This is the March 2011 version of the High School Mathematics Model Curriculum for the conceptual category Number and Quantity. (Note: The conceptual categories Number and Quantity, Algebra, Functions, Modeling, Geometry, and Statistics and Probability do not directly align with high school mathematics courses.) The current focus of this document is to provide instructional strategies and resources, and identify misconceptions and connections related to the clusters and standards. The Ohio Department of Education is working in collaboration with assessment consortia, national professional organizations and other multistate initiatives to develop common content elaborations and learning expectations.

Number and Quantity

Domain	Cluster
Real Number System	Extend the properties of exponents to rational exponents. Use properties of rational and irrational numbers.
Quantities	Reason quantitatively and use units to solve problems.
Complex Number System	Perform arithmetic operations with complex numbers. (+) Represent complex numbers and their operations on the complex plane. Use complex numbers in polynomial identities and equations.
Vector and Matrix Quantities	(+) Represent and model with vector quantities. (+) Perform operations on vectors. (+) Perform operations on matrices and use matrices in applications.

Domain	***The Real Number System***
Cluster	***Extend the properties of exponents to rational exponents***
Standards	1. Explain how the definition of the meaning of rational exponents follows from extending the properties of integer exponents to those values, allowing for a notation for radicals in terms of rational exponents. 2. Rewrite expressions involving radicals and rational exponents using the properties of exponents.

Content Elaborations (in development)

This section will provide additional clarification and examples to aid in the understanding of the standards. To support shared interpretations across states, content elaborations are being developed through multistate partnerships organized by CCSSO and other national organizations. This information will be included as it is developed.

Expectations for Learning (in development)

As the framework for the assessments, this section will be developed by the CCSS assessment consortia (SBAC and PARCC). Ohio is currently participating in both consortia and has input into the development of the frameworks. This information will be included as it is developed.

Instructional Strategies and Resources

Instructional Strategies

The goal is to show that a fractional exponent can be expressed as a radical or a root. For example, an exponent of 1/3 is equivalent to a cube root; an exponent of ¼ is equivalent to a fourth root.

Review the power rule,)), for whole number exponents (e.g. $(7^2)^3 = 7^6$.

Compare examples, such as $(7^{1/2})^2 = 7^1 = 7$ and √) , to help students establish a connection between radicals and rational exponents: √ and, in general, √ .

Provide opportunities for students to explore the equality of the values using calculators, such as $7^{1/2}$ and √ .
Offer sufficient examples and exercises to prompt the definition of fractional exponents, and give students practice in converting expressions between radical and exponential forms.
When n is a positive integer, generalize the meaning of √ and then to √ , where n and m are integers and n is greater than or equal to 2. When m is a negative integer, the result is the reciprocal of the root √

Stress the two rules of rational exponents: 1) the numerator of the exponent is the base's power and 2) the denominator of the exponent is the order of the root. When evaluating expressions involving rational exponents, it is often helpful to break an exponent into its parts – a power and a root – and then decide if it is easier to perform the root operation or the exponential operation first.

Model the use of precise mathematics vocabulary (e.g., base, exponent, radical, root, cube root, square root etc.). The

rules for integer exponents are applicable to rational exponents as well; however, the operations can be slightly more complicated because of the fractions. When multiplying exponents, powers are added). When dividing exponents, powers are subtracted). When raising an exponent to an exponent, powers are multiplied).

Instructional Resources/Tools

Graphing calculator Computer algebra systems The Ohio Resource Center

The National Council of Teachers of Mathematics, Illuminations

Common Misconceptions

Students sometimes misunderstand the meaning of exponential operations, the way powers and roots relate to one another, and the order in which they should be performed. Attention to the base is very important. Consider examples:

( ) and ( ). The position of a negative sign of a term with a rational exponent can mean that the rational exponent should be either applied first to the base, 81, and then the opposite of the result is taken ( ), or the rational exponent should be applied to a negative term ( ). The answer of √ will be not real if the denominator of the exponent is even. If the root is odd, the answer will be a negative number.

Students should be able to make use of estimation when incorrectly using multiplication instead of exponentiation.

Students may believe that the fractional exponent in the expression means the same as a factor in multiplication expression, and multiply the base by the exponent.

Diverse Learners

Information and instructional strategies for gifted students, English Language Learners (ELL), and students with disabilities is available in the Introduction to Universal Design for Learning document located on the Revised Academic Content Standards and Model Curriculum Development Web page. Additional strategies and resources based on the Universal Design for Learning principles can be found at www.cast.org.

Connections:

Integer exponents (both positive and negative) and radicals were studied in Grade 8. In this cluster, students expand the concept of exponent to include fractional exponents and make a connection to radicals. In more advanced courses, rational exponents will be extended to irrational exponents by means of exponential and logarithmic functions. For example, the definitions for integer and rational exponents will allow for the next step a definition of irrational exponents, such as √ or $2^{1.414213......}$ and then a new class of functions – exponential functions of the form $f(x) = b^x$, where $b \neq 1$, b › 0. The domain of this class of functions (the x values) is all real numbers (rational and irrational) and the range is the set of all positive real numbers.)

Domain	**The Real Number System**
Cluster	***Use properties of rational and irrational numbers***
Standards	3. Explain why the sum or product of two rational numbers is rational; that the sum of a rational number and an irrational number is irrational; and that the product of a nonzero rational number and an irrational number is irrational.

Content Elaborations (in development)

This section will provide additional clarification and examples to aid in the understanding of the standards. To support shared interpretations across states, content elaborations are being developed through multistate partnerships organized by CCSSO and other national organizations. This information will be included as it is developed.

Expectations for Learning (in development)

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Instructional Strategies and Resources

Instructional Strategies

This cluster is an excellent opportunity to incorporate algebraic proof, both direct and indirect, in teaching properties of number systems.

Students should explore concrete examples that illustrate that for any two rational numbers written in form a/b and c/d, where b and d are natural numbers and a and c are integers, the following are true:

– – —— represents a rational number, and

– – — represents a rational number.

Continue exploring situations where the sum of a rational number and an irrational number is irrational (e.g., a sum of rational number 2 and irrational number √3 is (2 + √3), which is an irrational).

Proofs are valid ways to justify not only geometry statements also algebraic statements.
Use indirect algebraic proof to generalize the statement that the sum of a rational and irrational number is irrational.

Assume that *x* is an irrational number and the sum of x and a rational number – is also rational and is represented as –

x + – –

x = – –

x = —— represents a rational number

Since the last statement contradicts a given fact that *x* is an irrational number, the assumption is wrong and a sum of a rational number and an irrational number has to be irrational. Similarly, it can be proven that the product of a non- zero rational and an irrational number is irrational.

Students need to see that results of the operations performed between numbers from a particular number set does not always belong to the same set. For example, the sum of two irrational numbers √ and √ is 4, which is a rational number.

Instructional Resources/Tools

The Ohio Resource Center
The National Council of Teachers of Mathematics, Illuminations
NCTM *Principles and Standards for School Mathematics*

Common Misconceptions

Some students may believe that both terminating and repeating decimals are rational numbers, without considering nonrepeating and nonterminating decimals as irrational numbers.

Students may also confuse irrational numbers and complex numbers, and therefore mix their properties. In this case, students should encounter examples that support or contradict properties and relationships between number sets (i.e., irrational numbers are real numbers and complex numbers are non-real numbers. The set of real numbers is a subset of the set of complex numbers).

By using false extensions of properties of rational numbers, some students may assume that the sum of any two irrational numbers is also irrational. This statement is not always true (e.g., (√‾) (√‾) , a rational number), and therefore, cannot be considered as a property.

Diverse Learners

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Connections

In high school, practicing operations with rational and irrational numbers helps students to understand the properties of real numbers and the relationships between number sets.

Algebraic manipulations and reasoning become a powerful tool for transfering students' experience in proofs from geometry to proofs in algebra.

