Ohio's Model Curriculum | Mathematics

with Instructional Supports

Algebra 1 Course

Ohio | Department of Education

Mathematics Model Curriculum
with Instructional Supports
Algebra 1 Course

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Introduction

PURPOSE OF THE MODEL CURRICULUM

Just as the standards are required by Ohio Revised Code, so is the development of the model curriculum for those standards. Throughout the development of the standards (2016-17) and the model curriculum (2017-18), the Ohio Department of Education (ODE) has involved educators from around the state at all levels, Pre-K–16. The model curriculum reflects best practices and the expertise of Ohio educators, but it is not a complete a curriculum nor is it mandated for use. The purpose of Ohio's model curriculum is to provide clarity to the standards, a foundation for aligned assessments, and guidelines to assist educators in implementing the standards. The model curriculum is not a collection of lessons nor a full curriculum; it does not suggest pace, sequence, or amount of time spent on topics. It provides information about a topic related to the standards including ideas for examples, strategies for teaching, and possible connections between topics in addition to highlighting some misconceptions.

COMPONENTS OF THE MODEL CURRICULUM

The model curriculum contains two sections: Expectations for Learning and Content Elaborations.

Expectations for Learning: This section begins with an introductory paragraph describing the cluster's position in the respective learning progression, including previous learning and future learning. Following are three subsections: Essential Understandings, Mathematical Thinking, and Instructional Focus.

- **Essential Understandings** are the important concepts students should develop. When students have internalized these conceptual understandings, application and transfer of learning results.
- **Mathematical Thinking** statements describe the mental processes and practices important to the cluster.
- **Instructional Focus** statements are key skills and procedures students should know and demonstrate.

Together these three subsections guide the choice of lessons and formative assessments and ultimately set the parameters for aligned state assessments.

Content Elaborations: This section provides further clarification of the standards, links the critical areas of focus, and connects related standards within a grade or course.

Introduction, continued

COMPONENTS OF INSTRUCTIONAL SUPPORTS

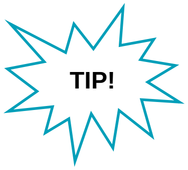
The Instructional Supports section contains the **Instructional Strategies** and **Instructional Tools/Resources** sections which are designed to be fluid and improving over time, through additional research and input from the field. The **Instructional Strategies** are descriptions of effective and promising strategies for engaging students in observation, exploration, and problem solving targeted to the concepts and skills in the cluster of standards. Descriptions of common misconceptions as well as strategies for avoiding or overcoming them and ideas for adapting instructions to meet the needs of all students are threaded throughout. The **Instruction Tools/Resources** are links to relevant research, tools, and technology.

There are several icons that help identify various tips in the Instructional Strategies section:

 = a common misconception

 = a technology tip

 = a career connection



= a general tip which may include diverse learner or English learner tips.

Standards for Mathematical Practice—Algebra 1

The Standards for Mathematical Practice describe the skills that mathematics educators should seek to develop in their students. These practices rest on important "processes and proficiencies" with long-standing importance in mathematics education. The descriptions of the mathematical practices in this document provide **examples** related to the Algebra 1 course of how student "processes and proficiencies" will change and grow as students engage with and master new and more advanced mathematical ideas across the grade levels. For a more detailed description of the Standards of Mathematical Practice see page 4 of Ohio's Learning Standards for Algebra 1. These examples just highlight a few specific concepts where the practices may be applied. These examples in no way encompass all the applications of the mathematical practices involved in the course.

MP.1 Make sense of problems and persevere in solving them.
Students learn that patience and concentration is often required to fully understand what a problem is asking. They discern between useful and extraneous information. They expand their repertoire of expressions and functions that can be used to solve problems.

MP.2 Reason abstractly and quantitatively.
Quantitative reasoning entails habits of creating a coherent representation of the problem at hand; of considering the units involved; of attending to the meaning of quantities, not just how to compute them; and of knowing and flexibly using different properties of operations.

MP.3 Construct viable arguments and critique the reasoning of others.
Students reason through mathematical procedures and concepts such as the solving of equations, recognizing that it involves more than simply following rote rules and steps. They may use language such as "If ___, then ___" when explaining steps in their thinking and provide justification for their reasoning.

MP.4 Model with mathematics.
Students also discover mathematics through experimentation and by examining data patterns from real-world contexts. They apply their new mathematical understanding of exponential, linear, and quadratic functions to real-world problems.

MP.5 Use appropriate tools strategically.
Students develop a general understanding of the graph of an equation or function as a representation of that object, and they use tools such as graphing calculators or graphing software to create graphs in more complex examples, understanding how to interpret results. They construct diagrams to solve problems.

Continued on next page

Standards for Mathematical Practice, continued

MP.6 Attend to precision.

Students use clear and precise definitions in discussion with others and in their own reasoning. They understand the meaning of the symbols they choose, including using the equal sign consistently and appropriately. They are careful about specifying units of measure and labeling to clarify the correspondence with quantities in a problem. They decide whether an equation represents a function by making sure that every input value corresponds to exactly one output value.

MP.7 Look for and make use of structure.

Students develop formulas such as $(a \pm b)^2 = a^2 \pm 2ab + b^2$ by applying the Distributive Property. Students see that the expression $5 + (n - 2)^2$ can be interpreted as the form of 5 plus "a quantity squared," and because "a quantity squared" must be positive or zero, the expression cannot be smaller than 5.

MP.8 Look for and express regularity in repeated reasoning.

Students see that one of the key features of a line in the plane is an equal difference in outputs over equal intervals of inputs, and that the result of evaluating the expression $\frac{y_2 - y_1}{x_2 - x_1}$ for any two points on the line is always equal to a certain number m. Therefore, if (x, y) is a generic point on this line, the equation $m = \frac{y - y_1}{x - x_1}$ will give a general form of the equation of that line.

Modeling

Modeling links classroom mathematics and statistics to everyday life, work, and decision-making. Modeling is the process of choosing and using appropriate mathematics and statistics to analyze empirical situations, to understand them better, and to improve decisions. Quantities and their relationships in physical, economic, public policy, social, and everyday situations can be modeled using mathematical and statistical methods. When making mathematical models, technology is valuable for varying assumptions, exploring consequences, and comparing predictions with data.

A model can be very simple, such as writing total cost as a product of unit price and number bought, or using a geometric shape to describe a physical object like a coin. Even such simple models involve making choices. It is up to us whether to model a coin as a three-dimensional cylinder, or whether a two-dimensional disk works well enough for our purposes. Other situations—modeling a delivery route, a production schedule, or a comparison of loan amortizations—need more elaborate models that use other tools from the mathematical sciences. Real-world situations are not organized and labeled for analysis; formulating tractable models, representing such models, and analyzing them is appropriately a creative process. Like every such process, this depends on acquired expertise as well as creativity.

Some examples of such situations might include the following:

- Estimating how much water and food is needed for emergency relief in a devastated city of 3 million people, and how it might be distributed.
- Planning a table tennis tournament for 7 players at a club with 4 tables, where each player plays against each other player.
- Designing the layout of the stalls in a school fair so as to raise as much money as possible.
- Analyzing the stopping distance for a car.
- Modeling a savings account balance, bacterial colony growth, or investment growth.
- Engaging in critical path analysis, e.g., applied to turnaround of an aircraft at an airport.
- Analyzing risk in situations such as extreme sports, pandemics, and terrorism.
- Relating population statistics to individual predictions.

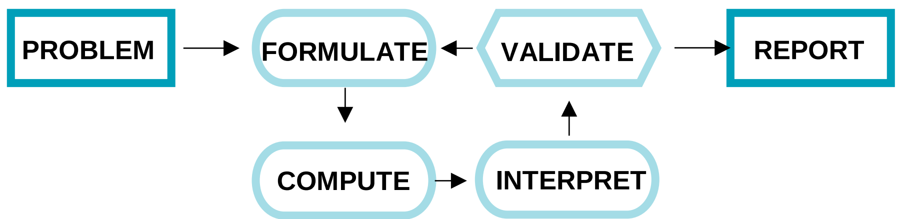
In situations like these, the models devised depend on a number of factors: How precise an answer do we want or need? What aspects of the situation do we most need to understand, control, or optimize? What resources of time and tools do we have? The range of models that we can create and analyze is also constrained by the limitations of our mathematical, statistical, and technical skills, and our ability to recognize significant variables and relationships among them. Diagrams of various kinds, spreadsheets and other technology, and algebra are powerful tools for understanding and solving problems drawn from different types of real-world situations.

Continued on next page

Modeling, continued

One of the insights provided by mathematical modeling is that essentially the same mathematical or statistical structure can sometimes model seemingly different situations. Models can also shed light on the mathematical structures themselves, for example, as when a model of bacterial growth makes more vivid the explosive growth of the exponential function.

The basic modeling cycle is summarized in the diagram. It involves (1) identifying variables in the situation and selecting those that represent essential features, (2) formulating a model by creating and selecting geometric, graphical, tabular, algebraic, or statistical representations that describe relationships between the variables, (3) analyzing and performing operations on these relationships to draw conclusions, (4) interpreting the results of the mathematics in terms of the original situation, (5) validating the conclusions by comparing them with the situation, and then either improving the model or, if it is acceptable, (6) reporting on the conclusions and the reasoning behind them. Choices, assumptions, and approximations are present throughout this cycle.



In descriptive modeling, a model simply describes the phenomena or summarizes them in a compact form. Graphs of observations are a familiar descriptive model—for example, graphs of global temperature and atmospheric CO_2 over time. Analytic modeling seeks to explain data on the basis of deeper theoretical ideas, albeit with parameters that are empirically based; for example, exponential growth of bacterial colonies (until cut-off mechanisms such as pollution or starvation intervene) follows from a constant reproduction rate. Functions are an important tool for analyzing such problems.
Graphing utilities, spreadsheets, computer algebra systems, and dynamic geometry software are powerful tools that can be used to model purely mathematical phenomena, e.g., the behavior of polynomials as well as physical phenomena.

Modeling Standards

Modeling is best interpreted not as a collection of isolated topics but rather in relation to other standards. Making mathematical models is a Standard for Mathematical Practice, and specific modeling standards appear throughout the high school standards indicated by a star symbol (★).

Mathematics Model Curriculum
with Instructional Supports
Algebra 1 Course

<table>
<tr><th>STANDARDS</th><th>MODEL CURRICULUM (N.Q.1-3)</th></tr>
<tr><td>

Number and Quantity

QUANTITIES

Reason quantitatively and use units to solve problems.

N.Q.1 Use units as a way to understand problems and to guide the solution of multi-step problems; choose and interpret units consistently in formulas; choose and interpret the scale and the origin in graphs and data displays. ★

N.Q.2 Define appropriate quantities for the purpose of descriptive modeling. ★

N.Q.3 Choose a level of accuracy appropriate to limitations on measurement when reporting quantities. ★

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Expectations for Learning

In elementary grades, students use units for distance, time, money, mass, etc. In grades 6, 7, and 8, students work with rates, especially speed, as a quotient of measurements. In this cluster, students extend the use of units to more complicated applications including rates, formulas, interpretation of scale and origin in graphs, data displays, and related applications. Next, students will apply modeling within the context of the algebra concepts studied and begin to develop strategies to solve more complicated mathematical problems.

ESSENTIAL UNDERSTANDINGS

- Units are necessary when representing quantities in a modeling situation to make sense of the problem in context.
- A particular quantity can be represented with units from multiple systems of measurement.
- Quantities in different units of measure can be compared using equivalent units.
- Derived quantities are calculated by multiplying or dividing known quantities, along with their units, e.g., 40 miles in 8 hours is 5 miles per hour.
- Quantities can be converted within a system of units, e.g., feet to inches, and between two systems of units, e.g., feet to meters.
- There are some contexts in which the origin of a graph or data display is essential to show, and there are other contexts in which the origin of a graph or data display where it is common to omit the origin, e.g., stock prices over time.

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STANDARDS	MODEL CURRICULUM (N.Q.1-3)
N.Q.1-3, continued	**Expectations for Learning, continued** **MATHEMATICAL THINKING** • Determine reasonableness of results. • Attend to the meaning of quantities. • Consider mathematical units involved in a problem. **INSTRUCTIONAL FOCUS** • When modeling, consider the scale when choosing or deriving suitable units. • Choose a level of accuracy appropriate for the given context. • Convert measurements within a system of units, e.g., convert 4.6 feet to inches and feet, or between a two systems of units, e.g., convert 4.6 feet to meters. **Content Elaborations** **OHIO'S HIGH SCHOOL CRITICAL AREAS OF FOCUS** • Algebra 1, Number 1, pages 3-4 **CONNECTIONS ACROSS STANDARDS** • Create equations that describe numbers or relationships (A.CED.1-4).

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (N.Q.1-3)

Instructional Strategies

Note: The model curriculum is not a collection of lessons nor a full curriculum; it does not suggest pace, sequence, or amount of time spent on topics. It provides information about a topic related to the standards including ideas for examples, strategies for teaching, common misconceptions, and possible connections between topics. In addition, the emphasis of the model curriculum is on instruction not on assessment; therefore, the examples in the Instructional Strategies section are written for instructional purposes not necessarily as assessment items. Examples may need to be rewritten to accommodate the needs of each individual classroom.

Standards for Mathematical Practice

This cluster focuses on but is not limited to the following practices:

***MP.2** Reason abstractly and quantitatively.*

***MP.4** Model with mathematics.*

***MP.6** Attend to precision.*

Units are central to applied mathematics and everyday life. In real-world situations, answers are usually represented by numbers associated with units, for example, acceleration, currency, people-hours, energy, power, concentration, and density. Therefore this cluster should not be taught in isolation, but should be combined with other standards that use units and graphs in real-world situations.

QUANTITIES

Numerical values without units are not quantities. To be added or subtracted, quantities must be measurements of the same attribute (length, area, speed, etc.) and expressed in the same units. Converting quantities expressed in different units to having the same unit is like converting fractions to have a common denominator before adding or subtracting. "Depending on context, quantities are called by different names, such as 'measure' (e.g., productivity measure) or 'index' (e.g., Consumer Price Index). In situations where quantities are represented as variables, quantities are often referred to as 'variables'" (Arizona High School Progression on Quantity, page 4).

Reasoning quantitatively includes the following:

- knowing when and how to convert units;
- analyzing the units in a calculation;
- using units to check work;
- making sound choices for the scale and origin of a graph or a display;
- choosing an appropriate level of accuracy when reporting quantities;
- specifying units when defining variables; and
- attending to units when writing expressions or equations.

To accomplish this goal, teachers should provide students with a broad range of contextual problems that offer opportunities for performing operations with quantities. These should involve units where students develop math reasoning skills to make judgements about the correctness of their answers. These problems should be connected to science, engineering, economics, finance, medicine, etc.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (N.Q.1-3)

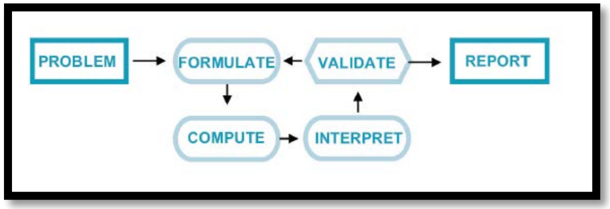
TIP! The seven units in the International System of Units (S.I.) were defined by actual material objects locked up in the International Bureau of Weights and Measures in Paris. The kilogram was defined by a cylinder of platinum-iridium locked in a vault in Paris. The meter was defined by the distance between two scratched on a hallowed bar of platinum-iridium metal. Copies would be distributed to various countries to keep the standardization of the units consistent. The problem is that physical objects could be broken, scratched, or chipped, so scientists have been working on redefining the units in terms of fundamental concepts of nature. Therefore in 1983 the meter was redefined as the distance traveled by the speed of light in a vacuum in $\frac{1}{299,792,458}$ of a second. The second has been defined as the amount of time it takes an atom of cesium-133 to vibrate 9,192,631,770 times. On May 20, 2019 the kilogram, ampere, kelvin, and mole will also be redefined. Discuss with students the importance of standardization with respect to units of measure. For more information https://www.nytimes.com/2018/11/16/science/kilogram-physics-measurement.html?smid=fb-nytimes&smtyp=cur&fbclid=IwAR0pSkhwRVla9MkkUR10B0ggQZKFyEjnXKPJ4LsRQWFdqAr9ec5n8124388

Some measures where the units are not specified have to be understood in the way the measure is defined. For example, the S&P 500 stock index is a measure derived from the quotient of the value of 500 companies now and in 1940–42.

MODELING

Throughout the modeling process, units are critical for several reasons:

- Units guide calculations.
- Units communicate the results of a model since real-world examples are usually quantities (numbers with units).
- Units help make or evaluate assumptions.
- Units show the reasonableness of an answer.
- Units help define the quantity.



Sometimes in the modeling process students will need to make their own decisions about units. See page 13 for more information about modeling.

MISLEADING GRAPHS

Graphical representations serve as visual models for understanding phenomena that take place in our daily surroundings. The use of different kinds of graphical representations along with their units, labels, and titles demonstrates the level of students' understanding and fosters the ability to reason, prove, self-check, examine relationships and establish the validity of arguments. Students can benefit from examples of misleading graphs where they must analyze data from the graph. It may be helpful to initially choose uniform units and scales in order to create a correct representation and make correct conclusions of a situation. This should also be applied to line plots, histograms, and box plots in S.ID.1-3 and S.ID.6.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (N.Q.1-3)

PRECISION VS ACCURACY

Measurement in mathematics and science involves both precision and accuracy. *Precision* refers to the closeness of two or more values to each other. *Accuracy* refers to the closeness of the measured value to a standard or known value. Values can be accurate but imprecise or precise but inaccurate; however it is better to be precise and accurate. Accuracy depends on the instrument being used.

Intermediate Rounding

Teachers at the high school level should discourage intermediate rounding when solving problems. Students should be encouraged to use the π button on the calculator whenever possible for greater precision. Give students a problem where intermediate rounding provides two very different answers and have a discussion on which answer is more precise and the consequences of less precision.

UNIT CONVERSION

Measurement involves units and often requires a unit conversion. Some contextual problems may require an understanding of derived measurements and capability in unit analysis.

EXAMPLE

While driving in the United Kingdom (UK), a U.S. tourist puts 60 liters of gasoline in his car. The gasoline cost is £1.28 per liter. The current exchange rate is £ 0.62978 for each $1.00. The price for a gallon of a gasoline in the United States is $3.05. Compare the costs for the same amount and the same type of gasoline when he/she pays in UK pounds.

Discussion:

- Making reasonable estimates should be encouraged prior to solving this problem. If the current exchange rate has inflated the UK pound to more than the U.S. dollar, the driver will pay more for the same amount of gasoline. By dividing $3.05 by 3.79L (the number of liters in one gallon), students can see that 80.47 cents per liter of gasoline in US is less expensive than £1.28 or $ 2.03 per liter of the same type of gasoline in the UK when paid in U.S. dollars. The cost of 60 liters of gasoline in UK is $£\,76.8$ $(\frac{£1.28}{1\,L} \times 60L = UK\,£\,76.8)$.
- In order to compute the cost of the same quantity of gasoline in the United States in UK currency, it is necessary to convert between both monetary systems and units of volume. Based on UK pounds, the cost of 60 liters of gasoline in the U.S. is $£30.41$ $(\frac{US\$3.05}{1\,gal} \times \frac{1\,gal}{3.79\,L} \times 60L \times \frac{UK\,£0.62978}{US\,\$1.00} = UK\,£30.41)$.
- The computation shows that the gasoline is less expensive in the United States and shows how an analysis can be helpful in keeping track of unit conversations. Students should be able to correctly identify the degree of precision of the answers, understanding that the degree of precision should not be greater than the actual accuracy of measurement. The use of significant digits and unit analysis is an emphasis in high school science classes, and efforts could be made to collaborate with science teachers.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (N.Q.1-3)

Students will compare the cost of gasoline across two countries with different monetary systems and units of measure. Host a career speaker in the classroom, where students can ask questions of related to the work-based applications of these concepts (e.g., economics, engineering, finance, health). See also "Corn and Oats" and "Dental Impressions" lessons in the Instructional Resources section.

Students may not realize the importance of the units' conversions in conjunction with the computation when solving problems involving measurements. Since today's calculating devices often display 8 to 10 decimal places, students frequently express answers to a much greater degree of precision than required or reasonable for the given problem situation.

TIP!

Instructional Tools/Resources

These tools and resources may be helpful for instruction. Before use, they should be evaluated at the local level to determine their appropriateness for instruction.

Understanding Units

- Scale of the Universe: Putting the Universe into Perspective is an applet that shows the perspective of the universe. Teachers can use this as an introduction to talk about units.
- Leaky Faucet is lesson by Dan Meyer that aligns to 6.RP.3, but could be used at the high school level to start a discussion about units and why they are necessary to solve problems.
- The Largest Loser is the first link on the Georgia Mathematics Design Collaborative Formative Assessment Lessons. It is a lesson that has students interpret graphs based on differing scales and labels.

Converting Units

- Dental Impressions is a lesson connecting math standards to CTE by Achieve the Core. This lesson has students converting units to make a stone model from dental impressions.
- Corn and Oats is a lesson connecting math standards to CTE by Achieve the Core. The lesson has the students assist Producer Bob with management tasks regarding the planting of corn and oats.
- Dimensional Analysis is a lesson by Rachel Meisner on dimension analysis with a video, PowerPoint, and worksheet using dimensional analysis. In addition to some cute animation in the video, it ties in real-world problems and significant figures.
- Dimensional Analysis is a lesson by Leslie Gushwa that includes three resources on dimensional analysis. The first two Word documents could be used in a math classroom. The third resource deals with moles in conjunction with science.

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INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (N.Q.1-3)

Precision and Accuracy

- Precision and Accuracy is a lesson on the difference between precision and accuracy by Annenberg Learner. It includes a video, some problems, and an interactive applet. It also has some homework questions.
- Measurement, Accuracy, and Precision is an activity that discusses accuracy and precision with respect to measurement and discusses the importance of accuracy. The first activity has students weigh Mars Bars and the graph the results. The second lesson shows how equipment can affect accuracy and precision. The third lesson discusses the importance of accuracy.

General Resources

- Arizona High School Progression on Number and Quantity is an informational document for teachers. This cluster is addressed on pages 2-6
- Arizona High School Progression on Modeling is an informational document for teachers. This cluster is addressed on pages 7-10.
- High School Coherence Map by UnboundED is a map of the relationships among high school mathematics standards. Please note: This resource is aligned to the Common Core not Ohio's Learning Standards. Although the tool is useful, it needs to be checked with respect to Ohio's Learning Standards.

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STANDARDS	MODEL CURRICULUM (A.SSE.1-2)
Algebra **SEEING STRUCTURE IN EXPRESSIONS** **Interpret the structure of expressions.** **A.SSE.1**. Interpret expressions that represent a quantity in terms of its context. ★ **a.** Interpret parts of an expression, such as terms, factors, and coefficients. **b.** Interpret complicated expressions by viewing one or more of their parts as a single entity. **A.SSE.2** Use the structure of an expression to identify ways to rewrite it. *For example, to factor $3x(x - 5) + 2(x - 5)$, students should recognize that the "$x - 5$" is common to both expressions being added, so it simplifies to $(3x + 2)(x - 5)$; or see $x^4 - y^4$ as $(x^2)^2 - (y^2)^2$, thus recognizing it as a difference of squares that can be factored as $(x^2 - y^2)(x^2 + y^2)$.*	**Expectations for Learning** Students build expressions in grades K-5 with arithmetic operations. As they move into the middle grades and progress through high school, students build expressions with algebraic components, beginning with linear, exponential, and quadratic expressions. In later courses, they build algebraic expressions with polynomial, rational, radical, and trigonometric expressions. In this cluster, they focus on interpreting the components of linear, exponential, and quadratic expressions and their meaning in mathematical and real-world contexts. Also, students determine when rewriting or manipulating expressions is helpful in order to reveal different insights into a mathematical or real-world context. **ESSENTIAL UNDERSTANDINGS** • An expression is a collection of terms separated by addition or subtraction. • A term is a product of a number and a variable raised to a nonnegative integer exponent. • Components of an expression or expressions within an equation may have meaning in a mathematical context, e.g., $y = mx + b$, b represents the y-intercept; $b^2 - 4ac$ in the quadratic formula indicates the number and nature of solutions to the equation. • Components of an expression may have meaning in a real-world context, e.g., in data surcharges, $60 + 0.05x$ the 60 represents the fixed costs and the 0.05 represents the cost per unit of data. • Expressions may potentially be rearranged or manipulated in ways to reveal different insights into the mathematical or real-world context. **MATHEMATICAL THINKING** • Attend to the meaning of quantities. • Use precise mathematical language. • Apply grade-level concepts, terms, and properties. • Look for and make use of structure. *Continued on next page*

<table>
<tr><th>STANDARDS</th><th>MODEL CURRICULUM (A.SSE.1-2)</th></tr>
<tr><td>A.SSE.1-2, continued</td><td>
<h2>Expectations for Learning, continued</h2>
<h3>INSTRUCTIONAL FOCUS</h3>

Identify the components, such as terms, factors, or coefficients, of an expression and interpret their meaning in terms of a mathematical or real- world context.
Explain the meaning of each part of an expression, including linear, simple exponential, and quadratic expressions, in a mathematical or real-world context.
Analyze an expression and recognize that it can be rewritten in different ways.

<h2>Content Elaborations</h2>
<h3>OHIO'S HIGH SCHOOL CRITICAL AREA OF FOCUS</h3>

Algebra 1, Number 1, page 3
Algebra 1, Number 4, pages 9-10

<h3>CONNECTIONS ACROSS STANDARDS</h3>

Write expressions in equivalent forms (A.SSE.3).
Create equations in one or two variables (A.CED.1-2).
Interpret expressions for functions in terms of the situations they model (F.LE.5).
Interpret linear models (S.ID.7).

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INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (A.SSE.1-2)

Instructional Strategies

Note: The model curriculum is not a collection of lessons nor a full curriculum; it does not suggest pace, sequence, or amount of time spent on topics. It provides information about a topic related to the standards including ideas for examples, strategies for teaching, common misconceptions, and possible connections between topics. In addition, the emphasis of the model curriculum is on instruction not on assessment; therefore, the examples in the Instructional Strategies section are written for instructional purposes not necessarily as assessment items. Examples may need to be rewritten to accommodate the needs of each individual classroom.

Standards for Mathematical Practice
This cluster focuses on but is not limited to the following practices:
MP.1 *Make sense of problems and persevere in solving them.*
MP.4 *Model with mathematics.*
MP.7 *Look for and make use of structure.*

The focus of this cluster is in "seeing" how each part of an algebraic expression interplays with the other parts of the expression to create meaning. Since the goal is on the interpretation of the components, the focus should not be simplifying expressions or solving equations. As the expressions become more complex, students should be able to see them built out of basic operations such as sums of terms or products of factors.

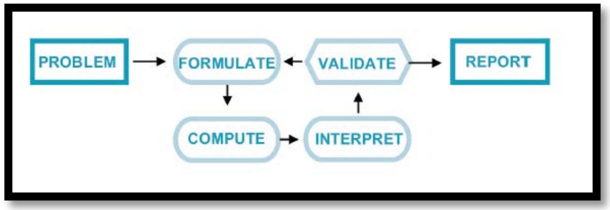
Development and proper use of mathematical language is an important building block for future content. For example, a student should recognize that in the expression $2x + 1$, "$2x$" and "1" are *terms* of the binomial, "2" is the coefficient, "2" and "x" are factors, and "1" is a constant.

A student should also be able to see that more complicated expressions can be built from simpler ones. For example, the expression $3 + (y - 2)^2$ can be viewed as the sum of the constant term 3 and the squared term $(y - 2)^2$; therefore viewing the expression in this manner allows a student to recognize that it is always greater than or equal to 3, because $(y - 2)^2$ is nonnegative and the sum of 3 and a nonnegative number is greater than 0. A student can also recognize that inside the squared term is the expression $y - 2$, so the square term is 0 when $y = 2$.

TIP! Students may believe that the use of algebraic expressions is merely the abstract manipulation of symbols. To counter this idea, the use of real-world examples is very helpful. Students can be asked to explain the meaning of the parts of algebraic expression that represent the situation, and provide a rationale for why one form of the expression is more beneficial than another.

MODELING

A.SSE.1 is a modeling standard. See page 13 for more information about modeling.



INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (A.SSE.1-2)

LINEAR, EXPONENTIAL, AND OTHER EXPRESSIONS AND EQUATIONS

Building upon prior knowledge of properties and structure of expressions, students can apply mental manipulations to solve problems.

EXAMPLE

For which values of *m* are the following inequalities true?

- **a.** $m^2 + m < m^2 + m + 3$
- **b.** $m^4 - m + 9 \geq -m + m^2 + m^4$

Discussion: For Part **a**., students should realize that $m^2 + m$ appears on both sides of the inequality. Therefore, the inequality is equivalent to $0 < 3$, which is a true statement, so any real number will make the inequality true. For Part **b**., students should realize that $m^4 - m$ is present on both sides of the inequality; therefore the inequality is equivalent to $9 \geq m^2$ and $-3 \leq m \leq 3$.

EXAMPLE

Evaluate.

$$6.7 \times \frac{2}{3}\left(\frac{5.2 \times 4.3}{9.6^2}\right)(0)$$

Discussion: Students should recognize that the actual multiplication is pointless because the Zero Product Property can be applied. Therefore, the expression is equal to 0.

EXAMPLE

Solve for *b*.

- **a.** $\frac{3}{5}(y-7)(y+2) = b(y-7)(y+2)$
- **b.** $3(x-2) - b = 3x - 6$

Discussion: In Part **a**., students should recognize that $(y-7)(y+2)$ appears on both side of the equation, so $b = \frac{3}{5}$. In Part **b**., students should realize that $3(x-2) = 3x - 6$ by the Distributive Property, so $b = 0$.

Explore the nature of algebraic equations and systems of algebraic equations using mathematical and real-world contexts.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (A.SSE.1-2)

EXAMPLE

The equations below represent Mario's trip to the store to buy school supplies, where *m* represent binders and *n* represents folders.

$m + n = 15$
$0.50n + 4.69m + 2.28 = 34.92$

a. Determine how much each binder costs.
b. What could 2.28 represent in the second equation?
c. What does $0.50n + 4.69m + 2.28$ in the second equation represent?
d. How many different items did he buy?
e. What does $0.50n$ represent?

EXAMPLE

Create two equivalent expressions. The first expression should have two terms, and the second expression should have three terms. Then, write an appropriate context for each equivalent expression and explain why the context best represents that form.

EXAMPLE

What information is true about the expression?

$7 + (a - 1)^2$

a. It is always less than or equal to 1.
b. It is always greater than or equal to 7.
c. There will be two numbers for *a* that will make equivalent expressions.
d. There will be no numbers for *a* that will make equivalent expressions.
e. There will be three numbers for *a* that will make equivalent expressions.

Offer multiple real-world examples of exponential functions. For instance, students have to recognize that in an equation representing automobile cost $C(t) = 20{,}000(0.75)^t$, since the base of the exponential factor, 0.75, is positive and smaller than 1, it represents an exponential decay or a yearly 25% (1 – 0.75 = 0.25) depreciation of the initial $20,000 value of automobile over the course of *t* years. On the contrary, in an exponential equation representing the amount of investment $A(t) = 10{,}000(1.03)^t$, over *t* years, since the base of exponential factor, 1.03, is greater than 1, it represents exponential growth or a yearly 3% increase of the initial investment of $10,000 over the course of *t* years.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (A.SSE.1-2)

EXAMPLE

The equation below represents the amount of money Maria deposited in the bank at a fixed annual interest rate that is compounded monthly.

$$P = 2500\left(1 + \frac{0.06}{12}\right)^{12t}$$

- **a.** What does the "1 +" in the equation indicate? How would the equation change if it was "1 – "?
- **b.** What does 2500 represent in terms of the context of the situation?
- **c.** What does 0.06 represent in terms of the context of the situation?
- **d.** What does $\left(1 + \frac{0.06}{12}\right)$ represent in terms of the context of the situation?
- **e.** What does $12t$ represent in terms of the context of the situation?
- **f.** What is the contextual meaning of $\frac{0.06}{12}$?

EXTENDING TO QUADRATIC EXPRESSIONS AND EQUATIONS

This expressions in this cluster should be extended to quadratics. Quadratics should be in the various forms: standard form, factored form, and vertex form. Have students create their own expressions that meet specific criteria (e.g., number of terms factorable, difference of two squares, etc.) and verbalize how they can be written and rewritten in different forms. Additionally, pair/group students to share their expressions and rewrite one another's expressions.

Students may have a misconception that an expression cannot be factored because it does not fit into a form they recognize. Provide students with examples where they need to apply certain algebraic manipulations to transform the expression into a recognizable factorable form.

EXAMPLE

Factor the binomial completely.

$3x^2 - 12$

$3(x^2 - 4)$

$3(x - 2)(x + 2)$

EXAMPLE

Factor the trinomial completely.

$-2x^2 + 4x - 2$

$-2(x^2 - 2x + 1)$

$-2(x - 1)^2$

Discuss how to choose the most appropriate form of a quadratic equation to find the coordinate pair of the vertex of its graph; to determine an efficient method to obtain the form; and to use this form for determining the type of transformations applied to the graph of $y = x^2$ to attain a graph of the given quadratic equation.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (A.SSE.1-2)

Have students find the horizontal and vertical shift by rewriting a binomial using completing the square. Have them explain why the vertex form is useful.

Factoring expressions like $3x(x – 5) + 2(x – 5)$ is another opportunity to recognize the structure where the Distributive Property is applicable. Students should understand that the Distributive Property, $a(b + c) = ab + ac$ works in both directions and the expression $ab + ac$ factors as $a(b + c)$ because "a" is a common factor for terms "ab" and "ac". Similarly, in the expression $3x(x – 5) + 2(x – 5)$, the difference $(x – 5)$ is a common factor and the expression factors as $(x – 5)(3x + 2)$.

Technology may be useful to help students recognize that two different expressions represent the same relationship. For example, since $(x – y)(x + y)$ can be rewritten as $x^2 – y^2$, they can put both expressions into a graphing calculator (or spreadsheet) and have it generate two tables (or two columns of one table), displaying the same output values for each expression.

EXAMPLE

Multiply each pair of factors. Combine like terms where appropriate. Make a conjecture relating the terms in the resulting polynomials to the structure of the factors.

a. $(h + 2)(h + 3)$; $(h + 2)(h – 3)$; $(h – 2)(h + 3)$; $(h – 2)(h – 3)$

b. $(j + 5)(j + 7)$; $(j + 5)(j – 7)$; $(j – 5)(j + 7)$; $(j – 5)(j – 7)$

c. $(w + 9)(w + 9)$; $(w + 9)(w – 9)$; $(w – 9)(w + 9)$; $(w – 9)(w – 9)$;

d. $(2g + 5)(g + 3)$; $(2g + 5)(g – 3)$; $(2g – 5)(g + 3)$; $(2g – 5)(g – 3)$;

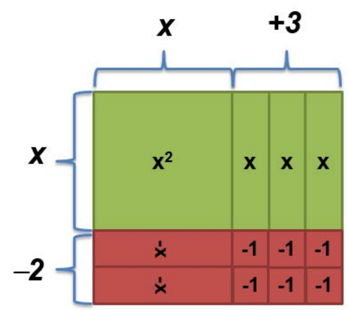
Discussion: Students should make the connection between the signs in the terms of the two binomials and the resulting trinomial. Once students make these connections, they should be able to apply that knowledge to factoring trinomials.

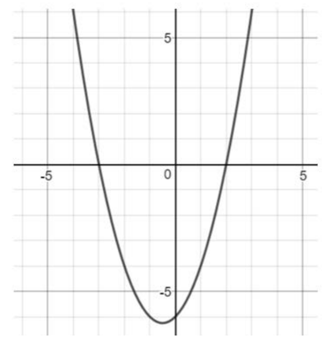
When a quadratic is factorable over integers, it can be modeled as a rectangle using algebra tiles. Explain patterns in factoring trinomials including those that can form squares—perfect square binomials and difference of squares. (See cluster A.SSE.3 for more information on factoring trinomials with algebra tiles.) After factoring a quadratic expression using algebra tiles, connect the factors represented by Algebra tiles to the graph of its corresponding parabola in order to make sense of the structure of a factored quadratic. Include examples that are perfect trinomials, a difference of squares, and unfactorable.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (A.SSE.1-2)

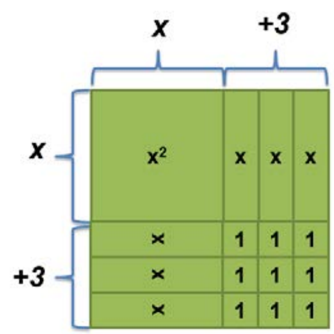
EXAMPLE

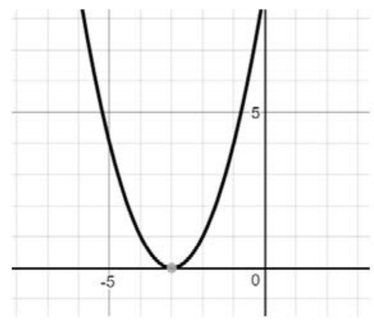
- Factor $x^2 + x - 6$ using algebra tiles, and then graph the quadratic. What connections do you see between the graph and the factored form of the quadratic?





- Factor $x^2 + 6x + 9$ using algebra tiles, and then graph the quadratic. What connections do you see between the graph and the factored form of the quadratic?





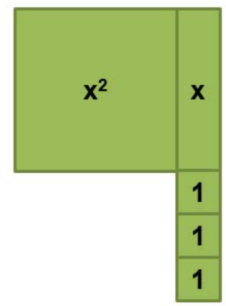
Example continued on next page

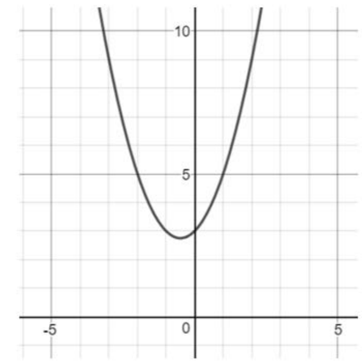
INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (A.SSE.1-2)

EXAMPLE, CONTINUED

- Factor $x^2 + x + 3$ using algebra tiles, and then graph the quadratic. What connections do you see between the graph and the factored form of the quadratic?

Discussion: In part **a**., the students should write the factored expression $(x + 3)(x – 2)$. Connect how changing the structure of the quadratic trinomial will reveal the zeros, –3 and 2, in the graph. Students should see that the polynomial in factored form will allow one to "see" the solutions to a quadratic equation without graphing. In part **b**., students should realize that since the quadratic forms a perfect square, there is only one zero, –3. In part **c**., students should realize that the quadratic cannot be factored since the Algebra tiles cannot form a rectangle, and they can make a connection between the model and the parabola which does not have any zeros.





