraic expressions for them*. 4. Find inverse functions. a. Solve an equation of the form f(x) = c for a simple function f that has an inverse and write an expression for the inverse. *For example,* $f(x) = 2x^3$ *for x>0 or f(x) = (x+1)/(x-1) for x ≠1.* b. (+) Verify by composition that one function is the inverse of another. c. (+) Read values of an inverse function from a graph or table, given that the function has an inverse. d. (+) Produce an invertible function from a non-invertible function by restricting the domain. 5. (+) Understand the inverse relationship between exponents and logarithms and use this relationship to solve problems involving logarithms and exponents.

Content Elaborations (in development)

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Instructional Strategies and Resources

Instructional Strategies

Use graphing calculators or computers to explore the effects of a constant in the graph of a function. For example, students should be able to distinguish between the graphs of $y = x^2$, $y = 2x^2$, $y = x^2 + 2$, $y = (2x)^2$, and $y = (x + 2)^2$. This can be accomplished by allowing students to work with a single parent function and examine numerous parameter changes to make generalizations.

Distinguish between even and odd functions by providing several examples and helping students to recognize that a function is even if *f(-x)* = *f(x)* and is odd if *f(-x)* = *-f(x)*. Visual approaches to identifying the graphs of even and odd functions can be used as well.

Provide examples of inverses that are not purely mathematical to introduce the idea. For example, given a function that names the capital of a state, *f*(Ohio) = Columbus. The inverse would be to input the capital city and have the state be the output, such that f^{-1}(Denver) = Colorado.

Some information below includes additional mathematics that students should learn in order to take advanced courses such as calculus, advanced statistics, or discrete mathematics and goes beyond the mathematics that all students should study in order to be college- and career-ready:

Students should also recognize that not all functions have inverses. Again using a nonmathematical example, a function could assign a continent to a given country's input, such as *g*(Singapore) = Asia. However, g^{-1}(Asia) does not have to be Singapore (e.g., it could be China).

Exchange the *x* and *y* values in a symbolic functional equation and solve for *y* to determine the inverse function. Recognize that putting the output from the original function into the input of the inverse results in the original input value.

Also, students need to recognize that exponential and logarithmic functions are inverses of one another and use these functions to solve real-world problems.

Nonmathematical examples of functions and their inverses can help students to understand the concept of an inverse and determining whether a function is invertible.

Instructional Resources/Tools

Graphing calculator that can be used to explore the effects of parameter changes on a graph

Common Misconceptions

Students may believe that the graph of $y = (x - 4)^3$ is the graph of $y = x^3$ shifted 4 units to the left (due to the subtraction symbol). Examples should be explored by hand and on a graphing calculator to overcome this misconception.

Students may also believe that *even* and *odd* functions refer to the exponent of the variable, rather than the sketch of the graph and the behavior of the function.

Additionally, students may believe that all functions have inverses and need to see counter examples, as well as examples in which a non-invertible function can be made into an invertible function by restricting the domain. For example, $f(x) = x^2$ has an inverse ($f^{-1}(x) = \sqrt{x}$) provided that the domain is restricted to $x \geq 0$.

Diverse Learners

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Specific strategies for mathematics may include:

Allow students to initially make tables of values by hand for some simple examples, such as $y = x^2$ and $y = x^2 + 3$ to examine the effects of changing the constant, including the existence of inverses. Then, students can examine additional effects and more complicated functions with technology.

Use real-world examples of functions and their inverses. For example, students might determine that folding a piece of paper in half 5 times results in 32 layers of paper, but that if they are given that there are 32 layers of paper, they can solve to find how many times the paper would have been folded in half.

Provide applied examples of exponential and logarithmic functions, such as the use of a logarithm to determine pH or the strength of an earthquake on the Richter Scale. Both pH and Richter Scale values are powers of 10 and are, therefore, logarithms. For example, the magnitude of an earthquake, M, on the Richter Scale can be calculated using the formula $M = \log_{10} A$, where A represents the amplitude of measured seismic waves.