Domain	Quantities
Cluster	***Reason quantitatively and use units to solve problems***
Standards	1. Use units as a way to understand problems and to guide the solution of multi-step problems; choose and interpret units consistently in formulas; choose and interpret the scale and the origin in graphs and data displays. 2. Define appropriate quantities for the purpose of descriptive modeling. 3. Choose a level of accuracy appropriate to limitations on measurement reporting quantities

Content Elaborations (in development)

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Expectations for Learning (in development)

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Instructional Strategies and Resources

Instructional Strategies

In real-world situations, answers are usually represented by numbers associated with units. Units involve measurement and often require a conversion. Measurement involves both precision and accuracy. Estimation and approximation often precede more exact computations.

Students need to develop sound mathematical reasoning skills and forms of argument to make reasonable judgments about their solutions. They should be able to decide whether a problem calls for an estimate, for an approximation, or for an exact answer. To accomplish this goal, teachers should provide students with a broad range of contextual problems that offer opportunities for performing operations with quantities involving units. These problems should be connected to science, engineering, economics, finance, medicine, etc.

Some contextual problems may require an understanding of derived measurements and capability in unit analysis. Keeping track of derived units during computations and making reasonable estimates and rational conclusions about accuracy and the precision of the answers help in the problem-solving process.

For example, while driving in the United Kingdom (UK), a U.S. tourist puts 60 liters of gasoline in his car. The gasoline cost is £1.28 per liter The exchange rate is £ 0.62978 for each $1.00. The price for a gallon of a gasoline in the United States is $3.05. The driver wants to compare the costs for the same amount and the same type of gasoline when he/she pays in UK pounds. Making reasonable estimates should be encouraged prior to solving this problem. Since the current exchange rate has inflated the UK pound at almost twice the U.S. dollar, the driver will pay more for less gasoline.

By dividing $3.05 by 3.79L (the number of liters in one gallon), students can see that 80.47 cents per liter of gasoline in US is less expensive than £1.28 or $ 2.03 per liter of the same type of gasoline in the UK when paid in U.S. dollars. The cost of 60 liters of gasoline in UK is ( )

In order to compute the cost of the same quantity of gasoline in the United States in UK currency, it is necessary to convert between both monetary systems and units of volume. Based on UK pounds, the cost of 60 liters of gasoline in the U.S. is).

The computation shows that the gasoline is less expensive in the United States and how an analysis can be helpful in keeping track of unit conversations. Students should be able to correctly identify the degree of precision of the answers which should not be far greater than the actual accuracy of the measurements.

Graphical representations serve as visual models for understanding phenomena that take place in our daily surroundings. The use of different kinds of graphical representations along with their units, labels and titles demonstrate the level of students' understanding and foster the ability to reason, prove, self-check, examine relationships and establish the validity of arguments. Students need to be able to identify misleading graphs by choosing correct units and scales to create a correct representation of a situation or to make a correct conclusion from it.

Instructional Resources/Tools

NCTM. *Focus in High School Mathematics (Reasoning and Sense Making)*
Mathematical Sciences Education Board. *High School Mathematics at Work*
NCTM. *Principles and Standards for School Mathematics*
Joint Committee of the MAA and NCTM. *A Sourcebook of Applications of School Mathematics*

Common Misconceptions

Students may not realize the importance of the units' conversions in conjunction with the computation when solving problems involving measurements.

Since today's calculating devices often display 8 to 10 decimal places, students frequently express answers to a much greater degree of precision than the required.

Diverse Learners

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Connections

Measuring commonly used object and choosing proper units for the measurements is part of the mathematics curriculum prior to high school. In high school, students experience a broader variety of units through real-world situations and modeling along with the exploration of the different levels of accuracy and precision of the answers.

High School Conceptual Category: Number and Quantity

Domain	**The Complex Number System**
Cluster	***Perform arithmetic operations with complex numbers***
Standards	1. Know there is a complex number *i* such as $i^2 = -1$, and every complex number has the form a +b*i* with a and b real. 2. Use the relation $i^2 = -1$ and the commutative, associative, and distributive properties to add, subtract, and multiply complex numbers. 3. (+) Find the conjugate of complex number; use conjugates to find moduli and quotients of complex numbers. (*This standard is suggested to be considered part of the next cluster*).

Content Elaborations (in development)

This section will provide additional clarification and examples to aid in the understanding of the standards. To support shared interpretations across states, content elaborations are being developed through multistate partnerships organized by CCSSO and other national organizations. This information will be included as it is developed.

Expectations for Learning (in development)

As the framework for the assessments, this section will be developed by the CCSS assessment consortia (SBAC and PARCC). Ohio is currently participating in both consortia and has input into the development of the frameworks. This information will be included as it is developed.

Instructional Strategies and Resources

Instructional Strategies

Before introducing complex numbers, revisit simpler examples demonstrating how number systems can be seen as "expanding" from other number systems in order to solve more equations. For example, the equation $x + 5 = 3$ has no solution as a whole numbers, but it has a solution $x = -2$ as an integers. Similarly, although $7x = 5$ has no solution in the integers, it has a solution $x = 5/7$ in the rational numbers. The linear equation $ax + b = c$, where a, b, and c are rational numbers, always has a solution x in the rational numbers: ——).

When moving to quadratic equations, once again some equations do not have solutions, creating a need for larger number systems. For example, $x^2 - 2 = 0$ has no solution in the rational numbers. But it has solutions √‾ in the real numbers. (The real number line augments the rational numbers, completing the line with the irrational numbers.)

Point out that solving the equation $x^2 - 2 = 0$ in terms of x is equivalent to finding x-intercepts of a graph of $y = x^2 - 2$, which crosses the x-axis at √‾) and √‾). Thus, the graph illustrates that the solutions are √‾.

Next, use an example of a quadratic equation with real coefficients, such as $x^2 + 1 = 0$, which can be written equivalently as $x^2 = -1$. Because the square of any real number is non-negative, it follows that $x^2 = -1$ has no solution in the real numbers. One can see this graphically by noticing that the graph of $y = x^2 + 1$ does not cross the x-axis.

The "solution" to this "impasse" is to introduce a new number, the imaginary unit *i*, where $i^2 = -1$, and to consider complex numbers of the form a +b*i*, where a and b are real numbers and *i* is not a real number. Because *i* is not a real number, expressions of the form a +b*i* cannot be simplified.

The existence of *i,* allows every quadratic equation to have two solutions of the form a + b*i* – either real when b = 0, or complex when b ≠ 0. Have students observe that if a quadratic equation (with real coefficients) has complex solutions, the solutions always appear in conjugate pairs, in the form a + b*i* and a – b*i.* Particularly, for an equation $x^2 = -9$, a conjugate pair of solutions are 0 +3*i* and 0 – 3*i*.

In order to find solutions of quadratic equations or to create quadratic equations from its solutions, introduce students to the condition of equality of complex numbers, with addition, subtraction and multiplication of complex numbers.

Stress the importance of the relationships between different number sets and their properties. The complex number system possesses the same basic properties as the real number system: that addition and multiplication are commutative and associative; the existence of additive identity and multiplicative identity; the existence of an additive inverse for every complex number and the existence of multiplicative inverse or reciprocal for every non-zero complex number; and the distributive property of multiplication over the addition. An awareness of the properties minimizes students' rote memorization and links the rules for manipulations with the complex number system to the rules for manipulations with binomials with real coefficients of the form a + bx.

Instructional Resources/Tools

Computer algebra systems

Graphing calculator

http://www.learner.org/courses/learningmath/number/session2/index.html

This session focuses on exploration of the number sets that make up the real number system and on the concept of infinity and the importance of zero.

http://www.clarku.edu/~djoyce/complex/numberi.html

This article is a short tour into the history of number *i*

http://www.math.hmc.edu/calculus/tutorials/complex/

Complex numbers: forms and operations

The Ohio Resource Center

The National Council of Teachers of Mathematics, Illuminations

Common Misconceptions

If irrational numbers are confused with non-real or complex numbers, remind students about the relationships between the sets of numbers.

If an imaginary unit *i* is misinterpreted as -1 instead of √, re-establish a definition of *i*.