EXAMPLE

A baseball is thrown into the air which is illustrated by the equation below. Rewrite in vertex form, and then answer the questions.

$h = 75t + 5 – 16t^2$

a. Which form is better to "see" the initial height?
b. Determine the initial height of the rocket before launching.
c. Which form is better to "see" the maximum height?
d. Determine the maximum height of the rocket.

Factoring by grouping is another example of how students might analyze the structure of an expression. To factor $3x(x – 5) + 2(x – 5)$, students should recognize that the "$x – 5$" is common to both expressions being added, so it simplifies to $(3x + 2)(x – 5)$. Students should become comfortable with rewriting expressions in a variety of ways until a structure emerges.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (A.SSE.1-2)

Instructional Tools/Resources

These tools and resources may be helpful for instruction. Before use, they should be evaluated at the local level to determine their appropriateness for instruction.

Manipulatives/Technology

- Hands-on materials, such as algebra tiles, can be used to establish a visual understanding of algebraic expressions and the meaning of terms, factors and coefficients.
- Desmos is a free graphing calculator that is available to students as website or an app.
- Geogebra is a free graphing calculator that is available to students as website.
- Wolframalpha is dynamic computing tool.

Factoring with Algebra Tiles

- Factoring Trinomials –With Algebra Tiles by Braining Camp is a YouTube video explain factoring trinomials with algebra tiles.
- Factoring Polynomials with Algebra Tiles (2) by Tom Horn is a YouTube video explain factoring trinomials with algebra tiles.
- Algebra Tiles 5: Factoring Trinomials by Simpson Math is a YouTube video explain factoring trinomials with algebra tiles.
- Algebra tile templates on the SMART Exchange has a variety of useful models that teacher can use if they have access to a SMART Board.
- Multiplying Binomials and Factoring Trinomials using Algebra Tiles and Generic Rectangle is a worksheet from West Contra Costa Unified School District about using algebra tiles.
- Advanced Algebra Tile Factoring is an applet by eMathLab.com that allows for factoring trinomials when *a* does not equal 1.

Interpreting Linear or Exponential Expressions and/or Equations

- Kitchen Floor Tiles is an Illustrative Mathematics task that has students interpret algebraic expressions in connection with geometric patterns. This task integrates modeling.
- Animal Populations is an Illustrative Mathematics task that has students reason which expression is larger based on its structure.
- Delivery Trucks is an Illustrative Mathematics task that has students relate structure to a context without any algebraic manipulation.
- Exponential Parameters is an Illustrative Mathematics task that has students interpret the parameter of an exponential function.
- Mixing Candies is an Illustrative Mathematics task that has students interpret a system of equations.
- The Bank Account is an Illustrative Mathematics task that explores the structure of a real-world exponential equation dealing with interest rates.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (A.SSE.1-2)

Interpreting Quadratic Expressions and/or Equations

- Seeing Dots is an Illustrative Mathematics task has students interpret two algebraic expressions in terms of quadratics and a geometric context. This task integrates modeling.
- Throwing Horseshoes is an Illustrative Mathematics task explores the structure of a real-world quadratic equation.
- Profit of a Company, Assessment is an Illustrative Mathematics task explores the structure of a real-world quadratic equation with respect to profit.
- Generating Polynomials from Patterns is a Formative Assessment Lesson from Mathematics Assessment Project that uses visual models to help students manipulate polynomials. This lesson should be discussed in context of structure of quadratic expressions with respect to which forms make the most sense in the context of the problem.

Interpreting Other Expressions and/or Equations

- Mixing Fertilizer is an Illustrative Mathematics task that applies ratio and proportions to mixture problems.
- The Physics Professor is an Illustrative Mathematics task has students drawing conclusions about expressions using information they already know.

Curriculum and Lessons from Other Sources

- EngageNY, Algebra 1, Module 1, Topic B, Lesson 6: Algebraic Expressions—The Distributive Property and Lesson 7: Algebraic Expressions—The Commutative and Associative Property are lessons that pertain to this cluster.
- EngageNY, Algebra 1, Module 1, Topic D, Lesson 25: Solving Problems in Two Ways—Rates and Algebra is a lesson that pertains to this cluster.
- Mathematics Vision Project, Algebra 1, Module 6: Quadratic Functions has lessons that pertain to this cluster.

General Resources

- Arizona High School Progressions on Algebra is an informational document for teachers. This cluster is addressed on pages 4-6 and pages 11-12.
- Arizona High School Progression on Modeling is an informational document for teachers.
- High School Coherence Map by UnboundED is a map of the relationships among high school mathematics standards. Please note: This resource is aligned to the Common Core not Ohio's Learning Standards. Although the tool is useful, it needs to be checked with respect to Ohio's Learning Standards.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (A.SSE.1-2)

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- Common Core Standards Writing Team. (2013, July 4). *Progressions for the Common Core State Standards in Mathematics (draft). High School, Modeling*. Tucson, AZ: Institute for Mathematics and Education, University of Arizona.
- Gurl, T., Artzt, A., Sultan, A., & Curcio, F. (2012). *Implementing the Common Core State Standards through Mathematical Problem Solving*. Reston, VA: National Council of Teachers of Mathematics.
- Kieran, C. (2007). Learning and teaching algebra at the middle school level through college levels. In F. K. Lester (Ed.), *Second Handbook of Research on Mathematics Teaching and Learning* (pp. 707-762). Charlotte, NC: Information Age Publishing.

STANDARDS	MODEL CURRICULUM (A.SSE.3)
Algebra **SEEING STRUCTURE IN EXPRESSIONS** **Write expressions in equivalent forms to solve problems.** **A.SSE.3** Choose and produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression. ★ **a.** Factor a quadratic expression to reveal the zeros of the function it defines. **b.** Complete the square in a quadratic expression to reveal the maximum or minimum value of the function it defines. **c.** Use the properties of exponents to transform expressions for exponential functions. *For example, 8^t can be written as 2^{3t}.*	**Expectations for Learning** Previously, students combine like terms and recognize equivalent expressions. In this cluster, students focus on the form of an expression that is most useful for a particular purpose. Students rewrite quadratic expressions by factoring and completing the square, and they use these forms to analyze the graphs of the functions they define. Students also rewrite exponential expressions using properties of exponents using integer exponents. In Algebra 2, students use these skills to analyze higher degree polynomial functions. **ESSENTIAL UNDERSTANDINGS** • Expressions may potentially be rearranged or manipulated in ways to reveal different insights into the mathematical or real-world context. • The factored form of a quadratic expression reveals the zeros of the function it defines. • The vertex form of a quadratic expression reveals the vertex and the maximum or minimum value of the function it defines. • Completing the square of a quadratic expression generates the vertex form of a quadratic expression. • Understanding the properties of exponents is essential for rewriting exponential expressions. **MATHEMATICAL THINKING** • Plan a solution pathway. • Determine the appropriate form of an expression in context. *Continued on next page*

STANDARDS	MODEL CURRICULUM (A.SSE.3)
A.SSE.3, continued	**Expectations for Learning, continued** **INSTRUCTIONAL FOCUS** **For quadratic functions, students should work with expressions in which the leading coefficient can be any real number, but assessment questions should focus on expressions with leading coefficients of 1 with occasional questions using other simple leading coefficients, and limit exponential expressions to expressions with integer exponents.* • Determine the appropriate equivalent form of an expression for a given purpose. • Factor a quadratic expression so that the zeros of the function it defines can be identified. • Complete the square for a quadratic expression to identify the vertex and maximum or minimum value of the function it defines. • Rewrite exponential expressions by using properties of exponents. **Content Elaborations** **OHIO'S HIGH SCHOOL CRITICAL AREA OF FOCUS** • Algebra 1, Number 4, pages 9-10 **CONNECTIONS ACROSS STANDARDS** • Interpret key features of graphs (F.IF.4). • Interpret the structure of expressions (A.SSE.1-2). • Analyze functions using different representations (F.IF.8).

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (A.SSE.3)

Instructional Strategies

Note: The model curriculum is not a collection of lessons nor a full curriculum; it does not suggest pace, sequence, or amount of time spent on topics. It provides information about a topic related to the standards including ideas for examples, strategies for teaching, common misconceptions, and possible connections between topics. In addition, the emphasis of the model curriculum is on instruction not on assessment; therefore, the examples in the Instructional Strategies section are written for instructional purposes not necessarily as assessment items. Examples may need to be rewritten to accommodate the needs of each individual classroom.

Standards for Mathematical Practice

This cluster focuses on but is not limited to the following practices:

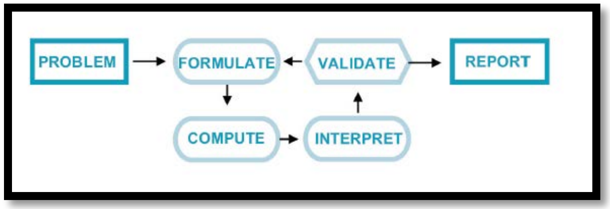
***MP.4** Model with mathematics.*

***MP.7** Look for and make use of structure.*

***MP.8** Look for and express regularity in repeated reasoning.*

This cluster extends the previous cluster (A.SSE.1-2) from analyzing structure to now using the most efficient equivalent form to solve a problem. It focuses specifically on transforming exponential equations using properties and rewriting quadratics in factored form and vertex form. Students are also connecting the various forms of an expression to the context of the problem and to the analysis of its corresponding graph.

Students must understand the idea that changing the forms of expressions, such as factoring, completing the square, or transforming expressions from one quadratic form to another are not independent algorithms that are learned for the sake of symbolic manipulations. They are processes that are guided by goals (e.g., investigating properties of families of functions, solving contextual problems, finding roots, and identifying maximum or minimum values). An expression can be written in many forms that may look different but are in fact equivalent, as each form still represents the given expression. Rewriting an expression in simplified form may not always be the best form for all situations. It is much more advantageous for students to think about which equivalent form would be the most useful for a particular context instead of always immediately simplifying.



MODELING

A.SSE.3 is a modeling standard. See page 13 for more information about modeling.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (A.SSE.3)

PROPERTIES OF EXPONENTS

This is the first time that students are expected to formally know the properties of exponents. In Grade 8 students informally gained experience with the properties of exponents and applied exponent reasoning to scientific notation. They explored some of the properties of exponents using patterns and structure involving negative integer and zero exponents, but were limited to numerical bases. Now they need to extend their understanding of the properties of powers to exponential expressions that have variables as bases. It may be helpful for students to do explorations that revisit the patterns of exponents with the purpose of deriving the more formal properties of exponents. Building on this, students can create and manipulate more advanced exponential expressions and equations. An activity such as Properties of Exponents from S^{2}TEM Centers might be a helpful bridge. Students should continue to write out the expanded form when simplifying exponents until they are able to internalize the properties.

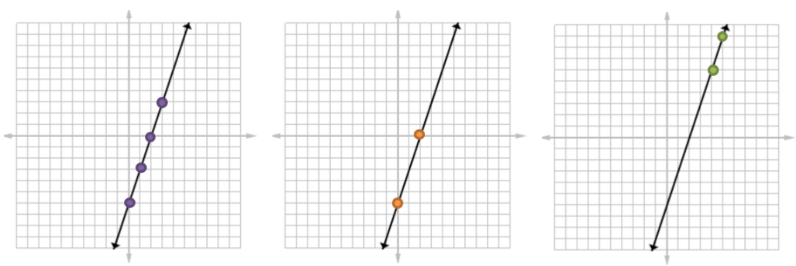
Properties of Integer Exponents

For any nonzero real numbers *a* and *b* and integers *n* and *m*:

1. $a^0 = 1$
2. $a^{-n} = 1/a^n$
3. $a^n a^m = a^{n+m}$
4. $(a^n)^m = a^{nm}$
5. $a^n b^n = (ab)^n$
6. $\frac{a^n}{a^m} = a^{n-m}$

LINEAR EXPRESSIONS AS PARTS OF A LINEAR EQUATION

Slope-intercept form	Standard form	Point-slope form
$y = 3x - 6$	$3x - y = 6$	$y - 6 = 3(x - 4)$



Slope-intercept form reveals the slope and the y-intercept.	Standard form allows one to find both the x- and y-intercepts easily.	Point-slope form allows one to see a specific point and the slope.

Students should be able to write an equation in equivalent forms to reveal and explain different properties of the equation. Writing an equation in different ways can reveal different features of the graph of a function. Students should be given the opportunity to come up with equivalent forms of the same equation, and then explore why the functions are the same both algebraically and graphically. The process of rewriting equations to reveal key features should be used to explain/reveal features in the context of real-world scenarios.

Students should be able to produce equivalent forms for the equation to reveal and explain different features of the graph represented by this equation. After students are provided with the opportunity to explore equivalent forms of the same equation, they will realize that those forms represent the same equation both algebraically and graphically.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (A.SSE.3)

QUADRATIC EXPRESSIONS

Students should apply the understandings of the structure of polynomials in factored form and vertex form from the previous cluster (A.SSE.1-2) to this cluster and choose the best form of a quadratic expression needed to solve real-world and mathematical problems.

To give a student a clearer picture of what method to use and when to use it, stress that each chosen method results in different information:

- The solutions of quadratic equations solved by factoring are the x-intercepts of the parabola or zeros of quadratic functions.
- A pair of coordinates (h, k) from the vertex form $y = a(x - h)^2 + k$ represents the vertex of the parabola, where h is a horizontal shift and k is a vertical shift of the "parent" parabola $y = x^2$ from its original position at the origin.
- A vertex (h, k) is the minimum point of the graph of the quadratic function if $a > 0$ and is the maximum point of the graph of the quadratic function if $a < 0$. Understanding the algorithm of completing the square provides a solid foundation for deriving the quadratic formula.

Students may incorrectly think that the vertex (minimum) of the graph of $y = (x + 5)^2$ is shifted to the right of the vertex (minimum) of the graph $y = x^2$ due to the addition sign. Students should explore examples both analytically and graphically to overcome this misconception.

Comparing different forms of expressions, equations, and graphs helps students to understand key connections among arithmetic, algebra, and applications of geometry. Have students derive information about a function's equation, represented in standard, factored, or vertex form, by investigating its graph. The reverse thinking technique (a process that allows working backwards from the answer to the starting point) can be very effective.

Some students may incorrectly believe that the minimum of the graph of a quadratic function always occurs at the y-intercept. Students should be provided with plenty of opportunities to investigate graphs and make connections between the vertices, vertex forms, and transformations of parabolas.

Some students may believe that factoring and completing the square are isolated techniques within a unit of quadratic equations. Teachers should help students see the value of these skills for solving real-world problems and the connections between forms.

Factoring

For quadratic functions, students should work with expressions in which the leading coefficient can be any real number, but assessment questions should focus on expressions with leading coefficients of 1 with occasional questions using other simple leading coefficients.

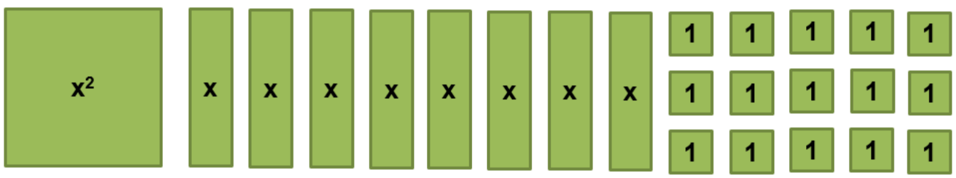
Factoring can be a challenging concept for students. Make connections between factoring of whole number, area models such as algebra tiles, and factoring of polynomials. Students should be familiar with multiplying and dividing using area models from earlier grades. Area models could also help students to make the connection between factoring a polynomial and division.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (A.SSE.3)

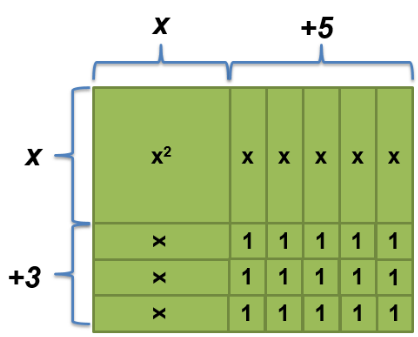
EXAMPLE

Factor $x^2 + 8x + 15$ using a model.

Represent $x^2 + 8x + 15$.



Arrange tiles into a rectangle and find its length and width.

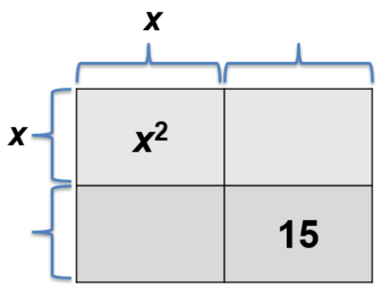


This can be modeled using algebra tiles. Since $x^2 + 8x + 15$ is the area, have students arrange the tiles into a rectangle.

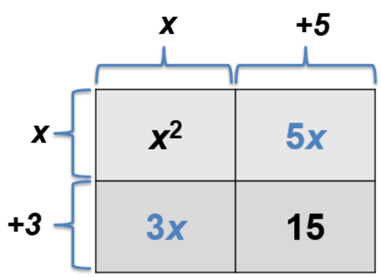
The length $(x + 5)$ and the width $(x + 3)$ of the rectangle are also the polynomial's factors.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (A.SSE.3)

Eventually students can move away from the actual tiles to rectangles, where they fill in the bottom left and top right corners.



They can then guess and check using factors of b in $ax^2 + bx + c$ to figure out the expressions in the empty of the corners of the rectangles.



$$\frac{15}{\begin{array}{c} 5 \cdot 3 \\ 3 \cdot 5 \\ 15 \cdot 1 \\ 1 \cdot 15 \end{array}}$$

More detailed explanations on factoring with algebra tiles as well as applets can be found in the Instructional Resources/Tools section.

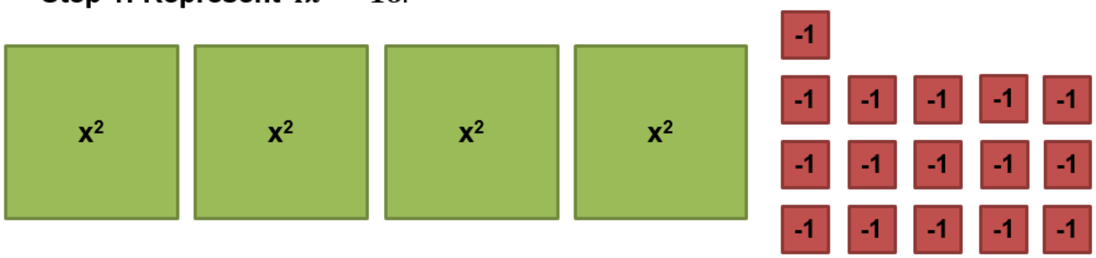
Students should also work with perfect square trinomials and differences of squares.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (A.SSE.3)

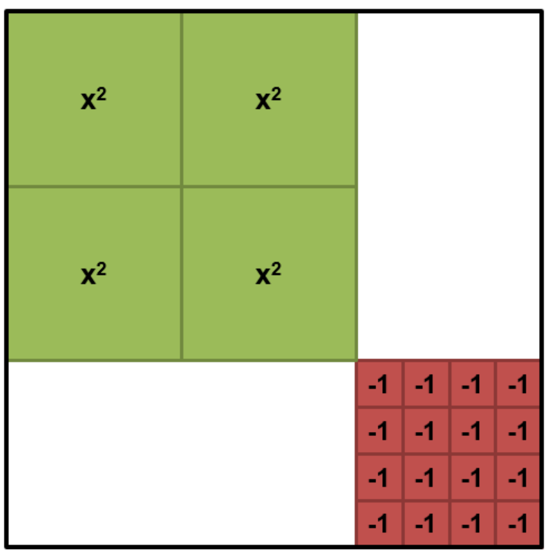
EXAMPLE

Factor $4x^2 - 16$ using a model.

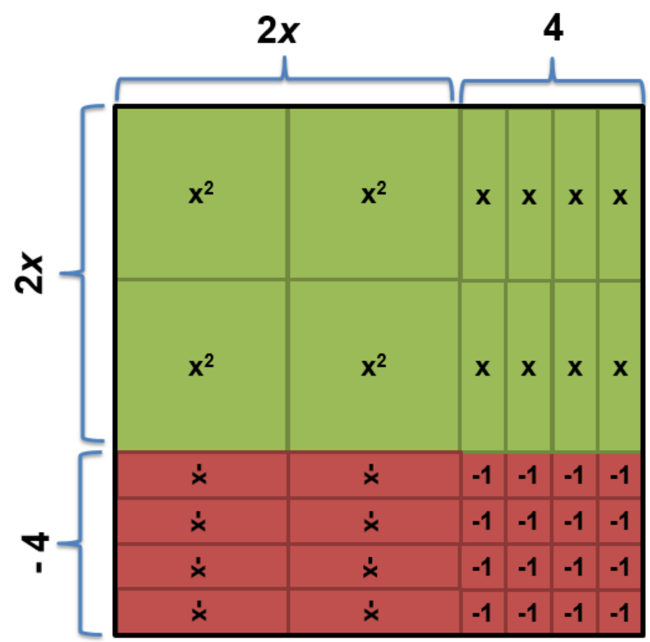
Step 1: Represent $4x^2 - 16$.



Step 2: Create a rectangle.



Step 3: Fill in the missing tiles and find the factors.



INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (A.SSE.3)

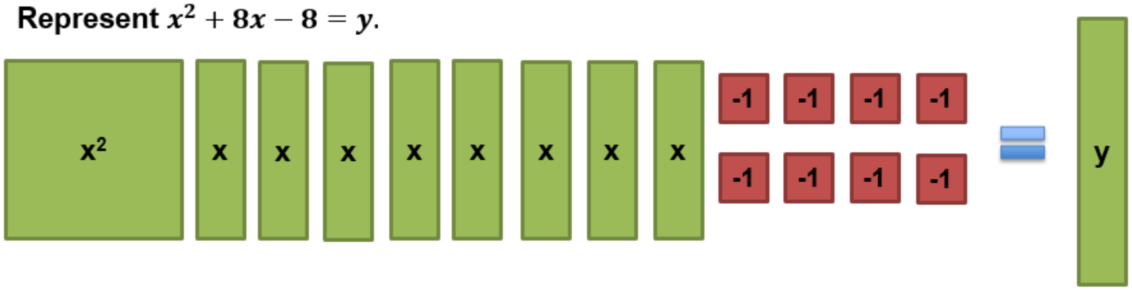
Completing the Square

Using algebra tiles for factoring should make an easy transition for students to conceptually understand completing the square.

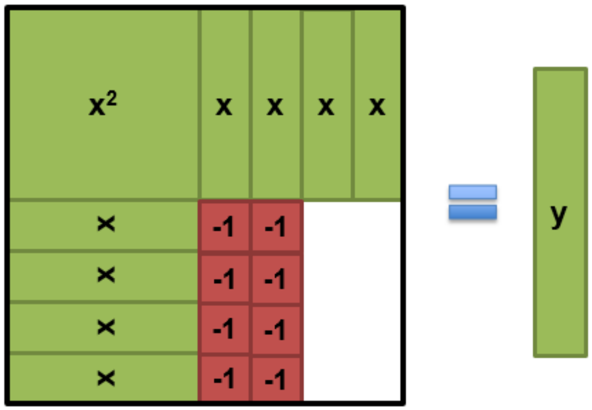
EXAMPLE

Rewrite $x^2 + 8x - 8 = y$ in vertex form by completing the square.

Represent $x^2 + 8x - 8 = y$.



Then rearrange the tiles (excluding the y) to make a square.

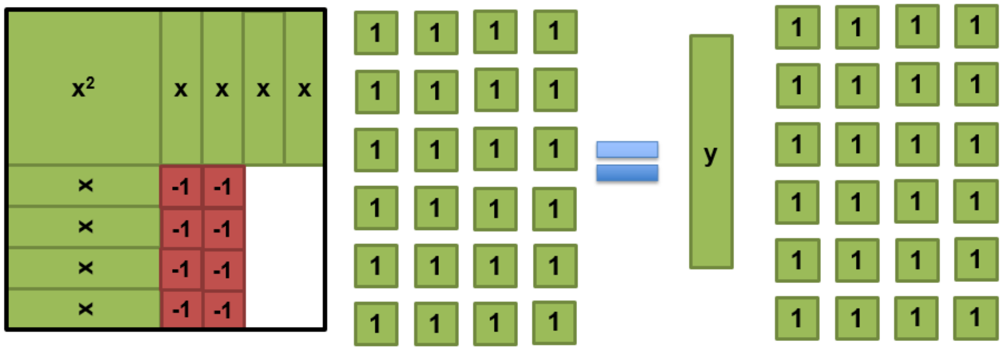


However, a problem arises because the units do not make a square. The student has to figure out how many more units are needed to make a square. The fact that the units that are present are negative is also a problem. For if all the x-term tiles are positive, the units also have to be positive in order to make a perfect square.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (A.SSE.3)

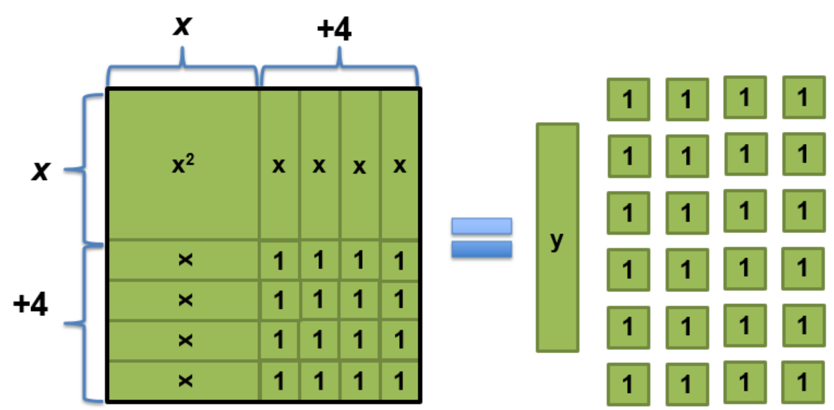
Students should realize that there needs to be 16 positive unit tiles to make a square; however they also need to take into account the –8 unit tiles already present. Therefore, the student need to add 16 + 8 or 24 unit tiles to the problem. But to maintain equivalency, 24 has to be added to both sides of the equation:

$x^2 + 8x - 8 + \boxed{24} = y + \boxed{24}$
$x^2 + 8x + 16 = y + 24$



Now he or she can finish factoring the equation.

So, the student gets $(x + 4)^2 = y + 24$ which can be rewritten in vertex form $y = (x + 4)^2 - 24$.



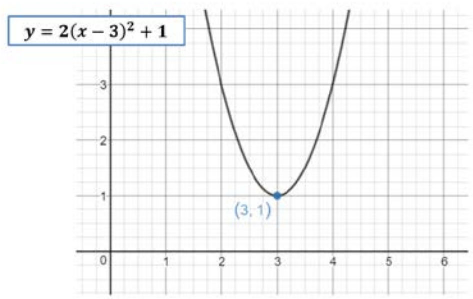
Discussion: After working with the algebra tiles, students should be able to generalize the rule of adding $\left(\frac{b}{2}\right)^2$ to both sides of the equation when completing the square in the form of $ax^2 + bx + c = 0$, where $a = 1$. Discuss with students how to approach problems when $a \neq 1$. Students may want to begin with dividing both sides by a to get a coefficient of 1 in front of the leading term to get $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$. This allows students to further generalize the rule that $\left(\frac{b}{2a}\right)^2$ is really the term that is being added to both sides of the equation. This concept could be further extended towards deriving the quadratic formula. See Completing the Square.ogv which is an applet from Wikipedia Commons that shows how to complete the square in the form of $ax^2 + bx + c = 0$ when $a \neq 1$.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (A.SSE.3)

Students should be fluent in expanding binomials of the form $(a + b)^2$, $(a – b)^2$ and $(a – b)(a + b)$ which will help them recognize perfect square and difference of squares trinomials when they need to factor them.

Vertex Form

Connect completing the square to the vertex form of an equation. Draw attention to the fact that the vertex form, $y = a(x – h)^2 + k$, of a quadratic expression reveals the vertex (h, k) and the maximum or minimum of the function it defines.



Instructional Tools/Resources

These tools and resources may be helpful for instruction. Before use, they should be evaluated at the local level to determine their appropriateness for instruction.

Manipulatives/Technology

- Graphing utilities to explore the effects of parameter changes on a graph
- Desmos is a free graphing calculator that is available to students as website or an app.
- Geogebra is a free graphing calculator that is available to students as website.
- Wolframalpha is dynamic computing tool.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (A.SSE.3)

Factoring with Algebra Tiles

- Factoring Trinomials –With Algebra Tiles by Braining Camp is a YouTube video that explains factoring trinomials with algebra tiles.
- Factoring Polynomials with Algebra Tiles (2) by Tom Horn is a YouTube video that explains factoring trinomials with algebra tiles.
- Algebra Tiles 5: Factoring Trinomials by Simpson_Math is a YouTube video that explains factoring trinomials with algebra tiles.
- Algebra tile templates on the SMART Exchange has a variety of useful models that teacher can use if they have access to a SMART Board.
- Multiplying Binomials and Factoring Trinomials using Algebra Tiles and Generic Rectangle is a worksheet from West Contra Costa Unified School District about using algebra tiles.
- Advanced Algebra Tile Factoring is an applet by eMathLab.com that allows for factoring trinomials when *a* does not equal 1.
- Lesson Plan 1: The X Factor – Trinomials and Algebra Tiles is a website that contains a lesson and a workshop that showcases ways that teachers can help students explore mathematical properties studied in Algebra. The activities use a variety of techniques to help students understand concepts of factoring quadratic trinomials.

Difference of Squares

- Difference of Squares from the National Council of Teachers of Mathematics, Illuminations is an activity that uses a series of related arithmetic experiences to prompt students to explore arithmetic statements leading to a factoring pattern for the difference of two squares. A geometric interpretation of the familiar formula is also included.

Completing the Square

- 3.3: Part 1, lesson 1 (Completing the Square using Algebra Tiles) is a YouTube video by martensmath that shows how to use algebra tiles to complete the square.
- Completing the Square-With Algebra Tiles is a YouTube video by Brainingcamp that shows how to use algebra tiles to complete the square.
- Using Algebra Tiles to Complete the Square is a YouTube video by Stacie Bender that shows how to use algebra tiles to complete the square.

Different Forms of Functions

- Graphs of Quadratic Functions is an Illustrative Mathematics task that explores graphing quadratics. It is most effective after students have graphed parabolas in vertex form, but have not yet explored graphs in different forms.
- Profit of a Company is an Illustrative Mathematics Task that compares the usefulness of different forms of quadratic expressions.
- Forms of Exponential Expressions is an Illustrative Mathematics Task that investigates usefulness of different exponential expressions.
- Representing Quadratic Forms Graphically by Mathematics Assessment Project is a lesson where students interpret different algebraic

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (A.SSE.3)

forms of a quadratic function to interpret a graph.

Curriculum and Lessons from Other Sources

- Mathematics Vision Project, Algebra 1, Module 7: Structure of Expressions has many lessons that pertain to this cluster.
- Georgia Standards of Excellence Curriculum Frameworks, Algebra 1, Unit 3: Modeling and Analyzing Quadratic Functions has several tasks that are related to this cluster. This cluster is addressed on pages 32-61, 73-92, and 148-173.
- EngageNY, Algebra 1, Module 3, Topic D, Lesson 22: Modeling an Invasive Species Population and Lesson 23: Newton's Law of Cooling are lessons that pertain to this cluster.
- EngageNY, Algebra 1, Module 4, Topic A, Lesson 1: Multiplying and Factoring Polynomial Expressions, Lesson 2: Multiplying and Factoring Polynomial Expressions, Lesson 3: Advanced Factoring Strategies for Quadratic Expressions, Lesson 4: Advanced Factoring Strategies for Quadratic Expressions, Lesson 9: Graphing Quadratic Function from Factored Form, $f(x) = a(x - m)(x - n)$ are lessons that pertain to this cluster.
- EngageNY, Algebra 1, Module 4, Topic B, Lesson 11: Completing the Square, Lesson 12: Completing the Square, and Lesson 16: Graphing Quadratic Equations from the Vertex Form, $y = a(x - h)^2 + k$ are lessons that pertain to this cluster.

General Resources

- Arizona High School Progression on Algebra is an informational document for teachers. This cluster is address on pages 4-6 and pages 11-12.
- Arizona High School Progression on Modeling is an informational document for teachers. This cluster is addressed on pages 6-7 and 13.
- High School Coherence Map by UnboundED is a map of the relationships among high school mathematics standards. Please note: This resource is aligned to the Common Core not Ohio's Learning Standards. Although the tool is useful, it needs to be checked with respect to Ohio's Learning Standards.

References

- Common Core Standards Writing Team. (2013, March 1). *Progressions for the Common Core State Standards in Mathematics (draft). High School, Algebra*. Tucson, AZ: Institute for Mathematics and Education, University of Arizona.
- Common Core Standards Writing Team. (2013, July 4). *Progressions for the Common Core State Standards in Mathematics (draft). High School, Modeling*. Tucson, AZ: Institute for Mathematics and Education, University of Arizona.
- Davis, B. (August 2015). Exponentiation: A new basic? *Mathematics Teaching in the Middle School*, *21(1),* 34-41.
- Griffith, L. (February 2016). Exponent rummy. *Mathematics Teaching in the Middle School, 21(6),* 376-377.

STANDARDS	MODEL CURRICULUM (A.APR.1)
Algebra **ARITHMETIC WITH POLYNOMIALS AND RATIONAL EXPRESSIONS** **Perform arithmetic operations on polynomials.** **A.APR.1** Understand that polynomials form a system analogous to the integers, namely, that they are closed under the operations of addition, subtraction, and multiplication; add, subtract, and multiply polynomials. **a.** Focus on polynomial expressions that simplify to forms that are linear or quadratic. (A1, M2)	**Expectations for Learning** In previous courses, students develop an understanding of the properties of integers as a number system under the operations of addition, subtraction, and multiplication. They also learn to combine like terms and simplify linear expressions. In this cluster, students explore the commonalities and differences between integers and polynomials regarding the operations of addition, subtraction, and multiplication. Students will also simplify linear and quadratic expressions, or those that simplify to linear or quadratic. In Algebra 2/Math 3, students extend these ideas to include higher-degree polynomials. **ESSENTIAL UNDERSTANDINGS** • Polynomials form a system (like the integers) in which addition, subtraction, and multiplication always result in another polynomial, but sometimes division does not. **MATHEMATICAL THINKING** • Compute accurately and efficiently. • Use different properties of operations flexibly. • Recognize and apply mathematical concepts, terms, and their properties. • Draw a picture or create a model to represent mathematical thinking. **INSTRUCTIONAL FOCUS** • Add, subtract, and multiply polynomial expressions, focusing on those that simplify to linear or quadratic expressions. **Content Elaborations** **OHIO'S HIGH SCHOOL CRITICAL AREA OF FOCUS** • Algebra 1, Number 4, pages 9-10 **CONNECTIONS ACROSS STANDARDS** • Interpret the structure of expressions (A.SSE.1).

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (A.APR.1)

Instructional Strategies

Note: The model curriculum is not a collection of lessons nor a full curriculum; it does not suggest pace, sequence, or amount of time spent on topics. It provides information about a topic related to the standards including ideas for examples, strategies for teaching, common misconceptions, and possible connections between topics. In addition, the emphasis of the model curriculum is on instruction not on assessment; therefore, the examples in the Instructional Strategies section are written for instructional purposes not necessarily as assessment items. Examples may need to be rewritten to accommodate the needs of each individual classroom.

Standards for Mathematical Practice
This cluster focuses on but is not limited to the following practices:
***MP.6** Attend to precision.*
***MP.8** Look for and express regularity in repeated reasoning.*

The primary strategy for this cluster is to make connections between arithmetic of integers and arithmetic of polynomials. In order to understand this standard, students need to work toward both understanding and fluency with polynomial arithmetic. Furthermore, to talk about their work, students will need to use correct vocabulary, such as integer, monomial, polynomial, factor, and term.

CLOSURE

Closure is the new concept in this standard. A set of numbers is *closed* over a certain operation if an operation performed between two of the members of a set results in a member of the same set. For example, anytime two or more integers are added, the result is an integer. The same holds true for multiplication and subtraction of integers. It does not hold true for the division of integers since 3 ÷ 4 = 0.75 (a rational number). Therefore integers are closed under the operations of addition, subtraction, and multiplication, but they are not closed under division. Like integers, the system of polynomials behaves the same way. The emphasis here should not be on the term closure, but rather that a polynomial results if two or more polynomials, are added, subtracted, and/or multiplied together. Like integers, closure is not always true for the division of polynomials. For examples $(3x^2 + 6x – 2) + (4x – 9)$ can be added to obtain another polynomial $3x^2 + 10x – 11$, so polynomials are closed over addition. Students should use counterexamples to prove that polynomials are not closed under division. For example, $(x + 1) \div (2x^3 + x – 5)$ does not result in a polynomial, therefore polynomials are not closed under division. Of course, the quotient of two polynomials is sometimes a polynomial. For example, $(x^2 - 9) \div (x - 3) = x + 3$.

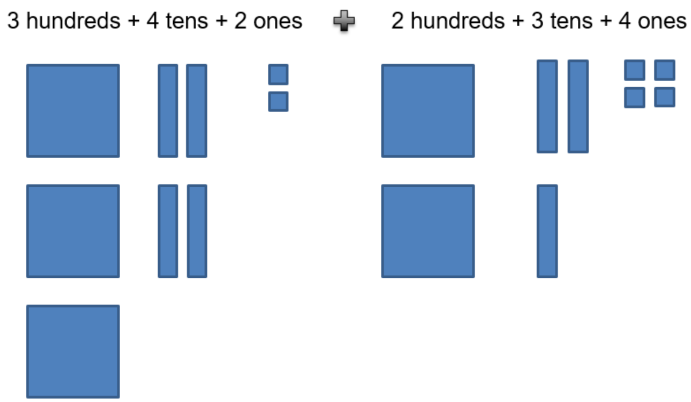
Students need to reason that the sum (difference or product) of any two polynomials is indeed a polynomial. At first, restrict attention to polynomials with integer coefficients. Later, students should consider polynomials with rational or real coefficients and reason that such polynomials are closed under these operations. Draw attention to the fact that rational expressions such as $\frac{2}{3}x^2 + \frac{1}{5x} - 6$ are not polynomials.

When dealing with fractional coefficients students may place the variable anywhere in the fraction (as can be done with the negative sign.) Draw attention to the fact that $\frac{2}{3}x^2 + \frac{1}{5}x - 6$ is not equivalent to $\frac{2}{3}x^2 + \frac{1}{5x} - 6$ since the second expression is a rational expression not a polynomial.