Connections:

Understanding functional relationships as input and output values that have an associated graph is introduced in Grade 8. In high school, changes in graphs is explored in more depth, and the idea of functions having inverses is introduced. Advanced students also expand their catalog of functions to include exponential and logarithmic cases.

Domain	Linear and Exponential Models
Cluster	***Construct and compare linear and exponential models and solve problems***
Standards	1. Distinguish between situations that can be modeled with linear functions and with exponential functions. a. Prove that linear functions grow by equal differences over equal intervals, and that exponential functions grow by equal factors over equal intervals. b. Recognize situations in which one quantity changes at a constant rate per unit interval relative to another. c. Recognize situations in which a quantity grows or decays by a constant percent rate per unit interval relative to another. 2. Construct linear and exponential functions, including arithmetic and geometric sequences, given a graph, a description of a relationship, or two input-output pairs (include reading these from a table.) 3. Observe using graphs and tables that a quantity increasing exponentially eventually exceeds a quantity increasing linearly, quadratically, or (more generally) as a polynomial function. 4. For exponential models, express as a logarithm the solution to $ab^{ct} = d$ where *a, c,* and *d* are numbers and the base *b* is 2, 10, or *e*; evaluate the logarithm using technology.

Content Elaborations (in development)

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Expectations for Learning (in development)

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Instructional Strategies and Resources

Instructional Strategies

Compare tabular representations of a variety of functions to show that linear functions have a first common difference (i.e., equal differences over equal intervals), while exponential functions do not (instead function values grow by equal factors over equal *x*-intervals).

Apply linear and exponential functions to real-world situations. For example, a person earning $10 per hour experiences a constant rate of change in salary given the number of hours worked, while the number of bacteria on a dish that doubles every hour will have equal factors over equal intervals.

Provide examples of arithmetic and geometric sequences in graphic, verbal, or tabular forms, and have students generate formulas and equations that describe the patterns.

Use a graphing calculator or computer program to compare tabular and graphic representations of exponential and polynomial functions to show how the *y* (output) values of the exponential function eventually exceed those of polynomial functions.

Have students draw the graphs of exponential and other polynomial functions on a graphing calculator or computer utility and examine the fact that the exponential curve will eventually get higher than the polynomial function's graph. A simple example would be to compare the graphs (and tables) of the functions $y = x^2$ and $y = 2^x$ to find that the *y* values are greater for the exponential function when $x > 4$.

Help students to see that solving an equation such as $2^x = 300$ can be accomplished by entering $y = 2^2$ and $y = 300$ into a graphing calculator and finding where the graphs intersect, by viewing the table to see where the function values are about the same, as well as by applying a logarithmic function to both sides of the equation.

Use technology to solve exponential equations such as $3*10^x = 450$. (In this case, students can determine the approximate power of 10 that would generate a value of 150.) Students can also take the logarithm of both sides of the equation to solve for the variable, making use of the inverse operation to solve.

Instructional Resources/Tools

Examples of real-world situations that apply linear and exponential functions to compare their behaviors

Graphing calculators or computer software that generate graphs and tables of functions. A graphing tool such as the one found at nlvm.usu.edu is one option.

Common Misconceptions

Students may believe that all functions have a first common difference and need to explore to realize that, for example, a quadratic function will have equal second common differences in a table.

Students may also believe that the end behavior of all functions depends on the situation and not the fact that exponential function values will eventually get larger than those of any other polynomial functions.

Diverse Learners

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Specific strategies for mathematics may include:

Explore simple linear and exponential functions by engaging in hands-on experiments. For example, students can measure the diameters and related circumferences of several circles and determine a linear function that relates the diameter to the circumference – a linear function with a first common difference. They can then explore the value of an investment when told that the account will double in value every 12 years – an exponential function with a base of 2.

Connections:

While students in Grade 8 examine some nonlinear situations, most of the functions explored are linear. In high school, basic understanding of functions is expanded to include exponential and other polynomial functions and how they compare to the behaviors of linear functions.