Some properties of radicals that are true for real numbers are not true for complex numbers.

In particular, for positive real numbers a and b, √ √ √) but √ √ √)) and √/√ √(−) but √/√ √)/). If those properties are getting misused, provide students with an example,

such as √ √)) √ √ that leads to a contradiction that a positive real number is equal to a negative real number.

Diverse Learners

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Specific strategies for mathematics may include:

Advanced students should explore the reciprocals of complex numbers using conjugates.

Connections

The use of complex numbers is spread throughout mathematics and its applications to science, such as electrical engineering, physics, statistics and aeronautical engineering.

The existence of complex numbers makes every quadratic equation with real coefficients solvable over the complex number system. This paves the way for the Fundamental Theorem of Algebra, which says that an n^{th} degree polynomial has n solutions in the complex numbers.

Domain	**The Complex Number System**
Cluster	***Represent complex numbers and their operations on the complex plane***
Standards	4. Represent complex numbers on the complex plane in rectangular and polar form (including real and imaginary numbers), and explain why the rectangular and polar forms of a given complex number represent the same number. 5. Represent addition, subtraction, multiplication, and conjugation of complex numbers geometrically on the complex plane; use properties of this representation for computation. For example, $(1 - \sqrt{3}i)^3 = 8$ because $(1 - \sqrt{3}i)$ has modulus 2 and argument 120˚. 6. Calculate the distance between numbers in the complex plane as the modulus of the difference, and the midpoint of a segment as the average of the numbers at its endpoints.

Content Elaborations (in development)

This section will provide additional clarification and examples to aid in the understanding of the standards. To support shared interpretations across states, content elaborations are being developed through multistate partnerships organized by CCSSO and other national organizations. This information will be included as it is developed.

Expectations for Learning (in development)

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Instructional Strategies and Resources

Instructional Strategies

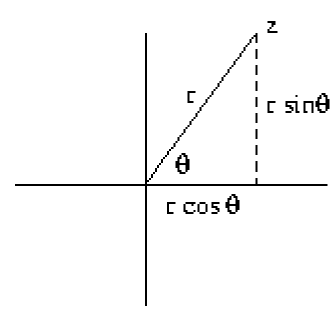
In this cluster, students are expected to understand the ways of representing complex numbers, relationships among representations of complex numbers, and the relationships between number systems. To reach these goals, teachers should emphasize that analogously to the one-to-one correspondence between every point on the real number line and a real number, there is a correspondence between every point on the rectangular coordinate plane and a complex number.

A complex plane is obtained by associating the ordered pairs of real numbers with points in a rectangular coordinate system. Students should see a correspondence between complex numbers written in rectangular form *(a +bi)* and a unique ordered pair of real numbers *(a, b),* where *a* represents a real number and *b* represents an imaginary number. They also need to view a pair of conjugate complex numbers (*a* - *bi*) and (*a* + *bi*) as a reflection of the point (a, b) to the point (a, -b) over the x- axis.

Similar to a vector, a complex number is characterized by length and direction. The length of a line segment connecting the origin and a coordinate point (*a, b*) in the coordinate complex plane is the modulus of the complex number (*a* + *bi*) and can be found using either the Pythagorean Theorem √) or by making use of conjugates:

√( ))) √)The results are equal and represent a modulus of either *a +bi* or *a – bi*. The angle between the line segment representing a complex number and the positive x-axis is called the argument of the complex number. The argument of (*a* + *bi*) is an angle whose tangent is b/a and can be found by applying an inverse tangent function to the ratio –

A complex number can be written in the alternative polar form, such that an arbitrary point P representing a complex number is associated with polar coordinates *(r, θ)*, where *r* is a modulus and *θ* is an argument. The coordinates of the complex numbers are transferable from one form to another.



Right triangle trigonometry is used to show that $a = r \cos\theta$ and $b = r \sin\theta$, so that a complex number $a + bi$ is $r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta)$. Since both cosine and sine are periodic functions with period 2π, $a + bi = r(\cos(\theta + 2\pi k) + i\sin(\theta + 2\pi k))$, where k is an integer and √).

The set of complex numbers is closed under addition, subtraction, multiplication and division. The conjugates are useful for finding the quotient of two complex numbers,

—— is also a complex number. The quotient of two quantities is the product of the dividend and the reciprocal of the divisor. The reciprocal of the divisor, —— is found by multiplying the numerator, 1, and the denominator *(c +di)* by the conjugate of the denominator or (*c - di*). The product of a complex number and its conjugate is always a nonnegative real number ($c^2 + d^2$). Dividing a complex number (*a + bi*)*(c - di)* by a real number ($c^2 + d^2$) results in a complex number.

Since a real number is also a complex number with $b = 0$, the argument of a real number is zero and a number itself can be placed in both the rectangular and polar coordinate planes along the x–axis or a polar axis.

Point out that, similar to a complex number in a coordinate plane associated with the ordered pair of real numbers, each geometric vector in the standard position on a coordinate plane is associated with the ordered pair of real numbers that are coordinates of its terminal point. Students need to realize that vectors and complex numbers as systems have some common properties and operations. For example, addition, subtraction or multiplication of complex numbers by a real number is performed by the same rules as addition, subtraction or multiplication of vectors by a real number. Multiplication of two complex numbers results in the cross product of two vectors.

Using a polar or an exponential form, a complex number can be raised to a whole number exponent. For example, to find (√) , students should raise the modulus √(√) to the third power ($2^3 = 8$) and multiply the argument 120˚ by three (120˚·3 = 360˚ or 0˚). The result is a complex number of the form 8 + 0i or a real number 8.

In the complex plane, a distance between two complex numbers, (*a, b*) and (*c, d*), is found either by the distance formula √))) or by the formula for the modulus of the difference between these complex numbers, √)))The midpoint of a line segment connecting two complex numbers, (*a, b*) and (*c, d*), is (—— ——) which represents the average of these complex numbers.

Instructional Resources/Tools

Dynamic geometric systems
Computer algebra systems
Graphing calculators

http://www.clarku.edu/~djoyce/complex/
This is an introduction to complex numbers that includes the mathematics and a little bit of history as well and is intended for a general audience. The necessary background in ordinary real numbers (all positive and negative numbers and zero), algebra and trigonometry is needed.

http://mathforum.org/library/drmath/sets/select/dm_demoivre.html
Powers of complex numbers; DeMoiver's Theorem

Common Misconceptions

Students may believe that complex numbers, as an area of study of mathematics, is completely isolated from other areas of study, such as vectors and matrices, and only has a connection to quadratic equations with negative discriminants. The variety of approaches to connect complex numbers, vectors and matrices can help students develop their understanding of important concepts of all three overlapping areas.

Diverse Learners

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Specific strategies for mathematics may include:
Some students may investigate the history of the so-called Argand Diagram that was invented almost simultaneously by the French mathematician Argand, the Norwegian surveyor Wessel, and the mathematical giant C. F. Gauss.

Connections:

Even though complex numbers, vectors and matrices are all independent areas of study and may be covered in any order, they overlap. Information from each area can be used to support the development of related concepts, skills and processes in the other areas of study.

A complex number can be viewed as a two-component number, somewhat like a two-component vector. For example, complex numbers and vectors overlap with the respect of vector representation of the additive structure of complex numbers in the complex plane (complex vectors). If two complex numbers are added, the resulting complex number is equivalent to the vector resulting from adding two vectors. If two complex numbers are multiplied, the resulting complex number is equivalent to the vector resulting from the cross product of the two vectors. Vectors and matrices have certain commonalities and can be compared by examining their laws. Matrices provide a natural representation of two-dimensional vectors when they are being considered in terms of translations of plane.

Matrices can be used to provide a model for complex numbers arising from consideration of certain transformations of the plane. Similarly, two-dimensional coordinate vectors in the complex plane provide a convenient geometric model for the additive structure of complex numbers.