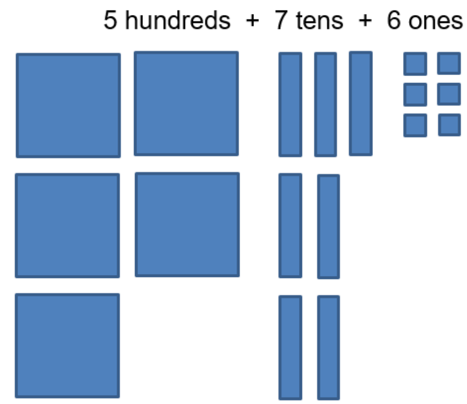
INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (A.APR.1)

CONNECTING OPERATIONS IN BASE TEN TO POLYNOMIALS

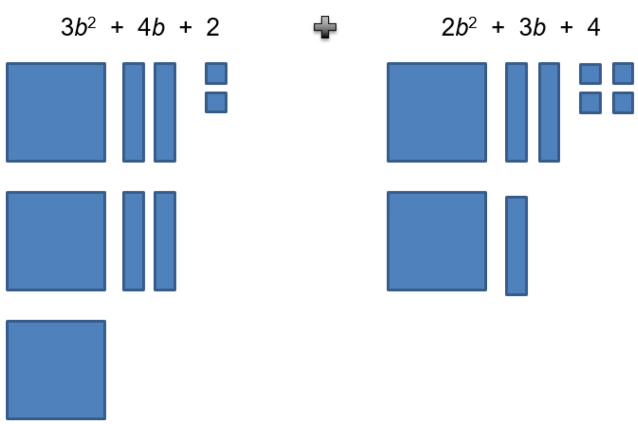
In elementary school students learn to add and subtract whole numbers with base ten blocks. For example, to add 342 and 234 students could write the numbers in expanded form such as the following:



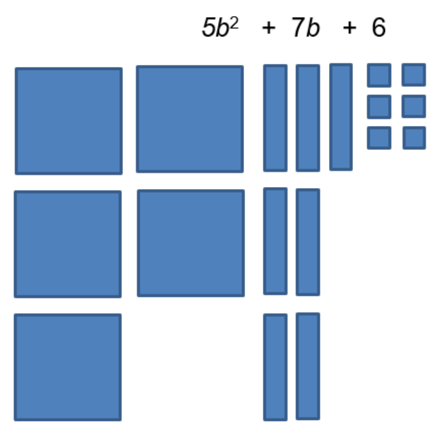
Which equals



In Algebra, students can apply the understanding that they developed with respect to place value of whole numbers to polynomials that have integer coefficients using dimensionality. For example, a 1 can be represented by a unit square, a b can be represented by a stick (1 unit by b units in lengths), a b^2 can be represented by a square (b units by b units in lengths), and a b^3 can be represented by a cube (b units by b units by b units in lengths). In addition by can be represented by a rectangle that is b units in width and y units in length.



Which equals



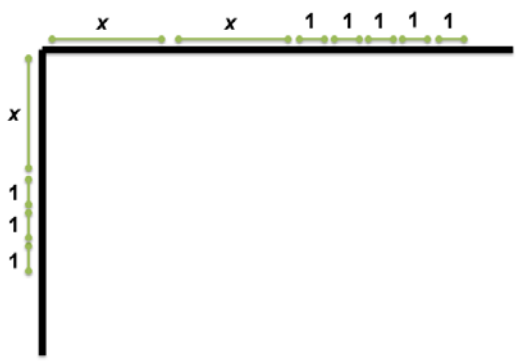
INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (A.APR.1)

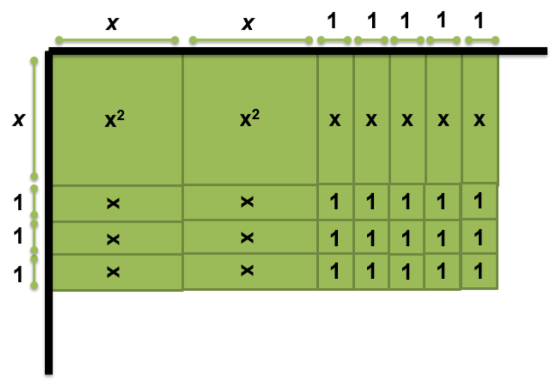
The use of algebra tiles in adding and subtracting polynomials allows students to make a connection between whole numbers and polynomials. It also allows students to see each term as a distinct entity. For example x-shaped pieces are different than x^2-shaped pieces which are also different then y^2-shaped pieces. This prevents the common error of misapplying exponent rules by adding x^2 and x. The dimensionality of the algebra tiles also allows for a geometric connection to area and even to volume. Negative integers can be represented by different color tiles just like integer chips.

Algebra tiles and/or area models can also be used to multiply polynomials. The benefit of using these models is that allows students to not only conceptualize multiplication of polynomials, but it also helps students make meaning out of factoring binomials by visualizing the connection to multiplication. For more information on factoring using algebra tiles, see A.SSE.3.

Example

Multiply $(2x + 5)(x + 3)$.





Discussion: This leads students to see that since there are two x^2-shaped pieces, eleven x-shaped pieces and fifteen square unit-shaped pieces, the product of $(2x + 5)(x + 3) = 2x^2 + 11x + 15$. This model can also be used with polynomials where x is multiplied by x^2 with the resulting piece being a cube or an x-term multiplied by a y-term with the resulting piece being a rectangle with an x-width and y-length.

TIP! Rearranging polynomials is a great opportunity to reinforce the Properties of Operations.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (A.APR.1)

Once students conceptually understand multiplying polynomials, they may choose to move more towards using a box instead of using the Algebra tiles. This follow's Bruner's levels of representation: concrete (enactive), pictorial (iconic), abstract. In addition this connects multiplying polynomials with the area models students used to understand multiplication in elementary school. Also, using the boxes as a strategy instead of algebra tiles allow for the use of coefficients involving rational numbers.

EXAMPLE

Multiply $(y - 4)(2y + 3)$

	y	–4
2y		
3		

=

	y	–4
2y	$2y^2$	$-8y$
3	$3y$	-12

Discussion: Therefore $(y - 4)(2y + 3)$ equals $2y^2 - 5y - 12$. This model is not limited to the multiplication of binomials, but can be used with the multiplication of any polynomials by extending the rectangle. After students have had experiences with the models, there should be a discussion about how the multiplication of polynomials can be done without using models. However, students should be able to use whichever method they feel the most comfortable.

In arithmetic of polynomials, emphasize that the central idea of using the Distributive Property is fundamental not only in polynomial multiplication but also in polynomial addition and subtraction. Do not use shortcuts such as FOIL since FOIL only applies to multiplying a binomial by a binomial. For example, when adding the monomials $3x$ and $2x$, the result can be explained with the Distributive Property as follows: $3x + 2x = (3 + 2)x = 5x$.

TIP!

Students previously taught FOIL, may be confused when the polynomials are trinomials or others polynomials larger than a binomial. Therefore emphasize the distributive property instead of the term FOIL.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (A.APR.1)

The connections between methods of multiplication can be generalized even further which can be seen by considering whole numbers in base ten place value to be polynomials in the base *b* = 10. For example, compare the product 213 x 47 with the product $(2b^2 + 1b + 3)(4b + 7)$:

$$\begin{array}{rrrrr} & 2b^2 & + \ 1b & + \ 3 \\ \times & & 4b & + \ 7 \\ \hline & 14b^2 & + \ 7b & + \ 21 \\ +\ 8b^3 & + \ 4b^2 & + \ 12b & \\ \hline 8b^3 & + \ 18b^2 & + \ 19b & + \ 21 \end{array} \qquad \begin{array}{rrrr} & 200 \ + & 10 \ + & 3 \\ \times & & 40 \ + & 7 \\ \hline & 1400 \ + & 70 \ + & 21 \\ +\ 8000 \ + & 400 \ + & 120 & \\ \hline 8000 \ + & 1800 \ + & 190 \ + & 21 \end{array} \qquad \begin{array}{rr} & 213 \\ \times & 47 \\ \hline & 1491 \\ + & 8520 \\ \hline & 10011 \end{array}$$

Discussion: Note how the distributive property is in play in each of these examples. In the left-most computation, each term in the factor $(4b + 7)$ must be multiplied by each term in the other factor, $(2b^2 + 1b + 3)$, just like the value of each digit in 47 must be multiplied by the value of each digit in 213, as in the middle computation, which is similar to "partial products methods" that some students may have used for multiplication in the elementary grades. The common algorithm on the right is merely an abbreviation of the partial products method.

Some students will apply the Distributive Property inappropriately. Emphasize that it is the *Distributive Property of Multiplication over Addition*. For example, the Distributive Property can be used to rewrite $2(x + y)$ as $2x + 2y$, because in this product the second factor is a sum (i.e., involving addition). But in the product $2(xy)$, the second factor, (xy), is itself a product, not a sum.

For Algebra 1, notice that the focus is on linear and quadratic expressions, but higher degrees polynomials may be used as long as the expression(s) simplify to a linear or quadratic. Therefore simplifying an expression like $x^3 - 2x^2 - (x^3 + 8x)$ is appropriate for this course.

Instructional Tools/Resources

These tools and resources may be helpful for instruction. Before use, they should be evaluated at the local level to determine their appropriateness for instruction.

Manipulatives/Technology

- Computer Algebra Systems
- Algebra tiles
- Area models

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (A.APR.1)

Algebra Tiles

- Virtual Algebra Tiles is an applet from Michigan Virtual University that allows students to use algebra tiles. This applet allows for positive and negative representation of the tiles.
- CPM Tiles is an applet from the CPM Educational Program that allows students to use algebra tiles. The benefit of this applet is that students can change the dimensionality of *x* and *y*. However, it is limited by not allowing for a negative representation of the tiles.
- Algebra tile templates on the SMART Exchange has a variety of useful models that teacher can use if they have access to a SMART Board.
- Adding and Subtracting Polynomials Using Algebra Tiles is lesson from Buffalo State University.
- Adding and Subtracting Polynomials Using Algebra Tiles is a lesson from the Virginia Department of Education.
- Algebra Tiles Workbook from Learning Resources has lessons that use Algebra tiles to add, subtract, multiply, and divide polynomials.

Multiplying Polynomials

- The Astounding Power of Area by G'Day Math is a website that has a section that focuses on how to use an area model with polynomials.

Curriculum and Lessons from Other Sources

- EngageNY, Algebra 1, Module 1, Topic B, Lesson 8: Adding and Subtracting Polynomials and Lesson 9: Multiplying Polynomials are lessons that pertain to this cluster.
- EngageNY, Algebra 1, Module 4, Topic A, Lesson 1: Multiplying and Factoring Polynomial Expressions pertains to this cluster.
- Georgia Standards of Excellence Framework, Unit 1: Relationships Between Quantities has a couple of lessons that pertain to this cluster. This cluster is addressed on pages 16-29.
- High School Coherence Map by UnboundED is a map of the relationships among high school mathematics standards. Please note: This resource is aligned to the Common Core not Ohio's Learning Standards. Although the tool is useful, it needs to be checked with respect to Ohio's Learning Standards.

General Resources

- Arizona High School Progression on Algebra is an informational document for teachers. The cluster is addressed in the first two paragraphs on page 7.

Research

- Common Core Standards Writing Team. (2013, March 1). *Progressions for the Common Core State Standards in Mathematics (draft). High School, Algebra*. Tucson, AZ: Institute for Mathematics and Education, University of Arizona.
- Gurl, T., Artzt, A., Sultan, A., & Curcio, F. (2012). *Implementing the Common Core State Standards through Mathematical Problem Solving*. Reston, VA: National Council of Teachers of Mathematics.

STANDARDS	MODEL CURRICULUM (A.CED.1-4)
Algebra **CREATING EQUATIONS** **Create equations that describe numbers or relationships.** **A.CED.1** Create equations and inequalities in one variable and use them to solve problems. *Include equations and inequalities arising from linear, quadratic, simple rational, and exponential functions.* ★ **a.** Focus on applying linear and simple exponential expressions. (A1, M1) **b.** Focus on applying simple quadratic expressions. (A1, M2) **A.CED.2** Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales. ★ **a.** Focus on applying linear and simple exponential expressions. (A1, M1) **b.** Focus on applying simple quadratic expressions. (A1, M2) **A.CED.3** Represent constraints by equations or inequalities, and by systems of equations and/or inequalities, and interpret solutions as viable or non-viable options in a modeling context. *For example, represent inequalities describing nutritional and cost constraints on combinations of different foods.* ★ (A1, M1) *Continued on next page*	**Expectations for Learning** In middle school, students create simple equations and simple inequalities and use them to solve problems. In this cluster, students extend this knowledge to write equations and inequalities for more complicated situations, focusing on linear, simple exponential, and quadratic equations. Students also rearrange formulas to highlight a particular variable. In Algebra 2, students model even more complicated situations. *Note: Simple exponential functions include integer exponents only.* **ESSENTIAL UNDERSTANDINGS** • Regularity in repeated reasoning can be used to create equations that model mathematical or real-world contexts. • The graphical solution of a system of equations or inequalities is the intersection of the graphs of the equations or inequalities. • Solutions to an equation, inequality, or system may or may not be viable, depending on the scenario given. • A formula relating two or more variables can be solved for one of those variables (the variable of interest) as a shortcut for repeated calculations. **MATHEMATICAL THINKING** • Create a model to make sense of a problem. • Represent the concept symbolically. • Plan a solution pathway. • Determine the reasonableness of results. • Consider mathematical units and scale when graphing. *Continued on next page*

STANDARDS	MODEL CURRICULUM (A.CED.1-4)
A.CED.4 Rearrange formulas to highlight a quantity of interest, using the same reasoning as in solving equations. ★ **a.** Focus on formulas in which the variable of interest is linear or square. *For example, rearrange Ohm's law $V = IR$ to highlight resistance R, or rearrange the formula for the area of a circle $A = (\pi)r^2$ to highlight radius r.* (A1)	**Expectations for Learning, continued** **INSTRUCTIONAL FOCUS** • Given a mathematical or real-world context, express the relationship between quantities by writing an equation or inequality that must be true for the given relationship. Focus on situations where the equations will be linear, exponential, and quadratic. • For equations or inequalities relating two variables, graph the relationships on coordinate axes with proper labels and scales. Focus on situations where the equations will be linear, exponential, and quadratic. • Identify the constraints implied by the scenario, and represent them with equations or inequalities. • Determine the feasibility (possibility) of a solution based upon the constraints implied by the scenario. • Solve formulas for a given variable. **Content Elaborations** **OHIO'S HIGH SCHOOL CRITICAL AREA OF FOCUS** • Algebra 1, Number 1, pages 3-4 • Algebra 1, Number 4, pages 9-10 **CONNECTIONS ACROSS STANDARDS** • Interpret the structure of expressions (A.SSE.1). • Solve equations and inequalities in one variable (A.REI.3). • Interpret parameters of linear or exponential functions (F.LE.5). • Represent and interpret equations and inequalities (including systems) with two variables graphically (A. REI.10). • Build a function that models a relationship between two quantities (F.BF.1). • Interpret the slope and intercept of a linear model (S.ID.7). • Solve systems of equations (A.REI.6). • Construct and compare linear, quadratic, and exponential models, and solve problems (F.LE.1).

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (A.CED.1-4)

Instructional Strategies

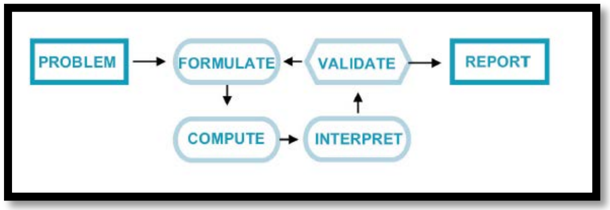
Note: The model curriculum is not a collection of lessons nor a full curriculum; it does not suggest pace, sequence, or amount of time spent on topics. It provides information about a topic related to the standards including ideas for examples, strategies for teaching, common misconceptions, and possible connections between topics. In addition, the emphasis of the model curriculum is on instruction not on assessment; therefore, the examples in the Instructional Strategies section are written for instructional purposes not necessarily as assessment items. Examples may need to be rewritten to accommodate the needs of each individual classroom.

Standards for Mathematical Practice
This cluster focuses on but is not limited to the following practices:
MP.2 *Reason abstractly and quantitatively.*
MP.4 *Model with mathematics.*
MP.5 *Use appropriate tools strategically.*
MP.6 *Attend to precision.*
MP.7 *Look for and make use of structure.*

MODELING

A.CED.1-4 is a modeling standard. See page 13 for more information about modeling.

The Arizona High School Progression on Algebra has a helpful statement with respect to modeling: "In high school, there is again a difference between directly representing the situation and finding a solution. The formulation of the equation tracks the text of the problem fairly closely, but requires more than a direct representation. The Compute node of the modeling cycle is dealt with in the next section, on solving equations. The Interpret node also becomes more complex. Equations in high school are also more likely to contain parameters than equations in earlier grades, and so interpreting a solution to an equation might involve more than consideration of a numerical value, but consideration of how the solution behaves as the parameters are varied. The Validate node of the modeling cycle pulls together many of the standards for mathematical practice, including the modeling standard itself." (Common Core Standards Writing Team, 2013)



CREATING EQUATIONS AND INEQUALITIES

Provide examples of real-world problems that can be modeled by writing an equation or an inequality. Students may believe that equations of linear, quadratic, and other functions are abstract and exist only "in a math book," without seeing the usefulness of these equations as modeling real-world phenomena. For example, use familiar contexts such as car depreciation to highlight linear and exponential equations. Begin with simple equations and inequalities and build up to more complex equations in two or more variables that may involve quadratic or exponential functions.

Make sure students have experience writing an equation of a line given two-points or given the slope and a point.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (A.CED.1-4)

Lack of Fractional Knowledge

Research has shown a link between students' knowledge of fractions and their ability to write equations. Although there has been some work done at the middle school level, some additional work may be needed for scaffolding at the high school level (See Grade 6 Model Curriculum 6.EE.5-8 and 6.EE.9 and Grade 7 Model Curriculum 7.EE.3-4 for scaffolding ideas.)

Key Words Strategy

Another issue in writing equations is that students are over reliant on key words in creating equations as they use key words in place of reasoning. This approach fails when problems become more complex and when there are several relationships between quantities. One of the problems with the Key Word approach is that it relies too heavily on numbers or values in the problem instead of the relationship between quantities, which ties into understanding the structure of an expression (A.SSE.1-2). To combat the misconceptions involving using key words, intentionally give students situations where aligning key words does not lend itself to writing equations that represents the situation. Drawing diagrams is one way to help students understand the structure of a problem. Another strategy is directing students to make sense of the situation by asking questions. Also, having students discuss their thinking in terms of quantities and relationships instead of values, calculations, and operations is another strategy for student success. Encourage students to explain "why" they did something in contrast to explaining "what" they did. It may be helpful for some students to write down all the quantities in the problem and to state the important aspects of a problem.

Writing Comparison Equations

Students also have a difficult time writing equations where the context is reversed such as "There are 8 times as many football players as cheerleaders." Many students incorrectly write $8F = C$ instead of correctly writing $F = 8C$. Studies show that this problem is not just limited to misunderstanding key words, but rather—

- incorrectly matching the order of the words in the situation to the equation;
- thinking that the larger number is placed next to the variable defining the larger group;
- treating variables as labels;
- treating variables as a fixed unknown rather than as a variable quantity; and/or
- treating the equal sign as representing equivalence but more like an association.

Students who are able to represent these problems correctly invent operations to establish equivalence thereby forcing unequal groups to be equal. They also look at the situation in terms of a function context instead of as two unrelated quantities. See Model Curriculum 8.F.4-5 for more information on writing comparison equations.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (A.CED.1-4)

TIP!

Students should always identify variables when creating equations. A reader should never have to assume anything. Students should be precise when identify variables rather than just stating x = apples, they should state x = number of apples or x = price per apple.

EXAMPLE

There are four times the number of pizza places in Centerville as there are Chinese restaurants. Write an equation to represent the number of pizza places and the number of Chinese restaurants. Make sure to define the variables.

EXAMPLE

There are twenty-seven more students in choir than in band. Write an equation to represent the number of students in choir and the number of students in band. Make sure to define the variables.

EXAMPLE

At the local steakhouse, for every 5 people who order steak, two order chicken. Write an equation to represent the number of steak entrees and the number of chicken entrees. Make sure to define the variables.

Guess and Check Strategy

Students often have difficulty writing equations and inequalities for given situations. Consider a strategy using guess and check as a process for writing an equation that must be true as described in Al Cuoco's blog entry in *Mathematical Musings*, "Teachers know that building is much harder for students than checking. The same practice of abstracting from numerical examples is useful here, too."

EXAMPLE

"Emilio drives from Tucson to Phoenix at an an average speed of 60 MPH and returns at an average speed of 50 MPH. If the total time on the road is 4.4 hours, how far is Tucson from Phoenix?"

Discussion: "The practice of abstracting regularity from repeated actions can be used to build an equation whose solution is the answer to the problem: One takes several guesses (for the distance) and checks them, focusing on the steps that are common to each of the checks. The goal isn't to stumble on (or approximate) an answer by "guess and check;" the goal is to come up with a general "guess checker" expressed as an algebraic equation:

$\frac{guess}{60}+\frac{guess}{50}=4.4$…Coherence comes from the fact that exactly the same mathematical practice is used to find an algebraic equation whose solution solves the problem." (Cuoco, 2017).

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (A.CED.1-4)

Using Arithmetic to Write a Generalized Equation Strategy

Students can use arithmetic to write an equation from a real-world context as a strategy for writing equations. They can start with an example using numbers and move towards a more general equation that is true.

EXAMPLE

Suppose a friend tells you she paid a total of $16,368 for a car, and you would like to know the car's list price (the price before taxes) so that you can compare prices at various dealers. Find the list price of the car if your friend bought the car in—

- **a.** Arizona, where the sales tax is 6.6%.
- **b.** New York, where the sales tax is 8.25%.
- **c.** A state where the sales tax is r.

Taken from Buying a Car by Illustrated Mathematics.

Discussion: Students progress from two equations that involve an actual value of sales tax to an equation that uses sales tax as a parameter. The goal is for students to start to notice regularity in solution procedures.

Using Tables to Write Equations

Students can use tables to help them notice patterns and write equations. Use quadratic contexts that lend themselves to tabular representations. They can create tables that represent this relationship simply by counting and use this table to write quadratic equations. See Model Curriculum F.BF.1-2 for more information about using tables with quadratics.

REPRESENTING CONSTRAINTS

Creating constraints involves interpreting the equation that represents the context accurately. There are different types of constraints that can be represented by equations or inequalities. A context can require a constraint to just be in one-variable $x > 3$ or in both variables $x > 3$ and $y \leq 20$. Constraints can also be written as an equation or inequality in two-variables such as $y < -2x - 40$ or even as a system of equations and/or inequalities depending on the context. They can also include identifying the number sets such as whole numbers.

Desmos Marbleslides may be useful tool to help students represent constraints.

Examine real-world graphs in terms of constraints that are necessary to balance a mathematical model with the real-world context. For example, a student writing an equation to model the maximum area when the perimeter of a rectangle is 12 inches should recognize that $A = x(6 - x)$, where A represents the rectangle's area and x represents its width, only makes sense when $0 < x < 6$. This restriction on the domain is necessary because the side of a rectangle under these conditions cannot be less than or equal to 0, but must be less than 6. Students can discuss the difference between the parabola that models the problem and the portion of the parabola that applies to the context.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (A.CED.1-4)

REARRANGING FORMULAS

Explore examples illustrating when it is useful to rewrite a formula by solving for one of the variables in the formula. For example, the formula for the area of a trapezoid, $\boldsymbol{A = \frac{1}{2}h(b_1 + b_2)}$, can be solved for *h* if the area and lengths of the bases are known, but the height needs to be calculated. This strategy of selecting a different representation has many applications in science and business when using formulas. *Note: In Algebra 1 exponential equations will not be rearranged, because this would introduce logarithms.*

TIP! "Variable of interest" means the variable or quantity a person is interested in solving for or looking for a solution or relationship. It is not interest in the sense of banks and rate of growth.

Provide examples of real-world problems that can be solved by writing an equation, and have students explore the graphs of the equations on a graphing calculator to determine which parts of the graph are relevant to the problem context.

Use a graphing calculator to demonstrate how dramatically the shape of a curve can change when the scale of the graph is altered for one or both variables.

Give students geometric formulas such as area and volume or formulas from science or business contexts, and have students solve the equations for each of the different variables in the formula.

Solving equations for the specified letter with coefficients represented by letters (e.g., $A = \frac{1}{2}h(b_1 + b_2)$ when solving for b_1) is similar to solving an equation with one variable. Provide students with an opportunity to apply the same kind of manipulations to formulas as they did to equations.

EXAMPLE

Solve for y

$\boldsymbol{\frac{a}{x} = \frac{b}{y}}$

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (A.CED.1-4)

TIP! Letters can be referred to as "variables," "parameters," or "constants," which can be helpful if they are used consistently as it may give insight into how students view a problem. However, for formulas such as Ohm's Law it may be best to avoid using those terms all together when working with this formula, because there are six different ways it can be viewed as defining one quantity as a function of the other with a third held constant. (Common Standards Core Writing Team, March 2013)

Students may believe that formulas are static, but formulas that are models may sometimes be readily transformed into functions that are models. For example, the formula for the volume of a cylinder can be viewed as giving volume as a function of area of the base and the height, or, rearranging, giving the area of the base as a function of the volume and height. Similarly, Ohm's law can be viewed as giving voltage as a function of current and resistance.

In Grade 7 students learned about proportional relationships and constants of proportionality: "7.RP.2 Recognize and represent proportional relationships between quantities." These concepts surface often in high school modeling situations. Students learn that many modeling situations begin with a statement like Ohm's Law or Newton's Second Law. In Ohm's Law, $\boldsymbol{V} = \boldsymbol{IR}$, $\boldsymbol{V}$ is the quantity of interest, $\boldsymbol{V}$ is directly proportional to $\boldsymbol{R}$ where $\boldsymbol{I}$ is the constant. The formula can be also rearranged to highlight a different quantity of interest, $\boldsymbol{I}$, where $\boldsymbol{I} = \frac{\boldsymbol{V}}{\boldsymbol{R}}$. In this case $\boldsymbol{I}$ is inversely proportional to $\boldsymbol{R}$ where $\boldsymbol{V}$ is the constant.

ADDITIONAL NOTES

To understand the differences among A.CED.1, A.CED.2, and A.CED.3, consider the following problem:

- We have 14 coins (nickels and dimes) and they are worth $0.95. How many of each coin?

For A.CED.1, students can write an equation in one variable (as shown below) and then solve:
Value of nickels + value of dimes = 95 cents
$5n + 10(14 - n) = 95$

Alternatively, for A.CED.2 and A.CED.3, students can write two equations using two variables (A.CED.2), creating a system of equations in two variables (A.CED.3) and then solve:
$n + d = 14$
$5n + 10d = 95$

All students should be able to understand both of these approaches and should be able to use them as appropriate without requiring a particular approach on a given problem.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (A.CED.1-4)

For A.CED.4, when rewriting the formula for the area of a circle to highlight radius *r* (for example), first ask students to figure out what the radius would be if the area is 10 square units. Then ask them what the radius would be if the area is 20 square units. Then if the area is 23 square units. Eventually, students should understand the rewritten equation solved for *r* is a general formula for finding the radius given any area, instead of going through the several steps to find *r* every time. This process is a way to encourage students look for and express regularity in repeated reasoning (Mathematical Practice 8).

Instructional Tools/Resources

These tools and resources may be helpful for instruction. Before use, they should be evaluated at the local level to determine their appropriateness for instruction.

Manipulatives/Technology

- Graphing calculators
- Computer software that generate graphs of functions
- Examples of real-world situations that lend themselves to writing equations that model the contexts.
- Desmos is a free graphing calculator that is available to students as website or an app.
- Geogebra is a free graphing calculator that is available to students as website.
- Wolframalpha is dynamic computing tool.
- Visual Patterns is a website that shows pictures of linear, exponential, and quadratic patterns.
- Patterns Posters for Algebra 1 from Finding Ways by Fawn Nguyen is a blog about a class activity where students are given patterns, and they have to make posters from them. She is the creator of the visual patterns link above.

Creating Equations

- Dairy Barn is a lesson connecting math standards to CTE by Achieve the Core. The lesson has students create equations in order to establish the amount of fill sand need to fill the barn stalls.
- Ivy Smith Grows Up is a lesson connecting math standards to CTE by Achieve the Core. The lesson has students evaluate the growth of newborns and infants. They write equations to model the situation.
- New and Improved Thinking Blocks by Math Playground has videos, a game, and an applet to use thinking blocks to solve system of equation problems. Thinking blocks are similar to tape diagrams.
- Buying a Car is a task from Illustrative Mathematics where students create equations that involve different values for sales tax but move towards representing sales tax as a parameter.
- Paula's Peaches-Writing Quadratics is a task from Georgia Standards of Excellence Framework, Algebra 1, Unit 3 where students create a quadratic equation from the context of growing peaches.
- Dirt Bike Dilemma is a two-part lesson by NCTM Illuminations where students write algorithms for linear programming.
- Interpreting Algebraic Expressions from Mathematics Assessment Project is a task where students translate algebraic expressions.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (A.CED.1-4)

Creating Inequalities

- How much Folate? is a task by Illustrative Mathematics that could be used as an introduction to writing and graphing linear inequalities.
- Rabbit Food is a lesson connecting math standards to CTE by Achieve the Core. The lesson has students write inequalities, equations, and applying constraints to the situation. This lesson aligns with A.CED.2-3.

Rearranging Formulas

- Forget the Formula is a task from Georgia Standards of Excellence Framework pages 49-55. In this task students will develop a formula to convert temperatures from Celsius to Fahrenheit, and then they will rearrange the formula. This aligns to A.CED.2-4.

Representing Constraints

- Cara's Candles Revisited is a task from Georgia Standards of Excellence Framework, Algebra 1, Unit 2: Reasoning with Linear Equations and Inequalities, pages 63-70. In this task students will create equations with constraints and then graph the equations.

Curriculum and Lessons from Other Sources

- EngageNY, Algebra 1, Module 1, Topic C, Lesson 19: Rearranging Formulas and Lesson 24: Applications of Systems of Equations and Inequalities are lessons that pertain to this cluster.
- EngageNY, Algebra 1, Module 1, Topic D, Lesson 25: Solving Problems in Two Ways—Rates and Algebra and Lesson 28: Federal Income Tax are lessons that pertain to this cluster.
- EngageNY, Algebra 1, Module 4: Topic A, Lesson 7: Creating and Solving Quadratic Equations in One Variable is a lesson that pertains to this cluster.
- EngageNY, Algebra 1, Module 4: Topic C, Lesson 24: Modeling with Quadratic Functions is a lesson that pertains to this cluster.
- Mathematics Vision Project , Algebra 1, Module 4: Solving Equations has lessons that pertain to this cluster.
- Exporing Symbols by Burrill, Clifford, Scheaffer is the teacher's edition of a textbook in the Data-Driven Mathematics series published by Dale Seymour Publications. There are several lessons that pertain to this cluster. The student edition can be found here.

General Resources

- Arizona High School Progression on Algebra is an informational document for teachers. This cluster is addressed on the last paragraph of pages 10-12.
- Arizona High School Progression on Modeling is an informational document for teachers. This cluster is addressed on the last paragraph of page 13 which continues on page 14 and is addressed under the Formulas as Models section on pages 16-17.
- High School Coherence Map by UnboundED is a map of the relationships among high school mathematics standards. Please note: This resource is aligned to the Common Core not Ohio's Learning Standards. Although the tool is useful, it needs to be checked with respect to Ohio's Learning Standards.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (A.CED.1-4)

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<table>
<tr><th>STANDARDS</th><th>MODEL CURRICULUM (A.REI.1)</th></tr>
<tr><td>
<h2>Algebra</h2>
REASONING WITH EQUATIONS AND INEQUALITIES
Understand solving equations as a process of reasoning and explain the reasoning.
A.REI.1 Explain each step in solving a simple equation as following from the equality of numbers asserted at the previous step, starting from the assumption that the original equation has a solution. Construct a viable argument to justify a solution method.
</td><td>
<h2>Expectations for Learning</h2>
In previous courses, students solve simple equations using a variety of methods and investigate whether a linear equation (8.EE.7) or a system of linear equations (8.EE.8) has one solution, infinitely many solutions, or no solutions. In this cluster, students explain the process for finding a solution for any type of simple equation. Similar to proofs in Geometry, students provide reasons for the steps they follow to solve an equation. In Algebra 2/Math 3, students solve simple rational and radical equations and explain why extraneous solutions may arise.
<h3>ESSENTIAL UNDERSTANDINGS</h3>

Solving equations is a process of reasoning based on properties of operations and properties of equality.
Assuming no errors in the equation-solving process,

A result that is false (e.g., 0 = 1) indicates the initial equation must have had no solutions; and
A result that is always true (e.g., 0 = 0) indicates the initial equation must have been an identity.

Adding or subtracting the same value or expression to both sides of an equation results in an equivalent equation.
Multiplying or dividing both sides by the same value or expression (except by 0) results in an equivalent equation.
The Addition Property of Equality and Subtraction Property of Equality can be used interchangeably, since subtracting a number is the same as adding its opposite.
The Multiplication Property of Equality and the Division Property of Equality can be used interchangeably (except when multiplying by 0), since dividing a number is the same as multiplying the number by its inverse.
Squaring both sides of an equation introduces new solutions; thus, when taking the square root of both sides of an equation two possible equations must be considered.

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STANDARDS	MODEL CURRICULUM (A.REI.1)
A.REI.1, continued	**Expectations for learning, continued** **MATHEMATICAL THINKING** • Explain mathematical reasoning. • Plan a solution pathway. **INSTRUCTIONAL FOCUS** *Note: Although, rote memorization of the names of the properties is not encouraged, it is expected for teachers to use formal language so that students gain familiarity and are able to recognize and apply the correct terminology.* • Justify the steps in solving an equation. • Solve equations in which there is one solution; equations in which there is no solution; and equations in which there are infinitely many solutions. **Content Elaborations** **OHIO'S HIGH SCHOOL CRITICAL AREA OF FOCUS** • Algebra 1, Number 1, pages 3-4 **CONNECTIONS ACROSS STANDARDS** • Solve linear equations and inequalities in one variable (A.REI.3). • Create equations that describe numbers or a relationship (A.CED.1).

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (A.REI.1)

Instructional Strategies

Note: The model curriculum is not a collection of lessons nor a full curriculum; it does not suggest pace, sequence, or amount of time spent on topics. It provides information about a topic related to the standards including ideas for examples, strategies for teaching, common misconceptions, and possible connections between topics. In addition, the emphasis of the model curriculum is on instruction not on assessment; therefore, the examples in the Instructional Strategies section are written for instructional purposes not necessarily as assessment items. Examples may need to be rewritten to accommodate the needs of each individual classroom.

Standards for Mathematical Practice

This cluster focuses on but is not limited to the following practices:

MP.1 *Make sense of problems and persevere in solving them.*

MP.2 *Reason abstractly and quantitatively.*

MP.3 *Construct viable arguments and critique the reasoning of others.*

MP.6 *Attend to precision.*

THE PROCESS OF REASONING IN EQUATION SOLVING

Solving equations is a process of reasoning. Each manipulation in an equation is based on a property that is known to be true. Understanding the reasoning behind each step in the manipulation of an equation helps students prevent errors or make up their own "rules."

The properties of operations (commutative, associative, distributive) should already be familiar and known to students. While the properties of equality will be used to solve equations, requiring students to use the formal names of these properties is not necessary; although students should be able to apply and recognize them. This is similar to Euclid's Common Notions where students demonstrate understanding without using formal names, but use appropriate informality. For example, Common Notion 1 states "Things which equal the same thing also equal one another" (Transitive and Symmetric Properties of Equality), and Common Notion 2 states "If equals are added to equals, then the wholes are equal" (Addition Property of Equality). Similarly, students do not need to know the formal names for the properties of equality, but should be able to recognize and apply them.

Note: The Distributive Property applied to the multiplication of two binomials should not be referred to as FOIL. Also, collecting like terms is an application of the Distributive Property (see A.APR.1). In addition, the Subtraction Property of Equality and Addition Property of Equality can be interchanged because subtraction is the same thing as adding the opposite. This also applies to the Multiplication and Division Properties of Equality.

Begin with simple, one-step equations and require students to write out a justification for each step used to solve the equation. Since students are not required to know the formal names of properties but only recognize them, provide a pool of properties for students to choose from. This type of reasoning will set the stage for proofs in Geometry.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (A.REI.1)

EXAMPLE

Step	Justification
$a + 4 = 12$	Given
$a + 4 - 4 = 12 - 4$	
$a + 0 = 12 - 4$	
$a = 12 - 4$	
$a = 8$	

Discussion: It may be helpful to give students a list of properties to choose from. In line 2 students could write the Addition or Subtraction Property of Equality or another sufficient explanation. Draw attention to the fact that the Addition and Subtraction Properties of Equality can be used interchangeably. In line 3 students could write the Additive Inverse Property or another sufficient explanation. In line 4 students could write the Additive Identity Property or other sufficient explanation. In line 5 students may write simplification or other sufficient explanation. Although students may not use the formal vocabulary, teachers should be modeling and using the correct vocabulary and encouraging it during discussion.

EXAMPLE

$6(4 - m)(3 + m) = -6m^2 + 8m$

a. Consider the equation and apply the Communtative Property of Multiplication to rearrange the equation having the same solution set.
b. Then add a value to both sides of the equation and explain why the new equation has the same solution set as part **a.**
c. Then multiply each side of the equation by a value. Discuss what would happen if you only multiplied part of one side of the equation by the value.

Discussion: The purpose of the above example is to reinforce the understanding of properties and equivalence. Draw students' attention to the fact that regardless of which property they apply, the original equation and the resulting equations have the same solution set. This example can be changed by having students apply different properties in part **a.** such as the Commutative Property of Addition to one side of the equation or applying the Distributive Property etc.

EXAMPLE

Do not solve! Determine which of the following equations have the same solution set by recognizing the properties.

a. $6(3y + 2) = 5y - 4$
b. $18y + 12 + 6y = 5y - 4 + 6y$
c. $12(6y + 4) = 10y - 8$
d. $18y + 12 = -4 + 5y$
e. $(3y + 2)6 = 5y - 4$
f. $18y + 2 = -4 + 5y$
g. $12(3y + 2) = 10y - 8$
h. $(3y + 2)6 = 5y - 4 + 6y$

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (A.REI.1)

Challenge students to justify each step of solving an equation. Transforming $2x - 5 = 7$ to $2x = 12$ is possible because $5 = 5$, so adding the same quantity to both sides of an equation makes the resulting equation true. Each step of solving an equation can be defended, much like providing evidence for steps of a geometric proof.

Provide examples for how the same equation might be solved in a variety of ways as long as equivalent quantities are added or subtracted to both sides of the equation. The order of steps taken will not matter.