Domain	Linear and Exponential Models
Cluster	*Interpret expressions for functions in terms of the situation they model*
Standards	5. Interpret the parameters in a linear or exponential function in terms of a context.

Content Elaborations (in development)

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Expectations for Learning (in development)

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Instructional Strategies and Resources

Instructional Strategies

Use real-world contexts to help students understand how the parameters of linear and exponential functions depend on the context. For example, a plumber who charges $50 for a house call and $85 per hour would be expressed as the function $y = 85x + 50$, and if the rate were raised to $90 per hour, the function would become $y = 90x + 50$. On the other hand, an equation of $y = 8{,}000(1.04)^x$ could model the rising population of a city with 8,000 residents when the annual growth rate is 4%. Students can examine what would happen to the population over 25 years if the rate were 6% instead of 4% or the effect on the equation and function of the city's population were 12,000 instead of 8,000.

Graphs and tables can be used to examine the behaviors of functions as parameters are changed, including the comparison of two functions such as what would happen to a population if it grew by 500 people per year, versus rising an average of 8% per year over the course of 10 years.

Instructional Resources/Tools

Graphing calculators or computer software that generates graphs and tables of functions.

Examples of real-world situations that apply linear and exponential functions to examine the effects of parameter changes.

Web sites and other sources that provide raw data, such as the cost of products over time, population changes, etc.

Common Misconceptions

Students may believe that changing the slope of a linear function from "2" to "3" makes the graph steeper without realizing that there is a real-world context and reason for examining the slopes of lines. Similarly, an exponential function can appear to be abstract until applying it to a real-world situation involving population, cost, investments, etc.

Diverse Learners

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Specific strategies for mathematics may include:

Provide students with opportunities to research raw data on the Internet (such as increases in gasoline consumption in China over the years) and to graph and generalize trends in growth, determining whether the growth is linear.

Working in pairs or small groups, students can be given different parameters of a function to manipulate and compare the results to draw conclusions about the effects of the changes.

Connections:

Understanding slope as a rate of change, as well as working with integral exponents, are important elements of the Grade 8 curriculum. In high school, knowledge is extended into a thorough study of functions that includes a contextual understanding of parameter changes in both linear and exponential function situations.

Domain	Trigonometric Functions
Cluster	***Extend the domain of trigonometric functions using the unit circle***
Standards	1. Understand radian measure of an angle as the length of the arc on the unit circle subtended by the angle. 2. Explain how the unit circle in the coordinate plane enables the extension of trigonometric functions to all real numbers, interpreted as radian measures of angles traversed counterclockwise around the unit circle. 3. (+) Use special triangles to determine geometrically the values of sine, cosine, tangent for π/3, π/4, and π/6, and use the unit circle to express the values of sine, cosines, and tangent for *x*, π + *x*, and 2π – *x* in terms of their values for *x*, where *x* is any real number. 4. (+) Use the unit circle to explain symmetry (odd and even) and periodicity of trigonometric functions.

Content Elaborations (in development)

This section will provide additional clarification and examples to aid in the understanding of the standards. To support shared interpretations across states, content elaborations are being developed through multi-state partnerships organized by CCSSO and other national organizations. This information will be included as it is developed.

Expectations for Learning (in development)

As the framework for the assessments, this section will be developed by the CCSS assessment consortia (SBAC and PARCC). Ohio is currently participating in both consortia and has input into the development of the frameworks. This information will be included as it is developed.

Instructional Strategies and Resources

Instructional Strategies

Use a compass and straightedge to explore a unit circle with a fixed radius of 1. Help students to recognize that the circumference of the circle is 2π, which represents the number of radians for one complete revolution around the circle. Students can determine that, for example, π/4 radians would represent a revolution of 1/8 of the circle or 45°.

Having a circle of radius 1, the cosine, for example, is simply the *x*-value for any ordered pair on the circle (adjacent/hypotenuse where adjacent is the *x*-length and hypotenuse is 1). Students can examine how a counterclockwise rotation determines a coordinate of a particular point in the unit circle from which sine, cosine, and tangent can be determined.