Domain	**The Complex Number System**
Cluster	***Use complex numbers in polynomial identities and equations***
Standards	7. Solve quadratic equations with real coefficients that have complex solutions 8. (+) Extend polynomial identities to the complex numbers. For example, rewrite $x^2 + 4$ as $(x +2i)(x - 2i)$ 9. (+) Know the Fundamental Theorem of Algebra; show that it is true for quadratic polynomials.

Content Elaborations (in development)

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Expectations for Learning (in development)

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Instructional Strategies and Resources

Instructional Strategies

Revisit quadratic equations with real coefficients and a negative discriminant and point out that this type of equation has no real number solution. Emphasize that with the extension of the real number system to complex numbers any quadratic equation has a solution. Since the process of solving a quadratic equation may involve the use of the quadratic formula with a negative discriminant, defining a square root of a negative number becomes critical √ ‾‾ √‾, where *N* is a positive real number; *i* is the imaginary unit and $i^2 = -1$). After the square root of a negative number has been defined, emphasize that the quadratic formula can be used without restriction.

While solving quadratic equations using the quadratic formula, students should observe that the quadratic equation always has a pair of solutions regardless of the value of the discriminant. If the discriminant, $b^2 – 4ac$, is positive, the equation has two unequal complex solutions that are real (the imaginary parts of complex numbers are zeros). If the discriminant is zero, the equation has a repeated real solution – a double root (two complex solutions with equal real parts and the imaginary parts equal to zero). If the discriminant is negative, the equation has two conjugate complex solutions that are not real.

Information below contains additional mathematics that students should learn in order to take advanced courses such as calculus, advanced statistics or discrete mathematics and goes beyond the mathematics that all students should study in order to be college- and career-ready:

Since the set of real numbers is a subset of the set of complex numbers, finding the complex solutions of a polynomial function requires finding all solutions of the form $(a +bi)$, which include solutions that have $b = 0$, or real solutions. The Fundamental Theorem of Algebra states that every complex polynomial function $f(x)$ (a function with complex coefficients and a complex variable x) of degree $n \geq 1$ has at least one complex zero. Point out that according to the Fundamental Theorem of Algebra, any quadratic equation with real coefficients (a subset of complex numbers) always has two complex solutions and can be factored into a product of two linear factors. If a pair of conjugate complex solutions of the quadratic polynomial is $(a + bi)$ and $(a – bi)$, the quadratic polynomial can be restated as the product of two linear factors as $f(x) = (x – (a + bi))\cdot(x – (a – bi))$. For example, since the quadratic equation $x^2 + 4 = 0$ has two complex solutions $2i$ and $(- 2i)$, the polynomial $x^2 + 4$ can be restated as $(x – (0 + 2i))\cdot(x – (0 – 2i))$ or $(x +2i)(x - 2i)$ and then the sum of the squares of two quantities becomes factorable over the set of complex numbers.

Instructional Resources/Tools

Graphing calculator
Computer algebra systems
Dynamic geometric systems
Complex number plane - explanation, history and visual representations of complex roots

http://www.nctm.org/eresources/article_summary.asp?from=B&uri=MT2006-01-366a
This site provides a specific instance when the textbook answer for finding a root of a complex number differed with the answer given by the TI-83. The site contains an expansion of the definition for the integral root of a complex number to

an arbitrary complex power of a complex number.

Common Misconceptions

Students may believe that a quadratic equation with the discriminant $b^2 - 4ac = 0$ has only one solution. For example, $x^2 - 10x + 25$ has a discriminant of 0 ($10^2 - 4 \cdot 1 \cdot 25 = 0$) and the only solution is $x = 5$. Students should refer to the Fundamental Theorem of Algebra which states that any quadratic equation has two solutions including those where the discriminant equals zero. These equations have a repeated real solution or a double root.

In the cases of quadratic equations, when the use of quadratic formula is not critical, students sometime ignore the negative solutions. For example, for the equation $x^2 = 9$, students may mention 3 and forget about (– 3), or mention 3i and forget about (- 3i) for the equation $x^2 = -9$. If this misconception persists, advise students to solve this type of quadratic equation either by factoring or by the quadratic formula. It is also beneficial to remind students about the Fundamental Theorem of Algebra that secures the existence of two complex solutions for any quadratic equation.

Diverse Learners

Information and instructional strategies for gifted students, English Language Learners (ELL), and students with disabilities is available in the Introduction to Universal Design for Learning document located on the Revised Academic Content Standards and Model Curriculum Development Web page. Additional strategies and resources based on the Universal Design for Learning principles can be found at www.cast.org.

Connections: A solid understanding of number systems, including complex numbers, and the Fundamental Theorem of Algebra is foundational for advancing in solving various types of equations, investigating functions and sketching their graphs.

Domain	***Vector and Matrix Quantities***
Cluster	***Represent and model with vector quantities***
Standards	1. (+) Recognize vector quantities as having both magnitude and direction. Represent vector quantities by directed line segments, and use appropriate symbols for vectors and their magnitudes (e.g., **v, lvl, llvll, v**). 2. (+) Find the components of a vector by subtracting the coordinates of an initial point from the coordinates of a terminal point. 3. (+) Recommended to move to the next cluster

Content Elaborations (in development)

This section will provide additional clarification and examples to aid in the understanding of the standards. To support shared interpretations across states, content elaborations are being developed through multistate partnerships organized by CCSSO and other national organizations. This information will be included as it is developed.

Expectations for Learning (in development)

As the framework for the assessments, this section will be developed by the CCSS assessment consortia (SBAC and PARCC). Ohio is currently participating in both consortia and has input into the development of the frameworks. This information will be included as it is developed.

Instructional Strategies and Resources

Instructional Strategies

Information below contains additional mathematics that students should learn in order to take advanced courses such as calculus, advanced statistics or discrete mathematics and goes beyond the mathematics that all students should study in order to be college- and career-ready:

Provide applied contextual examples to explore vectors. For example, in physics, a few vector quantities that are defined by both magnitude and direction are displacement, acceleration, velocity, angular velocity, force and momentum. They are typically represented by directed line segments (arrows). A specification of a vector quantity, denoted **v,** requires two numbers, one for the magnitude or amount, denoted **lvl** or **llvll**, and another for the direction.

Help students to recognize the difference between velocity and speed, and weight and mass. In physics, velocity is a vector quantity because it includes a direction, while the corresponding speed (the magnitude of the velocity) is a scalar quantity. Mass is also a scalar quantity and weight, the force of gravity exerted on the object, is a vector quantity.

Consider a weather report. The wind speed is 10 miles per hour, which is scalar, not a vector quantity. However, just reporting the wind speed does not give all the information about the wind. A more complete wind report might say that the wind is blowing 10 miles per hour towards the southwest. This wind velocity includes both the magnitude (10 miles per hour) and the direction (towards the southwest), so it is a vector quantity.

The use of graph paper will help students understand geometric and algebraic interpretations of vectors. On a coordinate plane, a vector is determined by the coordinates of its initial point (coordinate pair (*a, b*)) and a terminal point (coordinate pair (*c*, *d*)) or by its *x*- and *y*-components. A vector is often denoted as **‹c – a, d - b›**, where (*c* – *a*) is an *x*- component and (*d* – *b*) is a y-component. The magnitude of vector **v** is the length of the arrow and is found be the distance formula **lvl** √))).

Instructional Resources/Tools

Graph paper

Dynamic geometry software

Computer algebra systems

The Ohio Resource Center

The National Council of Teachers of Mathematics, Illuminations

http://www.euclideanspace.com/maths/algebra/vectors/index.htm

This article focuses on connections between vectors, matrices and complex numbers

Common Misconceptions

One of the most common misconceptions is equating vectors with line segments that lack directions. Students should be able to view a vector as the displacement of a plane (or a space) in a certain direction to a certain distance. A clarification of the fact that a vector **v** represents an equivalence class of distinct arrows. Arrows, pointing in the same direction and having the same length, all represented by a vector **‹a, b›** is beneficial for better understanding of vectors.