EXAMPLE

Have students work in groups. Instruct Group A to subtract 2 from both sides first. Instruct Group B to add 10 to both sides first. Instruct Group C to subtract *n* from both sides first. Then have them solve the equation. Have them compare solutions with your classmates.

Group A

$$\begin{array}{rcl} 3n + 2 & = & n - 10 \\ \underline{\quad -2} & \underline{=} & \underline{\quad -2} \\ 3n & = & n - 12 \\ \underline{-n} & \underline{=} & \underline{-n} \\ 2n & = & -12 \\ n & = & -6 \end{array}$$

Group B

$$\begin{array}{rcl} 3n + 2 & = & n - 10 \\ \underline{+10} & \underline{=} & \underline{+10} \\ 3n + 12 & = & n \\ \underline{-3n} & \underline{=} & \underline{-3n} \\ 12 & = & -2n \\ -6 & = & n \end{array}$$

Group C

$$\begin{array}{rcl} 3n + 2 & = & n - 10 \\ \underline{-n} & \underline{=} & \underline{-n} \\ 2n + 2 & = & -10 \\ \underline{-2} & \underline{=} & \underline{-2} \\ 2n & = & -12 \\ n & = & -6 \end{array}$$

Discussion: Use an example similar to this one to launch a discussion about equivalence and the properties of equality. Students should come to the conclusion that all groups arrived at the same solution regardless of the method. Draw attention to the fact that in certain situations some methods are easier to solve than others. See A.REI.3-4 for more information about efficiency and solving equations.

TIP!

Discuss the difference between mathematical errors and strategic errors. Strategic errors have mathematically correct steps but are not efficient strategies for solving an equation.

Students may believe that solving an equation such as $3x + 1 = 7$ involves "only removing the 1," failing to realize that the equation $1 = 1$ is being subtracted to produce the next step.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (A.REI.1)

Instructional Tools/Resources

These tools and resources may be helpful for instruction. Before use, they should be evaluated at the local level to determine their appropriateness for instruction.

- Reasoning with Linear Inequalities is an Illustrative Mathematics task that has students identify errors in mathematical reasoning.
- Same Solutions? is an Illustrative Mathematics task that has students evaluate several equations and determine which ones have the same solutions.
- Zero Product Property 1is the first in a series of four tasks that explore the Zero Product Property by Illustrative Mathematics. Understanding this property will help students solve quadratic equations by factoring. The other three tasks have links within this task.
- How Does the Solutions Change? is an Illustrative Mathematics task that has students reason about equations without explicitly solving them.

Curriculum and Lessons from Other Sources

- EngageNY, Algebra 1, Module 1, Topic C, Lesson 11: Solution Sets for Equations and Inequalities, Lesson 12: Solving Equations, and Lesson 13: Some Potential Dangers When Solving Equations are lessons that pertain to this cluster. Students are not required to know set notation, but it could be used as an extension for more advanced students.
- Jaden's Phone Plan from Georgie Standards of Excellence Curriculum Frameworks, Algebra 1, Unit 2: Reasoning with Linear Equations and Inequalities is a lesson that pertains to this cluster. This lesson is found on pages 39-46.
- Mathematics Vision Project, Algebra 1, Module 4: Equations and Inequalities has several lessons that pertain to this cluster.

General Resources

- Arizona High School Progression on Algebra
  This cluster is addressed on page 13, paragraphs 1 and 2.
- High School Coherence Map by UnboundED is a map of the relationships among high school mathematics standards. Please note: This resource is aligned to the Common Core not Ohio's Learning Standards. Although the tool is useful, it needs to be checked with respect to Ohio's Learning Standards.

References

- Common Core Standards Writing Team. (2013, March 1). *Progressions for the Common Core State Standards in Mathematics (draft). High School, Algebra*. Tucson, AZ: Institute for Mathematics and Education, University of Arizona.

STANDARDS	MODEL CURRICULUM (A.REI.3-4)
Algebra **REASONING WITH EQUATIONS AND INEQUALITIES** **Solve equations and inequalities in one variable.** **A.REI.3** Solve linear equations and inequalities in one variable, including equations with coefficients represented by letters. **A.REI.4** Solve quadratic equations in one variable. **a.** Use the method of completing the square to transform any quadratic equation in x into an equation of the form $(x - p)^2 = q$ that has the same solutions. **b.** Solve quadratic equations as appropriate to the initial form of the equation by inspection, e.g., for $x^2 = 49$; taking square roots; completing the square; applying the quadratic formula; or utilizing the Zero-Product Property after factoring. (+) **c.** Derive the quadratic formula using the method of completing the square.	**Expectations for Learning** In previous courses, students solve linear equations and inequalities. In this cluster, students extend this knowledge to solve equations with numeric and letter coefficients. Students also solve quadratic equations (with real solutions) using a variety of methods. In other standards, students learn to factor quadratics; this cluster builds on that idea to solve quadratic equations with the Zero Product Property. In Algebra 2, students use these skills to solve more complicated equations. **ESSENTIAL UNDERSTANDINGS** • An appropriate solution path can be determined depending on whether the equation is linear or quadratic in the variable of interest. • Quadratic equations and expressions can be transformed into equivalent forms, leading to different solution strategies, including inspection, taking square roots, completing the square, applying the quadratic formula, or utilizing the Zero Product Property after factoring. • When the coefficients of the variable of interest are letters, the solving process is the same as when the coefficients are numbers. • The discriminant can show the nature and number of solutions a quadratic has. • (+) The quadratic formula is derived from the process of completing the square. **MATHEMATICAL THINKING** • Generalize concepts based on properties of equality. • Solve routine and straightforward problems accurately. • Plan a solution pathway. • Solve math problems using appropriate strategies. • Solve multi-step problems accurately. • (+) Use formal reasoning with symbolic representation. *Continued on next page*

STANDARDS	MODEL CURRICULUM (A.REI.3-4)

A.REI.3-4, continued	**Expectations for Learning, continued**
	INSTRUCTIONAL FOCUS
	**For quadratic functions, students should work with expressions in which the leading coefficient can be any real number, but assessment questions should focus on expressions with leading coefficients of 1 with occasional questions using other simple leading coefficients.*
	• Recognize when an equation or inequality is linear in one variable, and plan a solution strategy.
	• Solve linear equations and inequalities with coefficients represented by letters.
	    o For inequalities, graph solutions sets on a number line.
	• Solve compound linear inequalities in one-variable.
	    o Graph solution sets on a number line.
	• Recognize when an equation is quadratic in one variable, and choose an appropriate solution strategy:
	    o using inspection, e.g., $(x-3)^2=0$ or $x^2=-5$;
	    o taking square roots, e.g., $x^2=8$;
	    o using Zero-Product Property after factoring;
	    o completing the square; or
	    o applying the quadratic formula.
	• Determine if a quadratic functions has one solution, two solutions, or no real solutions based on the discriminant.
	• (+) Formally derive the quadratic formula using completing the square.
	Continued on next page

STANDARDS	MODEL CURRICULUM (A.REI.3-4)
A.REI.3-4, continued	**Content Elaborations** **OHIO'S HIGH SCHOOL CRITICAL AREA OF FOCUS** • Algebra 1, Number 1, pages 3-4 • Algebra 1, Number 4, pages 9-10 **CONNECTIONS ACROSS STANDARDS** • Understand solving equations as a process of reasoning (A.REI.1). • Rearrange formulas to highlight a quantity of interest (A.CED.4). • Interpret the structure of expression (A.SSE.1). • Use the structure of an expression to identify ways to rewrite it (A.SSE.2). • Write expressions in equivalent forms to solve problems (A.SSE.3). • Create equations in one variable (A.CED.1). • Graph solutions to a linear inequality on a coordinate plane. (A.REI.12)

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (A.REI.3-4)

Instructional Strategies

Note: The model curriculum is not a collection of lessons nor a full curriculum; it does not suggest pace, sequence, or amount of time spent on topics. It provides information about a topic related to the standards including ideas for examples, strategies for teaching, common misconceptions, and possible connections between topics. In addition, the emphasis of the model curriculum is on instruction not on assessment; therefore, the examples in the Instructional Strategies section are written for instructional purposes not necessarily as assessment items. Examples may need to be rewritten to accommodate the needs of each individual classroom.

This cluster should not be taught in isolation; it should be joined with A.CED.1-4 where students create real-world problems and parameters for equations in context.

Standards for Mathematical Practice
This cluster focuses on but is not limited to the following practices:
MP.1 *Make sense of problems and persevere in solving them.*
MP.2 *Reason abstractly and quantitatively.*
MP.6 *Attend to precision.*
MP.7 *Look for and make use of structure.*

LINEAR EQUATIONS

Since students were required to solve linear equations in Grade 8 (8.EE.7), the focus in Algebra 1 should be on more complicated equations especially with respect to rational numbers. See Grade 8 Model Curriculum 8.EE.7-8 for ideas for scaffolding.

Equations

Students especially have difficulty with equations involving fractions. Discuss alternative methods to solving equations.

EXAMPLE

Solve for *h*.

$$\frac{3}{4}h + \frac{5}{12} = \frac{2}{3}$$

Method 1

$$\frac{3}{4}h + \frac{5}{12} - \frac{5}{12} = \frac{2}{3} - \frac{5}{12}$$
$$\frac{3}{4}h = \frac{8}{12} - \frac{5}{12}$$
$$\frac{3}{4}h = \frac{3}{12}$$
$$\left(\frac{4}{3}\right)\frac{3}{4}h = \frac{3}{12}\left(\frac{4}{3}\right)$$
$$h = \frac{1}{3}$$

Method 2

Clear the fractions by using the multiplication property of equality, and multiply each side by the common denominator of 12.

$$\frac{12}{1}\left(\frac{3}{4}h + \frac{5}{12}\right) = \frac{2}{3}\left(\frac{12}{1}\right)$$
$$9h + 5 = 8$$
$$9h = 3$$
$$h = \frac{1}{3}$$

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (A.REI.3-4)

TIP!

Make sure students have a clear understanding of the Multiplication Property of Equality. Many students misapply the strategy of clearing fractions in situations involving the Distributive Property. For example, in the equation $\frac{2}{3}\left(a-\frac{1}{2}\right)=\frac{5}{6}$, students may have difficulty in viewing $\frac{2}{3}\left(a-\frac{1}{2}\right)$ as a term, so they incorrectly multiply $\frac{2}{3}a, -\frac{1}{2}$, and $\frac{5}{6}$ by the common denominator of 6 instead of just $\frac{2}{3}$ and $\frac{5}{6}$. Students may benefit by distributing first before clearing fractions.

EXAMPLE

$$\frac{5}{6}\left(2q-\frac{1}{2}\right)=\frac{3}{4}$$

TIP!

Emphasize the multiplication identity $1 \cdot a = a$ in equations that do not appear to have a coefficient. Have your struggling students actually write out the coefficient 1, so they combine like terms effectively.

Regardless of the type of numbers or variables in the equation or inequality, students have to examine the validity of each step in the solution process. This is a connection to A.REI.1. Continually ask students for justifications when performing each step. This should help prevent students from making up their own rules. Students must be aware of what it means to check the solution of an equation of inequality.

Variables

Draw students' attention to equations containing several letters or variables with subscripts. The same variables with different subscripts (e.g., x_1 and x_2) should be viewed as different variables that cannot be combined as like terms. A variable with a variable subscript, such as a_n, must be treated as a single variable—the *n*th term, where letters *a* and *n* have different meaning.

Some students may incorrectly believe that subscripts can be combined as $b_1 + b_2 = b_3$ and that the capitalized and lowercase version of the same letter as *d* and D is 2D (d + D = 2D) can also be combined.

TIP!

Letters do not always represent variables. Letters can be unknowns, coefficients, parameters, names of functions, etc. Try not to use the word variable as a synonym of letter.

LINEAR INEQUALITIES

Solving inequalities can be taught either at the same time or after solving equations. If inequalities are taught after equations, it is important to highlight the similarities in the solving process. Examine the validity of each step in the solution process (A.REI.1).

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (A.REI.3-4)

EXAMPLE

Consider the values {–100, –1, 3, 4, 4½, 4.9999, 5, 5.00001, 6, 7, 101} to determine the solution set for the two inequalities. How do their solutions sets differ? Explain.

- $3x - 3 > x + 7$ *vs* $3x - 3 \geq x + 7$

Discussion: Consider the values {–100, –1, 3, 4, 4½, 4.9999, 5, 5.00001, 6, 7, 101}, have students determine which values are members of the solution set and how the presence of the symbol ≥ versus > signs changes the solution set. Have students graph the solutions on a number line and compare the graphs. Discuss why the > and < have an open circle and the ≥ and ≤ signs have a closed circle.

Flipping the Inequality Sign

Have each student choose two rational numbers and plot them on a number line. Then have each student write an inequality statement to compare the numbers. Write each students' statements on the board. Then have the students multiply each side of their inequality by –1 and plot it on the same number line they used initially. Then have them write a new inequality statement. Discuss why the inequality sign flips. Have them try multiplying by other negative numbers besides –1. Ask if flipping the inequality sign also is necessary for addition. Continue this process by having students write several numerical inequality expressions. Eventually extend to algebraic inequality expressions. Discuss the Multiplication Property of Inequality and why it always works. Here is a YouTube video that models a similar idea. See Grade 7 Model Curriculum 7.EE.3-4 for more ideas about scaffolding.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (A.REI.3-4)

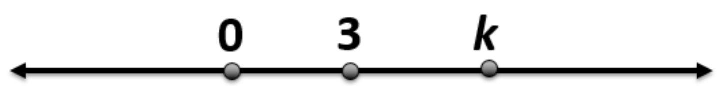
EXAMPLE

Part 1

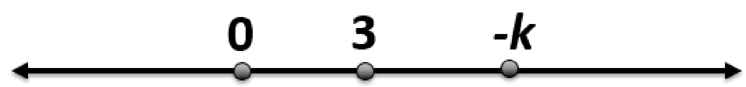
- Given that *k* is a number greater than 3, draw a number line with the points 0, *k*, and 3 that represent a possible situation. *(Shown in Line 1.)*
- What type of number is *k*? Positive? Negative? Rational? Integer? Explain.
- Write an inequality describing the relationship between 3 and *k*. *(Shown in Line 1.)*
- Write an inequality describing the relationship between 3 and –*k*. *(Still using Line 1.)*
- Given that –*k* is a number greater than 3, draw a number line with the points 0, –*k*, and 3 that represent a possible situation. *(Shown in Line 2.)*
- What type of number is *k*? Positive? Negative? Rational? Integer? Explain.
- Write an inequality describing the relationship between 3 and –*k*. *(Shown in Line 2.)*
- Write an inequality describing the relationship between 3 and *k*. *(Still shown in Line 2.)*

Example continued on next page

Line 1



Line 2



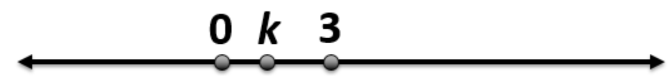
INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (A.REI.3-4)

Part 2

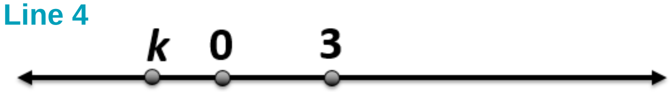
Given that $3 > k$, draw a number line with the points 0, k, and 3 that represent a possible situation.

- What types of numbers could k be? Positive? Negative? Rational? Integer? Explain.
- If k is a positive number, write an inequality comparing the situation. *(Shown in Line 3.)*
- If k is a negative number write an inequality comparing the situation. *(Shown in Line 4.)*
- Does it matter in this case if the value of k is positive or negative?
- Plot $-k$ in Lines 5-7. What do you notice about $-k$?
- If we still hold to $3 > k$, can we write an inequality comparing 3 to $-k$? Explain.
- However, if we still hold to $3 > k$, what do we know for sure about the value of $-k$ value? Explain.
- Write an equivalent inequality to $3 > k$ using $-k$. Explain why it is true.
- Using what you learned, write an equivalent inequality using $-p$ for the following:
  - o $p < 4$
  - o $2.5 > p$
  - o $-3 < p$
  - o $p > -\frac{1}{4}$
  - o $p \geq 7$
- Using what you learned write an equivalent inequality using b for the following:
  - o $-b > 5$
  - o $-b < 2.4$
  - o $-3 > -b$
  - o $\frac{5}{6} < -b$
  - o $-b < -6.1$
- Write a rule that explains to a friend how to change any inequality with a negative variable to an equivalent one with a positive variable. Make sure to explain why it works.

Line 3



Line 4



Lines 5-7

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (A.REI.3-4)

Parameters

Give students real-world contexts modeled by equations or inequalities where students need to incorporate parameters.

EXAMPLE

Write and solve an inequality to represent the following situation:

Dee is the school treasurer. She needs to buy a large number of pencils for a fundraiser at school, so she decides to join Costco. A Costco membership costs $60 and sells 96 pencils in a pack for $11.49. If she has $300 in the account, how many packs of pencils can she buy?

Discussion: The solution to this inequality is $p < 21$, where p is the set of integers because one cannot buy part of a pencil package. Also the number of packages cannot be negative which makes the solution a compound inequality $0 < p < 21$ that contains only natural numbers between 0 and 21.

Compound Inequalities

Compound inequalities need to be addressed in contextual situations to aid student understanding. It may be helpful to give students the opportunity when given an inequality to create their own conjunction (and) and disjunction (or) word problem situations. Begin by developing the concept that an "and" statement is an overlap an that an "or" statement is a "union" by using familiar categorical contexts. Then move towards using numbers and graphing compound inequalities on a number line. After students become familiar with the contexts of "and" and "or" move to real-world examples using numbers that can be graphed using compound inequalities.

EXAMPLE

Part 1

Have two students list their favorite fruits.

Beth:

Jennifer:

a. What are the favorite fruits of Beth *and* Jennifer?
b. What are the favorite fruits of Beth *or* Jennifer?

Discussion: Draw attention to the fact that the word "and" refers to the overlap between the two sets of fruit. Therefore the solution to part **a.** is bananas and oranges since both girls list those fruits as one of their favorites. The solution in part **b.** is each fruit listed either by Beth *or* Jennifer which would be strawberries, oranges, grapes, apples, bananas, cherries, pears, lemons. Notice that repeated fruit is only listed once, not twice.

Example continued on next page

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (A.REI.3-4)

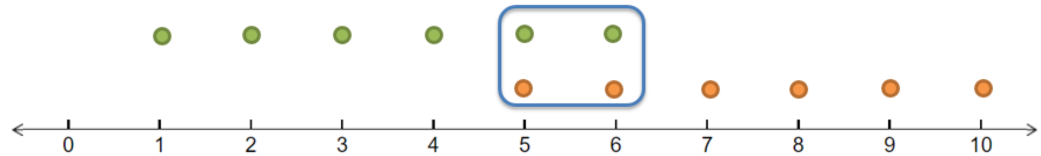
Part 2

- **a.** Use a number line to show the whole numbers between 1-10 that are bigger than 4 *and* smaller than 7?
- **b.** Use a number line to show the whole numbers between 1-10 that are bigger than or equal to 4 *and* smaller than or equal to 7?
- **c.** Use a number line to show the whole numbers between 1-10 that are bigger than 4 *or* smaller than 7?

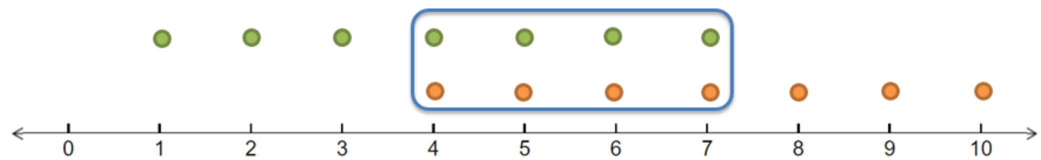
Discussion: Students should build off the idea behind "and" and "or" in the previous fruit context. Draw attention to the fact that "and" refers to the overlap of the solutions between the two sets and "or" refers to each number listed in either set. So the solution to part **a.** is 5 and 6. The solution to part **b.** is 4, 5, 6, and 7. The solution to part **c.** is 1, 2, 3, 4, 5, 6, 7, 8, 9, and 10. Notice that the numbers are listed only listed once, not twice.

Example continued on next page

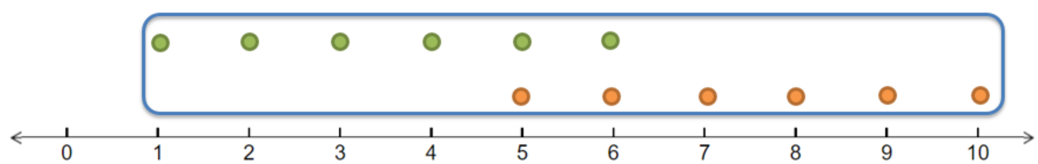
Bigger than 4 ***and*** smaller than 7



Bigger than or equal to 4 ***and*** smaller than or equal to 7



Bigger than 4 ***or*** smaller than 7



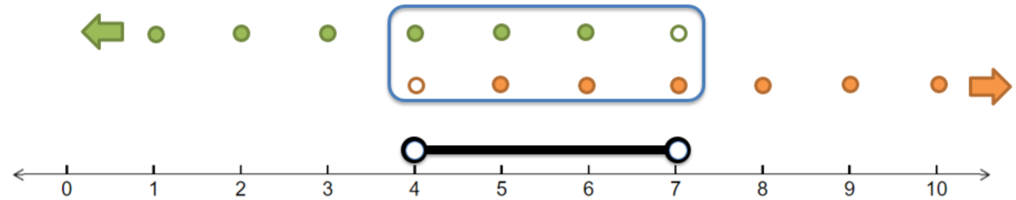
INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (A.REI.3-4)

Part 3

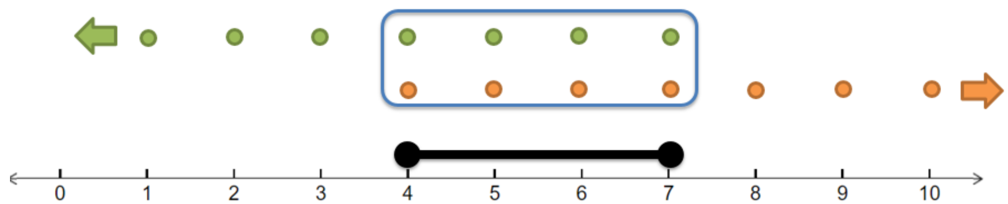
- **a.** Use a number line to show $x > 4 \; and \; x < 7$.
- **b.** Use a number line to show $x \geq 4 \; and \; x \leq 7$.
- **c.** Use a number line to show $x > 4 \; or \; x < 7$.

Discussion: Connect the inequalities to the previous two examples. Draw attention to the fact that "and" still mean an overlap or intersection of the two statements. The solution to part **a.** is $4 < x < 7$, and the solution to part **b.** is $4 \leq x \leq 7$. The solution to part **c.** is all real numbers since the graph covers the entire number line.

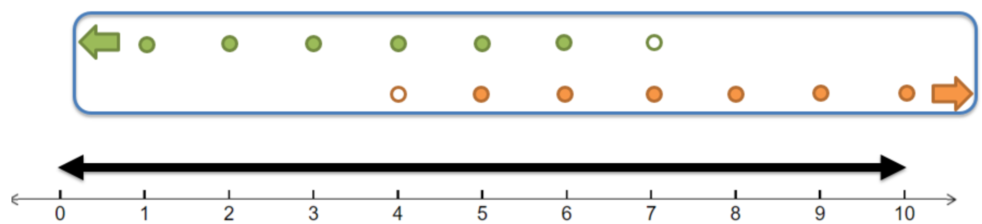
Greater than 4 ***and*** less than 7



Greater than or equal 4 ***and*** less than or equal to 7



Greater than 4 ***or*** *less* than 7



INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (A.REI.3-4)

QUADRATICS

All students need to be exposed to solving quadratic equations by using inspection, taking square roots, factoring, completing the square, and the quadratic formula.
Note: Students in Algebra 1 will only be expected to solve for real solutions.

Highlight and compare different approaches to solving the same problem. Use technology to recognize that two different expressions or equations may represent the same relationship. For example, since $x^2 + 2x - 8 = 0$ can be rewritten as $(x - 2)(x + 4) = 0$ or $(x + 1)^2 - 9 = 0$, these are all representations of the same equation that have the solutions $x = -4, 2$. Demonstrate that they are equivalent by setting all expressions $x^2 + 2x - 8$, $(x - 2)(x + 4)$, and $(x + 1)^2 - 9$ equal to y and graph them using technology. Compare their graphs and tables, displaying the same output values for each expression or by looking at their y-intercepts.

Inspection

Students should have some practice solving quadratics by inspection. For example, the solution for $x^2 = 9$ is 3 and -3. *Note: this does not work for every quadratic.*

Square Roots

Offer students examples of a quadratic equation, such as $x^2 - 9 = 0$. They could rewrite this in the form $x^2 = 9$ and square root both sides to get two solutions $x = 3$ and $x = -3$. Connect the solutions to the graph of its function. Then they should contrast their graph of $y = x^2 - 9$ to the graph of the quadratic function $y = x^2 + 9$ which is situated above the x–axis and opens upwards because it can be viewed as a vertical transformation of $y = x^2$. Since the graph of $y = x^2 + 9$ graph does not have x–intercepts, the quadratic equation does not have real solutions. This make sense since $x^2 = -9$ has no real solution. Students should realize that using square roots to solve quadratics may be used anytime an equation in x can be changed into the equation of the form $(x - p)^2 = q$ and then into the $x - p = \pm\sqrt{q}$ to be solved by taking a square root of both sides of the equation.

Students may incorrectly think that $y = (x - 5)^2$ is the graph of $y = x^2$ shifted 5 units to the left. Use the technology to explore examples to counter this thinking, and have students justify why this is not the case.

EXAMPLE

Solve using square roots.
$(y - 3)^2 - 16 = 0$

Discussion: Students should notice that there are two perfect squares in the equation: $(y - 3)^2$ and 16. Therefore, it may be wise to isolate $(y - 3)^2$ by adding 16 to both sides of the equation to get $(y - 3)^2 = 16$. Then the student can take the square root of both sides to get $y - 3 = \pm 4$ or $y = 7$ and -1. Draw attention to the fact that there are two solutions for y.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (A.REI.3-4)

Completing the square

Completing the square is usually introduced for several possible reasons:

- o to find the vertex of a parabola when an equation is written in standard or general form;
- o to look at the parabola through the lenses of transformations of a "parent" parabola $y = x^2$;
- o to derive the quadratic formula *(not required for all students).*

Connect completing the square to inspection:

- Start by inspecting equations such as $x^2 = 9$ that has two solutions, 3 and –3.
- Next, progress to equations such as $(x – 7)^2 = 9$ by substituting $x – 7$ for x.
- Then solve them either by inspection or by taking the square root on each side:

$$x - 7 = 3 \quad \text{and} \quad x - 7 = -3$$
$$x = 10 \qquad\qquad x = 4$$

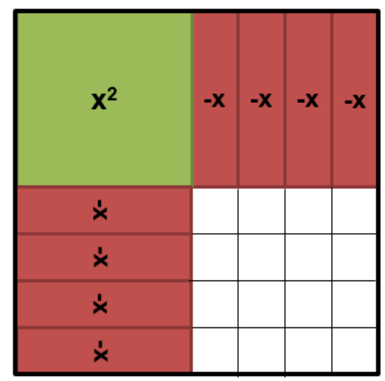
- Graph both pairs of solutions (–3 and 3, 4 and 10) on the number line and notice that 4 and 10 are 7 units to the right of –3 and 3.
- So, the substitution of $x – 7$ for x moved the solutions 7 units to the right.
- Next, graph the function $y = (x – 7)^2 – 9$, pointing out that the x-intercepts are 4 and 10, and emphasizing that the graph is the translation of 7 units to the right and 9 units down from the position of the graph of the parent function $y = x^2$ that passes through the origin (0, 0).
- Generate more equations of the form $y = a(x – h)^2 + k$ and compare their graphs using graphing technology.

EXPLORE

a. *Introduce completing the square using a visual model such as algebra tiles.*

- Create a perfect square trinomial with algebra tiles to represent $x^2 – 8x +$ _____.

Students should rearrange tiles to make a square. They should come to the realization that they cannot make a perfect square unless they add some unit tiles.



- How many more unit squares are needed to make a perfect square?
- Write the quadratic expression represented with the algebra tiles in standard form and factored form.

Give students several more examples such as $x^2 + 4x +$ ___, $x^2 – 6x +$ ___, and $x^2 + 2x +$ ___.

- Given a quadratic expression in the form of $x^2 + bx + c$, make up a rule to find c that makes a perfect square.

(Exploration continued on next page)

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (A.REI.3-4)

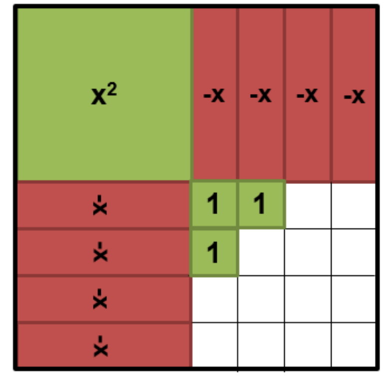
b. *Once a student can model creating a perfect square trinomial, they will be able to transform a quadratic equation into the equivalent vertex form $(x-p)^2 = q$.*

- Create an equivalent equation in vertex form of the equation $x^2 - 8x + 3 = 0$. First represent the equation in standard form with algebra tiles.

Note: A student could rearrange the unit squares differently.

- How many more unit squares will it take to make a square?

The student should answer 13.



Remember, according to the Addition Property of Equality we can add any number to both sides of an equation and still maintain equivalency, so we can write the following equation:

$x^2 - 8x + 3 + 13 = 0 + 13$
$x^2 - 8x + 16 = 13$

Now if we factor the left side, we will get an equivalent equation in vertex form $(x-4)^2 = 13$.

After students have worked with several examples on completing the square using algebra tiles, have students generalize a rule for completing a square. Make sure to introduce examples where c is greater than b. A worksheet on Completing the Square using Algebra Tiles can be found [here](#).

A teacher can also guide students in transforming a quadratic equation in standard form, $0 = ax^2 + bx + c$, to the vertex form $0 = a(x-h)^2 + k$ by separating your examples into groups with $a = 1$ and $a \neq 1$ and have students guess the number that needs to be added to the binomials of the type $x^2 + 6x$, $x^2 - 2x$, $x^2 + 9x$, $x^2 - \frac{2}{3}x$ to form a complete square of the binomial $(x-m)^2$. Then they can generalize the process by showing the expression $\left(\frac{b}{2}\right)^2$ that has to be added to the binomial $x^2 + bx$ when $a = 1$. Completing the square for an expression where $a \neq 1$can be challenging for some students. Present multiple examples of the type $0 = 2x^2 - 5x - 9$ to emphasize the logic behind every step, keeping in mind that the same process will be used to complete the square in the proof of the quadratic formula. See A.SSE.3 for more information on completing the square.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (A.REI.3-4)

Quadratic Formula

Discourage students from giving a preference to a particular method of solving quadratic equations. Students need experience in analyzing a given problem to choose an appropriate solution method before their computations become burdensome. Point out that the Quadratic Formula, x = $\frac{-b\pm\sqrt{b^2-4ac}}{2a}$, is a universal tool that can solve any quadratic equation; however, it is not the most efficient method when the quadratic equation is missing either a middle term, *bx*, or a constant term, *c*. When it is missing a constant term, (e.g., $3x^2 - 10x = 0$) a factoring method becomes more efficient. If a middle term is missing (e.g., $2x^2 - 15 = 0$), a square root method is usually more appropriate. Stress both the benefit of memorizing the Quadratic Formula and the flexibility of using a factoring strategy. Introduce the concept of discriminants and their relationship to the number and nature of the roots of quadratic equation.

Make connections between the form of the quadratic formula $x = -\frac{b}{2a} \pm \frac{\sqrt{b^2-4ac}}{2a}$, the formula for axis of symmetry, and the distance from the axis of symmetry to the zeros. Connect this to the concept of the discriminant and why the equation has one solution, no solution, or zero solutions.

Even though deriving the quadratic formula is not a requirement for all students, teachers may want to demonstrate the process.

Factoring and the Zero Product Property

Equations that are in factored form or can easily be written in factored form can be solved by using the Zero Product Property. Students should come with the understanding that any number when multiplied by 0 is 0 (See A.REI. 1); this includes expressions such as monomials, binomials and trinomials.

In equations where the product of two factors equals 0 ($a \cdot b = 0$), either *a* or *b* must equal 0, for the equation to be true. Although, *a* and *b* should not be limited to a single variable or number, since they can be represented by expressions such as binomials.

To make sense of solving quadratics by factoring, connect the graph to two linear equations: $f(x) = x + 3$ and $f(x) = x - 1$. Then multiply the solutions of two linear equations that make up the quadratic equation.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (A.REI.3-4)

EXAMPLE

Part 1

Exploring $y = (x + 3)(x - 1)$

	x	$x + 3$	$x - 1$	$(x + 3)(x - 1)$
Points 1	–4	–1	–5	5
Points 2	–3			
Points 3	–2			
Points 4	–1			
Points 5	0			
Points 6	1			
Points 7	2			

Figure 1

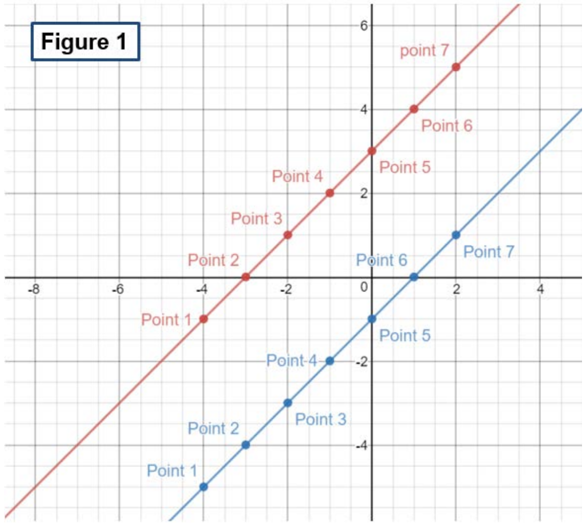
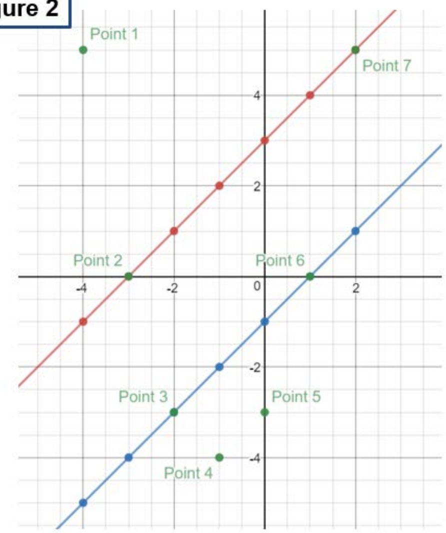


Figure 2



- Fill out Column 3 and Column 4 in the table for points 1-7.
- Use the points on the table to graph $y = x + 3$ in red and $y = x - 1$ in blue on the same coordinate plane (Figure 1).
- Looking at the graph multiply the y-coordinates of each point with the same x-coordinate. In green, plot the new points keeping the x-coordinate the same, but the y-value is the product of the y-values of the two lines (Figure 2).
- Fill out Column 5 in the table by multiplying the values in Column 3 and Column 4. How do the values in Column 5 connect to the green dots that you graphed? Why?
- How can you graph $y = (x + 3)(x - 1)$ using the binomials $(x + 3)$ and $(x - 1)$?
- Graph 3 more points on $y = (x + 3)(x - 1)$ using the binomials.
- Connect the green dots.
- What shape do you notice?

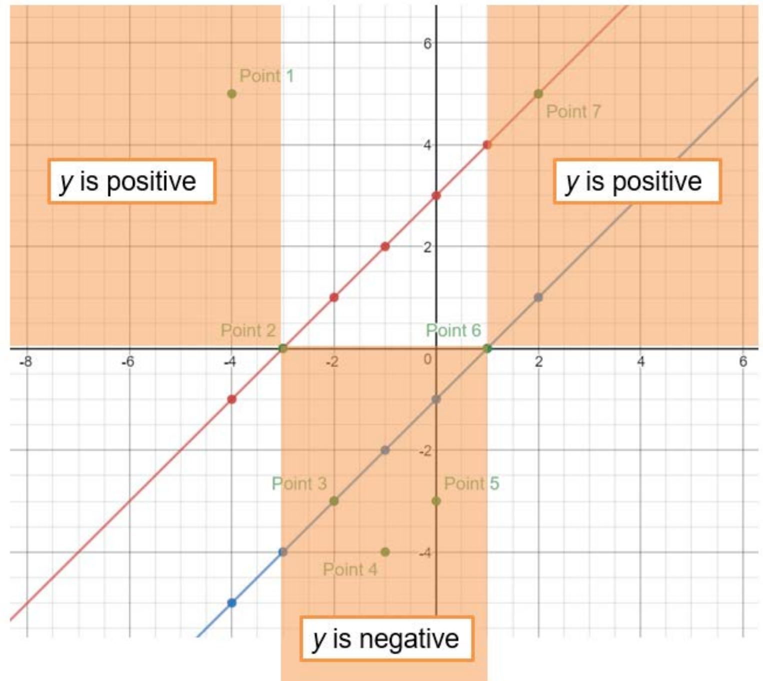
Discussion: The students should connect the multiplication of binomials to the solutions of the quadratic equation. Graph both linear equations on the same coordinate plan and have them reason about the lines to make conclusions about the graph of the quadratic equation. They should discover that the product of the two corresponding y-values is the y-coordinate of the of the product of the linear equations.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (A.REI.3-4)

Part 2

Exploring $y = (x + 3)(x – 1)$

	x	$x + 3$	$x – 1$	Is the product of $(x + 3)(x – 1)$, positive, negative, or 0?	Which quadrant is $(x + 3)(x – 1)$ in?
Points 1	–4	–1	–5	Positive	Quadrant 2
Points 2	–3				
Points 3	–2				
Points 4	–1				
Points 5	0				
Points 6	1				
Points 7	2				



- Fill out the table.
- Shade in the regions of the graph where the equation $y = (x + 3)(x – 1)$ could possibly lie based on analyzing if the product of $(x + 3)$ and $(x – 1)$ is positive, negative, or 0.
- Why does the product of two linear equations have to be a U-shaped graph? Explain.
- Will the product of any two linear equations result in a U-shaped graph? Explain.