Some information below includes additional mathematics that students should learn in order to take advanced courses such as calculus, advanced statistics, or discrete mathematics and goes beyond the mathematics that all students should study in order to be college- and career-ready:

Some students can use what they know about 30-60-90 triangles and right isosceles triangles to determine the values for sine, cosine, and tangent for π/3, π/4, and π/6. In turn, they can determine the relationships between, for example, the sine of π/6, 7π/6, and 11π/6, as all of these use the same reference angle and knowledge of a 30-60-90 triangle.

Provide students with real-world examples of periodic functions. One good example is the average high (or low) temperature in a city in Ohio for each of the 12 months. These values are easily located at weather.com and can be graphed to show a periodic change that provides a context for exploration of these functions.

Allow plenty of time for students to draw – by hand and with technology – graphs of the three trigonometric functions to examine the curves and gain a graphical understanding of why, for example, cos (π/2) = 0 and whether the function is even (e.g., cos(-*x*) = cos(*x*)) or odd (e.g., sin(-*x*) = -sin(*x*)). Similarly, students can generalize how function values repeat one another, as illustrated by the behavior of the curves.

Instructional Resources/Tools

Compass and straightedge to explore the unit circle and to draw sine and cosine curves and describe their periodicity.

Graphing calculators or computer graphing tools to determine radian measures and to find values of the sine, cosine, and tangent functions for any given *x* input value.

Common Misconceptions

Students may believe that there is no need for radians if one already knows how to use degrees. Students need to be shown a rationale for how radians are unique in terms of finding function values in trigonometry since the radius of the unit circle is 1.

Students may also believe that all angles having the same reference values have identical sine, cosine and tangent values. They will need to explore in which quadrants these values are positive and negative.

Diverse Learners

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Connections:

The study of trigonometry is reserved for high school students. In the Geometry conceptual category, students explore right triangle trigonometry, with advanced students working with laws of sines and cosines. In the conceptual category of Functions, students connect the idea of functions with trigonometry and see sine, cosine and tangent values as functions of angle values input in radians. Connections are made such as the cosine of an angle equaling the sine of its complement as well as to the Geometry Standards involving radian measures.

Domain	**Trigonometric Functions**
Cluster	***Model periodic phenomena with trigonometric functions***
Standards	5. Choose trigonometric functions to model periodic phenomena with specified amplitude, frequency, and midline. 6. (+) Understand that restricting a trigonometric function to a domain on which it is always increasing or always decreasing allows its inverse to be constructed. 7. (+) Use inverse functions to solve trigonometric equations that arise in modeling contexts; evaluate the solutions using technology, and interpret them in terms of context.

Content Elaborations (in development)

This section will provide additional clarification and examples to aid in the understanding of the standards. To support shared interpretations across states, content elaborations are being developed through multi-state partnerships organized by CCSSO and other national organizations. This information will be included as it is developed.

Expectations for Learning (in development)

As the framework for the assessments, this section will be developed by the CCSS assessment consortia (SBAC and PARCC). Ohio is currently participating in both consortia and has input into the development of the frameworks. This information will be included as it is developed.

Instructional Strategies and Resources

Instructional Strategies

Allow students to explore real-world examples of periodic functions. Examples include average high (or low) temperatures throughout the year, the height of ocean tides as they advance and recede, and the fractional part of the moon that one can see on each day of the month. Graphing some real-world examples can allow students to express the amplitude, frequency, and midline of each.

Help students to understand what the value of the sine (cosine, or tangent) means (e.g., that the number represents the ratio of two sides of a right triangle having that angle measure).

Using graphing calculators or computer software, as well as graphing simple examples by hand, have students graph a variety of trigonometric functions in which the amplitude, frequency, and/or midline is changed. Students should be able to generalize about parameter changes, such as what happens to the graph of $y = \cos(x)$ when the equation is changed to $y = 3\cos(x) + 5$.