Some students associate the coordinates of the vector's initial point with the origin (0, 0), even though the initial point is elsewhere. As a result, a vector is always denoted by the coordinates of its terminal point **‹c, d›.** Using coordinate geometry will help clarify this misconception.

Diverse Learners

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Connections:

Initially, a basic understanding of vectors comes from distinguishing between scalar and vector quantities. The movement from an original position to a final position of a translation represents a vector. Vectors are introduced geometrically in a plane and then algebraically, which makes them applicable in many areas of science, engineering and applied mathematics. Later, the concept of vector merges with the concept of complex numbers, matrices and translations.

Domain	**Vector and Matrix Quantities**
Cluster	***Perform operations on vectors***
Standards	3. (+) Solve problems involving velocity and other quantities that can be represented by vectors (Moved from N-VM cluster 1) 4. (+) Add and subtract vectors. a. Add vectors end – to – end, component wise and by the parallelogram rule. Understand that the magnitude of a sum of two vectors is typically not the sum of the magnitudes. b. Given two vectors in magnitude and direction form, determine the magnitude and direction of their sum. c. Understand vector subtraction v – w as v + (-w) is the additive inverse of w, with the same magnitude as w and pointing in the opposite direction. Represent vector subtraction graphically by connecting the tips in the appropriate order, and perform vector subtraction component - wise. 5. (+) Multiply a vector by a scalar a. Represent scalar multiplication graphically by scaling vectors and possibly reversing their directions; perform scalar multiplication component – wise, e.g., as $c(v_x, v_y) = (cv_x, cv_y)$. b. Compute the magnitude of a scalar multiple c***v*** using IIc***v***II = IcI***v***. Compute the direction of c***v*** knowing that when IcI***v*** ≠ o, the direction of c***v*** is either along ***v*** (for c›0) or against ***v*** (for c‹0)

Content Elaborations (in development)

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Instructional Strategies and Resources

Instructional Strategies

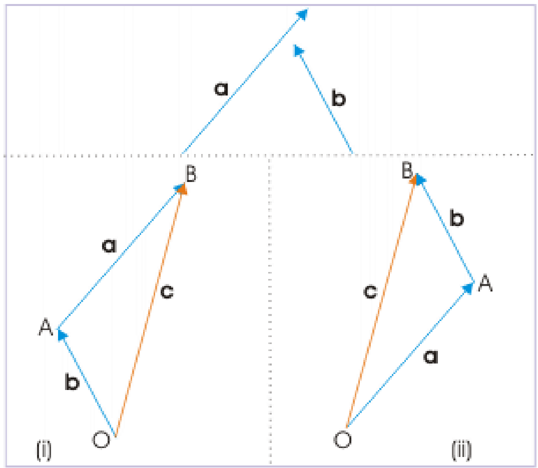
Information below contains additional mathematics that students should learn in order to take advanced courses such as calculus, advanced statistics or discrete mathematics and goes beyond the mathematics that all students should study in order to be college- and career-ready:

Point out that a vector (or a displacement) of a moving object is its final position relative to its starting position. Support this idea by example such as leaving from home, going to the store, stopping at the library, and then finally arriving to the friends' house 2 blocks away from the starting point. The whole trip would yield to a total displacement of only 2 blocks, no matter what route was taken to get there or how many miles were walked. Therefore, the sum of displacements can be interpreted as a sequence of several successive walks.

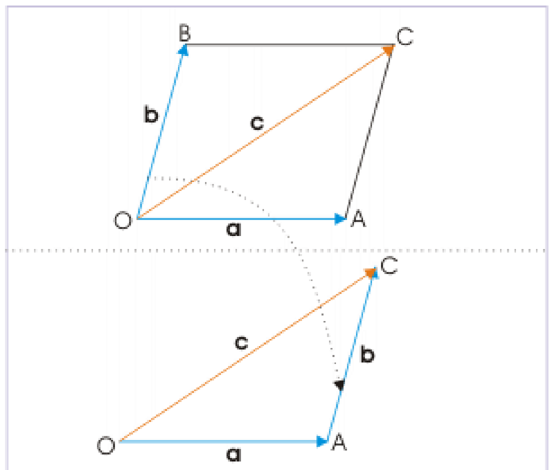
Illustrate this by showing a chain of geometric vectors (line segments with an arrowhead that have a magnitude and direction) with the vector sum being the vector distance from the beginning to the end point. The magnitude of the vector sum does not necessarily result in the sum of the magnitudes of the displacements, unless the displacements, represented by vectors, are parallel and have the same directions.

Just as there is arithmetic of numbers, there is arithmetic of vectors. There are two techniques available for vector addition: geometric and algebraic addition. Both yield the same result. Prior to introducing students to geometric addition of vectors, remind them that because a vector is defined by its magnitude and direction, changing its location without changing its direction or magnitude leaves it the same vector. The actual positions of the vector's head and tail do not matter, only their positions relative to each other. Different arrows represent the same vector if they have the same direction and the same length.

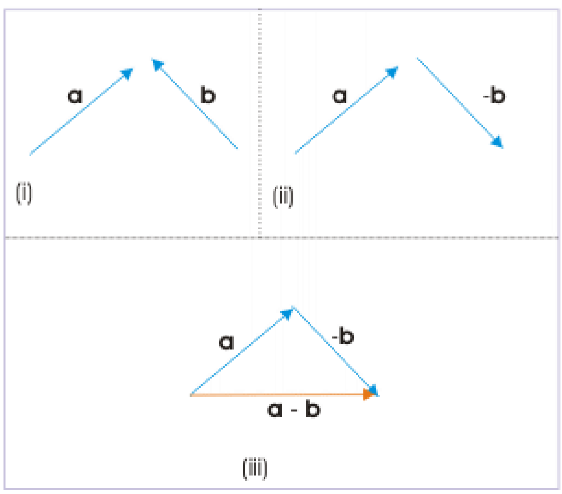
To add vectors geometrically, introduce students to the Triangle and Parallelogram Rules. If two displacements are to be combined, the obvious strategy is to place the tail of one vector at the head of the other vector. This way vectors represent two sides of a triangle (the Triangle Rule). The third closing side of the triangle, a vector from the tail of the first vector to the head of the last, represents the sum or resultant of the two vectors in both magnitude and direction.



The Parallelogram Rule is an alternate statement of Triangle Rule of vector addition. If two vectors are represented by two adjacent sides of a parallelogram, then the diagonal of parallelogram through the common point represents the sum of the two vectors in both magnitude and direction.



To subtract two vectors, define the difference of vectors ***a*** and ***b*** as ***a*** – ***b*** = ***a*** + ***(- b),*** where (***-b)*** is the vector having the same magnitude as ***b***, but whose direction is opposite to ***b***. Illustrate the process of subtracting vector, ***b***, from, ***a***, by reversing the direction of vector, ***b***, and by obtaining the vector sum using the Triangle or Parallelogram Rules.

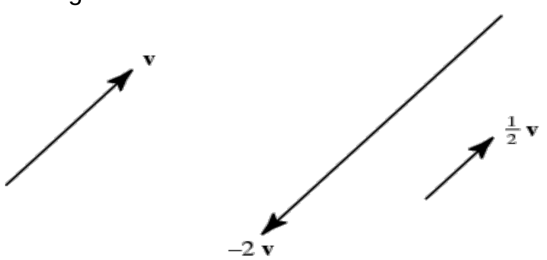


An algebraic vector is an ordered pair of real numbers **‹*x, y*›**, where ***x*** is the difference between the first coordinates of the terminal and the initial point of the vector ***a****,* and ***y*** is the difference between the second coordinates of the terminal and the initial point of the vector ***a***. The real numbers ***x*** and ***y*** are scalar components of the vector **‹*x, y*›.** To add (or subtract) algebraic vectors a = **‹x_1, y_1›** and b = **‹x_2, y_2›**, add (or subtract) the corresponding components, a ± b = **‹x_1 ± x_2; y_1 ± y_2›**.This definition of addition (or subtraction) of algebraic vectors is consistent with the Parallelogram and Triangle Rule for addition (or subtraction) of geometric vectors.