Discussion: Part 2 has students extend their understandings from Part 1 in order to make generalizations about all quadratics consisting of the product of two linear equations. Now they should see that the since the product of any two linear equations creates a U-shaped graph since the products of the *y*-coordinates at the ends of lines will be on the opposite top or bottom half of the figure compared to the products of the *y*-coordinate in the middle of the figure of the lines. Prompt students to explore a variety of quadratics consisting of the product of two linear equations such as $y = (3x + 1)(x + 3)$; $y = (2x – 2)(2x + 1)$; $y = (\frac{1}{3}x – 1)(2x – 1)$; $y = (–\frac{1}{2}x + 1)(x + 4)$; $y = (2x + 2)(–x – 3)$; $y = (–3x + 4)(–\frac{1}{2}x + 1)$. This example could also be extended to explore why larger coefficients lead to narrower parabolas. Students could discover that the as the coefficients get bigger the product becomes greater, therefore the parabola becomes narrower.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (A.REI.3-4)

Make sure students understand the Zero Product Property and its relationship to solving quadratic equations by factoring.

EXAMPLE

Given $x^2 – x = 2$

- First set the equation equal to zero: $x^2 – x – 2 = 0$.
- Factor the left side: $(x + 1)(x – 2) =0$.
- Highlight the Zero Product Property.
- Set each factor equal to zero and solve.
- The solution is $x = –1$ or $x = 2$.

Discussion: Highlight the conceptual understanding behind the process of using the Zero Product Property instead of emphasizing procedures. Explain that, although, we could set an equation equal to any number, it is more important to set the equation equal to 0, because of the Zero Product Property. Since, if one of the binomials equals 0, the whole side equals 0.

One misconception is that students will factor $x^2 – x = 2$ as $x(x – 1) = 2$ instead of setting the equation equal to 0, and then state the solutions to be $x = 2$ or 1 because they think $2 \cdot 1 = 2$. However, infinitely pairs of numbers can give a product of 2.

Choosing a Method to Solve Quadratics

Give students different situations where it is more efficient to choose a particular method for solving quadratics over another. Emphasize that they need to be strategic problem solvers.

Instructional Tools/Resources

These tools and resources may be helpful for instruction. Before use, they should be evaluated at the local level to determine their appropriateness for instruction.

Manipulatives/Technology

- Desmos is a free graphing calculator that is available to students as website or an app.
- Geogebra is a free graphing calculator that is available to students as website.
- Wolframalpha is dynamic computing tool.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (A.REI.3-4)

Algebra Tiles

- Algebra Tiles Applet by NCTM Illuminations is a link to a virtual algebra tiles applet.
- Virtual Algebra Tiles is an applet from Michigan Virtual University that allows students to use algebra tiles. This applet allows for positive and negative representation of the tiles.
- CPM Tiles is an applet from the CPM Educational Program that allows students to use algebra tiles. The benefit of this applet is that students can change the dimensionality of *x* and *y*. However, it is limited by not allowing for a negative representation of the tiles.
- Algebra tile templates on the SMART Exchange has a variety of useful models that teacher can use if they have access to a SMART Board.

Inequalities

- Compound Inequalities on the Number Line is a Desmos activity that introduces compound inequalities.
- Best Buy Tickets by Mathematics Assessment Project is a task where students write and solve inequalities.

Completing the Square

- Completing the square with Algebra Tiles is an activity from Ms. Hennessey's Classroom blog that connects to a visual model for completing the square. In this activity students draw algebra tiles.

Interpreting Quadratic Equations

- Projectile Motion by PhET is an applet that allows exploration of projectile motion.
- Weightless Wonder is a NASA activity that has students solve quadratic function involving the parabolic flights of NASA's Weightless Wonder jet.
- Quadratic Sequence 1 is an Illustrative Mathematics task that presents students with a sequence of figures for quadratic functions. Students are required to analyze the quadratics and rewrite them in different forms. Quadratic Sequence 2 and Quadratic Sequence 3 are part of this series.
- Why Does It Stay In Orbit? by YummyMath is a task that has students interpret quadratic equations.
- Throwing Up Again by Yummy Math is a task where students manipulate quadratic equations to establish a trajectory for snow throwing.
- My Teacher Says This Stream Is Parabolic. Is He Correct? by YummyMath has students rearrange an equation into vertex form.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (A.REI.3-4)

Curriculum and Lessons from Other Sources

- EngageNY, Module 1, Topic C, Lesson 10: True and False Equations, Lesson 11: Solution Sets for Equations and Inequalities, Lesson 12: Solving Equations, Lesson 13: Some Potential Dangers when Solving Equations, Lesson 14: Solving Inequalities, Lesson 15: Solution Sets of Two or More Equations (or Inequalities) Joined by "And" or "Or," Lesson 16: Solving and Graphing Inequalities Joined by "And" or "Or", Lesson 17: Equations Involving Factored Expressions are lessons that pertain to his cluster.
- EngageNY, Module 4, Topic A, Lesson 5: The Zero Product Property, Lesson 6: Solving Basic One-Variable Quadratic Equations, Lesson 7: Creating and Solving Quadratic Equations in One Variable are lessons that pertain to this cluster.
- EngageNY, Module 5, Topic B, Lesson 11: Completing the Square, Lesson 12: Completing the Square, Lesson 13: Solving Quadratic Equations by Completing the Square, Lesson 14: Deriving the Quadratic Formula, Lesson 15: Using the Quadratic Formula, Lesson 16: Graphing Quadratic Equations from the Vertex From, $y = a(x - h)^2 + k$ are lessons that pertain to this cluster. Note: Although some students may benefit from learning how to derive the quadratic formula, it is not required for all students.
- Mathematics Vision Project, Algebra 1, Module 4: Equations and Inequalities has several tasks that pertain to this cluster.
- Georgia Standards of Excellence Curriculum Frameworks, Algebra 1, Unit 3: Modeling and Analyzing Quadratic Functions has several tasks that pertain to this cluster. These tasks can be found on pages 62-84, 148-173, and 189-194.

General Resources

- Arizona High School Progressions on Algebra is an informational document for teachers. This cluster is addressed on page 13, paragraph 3.
- High School Coherence Map by UnboundED is a map of the relationships among high school mathematics standards. Please note: This resource is aligned to the Common Core not Ohio's Learning Standards. Although the tool is useful, it needs to be checked with respect to Ohio's Learning Standards.

Research

- Bazzini, L. & Tsamir, P. (2004). Algebraic equations and inequalities: Issues for research and teaching. *Proceedings of the 28th Conference of the International Group for the Psychology of Mathematics Education*, *1,* 137-166.
- Ceyanes, B., Lockwood, P., Gill, K. (December 2014). Three lessons on parabolas—what, where, why. T*he Mathematics Teacher*. *108(5),* 368-375.
- Common Core Standards Writing Team. (2013, March 1). *Progressions for the Common Core State Standards in Mathematics (draft). High School, Algebra*. Tucson, AZ: Institute for Mathematics and Education, University of Arizona.
- Garoflao, J. & Trinter, C. (November 2012). Tasks that make connections through representations. *Mathematics Teacher*, *106(4),* 302-307.
- Jimenez, B., Browder, D., & Courtade, G. (2008). Teaching an algebraic equation to high school students with moderate developmental disabilities. *Education and Training in Development Disabilities*, *43(2),* 266-274.
- Swartz, B. (February 1, 2016). Making Sense of Factoring (Part 1): Laying the Foundation [Blog Post]. Retrieved from https://www.nctm.org/Publications/Mathematics-Teacher/Blog/Making-Sense-of-Factoring-(Part-1)_-Laying-the-Foundation/.
- Weiss, M. (December 2016/January 2017). Solving and graphing quadratics with symmetry transformation. *Mathematics Teacher*, *110(5),* 394-397.

STANDARDS	MODEL CURRICULUM (A.REI.5-7)
Algebra **REASONING WITH EQUATIONS AND INEQUALITIES** **Solve systems of equations.** **A.REI.5** Verify that, given a system of two equations in two variables, replacing one equation by the sum of that equation and a multiple of the other produces a system with the same solutions. **A.REI.6** Solve systems of linear equations algebraically and graphically. **a.** Limit to pairs of linear equations in two variables. (A1, M1) **A.REI.7** Solve a simple system consisting of a linear equation and a quadratic equation in two variables algebraically and graphically. *For example, find the points of intersection between the line* $y = -3x$ *and the circle* $x^2 + y^2 = 3$.	**Expectations for Learning** In previous courses, students solve systems of linear equations graphically with an emphasis on the meaning of the solution. In this cluster, students solve systems of linear and quadratic equations in two variables graphically and algebraically, with a focus on the meaning of a solution to a system of equations. In Algebra 2, students solve systems of equations in three variables. Students who plan to take advanced mathematics courses (+) will represent systems of equations with matrices and use inverse matrices to solve the system. **ESSENTIAL UNDERSTANDINGS** • The graph of a linear equation is the set of ordered pairs that make the equation true. Therefore, multiplying that equation by a non-zero constant produces an equivalent equation, which has the same set of ordered pairs that make the equation true. • If a system of equations in two variables has a unique solution, then the sum of one equation and a (non-zero) multiple of the other equation also has that same solution. • The graphical solution to a system of equations in two variables is the intersection of the equations when graphed. • The solution to a system of equations in two variables is the set of ordered pairs that satisfies both equations. • A system of two linear equations can have no solutions, one solution, or infinitely many solutions. • A system of a linear equation and a quadratic equation can have no solutions, one solution, or two solutions. **MATHEMATICAL THINKING** • Determine reasonableness of results using informal reasoning. • Solve multi-step problems accurately. • Plan a solution pathway. • Use technology strategically to deepen the understanding. *Continued on next page*

STANDARDS	MODEL CURRICULUM (A.REI.5-7)
A.REI.5-7, continued	**Expectations for Learning, continued** **INSTRUCTIONAL FOCUS** *Note: For Algebra 1 – A.REI.7, the example in the standards is not appropriate, as students do not know equations of circles. Instead, use systems with an equation of a line and an equation of a parabola. In Geometry and Algebra 2, systems with an equation of a circle can be included.* • Substitute a solution into the original system and a manipulation of the system to show solutions are the same. • Solve a system of linear equations in two variables algebraically using substitution, algebraically using elimination, and by graphing. • Solve a system of a linear equation and a quadratic equation in two variables algebraically using substitution and by graphing. • Discuss the efficiency and effectiveness of various methods of solving systems of equations. **Content Elaborations** **OHIO'S HIGH SCHOOL CRITICAL AREA OF FOCUS** • Algebra 1, Number 2, pages 5-7 • Algebra 1, Number 4, pages 9-10 **CONNECTIONS ACROSS STANDARDS** • Solve linear and quadratic equations in one variable (A.REI.3-4). • Graph linear and quadratic models (F.IF.4, 7). • Rearrange formulas (A.CED.4). • Solve systems of equations and inequalities graphically (A. REI.10-12).

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (A.REI.5-7)

Instructional Strategies

Note: The model curriculum is not a collection of lessons nor a full curriculum; it does not suggest pace, sequence, or amount of time spent on topics. It provides information about a topic related to the standards including ideas for examples, strategies for teaching, common misconceptions, and possible connections between topics. In addition, the emphasis of the model curriculum is on instruction not on assessment; therefore, the examples in the Instructional Strategies section are written for instructional purposes not necessarily as assessment items. Examples may need to be rewritten to accommodate the needs of each individual classroom.

Standards for Mathematical Practice
This cluster focuses on but is not limited to the following practices:
MP.1 *Make sense of problems and persevere in solving them.*
MP.2 *Reason abstractly and quantitatively.*
MP.4 *Model with mathematics.*
MP.6 *Attend to precision.*

SYSTEMS OF LINEAR EQUATIONS

A linear system in two variables is a set of two equations that are joined by the word "and." There are several ways to present a system:

$$y = 5x + 40 \qquad \left\{\begin{array}{l} y = 5x + 40 \\ y = 9x \end{array}\right. \qquad y = 5x + 40 \text{ and } y = 9x$$
$$y = 9x$$

Since the meaning of "and" is intersection or overlap, then the solution set of a system is the intersection or overlap of the solution sets of two individual sentences (equations in this case). Therefore, a solution to the system of two linear equations in two variables is the set of ordered pair(s) that satisfies both equations. Students need to be aware that for a solution to be true for the set, it must be true for all equations in the system simultaneously.

Begin by presenting systems of equations in the context of real-world problems. It is preferable to pose problems before presenting procedures.

In Grade 8 students solved systems of equations graphically only. This is the first time they are expected to solve systems of equations algebraically. Point out that there is a need for algebraic methods because it is often difficult to get an exact solution using a graphical representation. Connections should still be made between the graphical representations of a system of equations and the algebra methodology.

From middle school, students should be familiar with the idea that a system of linear equations can have one solution, no solution, or infinitely many solutions. (Although not required, students should become familiar with terminology such as consistent, inconsistent, dependent, and independent.) They still need to be presented with problems that result in these types of systems algebraically. Students should be able understand why $a = a$ has infinitely many solutions, why $a = b$ has no solutions and should make connections to the corresponding graphical representation.

Students need to understand if a system has an infinite number of solutions, it does not mean that all real numbers make the system true, but only those that lie on the line of $y = mx + b$. Algebra 1 students should also understand that some systems that appear to have infinitely many solutions really have a finite number of solutions when there is a constraint on the function.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (A.REI.5-7)

EXAMPLE

Eduardo had 102 shirts and sold 3 shirts every 3 hours. Blake also had 102 shirts, but sold 6 shirts every 6 hours. When did Blake and Eduardo sell the same number of shirts?

Discussion: Blake and Eduardo's situation can be modeled by a system of equivalent (dependent) equations that has a finite (not an infinite) number of solutions. The graphs of the equations modeling both situations are discrete linear graphs that are alike, yet slightly different. Eduardo's graph consists of points graphed every 3 hours and Blake's graph consists of points graphed every 6 hours. It is difficult to tell what happens between the hours, but the amount of shirts they sell overlaps at the end of every 6 hours which means that Blake and Eduardo have sold the same number of shirts at the end of 6th hour, 12th hour, 18th hour, 24th hour and so on. They stopped selling shirts after 102 hours when they run out of shirts. Therefore, the graphs share 18 distinct data points that represents a finite number of solutions.

When solving systems of equation, students should be encouraged to always verify their solution by substituting their solutions into the original equations.

Systems of equations in two variables should include linear equations that are not in slope-intercept form. Encourage standard form in particular. In addition, they should be also able to solve a system given a table of values. Algebra 1 students should be limited to solving systems of equations in two variables.

Continuous and Discrete

Students should have practice with both continuous and discrete functions. They should be provided with some examples that appear to have a solution if the graphs were continuous that do not have a solution when correctly graphed as a discrete function according to context.

Inspection

Develop students' understanding of structure by encouraging them to "look" at the system before solving. For example, in a system which includes the equation $y = \frac{3}{5}x + 7$ and $y = \frac{3}{5}x - 2$, students should notice the equations have the same slope and different *y*-intercepts indicating that the lines are parallel and therefore the system has no solution.

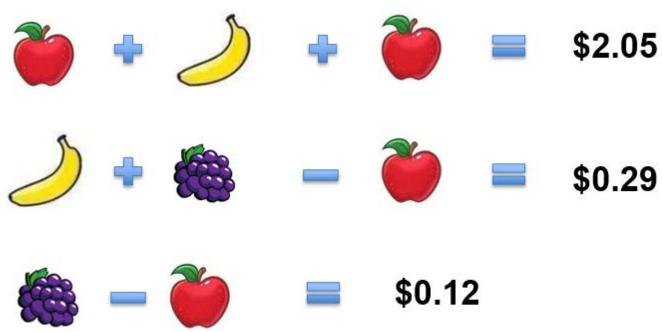
Substitution

Before solving systems of equations, pique students' interest with a system involving symbols such as they would see on social media.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (A.REI.5-7)

EXAMPLE

Find the value of each fruit.



Discussion: Discuss with students how they got the answer. They will most likely describe to you a strategy involving substitution. Capitalize on this discussion to launch into the substitution property by having students solve the same system with variables using substitution.

$2A + B = 2.45$

$B + G – A = 1.63$

$G – A = 0.12$

TIP! For kinesthetic learners have students physically cut out the equivalent expression from one side of the equal side of equation A and physically place it over the variable in equation B. This kinesthetic process allows students to make sense of substitution from the concrete to the abstract.

Elimination/Linear Combination

Before learning elimination techniques, students should be given the opportunity to tinker with a single equation and some ordered pairs that satisfy the equation. Then multiply the equation by a constant to build the concept that the same ordered pairs satisfy the new and old equations.

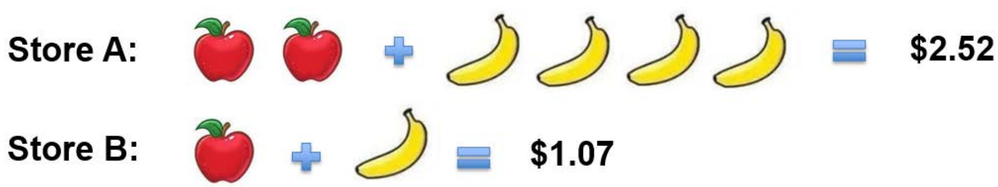
INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (A.REI.5-7)

Another approach is to explore the Multiplication Property of Equality using graphs. Have students graph equations such as $2x + 4y = 8$, $3x + 6y = 12$, $4x + 8y = 16$, $x + 2y = 4$, $0.5x + y = 2$, and $-2x - 2y = -8$. Draw connections between the Multiplication Property of Equality and the graphical representations of the equivalent equations. Have students verify their equality by using ordered pairs that satisfy the equations.

Students have experience with the Addition Property of Equality in the form if $a = b$, then $a + c = b + c$. Use a variation of the fruit problem to promote a discussion of the extension of the Addition Property of Equality to if $a = b$ and $b = c$, then $a + b = b + c$. This will lay the ground work for students to understand solving systems by elimination.

EXAMPLE

Mary bought apples and bananas at two different stores. John only needs to buy apples to make a pie. Which store should he purchase his apples from? How much will it cost him per apple?



Discussion: Students will most likely solve this problem in a variety of ways. Some students may subtract equation B (an apple, banana, and $1.07) from equation A. Discuss why this is allowed, and connect it with the property if $a = b$ and $b = c$, then $a + b = b + c$. (It may be wise to use variables other than a and b when discussing this property since the problem has apples and bananas in it.) Some students may double the apple and bananas in equation B and then subtract it from Equation A. Discuss why both strategies work and connect to elimination. Other students may use a substitution method. Discuss why substitution also works. Challenge students to graph the equations, and verify whether they get the same results.

Stress to students that efficiency and strategizing is important in solving systems of equations. This is especially true with respect to the elimination method. Have discussions with students about which term would be the most efficient to "eliminate" and why.

Choosing the Most Efficient Method

Before beginning solving systems of equations, students should analyze the structure of the equation and decide on the most efficient way to solve the system: graphing, substitution, inspection, or combination (elimination).

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (A.REI.5-7)

EXAMPLE

Fill in the table by choosing pairs of the equations that would highlight the efficiency of each method: inspection, graphing, substitution, elimination. An equation may be used more than once. Provide an explanation for your choice.

$y = \frac{2}{3}x - 4$	$3y = 3$	$3x + 4y = 8$	$y = \frac{2}{3}x + 7$	$4x = -2$
$y = \frac{3}{4}x + 1$	$x = 4y$	$y = 3x$	$6x + 8y = 16$	$3x + 4y = 9$
$x - y = 2$	$-3x + 2y = 20$	$21x - 7y = 14$	$0.4y = 2.8x - 8$	$3.5x = 24 + y$

	Inspection	Graphing	Substitution	Elimination
Pairs of Equations				
Explanation				
Pairs of Equations				
Explanation				
Pairs of Equations				
Explanation				

Real-world problems

Provide students opportunities to practice linear vs. non-linear systems; consistent vs. inconsistent systems; pure computational vs. real-world contextual problems (e.g., chemistry and physics applications encountered in science classes). A rich variety of examples can lead to discussions of the relationships between coefficients and consistency that can be extended to graphing and later to determinants and matrices.

EXAMPLE

Gym A had a joining fee of $99 and a single-club membership of $24.99 per month. Gym B has a flat yearly fee of $350. Write a system of equations to represent the situation.

- **a.** After how many months would the memberships be of equal value?
- **b.** What are the constraints on these functions?
- **c.** After how many months would Gym A be cheaper?
- **d.** After how many months would Gym B be cheaper?
- **e.** How would the problem change if Gym A only had a $50 joining fee?
- **f.** How would the problem change if Gym B kept the same rate, but did not lock you into a yearly contract?

Include a variety of real-world applications of systems such as mixture problems, motion problems, break even problems, and area models.

System of Linear and Quadratic Equations

Students should explore solving a system that involves a linear equation and a quadratic equation using the graphing method. Through exploration, they should realize that there could be 0, 1, or 2 solutions. Once they have an intuitive understanding of solving those types of systems by graphing, they should move toward solving a linear equation and a quadratic by substitution.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (A.REI.5-7)

Instructional Tools/Resources

These tools and resources may be helpful for instruction. Before use, they should be evaluated at the local level to determine their appropriateness for instruction.

Manipulatives/Technology

- Desmos is a free graphing calculator that is available to students as website or an app.
- Geogebra is a free graphing calculator that is available to students as website.
- Wolframalpha is dynamic computing tool.

Systems of Equations

- Suit Yourself is a lesson designed by NASA where students use systems of equation to evaluate the oxygen use of two astronauts.
- Ground Beef is a lesson by Achieve the Core that ties in CTE. Students use systems of equations and Pearson's square to determine the profit with respect to selling meat.
- Those Horrible Coin Problems (And What We Can Do About Them) is a task from Dan Meyer on making value problems more interesting by showing the need for systems in computation.
- Quinoa Pasta 2 is a task from Illustrative Mathematics that integrates modeling and systems of equations in the context of a pasta made of both quinoa and corn. It continues with Quinoa Pasta 3.
- Find a System is a task from Illustrative Mathematics that gives two points and has students create a system given two points. It encourages critical thinking by reversing the typical process.
- Mix It Up and Don't Freeze the Engine are lessons from NCTM's Illuminations that have students write equations in the context of a concentration. "Mix It Up" is a tactile lesson where students develop and use a formula to determine the final percent mix from two mixtures. In "Don't Freeze the Engine" the students use systems to analyze the antifreeze in a particular cooling system.
- Algebra 1-Mixture Problems is a teaching channel video on teaching mixture problems. It ties an informal understanding of a fulcrum and inverse variation which she calls the see-saw method. This could be contrasted with solving the same problem using a system of equations.
- Students use matrices and technology to solve the Meadows or Malls problem, a linear programming problem with six variables.
- Piling Up Systems of Linear Equations: How Much Does Each Weigh? by Tap Into Teen Minds is a 3-act task that has students write systems of linear equations using the weight of office supplies.
- [Makeover] Systems of Equations from Dan Meyer's Blog discusses how to make systems of equation word problems more meaningful.
- A Linear and Quadratic System by Illustrative Mathematics is a task where students make connections between equations and the geometry of their graphs to find the intersection point of a line and a parabola.
- A Mixture of Problems by Laurie Riggs, et. al has a variety of problems including mixture problems using different conceptual methods to solve equations.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (A.REI.5-7)

Curriculum and Lessons from Other Sources

- EngageNY, Algebra 1, Module 1, Topic C," Lesson 20: Solution Sets to Equations with Two Variables, Lesson 22: Solution Sets to Simultaneous Equation, Lesson 23: Solution Sets to Simultaneous Equation, and Lesson 24: Applications of Systems of Equations and Inequalities are lessons that pertain to this cluster.
- Mathematics Vision Project, Algebra 1, Module 5: Systems of Equations and Inequalities has many tasks that pertain to this cluster.
- Exploring Systems of Inequalities by Burrill and Hopfensperger is a pdf of the teacher's edition of the series Data-Driven Mathematics published by Dale Seymour Publications. It has many lessons that pertain to this cluster. The student edition can be found here.

General Resources

- Arizona High School Progression on Algebra is an informational document for teachers. This cluster is addressed on page 14 in paragraphs 3-6.
- High School Coherence Map by UnboundED is a map of the relationships among high school mathematics standards. Please note: This resource is aligned to the Common Core not Ohio's Learning Standards. Although the tool is useful, it needs to be checked with respect to Ohio's Learning Standards.

References

- Common Core Standards Writing Team. (2013, March 1). *Progressions for the Common Core State Standards in Mathematics (draft). High School, Algebra*. Tucson, AZ: Institute for Mathematics and Education, University of Arizona.
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<table>
<tr><th>STANDARDS</th><th>MODEL CURRICULUM (A.REI.10-12)</th></tr>
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Algebra

REASONING WITH EQUATIONS AND INEQUALITIES

Represent and solve equations and inequalities graphically.

A.REI.10 Understand that the graph of an equation in two variables is the set of all its solutions plotted in the coordinate plane, often forming a curve (which could be a line).

A.REI.11 Explain why the *x*-coordinates of the points where the graphs of the equation *y* = *f(x)* and *y* = *g(x)* intersect are the solutions of the equation *f(x)* = *g(x)*; find the solutions approximately, e.g., using technology to graph the functions, making tables of values, or finding successive approximations.

A.REI.12 Graph the solutions to a linear inequality in two variables as a half-plane (excluding the boundary in the case of a strict inequality), and graph the solution set to a system of linear inequalities in two variables as the intersection of the corresponding half-planes.

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Expectations for Learning

In prior courses, students graph linear relationships and identify slope and intercepts. In this cluster, students extend this knowledge to include the idea that a graph represents all of the solutions of an equation. Students use graphs and tables of equations in two variables to approximate solutions to equations in one variable. They also graph solutions to linear inequalities in two variables. In Algebra 2 or Math 3, students similarly study the relationship between the graph and the solutions of rational, radical, absolute value, polynomial, and exponential equations.

ESSENTIAL UNDERSTANDINGS

- A point of intersection of any two graphs represents a solution of the two equations that define the two graphs.
- An equation in one variable can be rewritten as a system of two equations in two variables, by thinking of each side of the equation as a function, i.e., writing *y* = left hand side and *y* = right hand side.
    - The approximate solution(s) to an equation in one variable is the *x*-value(s) of the intersection(s) of the graphs of the two functions.
    - Two-variable graphical and numerical (tabular) techniques to solve an equation with one variable always work and are particularly useful when algebraic methods are not applicable, e.g., $x^2 - 3x + 2 = 2^x$.
- A half plane represents the solutions of a linear inequality in two variables.
- The intersection of two half planes represents the solution set to two inequalities in two variables.

MATHEMATICAL THINKING

- Use technology strategically to deepen understanding.
- Plan a solution pathway.
- Create a model to make sense of a problem.

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<tr><th>STANDARDS</th><th>MODEL CURRICULUM (A.REI.10-12)</th></tr>
<tr><td>A.REI.10-12, continued</td><td>

<h2>Expectations for Learning, continued</h2>
<h3>INSTRUCTIONAL FOCUS</h3>

Rewrite a one-variable equation as two separate functions and use the x-coordinate of their intersection point to determine the solution of the original equation.
Approximate intersections of graphs of two equations using technology, tables of values, or successive approximations (focus on equations with linear, exponential, and quadratic expressions).
Graph the solution set of a linear inequality in two variables.
Graph the solution set of a system of linear inequalities in two variables.

<h2>Content Elaborations</h2>
<h3>OHIO'S HIGH SCHOOL CRITICAL AREA OF FOCUS</h3>

Algebra 1, Number 2, pages 5-7

<h3>CONNECTIONS ACROSS STANDARDS</h3>

Solve equations in one variable (A.REI.3-4).
Create equations in two variables (A.CED.2).
Graph functions expressed symbolically (F.IF.7).
Solve systems of equations graphically (A.REI.6-7).
Analyze functions using different representations (F.IF.9).

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INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (A.REI.10-12)

Instructional Strategies

Note: The model curriculum is not a collection of lessons nor a full curriculum; it does not suggest pace, sequence, or amount of time spent on topics. It provides information about a topic related to the standards including ideas for examples, strategies for teaching, common misconceptions, and possible connections between topics. In addition, the emphasis of the model curriculum is on instruction not on assessment; therefore, the examples in the Instructional Strategies section are written for instructional purposes not necessarily as assessment items. Examples may need to be rewritten to accommodate the needs of each individual classroom.

Standards for Mathematical Practice
This cluster focuses on but is not limited to the following practices:
MP.2 *Reason abstractly and quantitatively.*
MP.4 *Model with mathematics.*
MP.5 *Use appropriate tools strategically.*
MP.6 *Attend to precision.*

REPRESENTING SOLUTIONS OF EQUATIONS GRAPHICALLY

Begin with solving simple linear equations by tracing graphs and using tables on a graphing calculator. Then, advance students to nonlinear situations, so they can see that even complex equations that might involve quadratics, absolute value, or rational functions can also be solved fairly easily using this same strategy. While a standard graphing calculator does not simply solve an equation for the user, it can be used as a tool to approximate solutions.

EXAMPLE

Help students to recognize a graph as a set of solutions to an equation. If the equation $y = 6x + 5$ represents the amount of money paid to a babysitter (for example, \$5 for gas to drive to the job and \$6/hour to do the work), then every point on the line represents an amount of money paid, given the amount of time worked.

Using technology, have students graph a function and use the trace function to move the cursor along the curve. Discuss the meaning of the ordered pairs that appear at the bottom of the calculator, emphasizing that every point on the curve represents a solution to the equation.

Students may incorrectly believe that the graph of a function is simply a line or curve "connecting the dots" without recognizing that the graph represents all solutions to the equation.

Converting an Equation to Two Equations

An equation in one variable such as $2x + 3 = x – 7$ can be solved by converting an equation to a system of two equations in two variables: $y = 2x + 3$ and $y = x – 7$ and then graphing the functions $y = 2x + 3$ and $y = x – 7$. They should recognize that the intersection point of the lines is at (–10, –17). They should be able to verbalize that the intersection point means that when $x = –10$ is substituted into both sides of the equation, each side simplifies to a value of –17. Therefore, –10 is the solution to the equation. This same approach can be used whether the functions in the original equation are linear, nonlinear, or both.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (A.REI.10-12)

EXAMPLE

Solve the equation $x^2 - 3x + 2 = 5$.

Discussion: In other standards, students have learned various strategies to solve this equation algebraically (factor, complete the square, quadratic formula). Notice this equation has only one variable, *x*, and there is no *y*. In A.REI.11, students gain another technique: Graph $y = x^2 - 3x + 2$ and $y = 5$ and find the *x*-values of the intersections of the graphs. Another approach is to set the equation equal to 0 to get $x^2 - 3x - 3 = 0$, and then have students graph the equations $y = x^2 - 3x - 3$ and $y = 0$ and connect the intersection points to the solutions (*x*-intercepts) of the quadratic.

EXAMPLE

Solve the equation $x^2 - 3x + 2 = 2^x$.

Discussion: Algebraic techniques for solving equations require that equations can be manipulated into particular forms. Some equations like the example cannot be manipulated into forms that can be solved with algebraic techniques. The advantage of using the technique of setting a one-variable equation into two equations in two variables and finding the intersection points is that always works, even for equations like $x^2 - 3x + 2 = 2^x$, for which there are no algebraic techniques.

EXAMPLE

Compare the graphs of $x^2 - 3x + 2 = 2^x$ to $x^2 - 3x + 2 = 1.1^x$.

Students graphed the first equation in the previous example. Now graphing the second equation as a system of two equations $y = x^2 - 3x + 2$ and $y = 1.1^x$ appears to have two solutions in a traditional window. However since the exponential function will eventually exceed the quadratic function, there is a third solution if an appropriate window is found.

Tables

Use the table function on a graphing calculator to solve equations. For example, to solve the equation $x^2 = x + 12$, students can examine the equations $y = x^2$ and $y = x + 12$ and determine that they intersect when $x = 4$ and when $x = -3$ by examining the table to find where the *y*-values are the same.

Students who make a table of values to find the solution to a system may start with evaluating each function at integer values to determine an approximate solution. Using technology, students can then zoom-in on a smaller window (more precise) of values that would include a solution of the system, and make a zoomed-in table of values. They can continue this process, recognizing when the solution is exact, and when the solution is approximate.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (A.REI.10-12)

INEQUALITIES

Inequalities should be taught in the context of real-world examples, so students can create meaning.

EXAMPLE

Before teaching students how to shade a graph with respect to inequalities, explore what the graph would look like for money earned when a person earns *at least* $6 per hour compared to a person who earns *exactly* $6/hour. Have students plot points for all the correct combinations of solutions. Using technology or a transparency, have students combine their dots (solutions) onto one graph.

Discussion: The graph for a person earning *exactly* $6/hour would be a linear function, while the graph for a person earning at least $6/hour would be a half-plane including the line and all points above it. Then compare the graphs to the graph of a person who earns *more than* $6 per hour.

EXAMPLE

Before teaching the graphical representation of solid and dotted lines, put students into pairs. Have one student write an inequality and plot solutions on a coordinate plane to represent a woman who deposits $100 in the bank with *at least* a 0.05% simple interest rate? Have one student write an inequality and plot solutions on a coordinate plane to represent a man who deposits $100 in the bank with a simple interest rate *greater than* 0.05%. Then have the students compare graphs.

Discussion: The students should realize that the woman who earns at least a 0.05% interest rate could earn exactly that, so her solutions would fall on the line. Use this as a discussion point for the need of representing the lines differently. Explain that mathematicians came up with the convention of using solid and dotted lines.

EXAMPLE

Before formally teaching systems of inequalities, put students into pairs. Have one student write an inequality and plot solutions on a coordinate plane to represent part **a.** of the example. Then have the second student do part **b.** of the example.

- Julia has two jobs. She earns $8 an hour babysitting and $11 an hour walking the neighbor's dog. She must make at least $100 a week to save up for her class trip. Explain which solutions make your inequality true.
- Julia both babysits and walks her neighbor's dogs to earn money. Julia's mom will not let her work more than 15 hours a week. Explain which solutions make your inequality true.

Discussion: Then put both students back together, and have them use transparencies, tracing paper, or technology to overlay the second student's inequality on top of the first student's. Have students discuss the meaning behind the overlapping section. Use this as a launching pad to discuss systems of inequalities.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (A.REI.10-12)

Optimization Problems and Linear Programming
The ability to solve systems of equations is useful for solving linear programming problems.
Applications such as linear programming can help students recognize how businesses use constraints to maximize profit while minimizing the use of resources. These situations often involve the use of systems of two variable inequalities. Have students explore and discover on their own that the optimization point (where profits are maximized and costs are minimized) occurs at the corners of the feasible region. See the Instructional Tools/Resources section for examples of linear programming problems.

Instructional Tools/Resources

These tools and resources may be helpful for instruction. Before use, they should be evaluated at the local level to determine their appropriateness for instruction.

Manipulatives/Technology

- Desmos is a free graphing calculator that is available to students as website or an app.
- Geogebra is a free graphing calculator that is available to students as website.
- Wolframalpha is dynamic computing tool.

Solutions on a Graph

- Collinear Points by Illustrative Mathematics is a task where students connect Algebra and Geometry in exploring solutions on a graph.

Inequalities

- Rabbit Food by Achieve the Core is a CTE lesson where students use linear inequalities to analyze rabbit food.
- Maximizing Profit: Selling Boomerangs by Mathematics Assessment Project that has students explore an optimization problem.
- Solution Sets by Illustrative Mathematics is a task that gives students solution sets and asks them to create a corresponding system.

Linear Programming

- Hassan's Pictures-Linear Programming and Profit Lines is an Annenberg Learner lesson where students find the feasible region in a linear programming problem and find the optimum point.
- Linear Programming is a Desmos activity that introduces linear programming.
- Building Cat Furniture: An Introduction to Linear Programming is a lesson by Tim Marley that uses Legos as an introduction to linear programming.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (A.REI.10-12)

Curriculum and Lessons from Other Sources

- Georgia Standards of Excellence Curriculum Frameworks, Algebra 1, Unit 2: Reasoning with Linear Equations and Inequalities has several lessons that pertain to this cluster. These can be found on pages 63-70, 95-98, 103-127, 130-146.
- Mathematics Vision Project, Algebra 1, Module 5: Systems of Equations and Inequalities has tasks that pertain to this cluster.
- EngageNY, Algebra 1, Module 1, Topic C, Lesson 21: Solution Sets to Inequalities with Two Variables and Lesson 24: Applications of Systems of Equations and Inequalities are lessons that pertain to this cluster.
- EngageNY, Algebra 1, Module 3, Topic C, Lesson 16: Graphs Can Solve Equations Too is a lesson that pertains to this cluster. This lesson extends A.REI.11 to absolute value equations which can be used as an extension but is not required in Ohio.
- Exploring Systems of Inequalities by Burrill and Hopfensperger is a pdf of the teacher's edition of the series Data-Driven Mathematics published by Dale Seymour Publications. It has many lessons that pertain to this cluster. The student edition can be found here.

General Resources

- Arizona High School Progression on Algebra is an informational document for teachers. This cluster is addressed on page 15.
- High School Coherence Map by UnboundED is a map of the relationships among high school mathematics standards. Please note: This resource is aligned to the Common Core not Ohio's Learning Standards. Although the tool is useful, it needs to be checked with respect to Ohio's Learning Standards.