Some information below includes additional mathematics that students should learn in order to take advanced courses such as calculus, advanced statistics, or discrete mathematics and goes beyond the mathematics that all students should study in order to be college- and career-ready:

Some students can explore the inverse trigonometric functions, recognizing that the periodic nature of the functions depends on restricting the domain. These inverse functions can then be used to solve real-world problems involving trigonometry with the assistance of technology.

Instructional Resources/Tools

A list of real-world applications of periodic situations that can be modeled by using trigonometric functions for students to explore.

Graphing calculators or computer software to generate the graphs of trigonometric functions.

Common Misconceptions

Students may believe that all trigonometric functions have a range of 1 to -1. Students need to see examples of how coefficients can change the range and the look of the graphs.

Students may also believe that restrictions to the domain of trigonometric functions are not necessary for defining inverse functions.

Students may also believe that $\sin^{-1} A = 1/\sin A$, thus confusing the ideas of inverse and reciprocal functions.

Additionally, students may not understand that when sin A = 0.4, the value of A represents an angle measure and that the function $\sin^{-1}(0.4)$ can be used to find the angle measure.

Diverse Learners

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Connections:

The study of trigonometry is reserved for high school students. In the Geometry conceptual category, students explore right triangle trigonometry, with advanced students working with laws of sines and cosines. In the conceptual category of Functions, students connect the idea of functions with trigonometry and explore the effects of parameter changes on the amplitude, frequency and midline of trigonometric graphs.

Domain	Trigonometric Functions
Cluster	***Prove and apply trigonometric identities***
Standards	8. Prove the Pythagorean identity $\sin^2(\theta)+\cos^2(\theta)=1$ and use it to calculate trigonometric ratios. 9. (+) Prove the addition and subtraction formulas for sine, cosine, and tangent and use them to solve problems.

Content Elaborations (in development)

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Expectations for Learning (in development)

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Instructional Strategies and Resources

Instructional Strategies

In the unit circle, the cosine is the *x*-value, while the sine is the *y*-value. Since the hypotenuse is always 1, the Pythagorean relationship $\sin^2(\theta)+\cos^2(\theta)=1$ is always true. Students can make a connection between the Pythagorean Theorem in geometry and the study of trigonometry by proving this relationship. In turn, the relationship can be used to find the cosine when the sine is known, and vice-versa. Provide a context in which students can practice and apply skills of simplifying radicals.

Some information below includes additional mathematics that students should learn in order to take advanced courses such as calculus, advanced statistics, or discrete mathematics and goes beyond the mathematics that all students should study in order to be college- and career-ready:

Some students can explore other trigonometric identities, such as the half-angle, double-angle, and addition/subtraction formulas to extend on the Pythagorean relationship. Formulas should be proven and then used to determine exact values when given an angle measure, to prove identities, and to solve trigonometric equations. For example, by dividing the formula $\sin^2(\theta)+\cos^2(\theta)=1$ by $\cos^2(\theta)$, a new formula is generated ($\tan^2(\theta)+1=\sec^2(\theta)$).

Instructional Resources/Tools

Drawings of the unit circle can be useful in showing why the Pythagorean relationship must be true.

Dynamic geometry software, such as Geometer's Sketchpad or Geogebra, can be used to demonstrate that, regardless of the angle measure, the Pythagorean relationship always holds in the unit circle.

Common Misconceptions

Students may believe that there is no connection between the Pythagorean Theorem and the study of trigonometry.

Students may also believe that there is no relationship between the sine and cosine values for a particular angle. The fact that the sum of the squares of these values always equals 1 provides a unique way to view trigonometry through the lens of geometry.

Additionally, students may believe that $\sin(A+B)=\sin A+\sin B$ and need specific examples to disprove this assumption.

Diverse Learners

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Connections:

Students in Grade 8 grade learn to use the Pythagorean Theorem, while high school students in a geometry unit (or course) develop right triangle trignometry. This cluster allows high school students to connect these ideas as they derive a P tha orean relationshi for the tri onometric functions.