An algebraic technique for finding the sum of two vectors requires finding the magnitude and the direction of the vector resultant and involves finding the components of vectors ***a*** and ***b*** by associating the given vectors with the two axis of rectangular coordinate system (a resolution of a vector in two components). It is also practical to introduce students to unit vectors ***i*** = ‹1, 0› and ***j*** **=** ‹0, 1› and demonstrate that any vector a = **‹*x, y*›** can be expressed as a linear coordinates of those two vectors, that is ***xi*** + ***bj.***

Calculating the components of each vector involves forming a right triangle from each vector and using standard right triangle trigonometry. Calculating the correspondent components of the vector resultant involves adding x-components (a_x = a·cosθ; b_x = b·cosθ) of the given vectors and adding y-components (a_y = a·sinθ; b_y = b·sinθ) of the given vectors. The magnitude of the vector resultant is found by using the Pythagorean Theorem and taking the square root of the sum of the squares of the x- and the y-components√( ) ( )). The direction of the vector resultant involves finding the tangent of the angle between the vector resultant and the x-axis using standard right triangle trigonometry.

Two common mathematical manipulations involving scalars and vectors are scalar multiplication and vector multiplication. Scalar multiplication refers to the multiplication of a vector by a constant (a scalar) to give another vector with different magnitude. This is similar to multiplying a number by a scale factor to increase or decrease its value in proportion to its original value. The geometric representation of scalar multiplication of vectors can be interpreted as scaling.



If ***v*** is a vector and if c is a positive number, then ***cv*** is a different vector whose direction is that of ***v*** and whose length is c**|*v*|**. It should be noted that a negative value for c will result in a vector with the opposite direction of ***v***. When a vector is multiplied by a scalar it can be made larger or smaller, or its direction can be reversed, but the angle of its direction relative to another vector will not change.

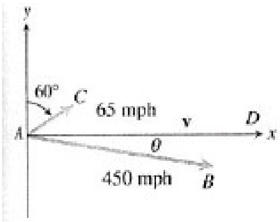
The algebraic definition of scalar multiplication of vector ***v*** = ‹v_x, v_y› by a scalar c is the product of each component of vector ***v*** by a scalar c or c***v*** = ‹cv_x, cv_y›. If the vector ***v*** is in the component form ***v*** = v_x***i*** + v_y***j*,** its product by a scalar c is c***v*** = (cv_x)***i*** + (cv_y)***j***

Similar to calculating the magnitude of vector ***v*** = ‹v_x, v_y› as IIvII √( ), the magnitude of vector c***v*** is calculated as IIc***v***II √) ( ) √ ( ) √( ) cII***v***II. The direction of vector ***v*** is represented by the angle θ in standard position and found as θ = $\tan^{-1}$(Iv $_y$/ v $_x$ I).The direction of vector c***v*** is either the same or opposite to the direction of vector ***v*** depending if scalar c is positive or negative.

There are two very important groups of vector applications extensively used in physics, engineering and applied mathematics. These are navigation applications and force applications involving physical quantities, such as velocity and forces, and may be conveniently represented by vectors. To solve these applications, it may be necessary to: draw pictures representing geometric operations of vectors; resolve vectors in components along the x- and y-axis; use right triangle trigonometry; and solve equations or system of equations for the indicated variable(s).

Example 1

A pilot plans to leave an airport and fly due east. There is a 65 mph wind with the bearing 60˚. Find the compass heading a pilot should follow, and determine airplane's ground speed, assuming that its speed with no wind is 450 mph.



In the picture, vector represents the velocity produced by the airplane alone, vector represents the velocity of the wind, and θ is angle DAB. Vector v = represents the resulting velocity.

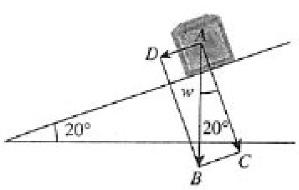
v = = +

By resolving the vectors in components, ‹65cos30˚, 65sin30˚› and = ‹450cosθ˚, 450sinθ˚›. Therefore, vector = ‹65cos30˚ + 450cosθ, 65sin30˚ + 450sinθ›. Because the plane travels due east, the second component of must be zero, or 65sin30˚ + 450sinθ = 0 and θ = $\sin^{-1}$ (———) ≈ -4.14˚. Thus the compass heading McCarty should follow is 90˚ + IθI ≈ 94.14˚.

The ground speed of the plane is IvI = √) = I I ≈ 505.12 mph. Different forces may have different effects on an ~~object. The next application ill~~ustrates the effect of forces acting on a block that rests on an incline plane. Students should take in consideration the force that pushes the block down the incline plane, the weight of the object and the force that presses the block against the incline plane, at the right angle to the incline.

Example 2

A force of 30 pounds just keeps the box from sliding down the ramp inclined at 20˚. Find the weight of the box.



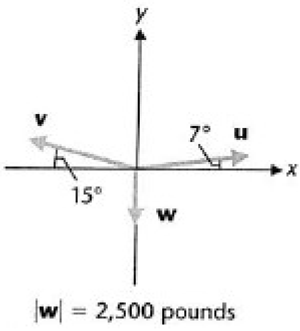
A force I I = 30 pounds. A vector represents the weight w. Since, sin 20˚ = —, the weight w = —— ≈ 87.71pounds.

Algebraic vectors can be used to solve applications involving static equilibrium. According to the principles regarding forces and objects subject to these forces, a coordinate system remains in static equilibrium, or at rest, if the vector sum of all the force vectors acting on the object is zero.

Example 3

A cable car used to ferry people and supplies across a river weighs 2, 500 pounds fully loaded. The car stops when part-way across and deflects the cable relative to the horizontal, as indicated in the picture below. What is the tension in each part of the cable running to each tower?

Consider a force diagram with all force vectors in standard position at the origin. The objective is to find I***u***I and I***v***I.



Each force vector can be written in terms of the ***i*** and ***j*** unit vectors as ***u*** = l***u***l(cos 7˚)***i*** + l***u***l(sin 7˚)***j*;** ***v*** = l***v***l(-cos 15˚)***i*** + l***v***l(sin 15˚) and ***w*** = - 2500***j.***

For the system to be in static equilibrium, the sum of the force vectors must be the zero vector. That is ***u*** + ***v*** + ***w*** = ***0.***

The same is represented as [l***u***l(cos 7˚)***i*** + l***u***l(sin 7˚)***j***] + [l***v***l(-cos 15˚)***i*** + l***v***l(sin 15˚)***j***] – 2,500j = 0***i*** +0***j***
[l***u***l(cos 7˚) + l***v***l(-cos 15˚)] ***i*** + [l***u***l(sin 7˚) + l***v***l(sin 15˚) – 2,500] ***j*** = 0***i*** +0***j***

Since two vectors are equal if and only if their corresponding components are equal, the following system of equations is true:

l***u***l(cos 7˚) + l***v***l(-cos 15˚) = 0
l***u***l(sin 7˚) + l***v***l(sin 15˚) – 2,500 = 0.

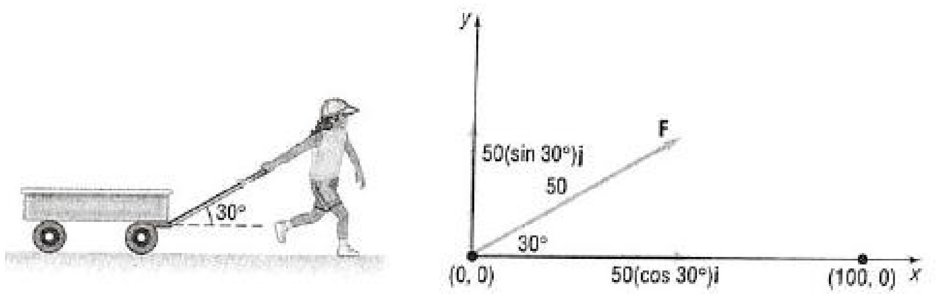
Solving this system by standard methods, l***u***l = 6,400 pounds and l***v***l = 6,600 pounds.