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- Common Core Standards Writing Team. (2013, March 1). *Progressions for the Common Core State Standards in Mathematics (draft). High School, Algebra*. Tucson, AZ: Institute for Mathematics and Education, University of Arizona.
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STANDARDS	MODEL CURRICULUM (F.IF.1-3)
Functions **INTERPRETING FUNCTIONS** **Understand the concept of a function, and use function notation.** **F.IF.1** Understand that a function from one set (called the domain) to another set (called the range) assigns to each element of the domain exactly one element of the range. If *f* is a function and *x* is an element of its domain, then $f(x)$ denotes the output of *f* corresponding to the input *x*. The graph of *f* is the graph of the equation $y = f(x)$. **F.IF.2** Use function notation, evaluate functions for inputs in their domains, and interpret statements that use function notation in terms of a context. **F.IF.3** Recognize that sequences are functions, sometimes defined recursively, whose domain is a subset of the integers. *For example, the Fibonacci sequence is defined recursively by* $f(0) = f(1) = 1, f(n + 1) = f(n) + f(n - 1)$ *for* $n \geq 1$.	**Expectations for Learning** In the eighth grade, students have learned a semi-formal definition of a function and know that a function pairs an input value with an output value. Eighth grade students do not use function notation nor the terms domain and range. In this cluster, students will now expand their understanding of functions to include function notation and the terms domain and range. Also, students will evaluate and interpret functions, including sequences as functions. Distinguishing between relations and functions is not a primary focus. This cluster is the foundation for all future work with functions. **ESSENTIAL UNDERSTANDINGS** • Function notation illustrates a formal connection between inputs and outputs. • Functions can be tied to real-world scenarios given by tables, graphs, equations, or verbal descriptions. • Function notation $f(x)$ is shorthand for the output of *f* when the input is *x*. • Function notation, $f(x)$, is a new representation for students and is articulated as "*f* of *x*", and it is not related to multiplication. • Sequences are functions whose domain is a subset of the integers, paying careful attention to how a sequence is indexed. For example, the sequence may be indexed from 0 to *n*, from 1 to $n - 1$, or something else. • An arithmetic sequence is a linear function, and a geometric sequence is an exponential function. **MATHEMATICAL THINKING** • Use accurate mathematical vocabulary to describe mathematical reasoning. • Represent a concept symbolically. • Determine reasonableness of results. • Make connections between concepts, terms, and properties within the grade level and with previous grade levels. *Continued on next page*

STANDARDS	MODEL CURRICULUM (F.IF.1-3)
F.IF.1-3, continued	**Expectations for Learning, continued** **INSTRUCTIONAL FOCUS** • Make connections among different representations (tables, graphs, symbols, and verbal descriptions) of functions, focusing on linear, quadratic, and exponential functions. • Solve problems with functions represented in tables, graphs, symbols, and verbal descriptions. • Explain function notation in a real-world context. For example, if *f(x)* represents the height of particle at *x* seconds, then *f*(1) represents the height of the particle at 1 second. • Interpret number patterns as sequences and their graphs as discrete points. When the number pattern arises from a context, consider whether it is appropriate to "connect the dots." • Use function notation to specify sequences, both explicitly and recursively. (Subscript notation is not required.) • Relate linear functions to arithmetic sequences, and relate exponential functions to geometric sequences. **Content Elaborations** **OHIO'S HIGH SCHOOL CRITICAL AREAS OF FOCUS** • Algebra 1, Number 2, pages 5-7 **CONNECTIONS ACROSS STANDARDS** • Build a function that models a relationship between two quantities (F.BF.1a, 2). • Build new functions from existing functions (F.BF.4a). • Interpret expressions for functions in terms of the situation they model (F.LE.5). • Construct and compare linear, quadratic, and exponential models, and solve problems (F.LE.2). • Represent and solve equations and inequalities graphically (A.REI.10).

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (F.IF.1-3)

Instructional Strategies

Note: The model curriculum is not a collection of lessons nor a full curriculum; it does not suggest pace, sequence, or amount of time spent on topics. It provides information about a topic related to the standards including ideas for examples, strategies for teaching, common misconceptions, and possible connections between topics. In addition, the emphasis of the model curriculum is on instruction not on assessment; therefore, the examples in the Instructional Strategies section are written for instructional purposes not necessarily as assessment items. Examples may need to be rewritten to accommodate the needs of each individual classroom.

Standards for Mathematical Practice
This cluster focuses on but is not limited to the following practices:
***MP.6** Attend to precision.*
***MP.7** Look for and make use of structure.*

The focus of this cluster is on understanding the concept of a function and reading and writing functions written in function notation. It also focuses on both discrete and non-discrete (continuous) functions in the form of linear, quadratic, and exponential functions.

THE CONCEPT OF A FUNCTION

The concept of functions, although central to high school mathematics, is one of the most challenging concepts for students. A function is a mathematical concept that describes how two values relate to one another. In a function every element in the domain must be mapped to one and exactly one element in the range.

A function can be thought of as—
- having two sets;
- having a correspondence between the two groups;
- meeting a special requirement where every input is matched or assigned to one and only one output.

Students may incorrectly believe that all relationships having an input and an output are functions, and therefore, misuse the function terminology.

Distinguish between relationships that are not functions and those that are functions (e.g., present a table or diagram in which one of the input values results in multiple outputs and contrast that with a functional relationship). See Model Curriculum 8.F.1-3 for scaffolding ideas about functions.

Students may incorrectly think that functions are always equations because that is usually how students see functions represented in an academic setting. However, some functions may have no algebraic representation at all. To prevent this misconception, it may be helpful to introduce the concept functions using non-algebraic contexts.

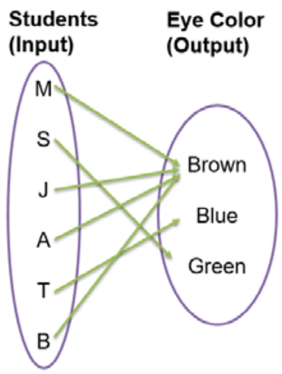
INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (F.IF.1-3)

EXAMPLE

In this Function Wall activity students are the inputs and the outputs are other categories assigned to spaces on the wall where students can physically go (or be assigned). This activity is adapted from Putting Essential Understanding of Functions Into Practice 9-12.

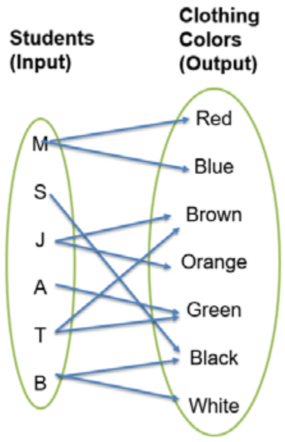
a. Put up signs on the wall of different eye colors (assuming no one in your class has two different colored eyes). Write a mapping diagram on the board, and then have students go to the appropriate place along the wall that represent each student's eye color. Complete the mapping diagram to represent their movement.

Students	**Eye Color (signs along the wall)**
Marsha	Brown
Shaqueesha	Blue
Jose	Green
Ashley	
Tim	
Brendan	



Discussion: Point out how each person can only go to one and exactly one place.

b. Put up signs on the wall to represent different clothing colors. Write a mapping diagram on the board, and then have students go to the appropriate place along the wall that represents the color of clothing each student is wearing.



Discussion: Discuss the difference between the two situations. Students should come to the realization that in part **a**., each student had only and exactly one place to go, but in part *b.* if a student was wearing more than one color, he or she had no clear cut place to go. Therefore part **a.** is a function, but part **b.** is just a correspondence (relation). Discuss how even if even just one student was wearing two colors, then the situations would not represent a function. After the discussion make a connection that the names of the students are the domain or input, and the clothing color is the range, or output. Also, discuss how limiting the domain of a set can affect whether a correspondence is a function or not. For example, if the class is limited to only students wearing monochromatic colors, then the situation would be a function.

c. Put up signs on the wall using only clothing colors black and white. Have students go to the appropriate place along the wall that represents the color of clothing each student is wearing.

Discussion: Students should come to the conclusion that many students will have no place to go, so the situation does not represent a function.

Example continued on next page

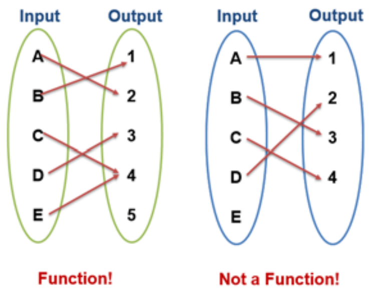
INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (F.IF.1-3)

d. Put up signs on the wall to represent different planets in the solar system. Have students go to the appropriate place along the wall on the planet that each student lives.

Discussion: Point out that even though no one in the class lives on Mars and Jupiter, it is still a function. This breaks the misconception that every output must have an input.

e. Reverse the situations in parts **a.-d.** Define the wall signs as the inputs and have several students stand along the wall as outputs. Move the signs to its assigned student.

Discussion: Discuss how determining which set is the input or output can determine whether a correspondence is a function or not. Explain that is some contexts the input (independent variable) and output (dependent variable) is clearly defined, but sometimes it is arbitrary. This activity is adapted from Ronau, et al., 2014



TIP! In a function, all the elements from the input set must be assigned or matched to one and exactly one element from the output set. In other words, there can be no inputs left unassigned. However, the reverse is not necessarily true; there can be outputs that exist without any assigned inputs.

Students may also incorrectly believe a mapping is not a function when multiple input values are paired with same output value. Exposure to real-world examples such as the mapping the amount of data used on an unlimited data plan to its corresponding monthly bill.

Functions should be explored in applied contexts. For example, examine the amount of money earned when given the number of hours worked at a fast food job, and contrast this with a situation such as riding an Uber where a single fee is paid by the "carload" of people, regardless of whether 1, 2, or 3 people ride.

Students may have the following misconceptions:

- They may incorrectly think a function consists of a single rule, so a split function is not a function.
- They may incorrectly think the graph of a function needs to be continuous.
- They may incorrectly think that the range must map back onto the domain creating a one-to-one correspondence.

Give students problems to confront their misconceptions.

Function Carnival by Desmos can be used to emphasize that in a function one element in the domain corresponds (or maps) to exactly one element in the range.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (F.IF.1-3)

Challenges Due to the Use of the Vertical Line Test

The vertical line test is not a universal tool and should be used very carefully or not at all. The emphasis should be on the definition of what a function is instead of just using the vertical line test. Instead of the vertical line test emphasize the concept of a function: For a given relation, does each input have a single output? In other words, does the input determine the output?" If the answer to the question is no, then it is not a function. Therefore the emphasis should be on determining whether a relation is a function by analyzing the correspondence between domain and range instead of using the vertical line test. (The vertical line test also makes it difficult to discuss functionality when *x* is a function of *y* if students interchange the *x*-axis as the output and the *y*-axis as the input.) In addition the vertical line test may not always work for inverse functions, polar functions, and parametric functions.

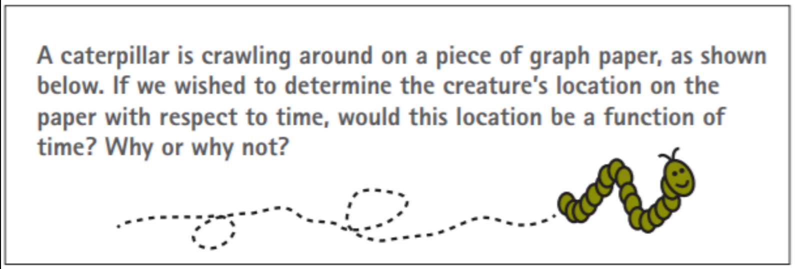


Fig. 1.6. Is a caterpillar's location a function of time? (Clement 2001, p. 745)

One reason to caution against the vertical line test is that even more advanced students struggle applying the vertical line test correctly. In one study over 60% of high school precalculus students misapplied the vertical line test to the Caterpillar Problem (shown on the left) where students were to determine whether the situation was a function. They confused the path of the caterpillar with a graph showing the correspondence between time and location. If you choose to use the vertical line test with your students, give the caterpillar students the problem after some exposure to the vertical line test, and see how they solve it. After discussion encourage them to graph it using time and distance. After students spend some time plotting the relation, they should see that it is indeed a function.

Function Machines

Use diagrams to help students visualize the idea of a function machine. Students can examine several pairs of input and output values and try to determine a simple rule for the function. Make connections of input and output values to independent and dependent variables which are used in science classes.

DOMAIN AND RANGE

Help students to understand that the word "domain" means the set of all possible input values and that "range" means the set of all possible output values.

Students need to understand that restricting the domain can create a function. For example, the equation $x = y^2$, where *x* is the input and *y* is the output, can be rewritten as $y = \pm\sqrt{x}$, and is therefore not a function. However, if the domain is restricted to $x \geq 0$, then the relation is a function. Items can also be added to the range of a relation to produce a function if an input is lacking a correspondence.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (F.IF.1-3)

The understanding of function and domain and range is helpful for people who design web surveys.

EXAMPLE

How long have you been working for the company?

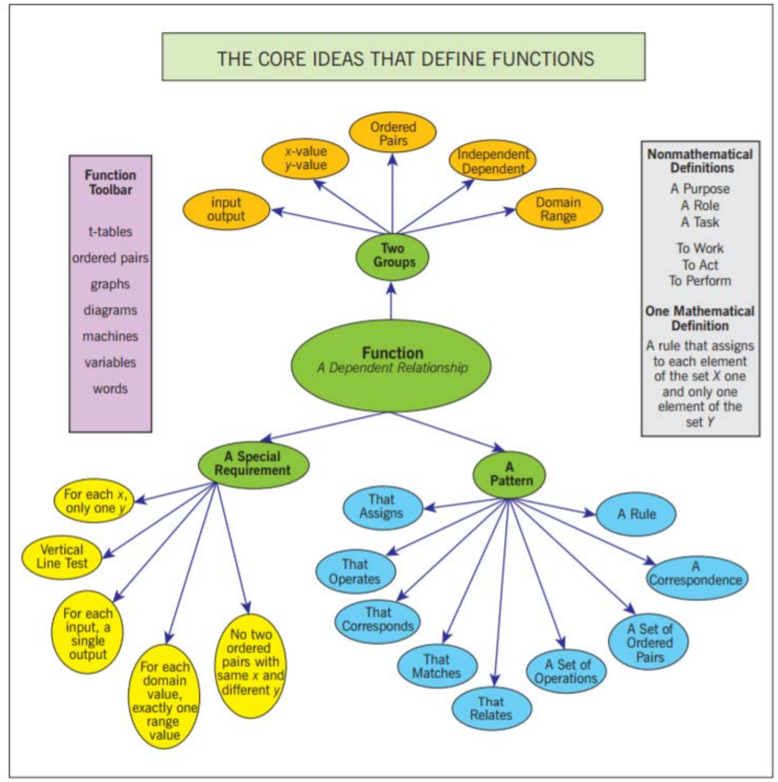
- More than 20 years
- More than 15 years
- More than 5 years
- More than 2 years
- More than 1 year
- Less than 6 months

Discussion: Students should recognize that a person who has worked at the company for 23 years should be able to select many of the statements on the survey and still answer truthfully. They also might notice that there can exist a person, such as an 8-month employee, who has nothing to select. Therefore it is important to carefully think about the wording of survey to ensure that the range is restricted so that the data will create a function.

Give students graphs and real-life situations that can be modeled by of various advanced relations such as circles, piecewise graphs, trigonometry curves etc., and have students write the domain and range.

Have students also consider what values could be added to the domain or range that would "break" the function by causing the relation to no longer be a function.

Connect to F.BF.1-2 which includes discrete vs continuous domains.



From Gregorio A. Ponce, *Mathematics Teacher*, 2007, "Critical Juncture Ahead"

Note: This graphic at this grade level is for teacher use not student use. Also, note that even though the vertical line test is shown in the graphic, using it is discouraged.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (F.IF.1-3)

FUNCTION NOTATION

The notation *f(x)* (pronounced *f* of *x*) represents the output of the function when *x* is in the input. While it is often correct to say *f(x)* is the same as *y*, students need to understand the significance of using function notation. One advantage is that it is a short way of saying that *h*(3) means "the output of the function *h* when the input value is 3". And more complicatedly, *h*(3) – *h*(1) is "the difference between the output values of function *h* from the corresponding input values of 3 and 1", which is a vertical distance between output values along the *y*-axis. Make sure that *f(x)* is not the only notation for functions used in the classroom. For example, use *g(x), k(x), r(t), h(t), v(t),* etc.

Students may incorrectly believe that the notation *f(x)* means to multiply some value *f* times another value *x*. The notation alone can be confusing and needs careful development. For example, *f*(2) means the output value of the function *f* when the input value is 2.

Students may incorrectly believe that *f*(2) within the example *f*(2) = 3 is a command to do a computation or a process given by a formula instead of imagining that *f*(2) is a specific number (in this case 3). When asked "Where is f(2)?", students who are exposed to mapping diagrams or function machines and have this misconception may refer to *f*(2) as "in-between," "on the way to 3," or "in process."

Students may incorrectly believe that *f(x)* refers to the entire relation, when it actually refers to the output value when the input value is *x*. It is better to use *f* to refer to the entire function.

The use of function notation is beneficial when situations have more than one function. Changing the independent variable that represents the same set of domain values, does not mean that the function is changed. For example, $f(x) = x^2 + 3$ is the same as $f(t) = t^2 + 3$. However, if two functions expressed by the same formula have different domains, they are not the same function. Also, it is important to have students interpret function notation within a context.

Function notation can be utilized for functions of two variables. For example, *A(b, h)* = *bh* is a function that identifies or determines the area of a rectangle given its base and height. Since the base and height can vary, it requires two input values. Also to be noted, *T(x, y)* = (*x* + 3, *y* + 2) is an algebraic way of representing a translation of the plane. Function notation *f(0)* is an alternative way of saying to evaluate the value of function *f* at *x* = 0 or to find the *y*-intercept of the graph *f(x).* Similarly function notation such as *f(x)* = 0 represents a condensed way to communicate the idea of finding the zeros of the function *f(x)* or the *x*-intercepts of the graph of *f(x)*.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (F.IF.1-3)

EXAMPLE

Given the function $g(b) = 2b + 5$, compare $g(3 + 4)$ to $g(3) + g(4)$.

Discussion: Students will most likely use trial and error and substitution to realize that the expression $g(3 + 4)$ is not equal to $g(3) + g(4)$. Some students may incorrectly use the Distributive Property when solving. Clarify that the Distributive Property applies to products of numbers or variables. Although function notation looks similar to multiplication, it is read "*g* of a sum of 3 and 4", so it means the output of the function when $b = 7$. Use this to launch a discussion of how $f(a + b)$ is usually not the same as $f(a) + f(b)$.

SEQUENCES

This cluster is recommended to be taught in conjunction with F.BF.1-2 and F.LE.1-3. The main difference between F.BF.1-2 and this cluster is that in this cluster the emphasis is on sequences being functions with integer domains, whereas F.BF.1-2 focuses on writing functions recursively and explicitly, and F.LE.1-3 is comparing the different types of functions including those formed by sequences and using the appropriate type to solve problems.

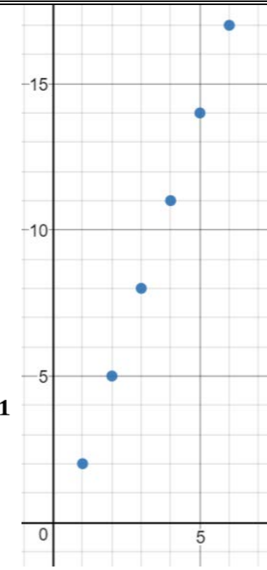
A sequence is a function whose domain is a subset of integers. (In Algebra 1, the domain is often counting numbers and whole numbers; in some mathematical contexts negative integers may be allowed in the domain.) A sequence can be thought of as ordered list of elements where each element in the list is called a term. A sequence is defined by a function *f* from a domain of a subset of integers to a range consisting of real numbers. Save sequence notation until more advanced courses. The use of subscript notation is not encouraged but can be used as an extension or saved for advanced courses. Pay careful attention to how a sequence is indexed. For example, the sequence may be indexed from 0 to n, from 1 to $n - 1$, or something else.

There are two main types of sequences students encounter in Algebra 1: an *arithmetic sequence* which has a *common difference* and a *geometric sequence* which has a *common ratio.* (Although most sequences are neither, and Algebra 1 students should have exposure to sequences that are neither arithmetic and geometric.) The goal is that at the end of Algebra 1 a student should know that an arithmetic sequence is a linear function, and a geometric sequence is an exponential function. Note that the converse of each statement is not always true. A linear function is an arithmetic sequence only when the domain is restricted to a subset of the integers. An exponential function is a geometric sequence only when the domain is restricted to a subset of the integers. Help students understand this concept by emphasizing the difference

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (F.IF.1-3)

between the graph of a linear function that is solid and that the graph of a sequence must be is discrete. An *explicit form* of an equation representing a sequence allows direct computation of any term in a sequence. A *recursive form* requires the preceding term to define the next term in the sequence. The recursive formula requires a starting value and a rule for computing the next terms; it should also include the parameters of the domain as part of the description. Sequences can be easily connected to the patterns that students learned in elementary school. For example, given the sequence 2, 5, 8, 11…, they will quickly recognize the pattern as plus 3. They should be able connect the pattern to the arithmetic sequence in order to make a transition to the recursive form *f(*1) = 2, *f*(*n* + 1) = *f*(*n*) + 3. Difficulty may lay in finding the explicit form. To do so, student can find the 0[th] term or the *y*-intercept, which may or may not be part of the domain of the sequence, and use that to figure out the rest of the equation in explicit form. They should be able to connect the common difference to the coefficient the independent variable in explicit form. Likewise, the connection can be established between the terms of geometric sequence, a common ratio, and the recursive form. Draw attention to the fact that the graph of a sequence has discrete points, because it is unknown what happens "between the dots." Have students discuss the advantage and disadvantage of each form. Note: If indexes are defined differently, the resulting formulas may be different, but the unindexed sequence would be the same.

Term (*n*)	*f(n)*	
0	-1	+3
1	2	+3
2	5	+3
3	8	+3
4	11	+3



Recursive Form

$f(1) = 2, f(n+1) = f(n) + 3,$ for integers $n \geq 1$

$f(3) = f(\boxed{2}) + 3 = 8$

$\boxed{5}$

Common difference

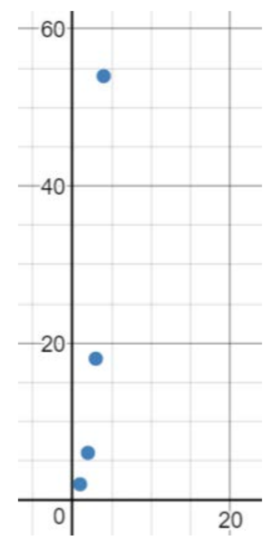
Explicit Form

$f(n) = 3n - 1$ for integers $n \geq 1$

Common difference

0th term

Term (*n*)	*f(n)*	
0	$\frac{2}{3}$	×3
1	2	×3
2	6	×3
3	18	×3
4	54	×3



Recursive Form

$f(1) = 2, f(n+1) = f(n) \times 3$, for integers $n \geq 1$

$f(3) = f(2) \times 3 = 18$

$\boxed{6}$

Common ratio

Explicit Form

$f(n) = \frac{2}{3} \cdot 3^n$, for integers $n \geq 1$

0th term

Common ratio

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INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (F.IF.1-3)

An *index* refers to the part of the sequence that is being discussed and serves as the input value. Students may wish to start their explicit formula with an index of $n = 1$ or $n = 0$.

Rewrite sequences of numbers in tabular form, where the input represents the term number (the position or index) in the sequence, and the output represents the number in the sequence.

EXAMPLE

a. Have students write an explicit form of an equation in function notation for the sequence 5, 8, 11, 14…

b. Then have the students graph the function.

Discussion:

1.) Students may have written an equation in explicit form for the sequence as $f(n) = 3n + 2$ or $f(n) = 3(n - 1) + 5$ when $n = 1$. Discuss the benefits and drawbacks of the equation in the form of $f(n) = 3n + 2$ versus $f(n) = 3(n - 1) + 5$ that arises from the formula for the *n*th term of the arithmetic sequence. The benefit of the first form, $f(n) = 3n + 2$, is that it connects the sequence to the graph of linear equation and writing linear equations is familiar to students. Using this form also avoids the $(n - 1)$ notation in the equation which may be difficult for some students; its drawback is that the first term of the sequence is not evident. Since traditionally the sequence begins with the 1^{st} term, to find this equation students would need to find the 0^{th} term ($n = 0$), or the *y*-intercept of the graph, before writing the equation in slope-intercept form. Remind students that the graph represents the entire set of infinite solutions of the equation $f(n) = 3n + 2$ but only particular points using the first coordinates 1, 2, 3, etc. would represent terms of the sequence. Point out that the graphs of sequences must be discrete. The drawback of the second form, $f(n) = 3(n - 1) + 5$ is that it is not useful for visualizing the graphs and some students have difficulty understanding that $(n - 1)$ refers to the preceding term. The benefit of using the second version of the explicit form is that it allows the first term in the sequence to be easily identified. Note: Although the form $f(n) = 3(n - 1) + 5$ makes sense to students who have been recently exposed to the formula for finding the *n*th term of an arithmetic sequences, the form $f(n) = 3n + 2$ is more intuitive arising from students' ideas of functions. Having an understanding of denoting the preceding term by $(n - 1)$ is especially beneficial for students intending to take advanced mathematics courses.

2.) Students can choose an index indicating where the sequence begins. Students may have written an equation in explicit form for the sequence as $f(n) = 3n + 2$ or $f(n) = 3n + 5$ depending if they choose an index of 1 or of 0 (or they could have chosen another index and would therefore have a different equation) for the beginning of the sequence. This forces them to make a decision about how they want to view the input-output relationship when viewing the sequence as a function. For example, for the explicit form $f(n) = 3n + 2$, for $n \geq 1$, when students choose $n = 1$, the input-output table will look like this:

n	1	2	3	4	…
f(n)	5	8	11	14	

When students choose $n = 0$, they will get the explicit form $f(n) = 3n + 5$, for $n \geq 0$, and the input-output table will look like this:

n	0	1	2	3	…
f(n)	5	8	11	14	

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (F.IF.1-3)

EXAMPLE

Describe the pattern for numbers 2, 3, 5, 8, …

Discussion: To encourage caution when guessing with patterns, give the students the following sequence 2, 3, 5, 8, … Most students will state the next number to be 12, by seeing the pattern +1, +2, +3, but it could also be 13 if the pattern really is adding the two previous terms. Another example illustrating a similar point is 1, 2, 4, 8, 16,… because both 31 or 32 can be legitimate numbers next in the sequence. See https://epicmath.org/2013/02/13/7-very-misleading-sequences/ for more misleading sequences.

EXAMPLE

Describe the pattern for numbers 1, 1, 2, 1, 1, 1, 2, 1, 1, 1, 1, 2, …..

Discussion: Students should be aware that many sequences in both mathematics and real-life are neither arithmetic nor geometric sequences.

Instructional Tools/Resources

These tools and resources may be helpful for instruction. Before use, they should be evaluated at the local level to determine their appropriateness for instruction.

Manipulatives/Technology

- Diagrams or drawings of function machines, as well as tables and graphs.
- Function Machine virtual manipulatives
- Desmos is a free graphing calculator that is available to students as website or an app.
- Geogebra is a free graphing calculator that is available to students as website.
- Wolframalpha is dynamic computing tool.
- Visual Patterns is a website that shows pictures of linear, exponential, and quadratic patterns.
- Patterns Posters for Algebra 1 from Finding Ways by Fawn Nguyen is a blog about a class activity where students are given patterns, and they have to make posters from them. She is the creator of the visual patterns link above.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (F.IF.1-3)

The Concept of a Function

- Card Sort Function by Desmos is an activity in which students sort graphs, equations, and contexts according to whether each one represents a function.
- Polygraph: Functions and Relations by Sam Wright from Desmos is an activity that is designed to spark vocabulary-rich conversations about discrete and continuous functions and relations. Key vocabulary that may appear in student questions includes: function, non-function, relation, discrete, continuous, input, output, *x*-value, and *y*-value.
- Points on a Graph is a task from Illustrative Mathematics that addresses a common confusion between independent and dependent variables.
- Domains is a task from Illustrative Mathematics that helps students consider the domain in terms of values for which each operation is invalid.
- Yam in the Oven is a task from Illustrative Mathematics that gives students practice interpreting statements in function notation. A similar task is Cell Phones.

Function Machines

- Function Machine is an applet by Math Playground of a function machine.
- Function Machine is an applet by Shodor of a function machine.
- Function Rules is a task by Illustrative Mathematics where students use a function machine to create a rule.

Function Notation

- Functioning Well is a lesson published in the Georgia Standards of Excellence Framework that is an introduction to functions and function notation. This lesson can be found on pages 147-155.
- Using Function Notation I is a task from Illustrative Mathematics that addresses a common misconception with respect to function notation.
- The task Interpreting the Graph connects interpreting function notation to a graph.

Sequences

- The Devil and Daniel Webster by NCTM Illuminations has students use recursive forms of a function to represent relationships.
- Snake on a Plane is a task from Illustrative Mathematics that has students approach a function using a recursive and algebraic definition.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (F.IF.1-3)

Curriculum and Lessons from Other Sources

- EngageNY, Algebra 1, Module 3, Topic B, Lesson 8: Why Stay with Whole Numbers?, Lesson 9: Representing, Naming, and Evaluating Functions, Lesson 10: Representing, Naming, and Evaluating Functions are lessons that pertain to this cluster. *Note: Ohio does not require the use of sequence notation.*
- Mathematics Vision Project, Algebra 1, Module 1: Sequences has many lessons that pertain to this cluster.
- Mathematics Vision Project, Algebra 1, Module 2: Linear and Exponential Functions has many lessons that pertain to this cluster.
- Unit 3: Functions from eMATHinstruction has materials that could be used for intervention. These documents can be used for individual students or for the entire class.
- Exporing Symbols by Burrill, Clifford, Scheaffer is the teacher's edition of a textbook in the Data-Driven Mathematics series published by Dale Seymour Publications. There are several lessons that pertain to this cluster. The student edition can be found here.

General Resources

- Arizona High School Progression on Functions is an informational document for teachers. This cluster is addressed on pages 7-8.
- High School Coherence Map by UnboundED is a map of the relationships among high school mathematics standards. Please note: This resource is aligned to the Common Core not Ohio's Learning Standards. Although the tool is useful, it needs to be checked with respect to Ohio's Learning Standards.

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<table>
<tr><th>STANDARDS</th><th>MODEL CURRICULUM (F.IF.4-5)</th></tr>
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Functions

INTERPRETING FUNCTIONS

Interpret functions that arise in applications in terms of the context.

F.IF.4 For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship. *Key features include the following: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; and periodicity.* ★ (A2, M3)

b. Focus on linear, quadratic, and exponential functions. (A1, M2)

F.IF.5 Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes. *For example, if the function h(n) gives the number of person-hours it takes to assemble n engines in a factory, then the positive integers would be an appropriate domain for the function.* ★

b. Focus on linear, quadratic, and exponential functions. (A1, M2)

</td><td>

Expectations for Learning

In eighth grade, students model linear relations between two quantities; analyze graphs to determine where they are increasing and decreasing; and determine if relations are linear or non-linear.

In this cluster, students interpret additional key features of the graphs and tables of linear, quadratic, and exponential functions only. They also determine the domain of a function by looking at a graph or table. In a real-life scenario students can find the restrictions on the domain.

In Algebra 2, students extend identifying and interpreting key features of functions to include periodicity. Students also have to select appropriate functions that model the data presented. Average rate of change over a specific interval will also be included in Algebra 2.

Note on differences between standards: In F.IF.4 and F.IF.5, the emphasis is on the context of the problem and on making connections among graphs, tables, and the context. In F.IF.7, the emphasis is on creating a graph given a symbolic representation, and then identifying the key features of the graph and connecting the key features to the symbols.

ESSENTIAL UNDERSTANDINGS

- Key features (as listed in the standard) of a function can be illustrated graphically and interpreted in the context of the problem.
- The sensible domain for a real-world context should be accurately represented in graphs, tables, and symbols.
- Functions can have continuous or discrete domains.
- A quadratic function is symmetrical about its axis of symmetry.

MATHEMATICAL THINKING

- Connect mathematical relationships to contextual scenarios.
- Attend to meaning of quantities.
- Determine reasonableness of results.

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STANDARDS	MODEL CURRICULUM (F.IF.4-5)
F.IF.4-5, continued	**Expectations for Learning, continued** **INSTRUCTIONAL FOCUS** **Remember, in this course, for exponential functions, assessments should focus on integer exponents only. For quadratic functions, students should work with expressions in which the leading coefficient can be any real number, but assessment questions should focus on expressions with leading coefficients of 1 with occasional questions using other simple leading coefficients.* • For linear functions represented as tables, graphs, or verbal descriptions, interpret intercepts and rates of change in the contexts of the problem. • For exponential functions, interpret intercepts, growth/decay rates, and end behaviors in the contexts of the problems, given tables, graphs, and verbal descriptions. • For quadratic functions, interpret intercepts; maximum or minimum; symmetry; intervals of increase or decrease; and end behavior, given tables, graphs, and verbal descriptions. • Use written descriptions or inequalities to describe intervals on which a function is increasing/decreasing and/or positive/negative (neither interval notation nor set builder notation are required). • Determine whether to connect points on a graph based on the context (continuous vs. discrete domain). • Demonstrate understanding of domain in the context of a real-world problem. • Compare the key features of quadratic functions to linear and exponential functions. For example: o Linear functions are either always increasing, decreasing, or constant. o Exponential functions are either always increasing or decreasing. o Quadratic functions increase to a maximum then decrease or decrease to a minimum then increase. *Continued on next page*

STANDARDS	MODEL CURRICULUM (F.IF.4-5)
F.IF.4-5, continued	**Content Elaborations** **OHIO'S HIGH SCHOOL CRITICAL AREAS OF FOCUS** • Algebra 1, Number 2, pages 5-7 • Algebra 1, Number 5, pages 11-12 **CONNECTIONS ACROSS STANDARDS** • Create equations that describe numbers or relationships (A.CED.2a, b). • Represent and solve equations and equations and inequalities graphically (A.REI.10). • Understand the concept of a function, and use function notation (F.IF.1-3). • Graph linear functions and indicate intercepts (F.IF.7a). • Graph quadratic functions and indicate maxima and minima (F.IF.7b). • Graph simple exponential functions, indicating intercepts, and end behavior (F.IF.7e). • Interpret expressions for functions in terms of the situation they model (F.LE.5). • Analyze functions using different representations (F.IF.9b). • Interpret linear models (S.ID.7).

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (F.IF.4-5)

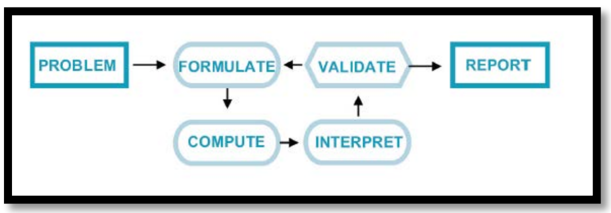
Instructional Strategies

Note: The model curriculum is not a collection of lessons nor a full curriculum; it does not suggest pace, sequence, or amount of time spent on topics. It provides information about a topic related to the standards including ideas for examples, strategies for teaching, common misconceptions, and possible connections between topics. In addition, the emphasis of the model curriculum is on instruction not on assessment; therefore, the examples in the Instructional Strategies section are written for instructional purposes not necessarily as assessment items. Examples may need to be rewritten to accommodate the needs of each individual classroom.

Standards for Mathematical Practice
This cluster focuses on but is not limited to the following practices:
***MP.1** Make sense of problems and persevere in solving them.*
***MP.2** Reason abstractly and quantitatively.*
***MP.4** Model with mathematics.*

Functions are often described in terms of their using key features. Graphs allow the behavior of the function to be more apparent.

MODELING



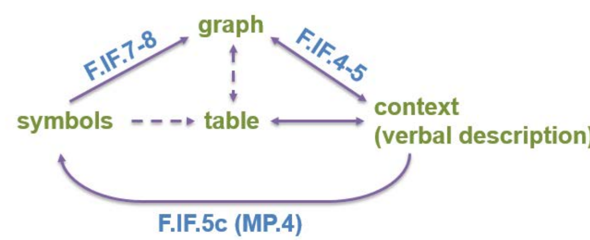
This cluster also includes the modeling standards. See page 13 for more information about modeling.

Begin instruction from a modeling standpoint. Start with a context and ask "Do one of the three functions—linear, exponential, quadratic—fit the behavior seen in the graph?" The answer sometimes needs to be "no," so that other function types can be explored within the context of the problem. For example, although not necessarily in Algebra 1, students should be aware that some scenarios are modeled with a periodic phenomenon that have graphs that repeat themselves after the particular interval along *x* – axis. Other situations are modeled by graphs with "wiggles" that are called polynomial functions.

Students should be given a formula that can be graphed using Desmos or other graphing technology and they should be able to reason about the graph after they can see it. Given a table of values students could then create a scatterplot, possibly fit a curve to it, and reason about it in the same way they reason about formulas. *Note: Draw attention to the fact that sometimes the function may not be able to be described by a formula; sometimes the best we can do to describe a function is by a graph or a table.*

INTERPRETING FUNCTIONS

Standards F.IF.4, F.IF.5, and F.IF.7 should be approached individually and in conjunction with one another. On a particular day, the focus could be on F.IF.4, F.IF.5, or F.IF.7 individually, but over the course of a few days, the standards should be woven together. See cluster F.IF.7-9 for an introductory activity about graphing stories. A follow-up activity should be connecting graphs to stories. Students should also be encouraged to write their own stories and then graph them. Then they could share their graphs with their classmates. Focus on graphs that are neither linear, quadratic, or exponential including piecewise scenarios.



INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (F.IF.4-5)

EXAMPLE

Put students into groups. Give each student a role such as walker, measurer, marker, recorder, and timer. To prepare give students sticky notes and have them label each sticky note in increments of 5 seconds from 0 seconds to 45 seconds. Have them make 4 sets of these sticky notes in advance. Place the 0 second sticky note on the ground as the starting point. Have one student walk the scenario below. Another student should time the students calling out the time out loud. Another student should place a sticky at the foot of the walker every time the timer calls out a multiple of 5 seconds. The measurer should measure the distance from the beginning to each sticky note, and the recorder should record the information in a distance/time table. Before doing the experiment, have students do a quick sketch of what they think the graph will look like. Many students will draw graphs that are not functions.

- **a.** Walk forward slowly for 10 seconds. Stop for 10 seconds. Walk quickly for 5 seconds. Walk backwards for 15 seconds. Stop for 5 seconds.
- **b.** Walk forward quickly for 10 seconds. Walk backwards slowly for 10 seconds. Stop for 5 seconds. Walk backwards quickly for 5 seconds. Stop for 10 seconds. Walk forward slowly for 5 seconds.
- **c.** Walk forward quickly for 5 seconds. Stop for 15 seconds. Walk backwards slowly for 5 seconds. Stop for 5 seconds. Walk forward slowly for 15 seconds.

Discussion: Have each student graph the three different situations. Then have a class discussion about the graphs. Some students will have a difficult time realizing that stopping creates a horizontal line and that going backwards is graphed by a line with a negative slope (Many will assume that the line actually goes backwards and is therefore not a function.) Then have a discussion with students about how a speed/time graph would differ from a distance/time graph given the same situation. It may also be helpful to have students jump vertically and measure the height along the wall, so they realize that time still moves forward and forms a parabola even though their distance jumped is vertical.

EXAMPLE

Have students do the activity found here about matching graphs to their stories. See the Instructional Tools/Resources section for more resources on matching graphs to their stories.

Desmos Function Bundle has more activities involving graphing stories.

Investigate real-world data in which—

- several outputs may be paired with one input, like height vs. arm span (two people may have the same height, but different arm spans), or
- one output is paired with one input, like city population over time.

Students should be able to reason about trends in the data.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (F.IF.4-5)

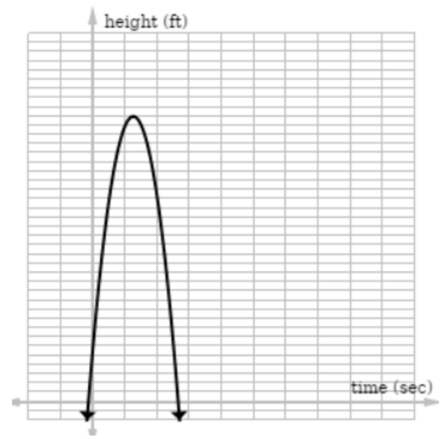
Have students flexibly move from examining a graph and describing its characteristics (e.g., intercepts, relative maximums, etc.) to using a set of given characteristics to sketch the graph of a function.