If a constant force F moves an object from a point A to a point B, then there is a work W being done by this force that equals to a product of magnitude of force F and a distance AB.

Example 4

The picture below shows a girl pulling a wagon with a force of 50 pounds. How much work is done in moving the wagon 100 feet if the handle makes an angle of 30˚ with the ground?

Position vectors in a coordinate system in such a way that the wagon is moved from (0, 0) to (100, 0).



Since the motion is from point A (0, 0) to point B (100, 0), then ***AB*** = 100***i.*** The vector force **F,** as shown above is

F = 50(cos 30˚***i*** + sin30˚***j***) = 50(√ –) = 25 √ **.** Since W = **F•** ***AB***, the work is W = (25√)• 100***i*** or W = 2500√ ft • lb

<u>Instructional Resources/Tools</u>

Graphing calculator
Computer algebra software
Dynamic geometry software
The Ohio Resource Center

The National Council of Teachers of Mathematics, Illuminations

http://www.teachers.ash.org.au/mikemath/mathsc/vectorgeometric/notes.pdf
This article concentrates on two- and three-dimensional geometric vectors, arithmetic of vector forces, and geometry methods for finding the result of forces acting at a point.

Common Misconceptions

Some students may interpret the coordinates of the algebraic vector a = **‹*x, y*›** as the coordinates of its terminal point. However, the coordinates of algebraic vector ***a*** is the difference between the x– and the y–coordinates of the terminal and the initial point of algebraic vector ***a*.** To overcome this misconception teachers should highlight the difference between algebraic vector (or vector), position vector (or vector in standard position) and a unit vector by offering clear definitions, contrasting distinctions, providing similarities, and presenting a sufficient number of applications of vectors.

Another common misconception is ignoring the possibility of an occurrence of the vector resultant in any quadrant of the rectangular system. Students need to pay attention to the signs of the sums of the x– and the y–components of the vector resultant when identifying a position of the vector resultant, and use the notion of the reference angle when finding the actual direction.

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Connections

Vectors and complex numbers as systems have a significant commonality. A two-component vector can be viewed as a complex number with two components – real and imaginary. The rules for addition and subtraction of vectors are the same as the rules for addition and subtraction of complex numbers. If two vectors are added, the vector resultant is equivalent to the complex number resulting from adding two complex numbers. In the complex plane, the magnitude of a vector a = **‹*x, y*›** is found by the Pythagorean Theorem which is similar to finding a magnitude of a complex number. The argument or the direction angle of a vector can be found using the inverse tangent function. Converting a complex number from rectangular to polar form replicates a resolution of a vector in two components.

Domain	**Vector and Matrix Quantities**
Cluster	***Perform operations on matrices and use matrices in applications***
Standards	6. (+) Use matrices to represent and manipulate data, e.g., to represent payoffs or incidence relationships in a network. 7. (+) Multiply matrices by scalars to produce new matrices, e.g., as when all of the payoffs in a game are doubled. 8. (+) Add, subtract, and multiply matrices of appropriate dimensions. 9. (+) Understand that, unlike multiplications of numbers, matrix multiplication for square matrices is not a commutative operation, but still satisfies the associative and distributive properties. 10. (+) Understand that the zero and identity matrices play a role in matrix addition and multiplication similar to the role of 0 and 1 in the real numbers. The determinant of a square matrix is nonzero if and only if the matrix has a multiplicative inverse. 11. (+) Multiply a vector (regarded as a matrix with one column) by matrix suitable dimensions to produce another vector. Work with matrices as transformations of vectors. 12. (+) Work with 2x2 matrices as transformations of the plane, and interpret the absolute value of the determinant in terms of area.

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Instructional Strategies and Resources

Instructional Strategies

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A common way to organize and manage data is to place it in a rectangular configuration called a matrix (spreadsheet). Present students with a variety of contextual problems that require storing information from the application into a matrix. Explain to students that care should be taken when translating the context of the problem in the matrix as an information organizer and then translating the result of matrix manipulations to the context of the problem. Direct them to write column labels above the matrix and row labels to its left to indicate the meaning of the data. Units of measure usually do not appear in matrices, but are established by the context in which the matrices are used.

Point out the existence of more than one type of matrix multiplication. Scalar multiplication refers to the multiplication of a matrix by a constant (a scalar) to produce another matrix of the same size. This is similar to multiplying a number by a scale factor to increase or decrease its value in proportion to its original value. For example, a consumer advocacy group has computed the retail prices for brand-name products and generic products at different stores in a major city. The prices are shown in the 3x2 matrix.

Brand Generic

[ ]

The city has a combined sales tax of 7.25%. A matrix showing the tax on each type of product at each store is

Brand Generic *Brand Generic*

0.0725 x [ ] = [ ]

The scalar multiplication is performed by multiplying each element of a matrix by the same constant.

To help students understand the operations with matrices, review properties of real number operations (commutative, associative, identity and inverse for addition and multiplication) prior to introducing operations with matrices. Matrix addition (subtraction) and multiplication are similar to real number addition (subtraction) and multiplication in many instances, but there are some important differences.

Begin with defining equality of matrices and emphasize the importance of the same size. Two matrices are equal if they have the same size and their corresponding elements are equal. The sum of two matrices of the same size is a matrix with elements that are the sums of the corresponding elements of the two given matrices. Addition is not defined for matrices of different sizes. Because two matrices are added by adding their corresponding elements, it follows from the properties of real numbers that matrices of the same size are commutative and associative relative to addition. When matrices are used in a context, they should not be added or subtracted unless their labels match. Provide students with contextual applications to demonstrate the proper use of matrix addition (subtraction).

A matrix with elements that are all 0's is a zero matrix and has properties similar to the real number zero. The zero matrixes play the role of the additive identity in matrix algebra.

In preparation for matrix multiplication and for understanding the connection between matrices with vectors, students should be introduced to the definition of special matrices - a row vector R = [ ] and column vector

C =[ ]. The product of R times C is defined as the number RC = [ ]•[ ] = $r_1c_1 + r_2 c_2 + \ldots + r_n c_n$. Notice

that a row and a column vector can be multiplied only if they contain the same number of entries. Thus, the definition for multiplying two matrices is based on the definition of a row vector and a column vector: if A = $[a_{ij}]$ is a m x r matrix and B = $[b_{ij}]$ *is* a r x n matrix, the product A•B = $[c_{ij}]$ is the m x n matrix where
$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{ir}b_{rj}$.

It is possible for both products A•B and B•A to be defined, yet both products are nearly always unequal. Provide students with a series of examples to demonstrate that even though *A* and *B* are both *n* x *n* matrices, and both products *A•B* and *B•A* are defined, the commutative property of multiplication is not shared by matrices, so matrix multiplication is not commutative, $A \cdot B \neq B \cdot A$. Also, *A•B* may be zero with neither *A* or *B* equal to zero. Thus, the zero property does not hold for matrix multiplication.

Next, demonstrate two of the properties of real numbers that are shared by matrices. Assuming each product and sum is defined, the associative property, $A \cdot (B \cdot C) = (A \cdot B) \cdot C$, and the distributive property, $A \cdot (B+C) = A \cdot B + A \cdot C$, are shared by matrices.

Help students explore additional dissimilarities between real number multiplication and matrix multiplication by offering them the following problems:

1. In real number multiplication, the only real number whose square is 0 is the real number 0 ($0^2 = 0$). In matrices, there is at least one 2 x 2 matrix *A* with all elements nonzero such that $A^2 = 0$, where 0 is not the 2 x 2 zero matrix. Find this matrix.
2. In real number multiplication, the only nonzero real number that is equal to its square is the real number 1 ($1^2 = 1$), there is at least one 2 x 2 matrix *A* with all elements nonzero such that $A^2 = A$. Find this matrix.

Offer students contextual matrix applications so that they explore possibilities not only to perform the assigned operations on matrices but also to choose the appropriate operations and interpret the results.