Some students incorrectly believe that a graph is a picture of the situation (such as the path of the basketball) rather than a representation of the relationship of two particular quantities in a situation. Attention needs to be paid to the quantities given in the context, such as time in seconds since the basketball was released and the height in centimeters of the basketball from the ground.

EXAMPLE

Using technology have students graph the relationship between the height of a basketball above the ground and the time the basketball is in the air: $y = -16x^2 + 40x + 6$. Have them complete a table using the trace button connecting key features of the graph to the quantities of the situation.

- What is the initial height of the ball above the ground? Why do you think the height does not start at 0?
- What is the maximum height of the ball?
- What time does the ball hit the 10ft hoop?
- Assuming there is no resistance, and the ball can follow its path, what time will the ball hit the ground?



Time (*s*)	Height (*cm*)

Draw attention to the fact that the height of the basketball is related to time. The horizontal access is not showing the distance of the hoop from the player nor does it trace the basketball's path. The shape would still be a parabola if the basketball was thrown straight up vertically into the air. Therefore the graph and the function represent that the time is moving forward not the basketball. To break this misconception have students jump straight up and down three times and graph their height and time. Point out that even though they did not move any distance horizontally, their graph creates 3 parabolas.

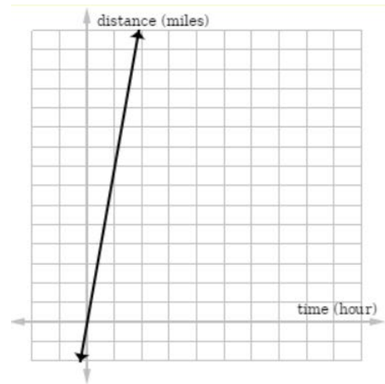
INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (F.IF.4-5)

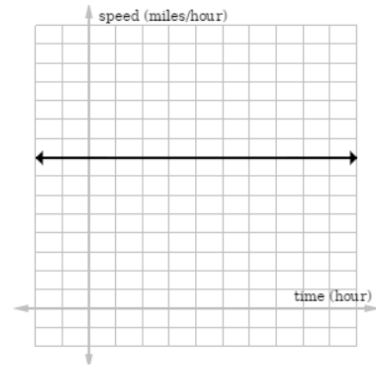
EXAMPLE

The teacher asked students to graph the following situation: Monica ran 8 miles per hour.

- Lawrence illustrated this situation by drawing the graph on the left.
- Michelle illustrated this situation by drawing the graph on the right.

Whose graph correctly describes the situation? Explain.





Discussion: Students should come to the realization that both graphs are correct. The first graph illustrates a distance/time graph and the second graph illustrates a speed/time graph. Since the distance increases 8 mile for every hour, the line in Lawrence's graph has a positive slope of 8. Since the speed is constant, the line in Michelle's graph is horizontal line through $y = 8$.

Emphasize that, for all functions, the x-intercepts of the graph of f are the solutions of the equation $f(x) = 0$. *Note: For quadratics, the methods of solving equations in A.REI.4 can be referenced.*

Examine a table of a function and identify its key features from the table, such as intervals on which the function increases, decreases, or exhibits periodic behavior.

EXAMPLE

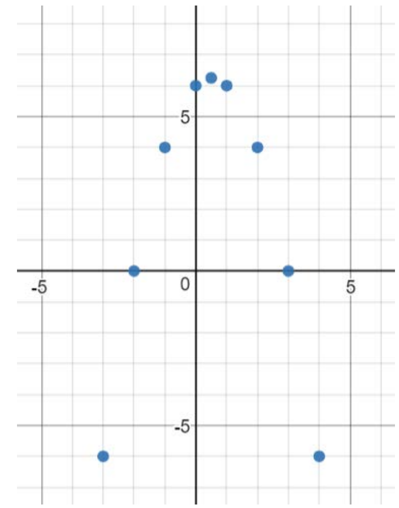
The table represents a continuous function defined on the interval $-2 < x < 3,$ where just some integer inputs being used are displayed. Identify the key features of a graph.

x	$f(x)$
-2	$\frac{2}{9}$
-1	$\frac{2}{3}$
0	2
1	6
2	18
3	54

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (F.IF.4-5)

EXAMPLE

The graph on the right represents a discrete quadratic function defined on the interval $-4 < x < 3$. Identify the key features of a graph.



KEY FEATURES OF FUNCTIONS

In Algebra 1, the majority of graphs representing functions may have the following key features:

- increasing intervals,
- decreasing intervals,
- relative maximums,
- relative minimums,
- *x*- and *y*-intercepts,
- symmetries,
- end behavior, and/or
- periodicity (not needed for Algebra 1)

There are limits to the key features depending on the function type. If the domain is restricted, additional key features are possible. Although the focus of Algebra 1 is on linear, exponential, and quadratic functions, students should be exposed to other function types and informally discuss their key features. Connect with concepts of parent functions and function families in F.IF.7-9. It might be helpful to start with a non-formula graph such as temperature over time.

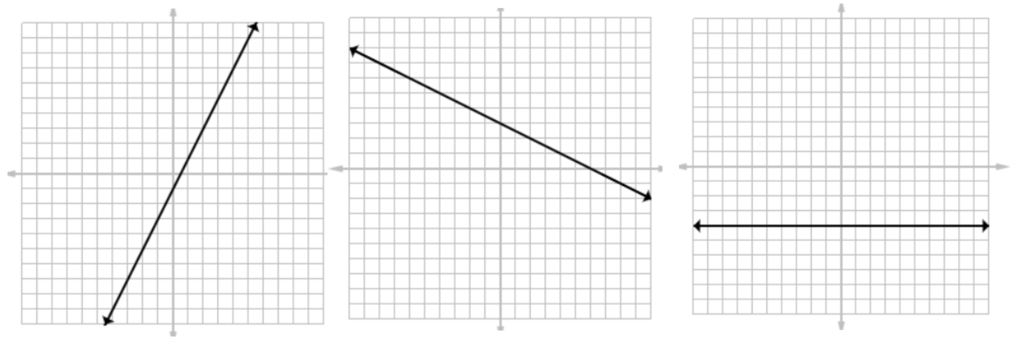
When discussing intervals, informal descriptions of end behaviors are acceptable. For example, "To the right, the graph goes to infinity and to the left, it is 'leveling off.'" (Interval notation and set builder notation are not necessary.) Also, written descriptions or inequalities are acceptable. For example,

- all x-values greater than 3 or x > 3;
- the function is increasing between 3 and 7, or is increasing 3 < x < 7.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (F.IF.4-5)

Linear Functions

A graph of a linear function with an unrestricted domain may have *x*-intercept or *y*-intercept, or both. The graph may be increasing or decreasing, or neither. Regardless how the graph of a linear function looks, it does not have a minimum or maximum. See 8th Grade Model Curriculum cluster 8.F.4-5 for ideas about scaffolding with linear functions.



Rate of Change/Slope

Although F.IF.6 has been moved to a later course, slope should still be discussed with respect to linear functions as it was a concept introduced in 8th grade. (See Grade 8 Model Curriculum 8.F.1-3 for scaffolding ideas.) Given a table of values, such as the height of a plant over time, students can estimate the rate of plant growth. Also, if the relationship between time and height is expressed as a linear equation, students should explain the meaning of the slope of the line in the context of the situation. In addition students should review the slope formula.

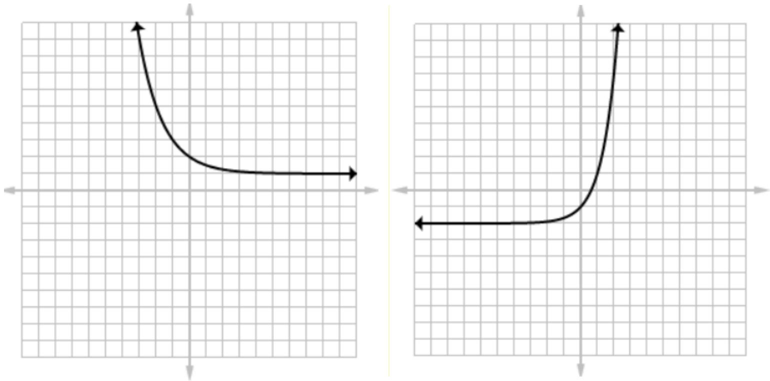
Students may believe that the slope of a linear function is merely a number used to sketch the graph of the line. In reality, slopes have real-world meaning, and the idea of a rate of change or related quantities is fundamental to understanding major concepts in mathematics.

Some students perceive slope as the tilt of a line instead of a relationship between two quantities on a graph. Correct this misconception by changing the scale so that a large slope appears "flat" or a fractional slope (between 0 and 1) appears "steep."

There are some students who incorrectly think that the first term, mx, of the linear equation $y = mx + b$ represents a slope. Instead point out that the slope, m, represents the covariance between y and x.

Exponential Functions

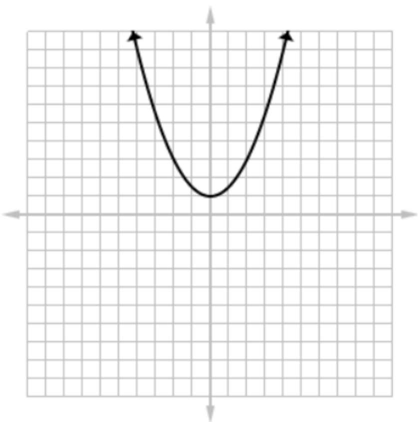
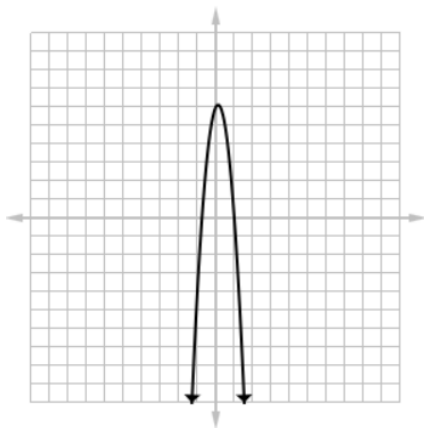
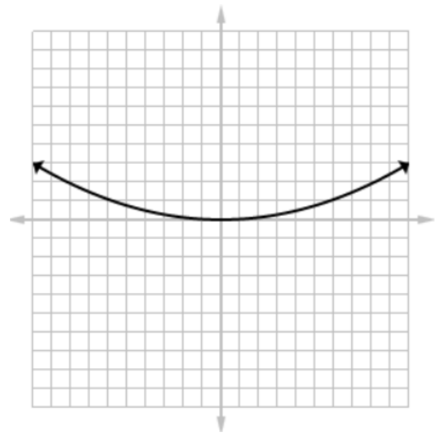
A graph of an exponential function may have an *x*-intercept or both *x*- and *y*-intercepts. If the definition of an exponential functions is that it is a function in which the values of the domain are exponents, then adding or subtracting a constant can make different *x*-intercepts possible. For example, a function such as $y = 2^x - 8$ has an *x*-intercept at 3 since the point (3, 0) is on the graph. The graph of an exponential function may be increasing or decreasing. It does not have relative maximums, minimums, or symmetry. However, it can be described by its end behavior.



INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (F.IF.4-5)

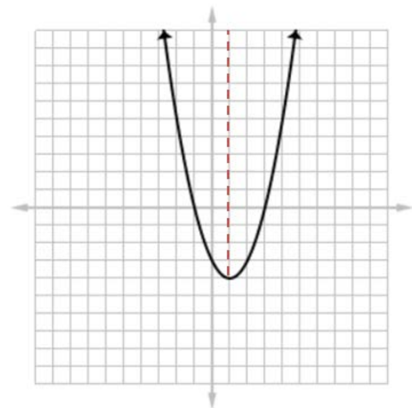
Quadratic Functions

The graph of a quadratic function may have either 0, 1, or 2 *x*-intercepts but only 1 y-intercept. The graph has either a decreasing interval followed by an absolute minimum and then an increasing interval, or it has an increasing interval followed by an absolute maximum and a decreasing interval. The graphical representation of a quadratic function is also symmetrical and is described by its axis of symmetry. In addition, a quadratic can be described by its end behavior. If the domain is a restricted, a quadratic may not have both an increasing and decreasing interval or necessarily an absolute minimum/maximum.

Axis of Symmetry

To develop an intuitive sense of the vertex formula ($h = \frac{-b}{2a}$ for $f(x) = ax^2 + bx + c$ with vetex $(h, k)$and then evaluating *y* for *h* to find *k*) for the axis of symmetry, *x* = *h*, a student may find the mean of the two x-intercepts. However, to be noted this method will only apply to quadratics such as found in Algebra 1 that have real roots; See F.BF.3-4 for an example on using transformations to find the vertex formula.



DOMAIN OF A FUNCTION

When choosing a function family, be sure to ask whether that function family makes sense within the context. Sometimes the answer is no, and other times the answer may be yes over a restricted domain. For example, an entire roller coaster cannot be defined by a single quadratic equation, but one hill may be modeled by one quadratic function, and the next hill could be modeled by a different quadratic function.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (F.IF.4-5)

Recognize appropriate domains of functions in real-world settings. For example, when determining a weekly salary based on hours worked, the hours (input) could be a rational number, such as 25.5, but not a negative number. Furthermore, there must be a maximum number of hours worked, determined based on reasonable assumptions. If a function relates the number of cans of soda sold in a machine to the money generated, the domain must consist of whole numbers.

Make sure students are exposed to functions in mathematical and real-world contexts that have both continuous and discrete domains.

Students may incorrectly believe that it is reasonable to input any *x*-value into a function, not understanding that context determines the domain. Therefore, they will need to examine multiple situations in which there are various limitations to the domains.

Instructional Tools/Resources

These tools and resources may be helpful for instruction. Before use, they should be evaluated at the local level to determine their appropriateness for instruction.

Manipulatives/Technology

- Tables, graphs, and equations of real-world functional relationships
- Graphing calculators to generate graphical, tabular, and symbolic representations of the same function for comparison
- Geogebra is a free graphing calculator that is available to students as website.
- Desmos is a free graphing calculator that is available to students as website or an app.
- Wolframalpha is dynamic computing tool.

Graphing Stories

- Graphing Stories is an Illustrative Math task that investigates the graphs of relationships between quantities using video clips.
- Stories from Graphs a lesson from Cobb Learning where students match distance/time graphs to stories and then create stories to go with other graphs.
- Graphing Stories is a collaboration between Dan Meyer and Buzz Math that has students graph stories based on a real-life video clip.
- Functions Bundle by Desmos has 7 activities that explore functions.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (F.IF.4-5)

Interpreting Functions

- Warming and Cooling is an Illustrative Mathematics task that could be used as an assessment of reading and interpreting graphs.
- Is the Gateway Arch a Parabola is a blog by Murray Bourne that discuss if the Gateway Arch is a parabola.
- Exponential Graph Characteristics is a lesson from Milwaukee Public Schools surrounding the key features of exponential functions using the Frayer Model.
- Lifespan of a Meme, the Harlem Shake by Yummy Math is an activity where students interpret a graph to explore a viral video.
- The Queen's Reward is an activity where students help a young mathematicin outwit the Queen's chief advisors by solving quadratic functions.
- Roller Coasting through Functions is a lesson by NCTM Illuminations where students use graphs and tables to analyze the falls of different roller coasters using a quadratic equations. *NCTM now requires a membership to view their lessons.*
- Protein Bar Toss Parts 1 and 2 is a task from the Georgia Standards of Excellence Framework where students find the maximums and minimums of quadratic functions. This task starts on page 93.
- Egg Launch Contest by NCTM Illuminations is a lesson where students represent and interpret quadratic functions as a table, with a graph, and with an equation.
- Weightless Wonder by NASA is a lesson where students investigate the characteristics of quadratic functions in the context of parabolic flights involving NASA's Weightless Wonder Jet.
- Parabolic Pee by Yummy math has students interpret parabolic motion functions.

Domain and Range

- Oakland Coliseum is a task from Illustrative Mathematics where students are asked to find the domain and range from a given context.
- The Restaurant is a task from Illustrative Mathematics where students are asked to find the domain and range from a given context.
- The Domain and Range Introduction and Finding Domain and Range are lessons by Desmos that explore domain and range.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (F.IF.4-5)

Curriculum and Lessons from Other Sources

- EngageNY, Module 1, Topic A, Lesson 1: Graphs of Piecewise Linear Functions, Lesson 2: Graphs of Quadratic Functions, Lesson 3: Graphs of Exponential Functions, Lesson 4: Analyzing Graphs—Water Usage During a Typical Day at School are lessons that pertain to this cluster.
- EngageNY, Module 3, Topic B, Lesson 13: Interpreting the Graph of a Function, Lesson 14: Linear and Exponential Models—Comparing Growth Rates are lessons that pertain to this cluster.
- EngageNY Module 4, Topic A, Lesson 8: Exploring the Symmetry in Graphs of Quadratic Functions, Lesson 10: Interpreting Quadratic Functions from Graphs and Tables are lessons that pertain to this cluster.
- EngageNY, Module 5, Topic A, Lesson 1: Analyzing a Graph, Lesson 2: Analyzing a Data Set, Lesson 3: Analyzing a Verbal Description are lessons that pertain to this cluster.
- EngageNY, Module 5, Topic B, Lesson 9: Modeling a Context from a Verbal Description is a lesson that pertains to this cluster.
- Mathematics Vision Project, Algebra 1, Module 2: Linear and Exponential Functions has many lessons that pertain to this cluster.
- Mathematics Vision Project, Algebra 1, Module 3: Features of Functions has many lessons that pertain to this cluster.
- Mathematics Vision Project, Algebra 1, Module 8: More Functions, More Features has many lessons that pertain to this cluster.
- Unit 3: Functions from eMATHinstruction has materials that could be used for intervention. These documents can be used for individual students or for the entire class.

General Resources

- Arizona High School Progression on Functions is an informational document for teachers. This cluster is addressed on pages 8-9.
- Arizona's Progression on High School Modeling is an informational document for teachers. This cluster is addressed on page 12.
- High School Coherence Map by UnboundED is a map of the relationships among high school mathematics standards. Please note: This resource is aligned to the Common Core not Ohio's Learning Standards. Although the tool is useful, it needs to be checked with respect to Ohio's Learning Standards.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (F.IF.4-5)

References

- Baron, L. (February 2015). An authentic task that models quadratics. *Mathematics Teaching in the Middle School*, *20(6),* 334-340.
- Common Core Standards Writing Team. (2013, March 1). *Progressions for the Common Core State Standards in Mathematics (draft). Grade 8, High School, Functions.* Tucson, AZ: Institute for Mathematics and Education, University of Arizona.
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- Nebesniak, A. & Burgoa, A. (February 2015). Developing the vertex formula meaningfully. *Mathematics Teacher*, *108(6),* 429-433.
- Paoletti, T., Stevens, I., & Moore, K. (February 2017). Tricks may inhibit students' reasoning. *Mathematics Teacher*, *110(6),* 446-453.
- Weiss, M. (December 2016/January 2017). Solving and graphing quadratics with symmetry transformation. *Mathematics Teacher*, *110(5),* 394-397.

STANDARDS	MODEL CURRICULUM (F.IF.7-9)
Functions **INTERPRETING FUNCTIONS** **Analyze functions using different representations.** **F.IF.7** Graph functions expressed symbolically and indicate key features of the graph, by hand in simple cases and using technology for more complicated cases. Include applications and how key features relate to characteristics of a situation, making selection of a particular type of function model appropriate. ★ **a.** Graph linear functions and indicate intercepts. (A1, M1) **b.** Graph quadratic functions and indicate intercepts, maxima, and minima. (A1, M2) **e.** Graph simple exponential functions, indicating intercepts and end behavior. (A1, M1) **F.IF.8** Write a function defined by an expression in different but equivalent forms to reveal and explain different properties of the function. **a.** Use the process of factoring and completing the square in a quadratic function to show zeros, extreme values, and symmetry of the graph, and interpret these in terms of a context. (A2, M3) **i.** Focus on completing the square to quadratic functions with the leading coefficient of 1. (A1) *Continued on next page*	**Expectations for Learning** In eighth grade, students graph and write linear functions, but their knowledge of key features of functions is limited to slope and *y*-intercept. They are exposed to non-linear functions and can distinguish between linear and non-linear functions. In this cluster, students graph linear, quadratic, and exponential functions given a symbolic representation and indicate intercepts and end behavior. They compare linear, quadratic, and exponential functions given various representations. In Algebra 2, students graph polynomial, square root, cube root, trigonometric, piecewise-defined, (+) rational, and (+) logarithmic functions. They identify and interpret key features (as applicable) including intercepts, end behavior, period, midline, amplitude, symmetry, asymptotes, maxima/minima, and zeros. *Note on differences between standards: In F.IF.4 and F.IF.5, the emphasis is on the context of the problem and on making connections among graphs, tables, and the context. In F.IF.7, the emphasis is on creating a graph given a symbolic representation, and then identifying the key features of the graph and connecting the key features to the symbols.* **ESSENTIAL UNDERSTANDINGS** • The graph of a linear function shows intercepts and rate of change. • The graph of an exponential function shows the *y*-intercept and end behaviors. • The graph of a quadratic function shows intercepts and maximum or minimum. • Function families have commonalities in shapes and features of their graphs. • The factored form of a quadratic function reveals the zeros of the function (i.e., the *x*-intercepts of the graph); the vertex form of a quadratic function reveals the maximum or minimum of the function; the standard form of a quadratic function reveals the *y*-intercept of the graph. • Different representations (graphs, tables, symbols, verbal descriptions) illuminate key features of functions and can be used to compare different functions. • More generally, writing a function in different ways can reveal different features of the graph of a function. *Continued on next page*

STANDARDS	MODEL CURRICULUM (F.IF.7-9)
b. Use the properties of exponents to interpret expressions for exponential functions. *For example, identify percent rate of change*[G] *in functions such as* $y = (1.02)^t$, *and* $y = (0.97)^t$ *and classify them as representing exponential growth or decay.* (A2, M3) **i.** Focus on exponential functions evaluated at integer inputs. (A1, M2) **F.IF.9** Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions). *For example, given a graph of one quadratic function and an algebraic expression for another, say which has the larger maximum.* (A2, M3) **b.** Focus on linear, quadratic, and exponential functions. (A1, M2)	**Expectations for Learning, continued** **MATHEMATICAL THINKING** • Make connections between concepts, terms, and properties within the grade level and with previous grade levels. • Analyze a mathematical model. **INSTRUCTIONAL FOCUS** **Remember, in this course, for exponential functions, assessments should focus on integer exponents only. For quadratic functions, students should work with expressions in which the leading coefficient can be any real number, but assessment questions should focus on expressions with leading coefficients of 1 with occasional questions using other simple leading coefficients.* • Given symbolic representations of linear, quadratic, and exponential functions, create accurate graphs showing all key features. • Identify the key features of the graph of a quadratic function by factoring, using the quadratic formula, or completing the square • Compare and contrast linear, quadratic, and exponential functions given by graphs, tables, symbols, or verbal descriptions. • Determine the zeros of a quadratic function by factoring, using the quadratic formula, or completing the square. • Use different forms of quadratic functions (standard form, vertex form, factored form) to reveal different features. • Explore the relationship of the symbolic representation of a function and its graph by adjusting parameters. *Continued on next page*

STANDARDS	MODEL CURRICULUM (F.IF.7-9)
F.IF.7-9, continued	**Content Elaborations** **OHIO'S HIGH SCHOOL CRITICAL AREAS OF FOCUS** • Algebra 1, Number 2, pages 5-7 • Algebra 1, Number 5, 11-12 **CONNECTIONS ACROSS STANDARDS** • Interpret functions that arise in applications in terms of the context (F.IF.4). • Understand that the graph of an equation in two variables is the set of all its solutions plotted in the coordinate plane, often forming a curve (which could be a line) (A.REI.10). • Construct and compare linear, quadratics, and exponential models and solve problems (F.LE.1-2). • Interpret expressions for functions in terms of the situation they model (F.LE.5). • Solve quadratic equations in one variable (A.REI.4). • Write expressions in equivalent forms to solve problems (A.SSE.3). • Build a function that models a relationship between two quantities (F.BF.1).

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (F.IF.7-9)

Instructional Strategies

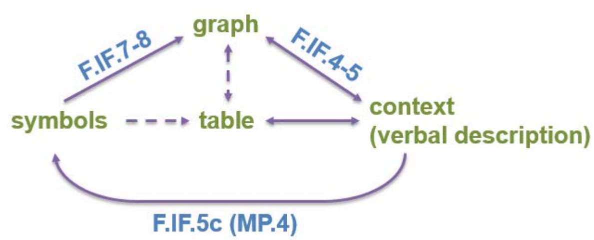
Note: The model curriculum is not a collection of lessons nor a full curriculum; it does not suggest pace, sequence, or amount of time spent on topics. It provides information about a topic related to the standards including ideas for examples, strategies for teaching, common misconceptions, and possible connections between topics. In addition, the emphasis of the model curriculum is on instruction not on assessment; therefore, the examples in the Instructional Strategies section are written for instructional purposes not necessarily as assessment items. Examples may need to be rewritten to accommodate the needs of each individual classroom.

Standards for Mathematical Practice
This cluster focuses on but is not limited to the following practices:
MP.4 *Model with mathematics.*
MP.5 *Use appropriate tools strategically.*
MP.7 *Look for and make use of structure.*

GRAPHING FUNCTIONS

Standards F.IF.4, F.IF.5, and F.IF.7 should be approached individually and in conjunction with one another. On a particular day, the focus could be on F.IF.4, F.IF.5, or F.IF.7 individually, but over the course of a few days, the standards should be woven together.

Introduce functions by having students create data from the walking and then graph the data to make a connection to distance/time graphs.



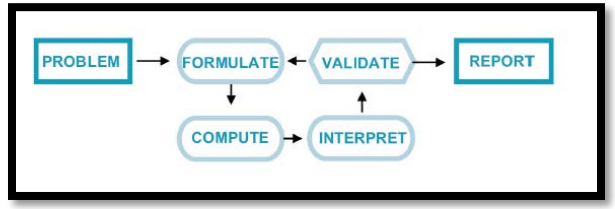
Some students may incorrectly believe that a function is a synonym for formula. Point out that some functions do not have formulas at all, and some formulas do not represent functions. For example, recoding the average daily temperature at the Columbus Airport cannot be represented by a formula, but can be represented by a table. There is a formula for the equation of circle, yet a circle is not a function.

Some students may incorrectly believe a piece-wise function is several different functions because it is represented by several different formulas. Emphasize that it is one function pieced together.

TIP!

Discuss the importance of using appropriate labels and scales on the axes when representing functions with graphs. Some students believe that the labels and scales on a graph are not important and can be assumed by a reader, and that it is always necessary to use the entire graph of a function when solving a problem that uses that function as its model.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (F.IF.7-9)



MODELING

This cluster is included in the modeling standards. See page 13 for more information about modeling.

FAMILIES OF FUNCTIONS

"Functions are often studied and understood as families, and students should spend time studying functions within a family, varying parameters to develop an understanding of how the parameters affect the graph of a function and formulas and its key features." (Common Core Standards Writing Team, March 2013)

A family of functions is a set of functions that are related by adjustable parameters. Students should develop an understanding of what the parameters in a family do. Explore various families of functions by graphing those families and helping students make connections in terms of the formulas and key features.

Have students explore and identify the function families: linear, quadratic, exponential, cubic, absolute value, square root, cubed root, sinusoidal among others but strive for fluency on linear, exponential, and quadratic functions with respect to representations and characteristics. However, for the other functions families, focus on shape. Introduce students to non-familiar functions to apply identification of key features. Also some functions, such as piece-wise functions, may not have formulas. This lends to connections with the knowledge about parent functions to model data. Students must be able to differentiate between linear, exponential, and quadratic functions and identify the parent function and interpret its key features. This should be driven by applications for modeling. Use domain and range values that are appropriate to the context.

TIP!

In a quadratic function $f(x) = ax^2 + bx + c$, many students forget to specify that $a \neq 0$. But this is critical, for if $a = 0$, then the function is not quadratic and its graph is not a parabola.

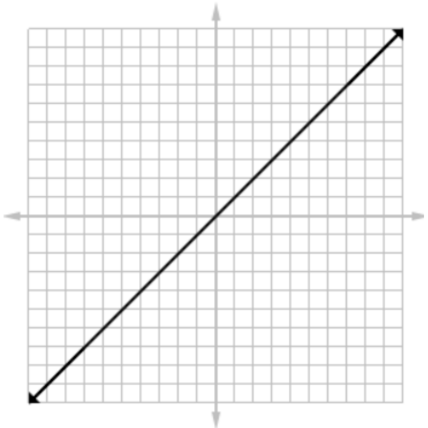
Real-world problems, such as maximizing the area of a region bound by a fixed perimeter fence, can help to illustrate applied uses of families of functions.

Students may believe that each family of functions (e.g., quadratic, square root, etc.) have no commonalities, so they may not recognize common aspects across the families of functions and their graphs such as *y*-intercepts and end behavior.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (F.IF.7-9)

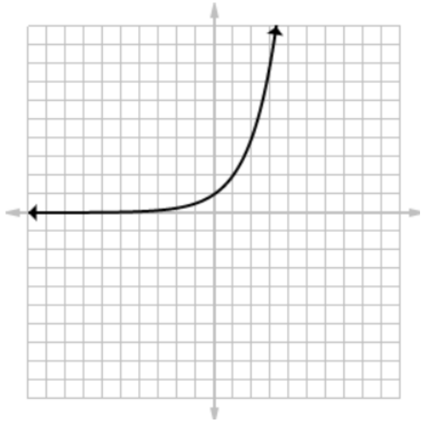
PARENT FUNCTIONS

Linear Functions



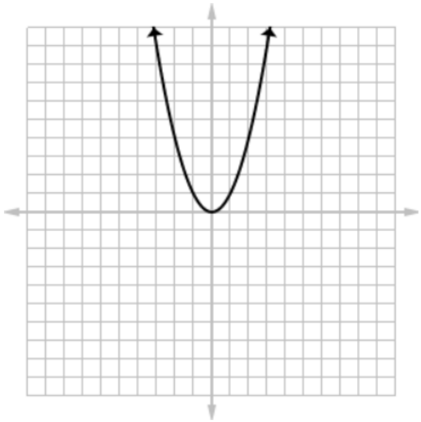
A linear function is a straight line with a constant rate of change and a *y*-intercept.

Exponential Functions



An exponential function is a curve that has a *y*-intercept and an end behavior approaching infinity at one end and approaching a number at the other end. It may or may not have an *x*-intercept.

Quadratic Functions



A quadratic function is a curve that is a U-shaped graph (or upside down U-shaped), *y*-intercepts, a maximum or minimum, and end behavior approaching toward either positive or negative infinity. It may or may not have an *x*-intercept(s).

EXAMPLE

Using technology have students create card pictures out of different function types. For example, give students a variety of exponential function and determine what all exponential functions have in common. This allows students to connect the graphs of functions with their corresponding algebraic representations.

Students should graph simple cases of functions by hand, but use technology for more complicated graphing done by students. Make connections to algebra work (from A.APR.6) to functions. Sometimes displaying a good graph means getting a good "window."

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (F.IF.7-9)

EXAMPLE

Identify a family of functions whose graphs are parabolas that are symmetric about the y-axis.

Discussion: Students should come to the conclusion that $f(x) = ax^2 + c$, with $a \neq 0$ is a family of functions. This is a legitimate family, but many students may not recognize this. This family can be considered a "subfamily" of the family of quadratic functions. What makes it a family is that (1) the functions all have the same "form" and (2) a particular member of the family is chosen by specifying the parameters.

FUNCTIONS IN EQUIVALENT FORMS

Writing a function in different ways can reveal different features of the graph of a function. Think of $f(x) = 4x^2$as a vertical stretch of $y = x^2$and the equivalent $f(x) = (2x)^2$ as a horizontal shrink that yields the same graph. Also, think of $g(x) = 2x + 6$ as a vertical shift of $y = 2x$, and the equivalent $g(x) = 2(x + 3)$ as a horizontal shift that yields the same graph. Think of $k(x) = 3^{x+2}$ as a horizontal shift of $y = 3^x$ and the equivalent $k(x) = 3^2 \cdot 3^x = 9 \cdot 3^x$ as a vertical stretch. Think of $h(x) = 1.02^{3x}$ as a horizontal shrink of $y = 1.02^x$, and the equivalent $h(x) = (1.02^3)^x = (1.061208)^x$ as a change of base of the exponential, but is not a vertical stretch. Students should be given the opportunity to come up with equivalent forms of the same function, and then explore why the functions are the same both algebraically and graphically. The process of rewriting functions to reveal key features should be used to explain/reveal features in the context of real-world scenarios.

Students may believe that the process of rewriting functions into various forms is simply an algebraic symbol manipulation exercise. Focus on the purpose of allowing different features of the function to be exhibited.

TIP!

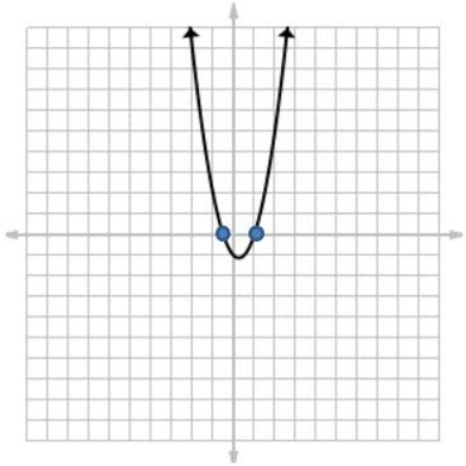
Use various representations of the same function to emphasize different characteristics of that function.

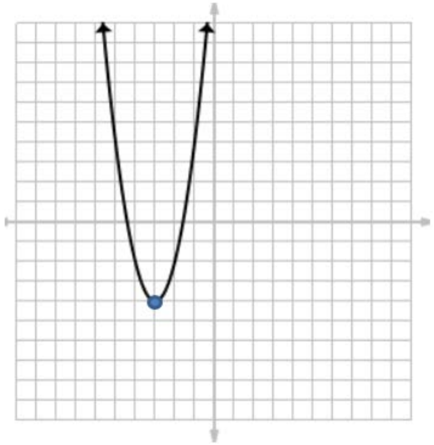
For quadratics in standard form, the *y*-intercept of the function $y = x^2 - 4x - 12$ is easy to recognize as (0, –12). However, rewriting the function as $y = (x - 6)(x + 2)$ reveals zeros at (6, 0) and at (–2, 0). Furthermore, completing the square allows the equation to be written as $y = (x - 2)^2 - 16$, which shows that the vertex (and minimum point) of the parabola is at (2, –16) and reveals transformations applied to the graph of parent quadratic function $y = x^2$. The same can be true for the various forms of linear equations.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (F.IF.7-9)

FACTORING QUADRATICS

A quadratic expression written in factored form helps students recognize the zeros of a function without having to graph the function. Since $f(x) = 2x^2 - x - 1$ can be rewritten as $f(x) = (2x + 1)(x - 1)$, students can easily deduce that the zeros or x-intercepts are $x = -\frac{1}{2}$ and $x = 1$ by setting each factor equal to 0 and then solving for x. Connect with factoring in A.SSE.3.





COMPLETING THE SQUARE

Completing the square is useful to rewrite a quadratic expression in vertex form, $f(x) = a(x - h)^2 + k$. Vertex form allows students to easily determine the maximum or minimum point (h, k) without having to graph. Since $g(x) = 2x^2 + 12x + 14$ can be rewritten in vertex form $g(x) = 2(x + 3)^2 - 4$, students can easily deduce that the minimum is $(-3, -4)$ without having to graph or use $h = \frac{-b}{2a}$ and then evaluating function $g(x)$ at $x = h$ to find k. Connect with completing the square in A.SSE.3.

EXPONENTIAL FUNCTIONS

For exponential functions in Algebra 1, it is acceptable to use continuous graphs as a part of problem solving, even though students will not know what is really going on for the non-integer domain values (if they have evaluated the function only at integer inputs). A continuous graph allows students to see trends and to make claims such as "the city population will reach 1 million between years 7 and 8."

Examine multiple real-world examples of exponential functions so that students recognize that a base between 0 and 1 (such as an equation describing depreciation of an automobile, $f(x) = 15{,}000(0.8)^x$, represents the value of a $15,000 automobile that depreciates 20% per year over the course of x years) results in an exponential decay, while a base greater than 1 (such as the value of an investment over time $f(x) = 15{,}000(1.07)^x$ represents the value of an investment of $5,000 when increasing in value by 7% per year for x years) illustrates growth. Connect to properties of exponents in A.SSE.3.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (F.IF.7-9)

COMPARING FUNCTIONS

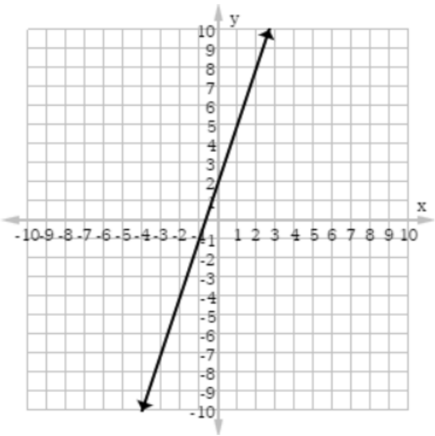
The purpose of F.IF.9 is so that students see key features across different representations of two functions.

EXAMPLE

Which function has a greater rate of change?

a. $y - 3 = -4(x + 2)$

b.



c.

x	y
5	19
8	28
9	31
11	37

Remind students that using a graphing tool such as a calculator or online applet does not always create an accurate graph. For example, technology may connects points when graphing a function that implies that the graph is continuous when in fact it is not (asymptotes drawn with lines, points of discontinuity are shown as complete points on graph unless traced). Window scale selection is key to show correct shape, features, and end behaviors of graphs.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (F.IF.7-9)

Instructional Tools/Resources

These tools and resources may be helpful for instruction. Before use, they should be evaluated at the local level to determine their appropriateness for instruction.

Manipulatives/Technology

- Graphing utilities on a calculator and/or computer can be used to demonstrate the changes in behavior of a function as various parameters are varied.
- Geogebra is a free graphing calculator that is available to students as website.
- Desmos is a free graphing calculator that is available to students as website or an app.
- Wolframalpha is dynamic computing tool.

Graphing Functions

- Graphing Stories by Desmos is a task that has students graph stories using function in the coordinate plane.
- Walking to Class: Modeling Students' Class schedules with Time-Distance Graphs is an NCTM Illuminations lesson where students use their class schedules to create time-distance graphs.
- Waterline by Desmos is a task that has students watch glasses filling with water and graph functions to uncover misconceptions about graphs.
- Marbleslides by Desmos is a good practice activity for functions and parameters but not the best initial instructional activity as it is not a delivery of content lesson.
- Should I Replace My Toilets? by Yummy Math has students create equations and graphs comparing a pre-1980s toilet and a high efficiency toilet.