Students should discover the existence of the identity and inverse properties for matrices. Revisit identity and inverse properties over addition and multiplication for real numbers before introducing students to the corresponding properties for matrices. Next, provide formal definitions of an identity matrix (an *n* x *n* square matrix whose diagonal entries, the entries located in row *i* and column *i*, are all 1's, while all other entries are 0's) and the matrix inverse A^{-1} (an n x *n* square matrix such that $A \cdot A^{-1} = A^{-1} \cdot A = I_n$). The identity matrix (the square matrix only) has properties analogous to those of the real number 1, i.e., $I_m \cdot A = A$ and $A \cdot I_n = A$, if *A* is *m* x *n* matrix.

The inverse matrix has a property analogous to the inverse property for multiplication of real numbers, i.e., $A \cdot A^{-1} = A^{-1} \cdot A = I_n$. However, not every matrix has an inverse. The inverses exist only for those matrices whose determinants are not zero.

A determinant of a square matrix is a unique number associated with any square matrix. Determinants have several important roles in operations with matrices. The determinant of a 2 x 2 square matrix is found by the formula det A = | | = – *bc*. Calculating determinants of higher order square matrices is more complicated and time consuming. The use of a graphing utility can be very helpful.

If the determinant of a square matrix is a non-zero number, then the square matrix has an inverse (see the formula for the inverse A^{-1} = —— • [ ] and notice that appears in denominator of the fraction). If a square matrix has an inverse that can be found using a graphing utility or by row-reduction procedures, then the determinant of the square matrix is the non-zero number.

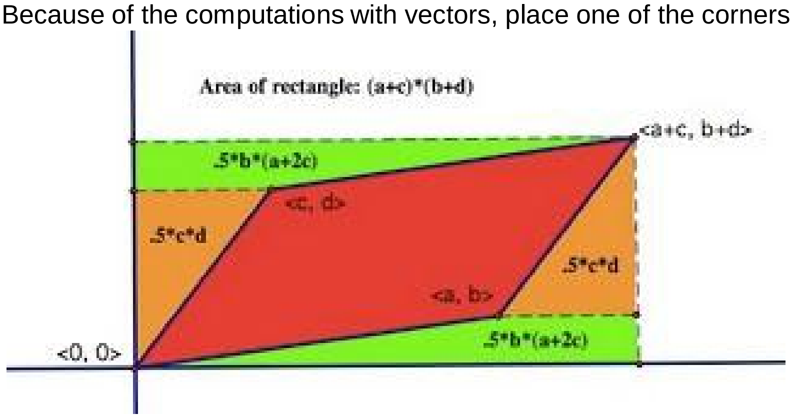
Matrices are often used to represent position vectors. The term vector is often applied to one-dimensional matrices. For example, the vector 2***i*** + 3***j*** + 5***k*** can be represented by either the 1x3 matrix [2, 3, 5] or the 3x1 matrix[ ]. Matrix multiplication can be used to perform linear transformations of the xy-plane that reflect the vector through the x-axis, the y-axis, the lines y = or the origin, to rotate the vector about the origin and to stretch it.

Because the reflection about the x-axis negates only the y-coordinates of the points, the matrix associated with this reflection is M_x = [ ] Since the reflection about the y – axis leaves the y coordinates invariant and negates the x-coordinates, the matrix associated with reflection over the y-axis is M_y = [ ] A reflection about the line y = x swaps x and y coordinates, thus the matrix associated with this reflection is M_{xy} = [ ] Finally, a reflection about line y = -x swaps x an –y, therefore, the matrix is M_{-xy} = [ ] Point out that each of these four reflections has determinant -1. A counterclockwise rotation in the xy – plane by the angle is also a linear transformation with a transformation matrix which is [ ].

A stretch in the xy-plane is a linear transformation which enlarges all distances in a particular direction by a constant factor k and does not change distances in the perpendicular direction. The matrix associated with a stretch by a factor k along the x-axis is T_x[k] = [ ] The matrix associated with a stretch by a factor k along the y-axis is T_y[k] = [ ]

Help students discover the meaning of a determinant using the formulas for the area of a parallelogram or triangle. Point out that the major disadvantage of the formulas for the area of a parallelogram (A = bh) and the area of a triangle (A = – bh) is that the formulas rely on the length of the height that is not truly a "part" of the shape and often needs to be determined. The formulas to be developed would use the lengths of their sides and utilize determinants to find the areas of the shapes, assuming that the vectors represent the sides and vertices have been given coordinates.

Because of the computations with vectors, place one of the corners of the parallelogram at the origin, as shown below.



The area of the parallelogram is computed by finding the area of the rectangle and then subtracting the area of the green trapezoids and orange triangles.

(*a* + *c*)(*b* + *d)* – 2[.5(*a* + 2*c*) + .5*cd*] = *ab* + *ad* + *cb* + *cd* – *b*(*a* + 2*c*) – *cd* = *ab* + *ad* + *cb* + *cd* – *ab* – *2cb* – *cd* =

ad + cb – 2cb + ab – ab + cd – cd = ad – cb = | |

This shows that the area of the parallelogram can be worked out from the vectors that define the parallelogram. Because triangles are half the size of the parallelogram we can compute the area of triangles in a similar manner.

$A_{parallelogram}$ = | |

$A_{triangle}$ = – | |

For example, the vector AB is <2, 6> and the vector AD is <8, 4>. Using these vectors the area is calculated as

A_{ABC} = | | = 8•6 – 2•4 = 40

Instructional Resources/Tools

Graphing calculators with CAS or at least matrix operations.
Spreadsheet software

http://www.facstaff.bucknell.edu/mastascu/elessonsHTML/Circuit/MatVecMultiply.htm
This lesson demonstrates how multiplying matrices and vectors can be used to solve a circuit related application.

http://www.kwon3d.com/theory/vect/vectmat.html
This articles views vectors as column matrices and discuses scalar and vector products.

http://www.euclideanspace.com/maths/algebra/matrix/index.htm
This article focuses on connections between matrices, vectors and complex numbers.
From the National Council of Teachers of Mathematics, Illuminations: Computer Animation - In this lesson, students transform images through rotation, reflection, dilation, and translation using matrix multiplication. After digitizing images by representing the images as matrices, they multiply image matrices by various transformation matrices, producing transformed images.

Common Misconceptions

A side effect of treating matrices as an isolated area of study with many symbol manipulation rules similar to operations with real numbers is that students may overestimate the similarity between operations with matrices and operations with real numbers. For example, some students may believe that: matrix multiplication is commutative; a product of matrices with the same sizes is a product of their corresponding elements; the identity matrix has all elements that are ones; and the inverse matrices have all entrees that are reciprocals of the elements of the original matrix. Teachers can remedy these misconceptions by offering students a wide range of applications that show connections between matrices, transformation, vectors, and systems of linear equations.

Diverse Learners

Information and instructional strategies for gifted students, English Language Learners (ELL), and students with disabilities is available in the Introduction to Universal Design for Learning document located on the Revised Academic Content Standards and Model Curriculum Development Web page. Additional strategies and resources based on the Universal Design for Learning principles can be found at www.cast.org.

Specific strategies for mathematics may include:
The use of technology is beneficial when dealing with matrices. Even though students need to understand and be fluent with operations with matrices by hand, they should verify results of matrix operations using graphing utility.

Connections:

Mathematics has developed the theory of matrices that helps manipulate growing information. Computational technology and computer programs can store and organize a large amount of information, often using matrix notation. Spreadsheets can be used by businesses in areas such as budgeting, sales projections and cost estimation. Scientists can use spreadsheets to analyze the results of an experiment. Teachers can use spreadsheets to record and average grades.

Linear algebra, as a branch of mathematics, deals with matrices in such a way that addition and multiplication of matrices, identity matrix, inverse matrix and matrix row operations are permitted. Matrices are connected to linear transformations and to the process of solving matrix equations that provides an alternative method of solving linear systems with the same number of variables as equations. Inverse matrices and determinants are used for solving systems of equations where the number of variables is the same as the number of equations.