Usefulness of Different Forms on a Function

- Which Function? is a task by Illustrative Mathematics that has students interpret different forms of a quadratic function.
- Graphs of Quadratic Functions is a task by Illustrative Mathematics that is an exploration of the usefulness of the different forms of a quadratic functions.

Factoring and Completing the Square

- Building Connections is a lesson by NCTM Illuminations that has students make connection among different classes of polynomial function by exploring their graphs. The last activity in the set, Higher Degree of Polynomials, could be an extension of the concepts.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (F.IF.7-9)

Comparing Properties of Functions Represented in Different Forms

- Throwing Baseballs is a task by Illustrative Mathematics that has students compare different representations of two different quadratic functions.

Curriculum and Lessons from Other Sources

- EngageNY, Module 4, Topic A, Lesson 9: Graphing Quadratic Functions in Factored Form is a lesson that pertains to this cluster.
- EngageNY, Module 4, Topic B, Lesson 16: Graphing Quadratic Equations from the Vertex Form, $y = a(x - h)^2 + k$, Lesson 17: Graphing Quadratic Functions from the Standard Form, $f(x) = ax^2 + bx + c$ are lessons that pertain to this cluster.
- EngageNY, Module 4, Topic C, Lesson 23: Modeling with Quadratic Functions and Lesson 24: Modeling with Quadratic Functions are lessons that pertain to this cluster.
- EngageNY, Module 5, Topic B, Lesson 4: Modeling a Context From a Graph, Lesson 5: Modeling From a Sequence, Lesson 6: Modeling a Context from Data, Lesson 8: Modeling a Context from a Verbal Description, Lesson 9: Modeling a Context from a Verbal Description are lessons that pertain to this cluster.
- Mathematics Vision Project, Algebra 1, Module 2: Linear and Exponential Functions has many lessons that pertain to this cluster.
- Mathematics Vision Project, Algebra 1, Module 3: Features of Functions has many lessons that pertain to this cluster.
- Mathematics Vision Project, Algebra 1, Module 6: Quadratic Functions has many lessons that pertain to this cluster.
- Mathematics Vision Project, Algebra 1, Module 8: More Functions, More Features has many lessons that pertain to this cluster.
- Unit 3: Functions from eMATHinstruction has materials that could be used for intervention. These documents can be used for individual students or for the entire class.

General Resources

- Arizona High School Progression on Functions is an informational text for teachers. This cluster is addressed on pages 9-10.
- Arizona's Progression on High School Modeling is an informational text for teachers. This cluster is addressed on page 12.
- High School Coherence Map by UnboundED is a map of the relationships among high school mathematics standards. Please note: This resource is aligned to the Common Core not Ohio's Learning Standards. Although the tool is useful, it needs to be checked with respect to Ohio's Learning Standards.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (F.IF.7-9)

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STANDARDS	MODEL CURRICULUM (F.BF.1-2)
Functions **BUILDING FUNCTIONS** **Build a function that models a relationship between two quantities.** **F.BF.1** Write a function that describes a relationship between two quantities. ★ **a.** Determine an explicit expression, a recursive process, or steps for calculation from context. **i.** Focus on linear and exponential functions. (A1, M1) **ii.** Focus on situations that exhibit quadratic or exponential relationships. (A1, M2) **F.BF.2** Write arithmetic and geometric sequences both recursively and with an explicit formula, use them to model situations, and translate between the two forms. ★	**Expectations for Learning** In the eighth grade, students create functions to model relationships between two quantities. In this cluster, students write linear, exponential, and quadratic functions symbolically given the relationship between two quantities. Relationships between quantities could be given as tables, graphs, or within a context. Students also write explicit and recursive rules for arithmetic and geometric sequences. In Algebra 2, students build functions from other functions allowing students to model more complex situations. This includes combining functions of various types using arithmetic operations or (+) composition. **ESSENTIAL UNDERSTANDINGS** • Functions can be written as explicit expressions, recursive processes, and in other ways. • An arithmetic sequence (informally, an addition pattern) has a starting term and a common difference between terms. • A geometric sequence (informally, a multiplication pattern) has a starting term and a common ratio between terms. • An arithmetic sequence is a linear function, and a geometric sequence is an exponential function. • Some sequences can be defined recursively or explicitly, while others cannot be defined by a formula. • The relationships between quantities can be modeled with functions that are linear, exponential, quadratic, or none of these. **MATHEMATICAL THINKING** • Make and modify a model to represent mathematical thinking. • Discern and use a pattern or structure. *Continued on next page*

STANDARDS	MODEL CURRICULUM (F.BF.1-2)
F.BF.1-2, continued	**Expectations for Learning, continued** **INSTRUCTIONAL FOCUS** • Model relationships with linear functions, which may be arithmetic sequences, using tables, graphs, symbols, and words within a context. • Model relationships with exponential functions, which may be geometric sequences, using tables, graphs, symbols, and words within a context. • Model relationships with quadratic functions using tables, graphs, symbols, and words within a context. • Model relationships that are not linear, exponential, or quadratic using tables, graphs, symbols, and words within a context. **Content Elaborations** **OHIO'S HIGH SCHOOL CRITICAL AREAS OF FOCUS** • Algebra 1, Number 2, pages 5-7 • Algebra 1, Number 5, pages 11-12 **CONNECTIONS ACROSS STANDARDS** • Create equations that describe numbers or relationships (A.CED.2). • Fit a linear function for a scatterplot that suggests a linear association (S.ID.6c). • Interpret linear models (S.ID.7). • Construct and compare linear, quadratic, and exponential models, and solve problems (F.LE.1). • Analyze functions using different representations (F.IF.8).

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (F.BF.1-2)

Instructional Strategies

Note: The model curriculum is not a collection of lessons nor a full curriculum; it does not suggest pace, sequence, or amount of time spent on topics. It provides information about a topic related to the standards including ideas for examples, strategies for teaching, common misconceptions, and possible connections between topics. In addition, the emphasis of the model curriculum is on instruction not on assessment; therefore, the examples in the Instructional Strategies section are written for instructional purposes not necessarily as assessment items. Examples may need to be rewritten to accommodate the needs of each individual classroom.

Standards for Mathematical Practice
This cluster focuses on but is not limited to the following practices:
MP.2 *Reason abstractly and quantitatively.*
MP.4 *Model with mathematics.*
MP.8 *Look for and express regularity in repeated reasoning.*

This cluster is recommended to be taught in conjunction with F.IF.3 and F.LE.1-3. The main difference between this cluster and F.IF.3 is that in F.IF.3 the emphasis is on sequences being functions with integer domains. Whereas the focus of this cluster is on writing, building, and interpreting functions recursively and explicitly. The focus of F.LE.1-3 is comparing the different types of functions, including those formed by sequences and using the appropriate type to solve problems. Although, one of the focuses of this cluster is sequences, it is not limited to writing functions that are sequences. Students should be able to write linear, exponential, and quadratic functions in different situations that are both discrete and continuous.

Note that subscript notation is not required. A major goal of Algebra 1 is that students understand the connection between linear functions and arithmetic sequences as well as between exponential functions and geometric sequences. Later students find a connection between second differences and quadratic functions. This is powerful mathematical connection that may be missed by many students if they are forced to do it while struggling with the subscript notation.

Students should be given pattern tasks where the tasks are—

- able to be modeled recursively and explicitly (flexible tasks);
- more easily modeled explicitly; or
- more easily modeled recursively.

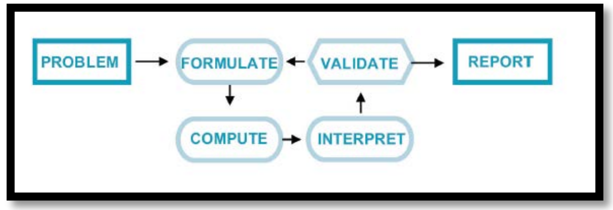
When using flexible tasks students should be asked the following:

- Which rule do you prefer and why?
- Does your preference depend on the situation? Explain.
- What advantages are there for using an explicit rule?
- What advantages are there for using a recursive rule?
- What are the connections between the recursive rule and the explicit rule?
- What are the different ways that slope is represented in the two rules?

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (F.BF.1-2)

MODELING

This cluster is included in the modeling standards. Geometric models that students can build are especially helpful to see patterns. See page 13 for more information about modeling.



ARITHMETIC AND GEOMETRIC SEQUENCES

An arithmetic sequence is a linear function, and a geometric sequence is an exponential function. Note that the converse of each statement is not quite true. A linear function is an arithmetic sequence only when the domain is restricted to a subset of the integers. An exponential function is a geometric sequence only when the domain is restricted to a subset of the integers.

When teaching sequences, do not emphasize formulas. One reason for not giving or memorizing formulas is that a sequence can be indexed from one to n, from 1 to $n – 1$, from 0 to n, or from 0 to $n – 1$, and formulas differ depending on the indexing. (Series are challenging and may be saved for later study.) Instead of using formulas try using words such as "start, next, current" or "next = now + common difference," and make sure that students define the starting term.

EXAMPLE

What are the next 4 numbers in the sequence 3, 6, 9, 12….

Discussion: Students may come up with 15, 18, 21, 24, or 3, 6, 9, 12, or 0, 3, 6, 9. Discuss why all of these answers could be true, but then as a class agree to have the rest of the discussion with respect to sequence continuing 15, 18, 21, 24. Have students come up with the 50^{th} term in the sequence and then write an expression in terms of n for the nth number in the sequence. Discuss how both $3n$ and $3(n – 1)$ can both be correct depending on how the starting term is defined. If the starting term is the 0^{th} term, $3n$ is correct and if the starting term is the 1^{st} term, then $3(n – 1)$ is correct. Because of this, tell students it is important to note the starting term: $f(n) = 3n$ starting with $n = 0$ (or for $n \geq 0$) or $f(n) = 3(n – 1)$ starting with $n = 1$ (or for $n \geq 1$). Discuss how and why the graphs vary depending on how the starting term is defined. Reinforce to students that $f(n)$ means the nth term in the sequence and does not mean f times n.

When creating Javascript arrays, computer programmers start with a term that is in the zero place rather than a term that stays in the first place.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (F.BF.1-2)

Explicit and Recursive Forms of Functions

An *explicit* rule allows one to take any input and find the corresponding output, whereas a *recursive* rule requires the previous term(s).

TIP! It may take some time for some students to realize that each term in the position *n* is defined by preceding term(s). Give students different sequences such as integers, odd integers, even integers, multiples of 3, etc, and have them pick any term as a starting point of the sequence and define the numbers going forward and backward.

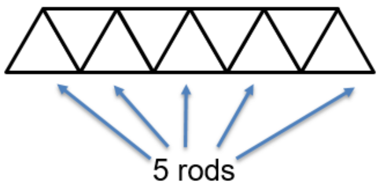
Provide real-world examples (e.g., a table showing how far a car has driven after a given number of minutes, traveling at a uniform speed). If time and distance are column headings, then examine the table by looking "down" the table to describe a relationship recursively, as well as "across" the table to determine an explicit formula to find the distance traveled if the number of minutes is known. (Changing the orientation of the table, swaps the "down" and the "across.")

Start with visual models (e.g., folding a piece of paper in half multiple times to compare the number of folds to the thickness of the paper), to generate sequences of numbers that can be explored and described with both recursive and explicit formulas. Perimeter and area problems that can be modeled with toothpicks or graph paper, could also be useful. As students are already familiar with function tables, use those to help build understanding.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (F.BF.1-2)

EXAMPLE

Beams, which are constructed from rods, are used to support bridges. The length of the beam is determined by the number of rods used to construct the bottom of the beam. Each rod is 1 meter long. An example of a beam is shown in the diagram.



Beam length *(n)*	Number of rods *f(n)*
1	3
2	7
3	11
4	14
20	79
100	399
n	$f(n) = 4n - 1$

+4 +4 +4

Recursive Rule

Explicit Rule

- How many rods will you need if you need a beam that is 20 meter long? What about 100 meters? For *n* meters?

Discussion: Students naturally reason recursively when looking at patterns. Most students will easily see that the recursive pattern is +4, so $f(n)$ = previous term + common difference or $f(n) = f(n-1) + 4$ starting at the 1st term which equals 3 or $f(n) = f(n-1) + 4$ for $n \geq 2$. Discuss how the recursive form could be used to find any number in the sequence, but it might take a really long time. Discuss how the same sequence can be described by an alternative recursive formula such as $f(n+1) = f(n) + 4$ for $n \geq 1$ and discuss why both forms are equivalent.

Notice that some students may come up with different explicit rules depending on how they view the pattern: $f(n) = n + 2n + (n-1)$; $f(n) = 4n - 1$; $f(n) = 3 + 4(n-1)$, $f(n) = 3n + (n-1)$. Discuss why all these expressions are equivalent and how each lends itself to seeing the structure slightly differently (A.SSE.2)

Some students may incorrectly think that $f(n+1)$ is f times $(n+1)$. To prevent this error have students translate sequences into words, e.g., $f(n) = f(n-1) + 4$ is a sequence where the nth term is four more than the one before it $(n-1)$, and $f(n+1) = f(n) + 4$ where the next term $(n+1)$ is the nth term plus four.

Tie sequences into the graph of a function. Discuss why working the sequences backwards and finding the 0^{th} term (the y-intercept), can also help write a explicit rule. Discuss how slope relates to both the recursive and explicit rule. Emphasize that there are times when one form to describe the function is preferred over the other.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (F.BF.1-2)

TIP!

Have students make a 3 or 4 column table, so they can see the patterns more clearly.

Beam length *n*	Pattern	Number of rods *f(n)*	Shorthand
1	Base + Middle + Top 1 + 2	3	Base + Middle + Top 1 + 2(1) + (1 – 1)
2	2 + 4 + 1	7	2 + 2(2) + (2 – 1)
3	3 + 6 + 2	11	3 + 2(3) + (3 – 1)
4	4 + 8 + 3	15	4 + 2(4) + (4 – 1)
20		79	
100		399	
n		*f(n)* = *n* + 2*n* + (*n* – 1)	

# of brown tiles
8
12
16
20
24
28

EXAMPLE

Johanna wants to make a square patio in her garden with brown and white tiles of the same size. The interior tiles are white and the border tiles are brown.

- Write either an explicit or recursive rule to describe the number of brown border tiles needed.
- Make sure to identify the domain of the chosen rule.
- Do you prefer the explicit or recursive rule? Explain.

Discussion: The domain of the sequence, {1, 2, 3, 4, …}, is the set of whole numbers representing the position of the term in the sequence. For example, the term 8 is in the position 1; term 12 is in the position 2, etc. Each number in the domain is also equal to the number of white tiles along one side of the patio. A rule in explicit form that describes the situation could be $f(x) = 4x + 4$ for $x \geq 1$ where *f(x)* is the number of brown tiles and *x* is the term in the sequence, and a rule in recursive form for the number of brown tiles could be $f(x) = f(x-1) + 4$ for $x \geq 1$. Discuss with students which form they prefer to write and why.

TIP!

Students may believe that the best (or only) way to generalize a table of data is by using a recursive formula. Students naturally tend to look "down" a table to find the pattern but need to realize that finding the 100th term requires knowing the 99th term unless an explicit formula is developed.

Using the recursive formula has become easier with the use of technology and tables in graphing calculators and spreadsheets.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (F.BF.1-2)

EXAMPLE

Your friend, Dominic, posts a meme to Facebook, and he asks you to not only share it with three people, but also to ask that the three people you share it with also share it with three people, and so on. Write a recursive and explicit rule for the situation. Discuss if this is an example of an arithmetic or geometric sequence and why.

Students may also incorrectly believe that arithmetic and geometric sequences are the same. Students need experiences with both types of sequences (and sequences that are neither) to be able to recognize the difference and more readily develop formulas to describe them.

Discrete vs Continuous Domains

When creating graphs of functions within a context, it is important to discuss the usage of a discrete versus a continuous domain. Make sure to present examples where it does not make sense to have a continuous domain and therefore the points on the graph should not be connected. For example, the profit after selling *n* tickets at $8 each and deducting $100 facility rental fee would have a domain of only whole numbers, so the dots would not be connected. In comparison, if someone is selling fudge at $8 per pound, the dots would be connected because it is possible to sell non-whole number pounds of fudge.

Students may incorrectly believe that they can always "connect the dots" in a graph. Spend time interpreting the ordered pairs in between the dots.Provide examples where the context of the sequence can be modified to make a continuous domain so that students can "connect the dots."

EXAMPLE

Melissa visits the grand canyon and drops a penny off the cliff. The penny has fallen 16 feet at the first second, 48 feet the next second, and 80 feet the third second continuing at that rate until it hits the ground. What is the total distance the penny will fall in 6 seconds?

- Write a function representing the situation in recursive form.
- Write a function representing the situation in explicit form.
- Graph the function.
- Would you "connect the dots" when graphing the function? Explain.
- What is the total distance the penny will fall in 6 seconds?

Discussion: Illustrate through examples that although most sequences are discrete, some situations depending on the context are continuous.

QUADRATIC FUNCTIONS

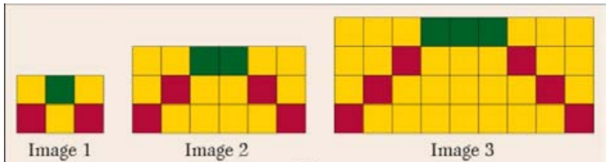
Some situations that give rise to quadratic functions are as follows: heights of objects in projectile motion, scaling situations about area (including dot diagrams that are area-like arrays), sum of *n* consecutive terms of an arithmetic sequence, and the profit when the relationship between price and demand is linear.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (F.BF.1-2)

EXAMPLE

In the examples below have students describe the patterns they see and then create two more images using the patterns. Have students create a table to describe the number of green, red, yellow, and total number of tiles including writing rules for describing the pattern. Then have students identify which patterns are linear or quadratic and explain why.

a.

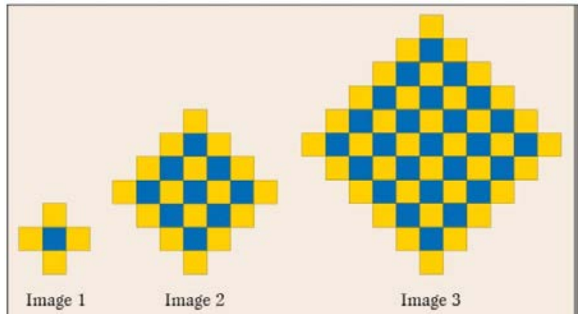


(a)

Number of Tiles				
Image Number	Green	Red	Yellow	Total
1	1	2	3	6
2	2	4	12	18
3	3	6	27	36
4	4	8	48	60
x	x	$2x$	$3x^2$	$3x^2 + 3x$

(b)

b.



(a)

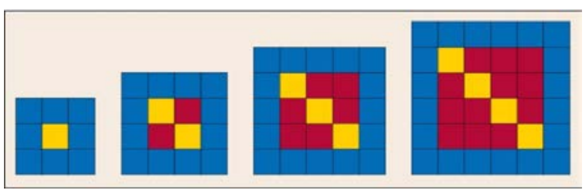
Number of Tiles			
Image Number	Blue	Yellow	Total
1	1	4	5
2	9	16	25
3	25	36	61
4	49	64	113
x	$4x^2 - 4x + 1$	$4x^2$	$8x^2 - 4x + 1$

(b)

From "Finding Linear & Quadratic Equations in Linear Patterns by Terri Kurz published in Mathematics Teacher, Volume 110, No. 6, February 2017.

c.
.

d.



e. Have students create their own quadratic patterns accompanied by a table and a rule. Have them explain which components in their patterns are linear or quadratic.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (F.BF.1-2)

OTHER TYPES OF FUNCTIONS

Give students examples of other functions represented in symbolic form that are not linear, exponential, or quadratic such as $V(s) = s^3$ or $f(n) = 1.04 \cdot f(n-1) + 500, f(0) = 500$.

Instructional Tools/Resources

These tools and resources may be helpful for instruction. Before use, they should be evaluated at the local level to determine their appropriateness for instruction.

Manipulatives/Technology

- Hands-on materials (e.g., paper folding, building progressively larger shapes using pattern blocks, etc.) can be used as a visual source to build numerical tables for examination.
- Desmos is a free graphing calculator that is available to students as website or an app.
- Geogebra is a free graphing calculator that is available to students as website.
- Wolframalpha is dynamic computing tool.
- Visual Patterns is a website that shows pictures of linear, exponential, and quadratic patterns.
- Patterns Posters for Algebra 1 from Finding Ways by Fawn Nguyen is a blog about a class activity where students are given patterns and they have to make posters from them. She is the creator of the visual patterns link above.

Continuous and Discrete Functions

- Continuous and Discrete Functions is a page by MathBitsNotebook that compare continuous and discrete functions.

Exponential Functions

- Game, Set, Flat by Desmos is an activity that helps students understand an exponential relationship to describe a "good" tennis ball. They will also construct an exponential equation.
- Lake Algae is a task from Illustrative Mathematics that introduces students to exponential growth.
- Compound with 100% Interest and Compounding with a 5% Interest from Illustrative Mathematics helps students develop the formulas for compound interest.
- To Fret or Not to Fret is an NCTM Illuminations two-part lesson where students explore geometric sequences and exponential functions by considering the placement of frets on stringed instruments. *NCTM now requires a membership to view their lessons.*

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (F.BF.1-2)

Recursive Reasoning

- Susita's Account is a task from Illustrative Mathematics that asks students to determine a recursive process from a context.
- Snake on a Plane is a task from Illustrative Mathematics that approaches a function recursively and by algebraic definition.
- The Devil and Daniel Webster is an NCTM Illuminations lesson that allows students to examine a recursive sequence. *NCTM now requires a membership to view their lessons.*
- Counting the Trains is an NCTM Illuminations three-part lesson where students investigate a relationship between recursive exponential functions. *NCTM now requires a membership to view their lessons.*

Quadratic Functions

- Skelton Tower is a task from the Mathematics Assessment Project that has students use cubes to build a tower of different heights which will lead to quadratic sequences.
- Generalizing Patterns: Table Tiles by Mathematics Assessment Project has students explore a quadratic sequence.

Curriculum and Lessons from Other Sources

- Although EnageNY, Algebra 1, Module 3, Topic A has good problems in Lessons 1 and 2 that might be used, it emphasizes subscript notation which Ohio does not.
- The Mathematics Vision Project, Algebra 1, Module 1: Sequences. Sections 1.3-1.8 align to this cluster.

General Resources

- Arizona High School Progressions on Functions is an informational document for teachers. This cluster is addressed on pages 11-12.
- Arizona High School Progressions on Modeling is an informational document for teachers.This cluster is addressed on pages 3 and 13.
- High School Coherence Map by UnboundED is a map of the relationships among high school mathematics standards. Please note: This resource is aligned to the Common Core not Ohio's Learning Standards. Although the tool is useful, it needs to be checked with respect to Ohio's Learning Standards.

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INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (F.BF.1-2)

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STANDARDS	MODEL CURRICULUM (F.BF.3-4)
Functions **BUILDING FUNCTIONS** **Build new functions from existing functions.** **F.BF.3** Identify the effect on the graph of replacing $f(x)$ by $f(x) + k$, $kf(x)$, $f(kx)$, and $f(x + k)$ for specific values of k (both positive and negative); find the value of k given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. Include recognizing even and odd functions from their graphs and algebraic expressions for them. (A2, M3) **a.** Focus on transformations of graphs of quadratic functions, except for $f(kx)$; (A1, M2) **F.BF.4** Find inverse functions. **a.** Informally determine the input of a function when the output is known. (A1, M1)	**Expectations for Learning** In eighth grade, students learn that functions map inputs to outputs. In this cluster, students informally reverse this to find the input of a function when the output is known. In later classes, (+) some students more fully develop the concepts, procedures, and notation for inverses of functions. In eighth grade, students attend to slope and intercepts for graphs of linear functions, without explicit attention to transformations of the graphs. In this cluster, students transform graphs of quadratic functions. Transformations of quadratic functions can be interpreted conveniently by observing the effect on the vertex and whether the parabola opens up or down. Students do not perform transformations of the form $f(kx)$. In Algebra 2, students perform all types of transformations for various function families and recognize even and odd functions. **ESSENTIAL UNDERSTANDINGS** • Sometimes the input of a function can be found when the output is given. • Vertical and horizontal transformations of $y = x^2$ are as follows: o horizontal shift: $g(x) = (x - h)^2$; o vertical stretch/shrink: $g(x) = ax^2$ when $a > 0$; o vertical shift: $g(x) = x^2 + k$; o reflection across the x-axis: $g(x) = -x^2$; and o a combination of transformations: $g(x) = a(x - h)^2 + k$. **MATHEMATICAL THINKING** • Explain mathematical reasoning. *Continued on next page*

STANDARDS	MODEL CURRICULUM (F.BF.3-4)
F.BF.3-4, continued	**Expectations for Learning, continued** **INSTRUCTIONAL FOCUS** ** Limit to situations where inverse values are unique. Exclude formal notation; exclude finding the inverse algebraically; exclude switching x and y; exclude reflecting about the line $y = x$. Transformations occur in the quadratic expression rather than inside the function notation.* • Use graphs and tables to find the input value of a function when given an output, and interpret the values in context. • Transform graphs of quadratic functions and interpret the transformations geometrically. **Content Elaborations** **OHIO'S HIGH SCHOOL CRITICAL AREAS OF FOCUS** • Algebra 1, Number 2, pages 5-7 • Algebra 1, Number 5, pages 11-12 **CONNECTIONS ACROSS STANDARDS** • Understand the concept of a function and use function notation (F.IF.1-2). • Analyze functions using different representations (F.IF.7a, b, and e).

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (F.BF.3-4)

Instructional Strategies

Note: The model curriculum is not a collection of lessons nor a full curriculum; it does not suggest pace, sequence, or amount of time spent on topics. It provides information about a topic related to the standards including ideas for examples, strategies for teaching, common misconceptions, and possible connections between topics. In addition, the emphasis of the model curriculum is on instruction not on assessment; therefore, the examples in the Instructional Strategies section are written for instructional purposes not necessarily as assessment items. Examples may need to be rewritten to accommodate the needs of each individual classroom.

Standards for Mathematical Practice
This cluster focuses on but is not limited to the following practices:
MP.1 *Make sense of problems and persevere in solving them.*
MP.8 *Look for and express regularity in repeated reasoning.*

TRANSFORMATIONS OF FUNCTIONS

Algebra 1 students need not use function notation to express transformation. Instead, for example, Algebra 1 students should see that $\boldsymbol{y = (x - h)^2}$ is a horizontal transformation of a parent quadratic function $\boldsymbol{y = x^2}$. Later, in Algebra 2, students should be able to see that the graph of the function $\boldsymbol{y = g(x - h)}$ is a result of a horizontal transformation of the graph of $\boldsymbol{y = g(x)}$.

Transformations with respect to linear functions should be postponed until after the study of transformations with quadratics because it is difficult to distinguish the horizontal and vertical transformations of a linear function. However, if choosing to discuss vertical translations of linear functions, relate them back to the *y*-intercept. In contrast to linear and exponential functions, transformations of quadratics are easier to observe and interpret because of the presence of the vertex of the parabola. Rather than asking students to memorize the connections between left/right, up/down, etc. encourage students to check their work with test points. The transformation $y = f(kx)$ is excluded until Algebra 2 because it is difficult to distinguish horizontal stretches from vertical stretches when dealing with quadratics. *Note: Fluency with absolute value functions is not expected until Algebra 2, so it is at the discretion of each district to determine how much emphasis to place on absolute value functions in Algebra 1.*

Use graphing calculators or computers to explore the effects of the position of a constant in an equation by the graph of its function. For example, students should be able to distinguish between the graphs of $y = x^2$, $y = 2x^2$, $y = x^2 + 2$, and $y = (x \pm 2)^2$. This can be accomplished by allowing students to work with a single parent function and examining numerous parameter changes in order to make generalizations. Websites such as Desmos Exploring Quadratic Transformations, Desmos Quadratics Explorations, Desmos, Quadratics Exploration 2, or WolframAlpha may be helpful.

Students may believe that the graph of $y = (x - 4)^2$ is the graph of $y = x^2$ shifted 4 units to the left (due to the subtraction symbol). Examples should be explored by checking a few points by hand or by using a graphing calculator to overcome this misconception.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (F.BF.3-4)

EXAMPLE

Part 1

a. Using a graphing calculator or computer program, graph the following on the same grid (preferable in different colors if possible):

- $y = x^2$
- $y = 2x^2$
- $y = 3x^2$
- $y = 4x^2$

b. Predict what you think $y = 100x^2$ looks like.

Part 3

a. What do you predict $y = -\frac{1}{10}x^2$ will look like? Justify your prediction.

b. Graph the following on the same grid:

- $y = -\frac{1}{2}x^2$
- $y = -\frac{1}{3}x^2$
- $y = -\frac{1}{4}x^2$

c. Do you think your prediction in **a.** was correct? Try it and see.

d. What do you predict $y = \frac{2}{3}x^2$ looks like? Justify your prediction.

e. What do you predict $y = -\frac{3}{4}x^2$ looks like? Justify your prediction.

Part 2

a. Graph the following on the same grid:

- $y = x^2$
- $y = -2x^2$
- $y = -3x^2$
- $y = -4x^2$

b. Predict what you think $y = -50x^2$ looks like.

Part 4

a. Graph the following on the same grid:

- $y = x^2$
- $y = x^2 + 1$
- $y = x^2 + 2$
- $y = x^2 + 3$

b. What do you predict $y = x^2 + 100$ looks like? Explain how you came up with your prediction.

c. What do you predict $y = x^2 - 3$ looks like? Explain how you came up with your prediction.

d. What do you predict $y = -4x^2 + 5$ looks like? Explain how you came up with your prediction.

Part 5

a. Graph the following on the same grid:

- $y = x^2$
- $y = (x - 1)^2$
- $y = (x - 2)^2$
- $y = (x - 3)^2$

b. What do you predict $y = (x - 9)^2$ looks like? Explain how you came up with your prediction.

c. What do you predict $y = (x + 4)^2$ looks like? Explain how you came up with your prediction.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (F.BF.3-4)

Using Properties of Transformations to Find the Vertex Formula

Students should also notice that a vertical shift does not affect the axis of symmetry, since the axis of symmetry is a vertical line. Because of this students should make connections between quadratics that are alike and their corresponding parabolas. For example, students should realize that $y = 3x^2 + 12x - 7$ is the same parabola as $y = 3x^2 + 12x$ except for the vertical shift; therefore both equations have the same axis of symmetry. It is quite simple to factor the second equation in order to find its x-intercepts: $3x(x + 4) = 0$, so $x = 0, -4$. Since the axis of symmetry is in fact a line of symmetry for a parabola, students should understand that they can find they x-coordinate of the vertex (or the axis of symmetry) by finding the mean of the two x-intercepts of the parabola: $\frac{(0-4)}{2} = -2$. The student can then use $x = -2$ and substitute it into the original equation to find the vertex of the graph of the graph of $y = 3x^2 + 12x - 7$ as $y = 3(-2)^2 + 12(-2) - 7 = -19$, so the vertex is $(-2, -19)$. Using the structure of expressions and transformations allows students to find the axis of symmetry and therefore the vertex formula ($x = \frac{-b}{2a}$ where $x = h$ and then evaluating y for h to find k where (h, k) is the maximum or minimum) without complex computations. See F.IF.4-5 for more information on the vertex formula and the axis of symmetry.

INVERSE OF FUNCTIONS

Provide examples of inverse relations that are not purely mathematical to introduce the idea of inverse of functions. For example, given a function that names the capital of a state, f(Ohio) = Columbus, ask questions such as what is x, when $f(x)$ = Denver. Students can conclude that x = Colorado. Build on this concepts by looking at numerical input and output values. Keep it simple, informal, and free of inverse function notation. Instead focus in on the idea of "going backwards." Ask question such as "What is the input when the output is known?"

The habit of immediately swapping x and y values may be confusing to some students. A better conceptual approach is to more clearly work backwards, keeping the letters the same. For example, suppose $f(x) = 3x + 5$. Call it $y = 3x + 5$, and solve for x to get $x = \frac{y-5}{3}$. Therefore if $x = g(y)$, and $g(y) = \frac{y-5}{3}$, then $f(x)$ and $g(y)$ are inverses of each other.

TIP! Swapping variables in an equation will lead to students misunderstanding notation in later mathematics courses when they will be required to use inverse function notation. For example, in later mathematics they may confuse the inverse of $y = f(x)$ with $y = f^{-1}(x)$. instead of the correct $x = f^{-1}(y)$. See "Inverse Functions: What Our Teachers Didn't Tell Us" article by Wilson, Adamson, Cox, and O'Bryan for further explanation.

For example, students might determine that folding a piece of paper in half 5 times results in 32 layers of paper. Then if they are given that there are 32 layers of paper, they can solve to find how many times the paper would have been folded in half.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (F.BF.3-4)

EXAMPLE

Mark earns $9 an hour working at Best Buy. Write a function to describe the situation using *a* to represent the number of hours worked, and *b* to represent the total money earned.

Discussion: Discuss why both $b = 9a$ and $a = \frac{b}{9}$ could both describe the situation. Discuss when it would be more useful to have the hours be the independent variable and when it would be more useful to have the total money earned be the independent variable. Connect with rearranging equations and formulas in A.CED.4. In some circumstances, it is more useful to have the hours be assigned as the independent variable, and in other situations it is more useful to have the total money earned be assigned as the independent variable.

Instructional Tools/Resources

These tools and resources may be helpful for instruction. Before use, they should be evaluated at the local level to determine their appropriateness for instruction.

Manipulatives/Technology

- Use the book, *The Sneeches*, by Dr. Seuss to introduce students to the concept of inverse functions.
- Desmos is a free graphing calculator that is available to students as website or an app.
- Geogebra is a free graphing calculator that is available to students as website.
- Wolframalpha is dynamic computing tool.

Transformations of Functions

- What's My Transformation? is a Desmos lesson that allows student to explore families of functions.
- Card Sort: Transformations is a Desmos lesson that has students match transformation of graphs to expressions using function notation.
- Marbleslides: Parabolas is a Desmos lesson that has students transform parabolas to send marbles through the stars.

Inverse of Functions

- Temperature in Degrees Fahrenheit and Celsius is an Illustrative Mathematics task that uses a real-world example of inverse functions.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (F.BF.3-4)

Curriculum and Lessons from Other Sources

- EngageNY, Algebra, Module 4, Topic C, Lesson 19: Translating Graphs of Functions, Lesson 20: Stretching and Shrinking Graphs of Functions, Lesson 21: Transformations of the Quadratic Parent Function, $f(x) = x^2$, Lesson 23: Modeling with Quadratic Functions, Lesson 24: Modeling with Quadratic Functions are lessons that align to this cluster. *Note: Some problems are above Ohio's expectations.*
- Georgia Standards of Excellence Framework, Algebra 1, Unit 3: Modeling and Analyzing Quadratic Functions has several lessons that address this cluster. These lessons can be found on pages 23-31, 62-72, 123-147, and 195-202.

General Resources

- Arizona High School Progressions on Functions is an informational text for teachers. This cluster is addressed on pages 12-13.
- High School Coherence Map by UnboundED is a map of the relationships among high school mathematics standards. Please note: This resource is aligned to the Common Core not Ohio's Learning Standards. Although the tool is useful, it needs to be checked with respect to Ohio's Learning Standards.

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STANDARDS	MODEL CURRICULUM (F.LE.1-3)
Functions **LINEAR, QUADRATIC, AND EXPONENTIAL MODELS** **Construct and compare linear, quadratic, and exponential models, and solve problems.** **F.LE.1** Distinguish between situations that can be modeled with linear functions and with exponential functions. ★ **a.** Show that linear functions grow by equal differences over equal intervals and that exponential functions grow by equal factors over equal intervals. **b.** Recognize situations in which one quantity changes at a constant rate per unit interval relative to another. **c.** Recognize situations in which a quantity grows or decays by a constant percent rate per unit interval relative to another. **F.LE.2** Construct linear and exponential functions, including arithmetic and geometric sequences, given a graph, a description of a relationship, or two input-output pairs (include reading these from a table). ★ **F.LE.3** Observe using graphs and tables that a quantity increasing exponentially eventually exceeds a quantity increasing linearly or quadratically. ★ (A1, M2)	**Expectations for Learning** In eighth grade, students interpret the rate of change and initial value of a linear function in terms of the situation it models, and in terms of its graph or a table of values. Students also see examples of non-linear functions and learn and apply the properties of integer exponents. In Algebra 1, students compare across linear, exponential, and quadratic functions. **ESSENTIAL UNDERSTANDINGS** • Linear functions have a constant additive change. • Exponential functions have a constant multiplicative change. • Linear and exponential functions both have initial values. • To highlight the constant growth/decay rate, *r*, often expressed as a percentage, exponential functions can be written in the form, $f(n) = a(1 + r)^n$. • To highlight the growth/decay factor, *b*, exponential functions can be written in the form, $f(n) = a(b)^n$. • An arithmetic sequence is a linear function, and a geometric sequence is an exponential function. • The phrase "eventually exceeds" (F.LE.3) directs the focus towards large values in the domain and the consideration of the base and *y*-intercept of the exponential function and the leading coefficient of the linear or quadratic function. • For large domain values, the growth of linear and quadratic functions is dominated by the leading term. **MATHEMATICAL THINKING** • Represent a concept symbolically. • Make and modify a model to represent mathematical thinking. • Make connections between concepts, terms, and properties within the grade level and with previous grade levels. *Continued on next page*

STANDARDS	MODEL CURRICULUM (F.LE.1-3)
F.LE.1-3, continued	**Expectations for Learning, continued** **INSTRUCTIONAL FOCUS** • Aim toward a multifaceted understanding of additive versus multiplicative change across different representations. • For linear functions (arithmetic sequences), focus on the constant rate of change across the tables, graphs, contexts, and the explicit and recursive representations. • For exponential functions (geometric sequences), focus on the constant growth/decay rate (or factor) across the tables, graphs, contexts, and the explicit and recursive representations. • Use graphs, tables, and contexts to see that as the domain value increases, the values of an exponential function will eventually exceed the corresponding values of a linear or quadratic function. **Content Elaborations** **OHIO'S HIGH SCHOOL CRITICAL AREAS OF FOCUS** • Algebra 1, Number 2, pages 5-7 • Algebra 1, Number 5, pages 11-12 **CONNECTIONS ACROSS STANDARDS** • Build a function that models a relationship between two quantities (F.BF.1a, 2). • Interpret functions that arise in applications in terms of the context (F.IF.4b, 5b). • Analyze functions using different representations (F.IF.7a, b, and e). • Summarize, represent, and interpret data on two categorical and quantitative variables (S.ID.6c). • Interpret linear models (S.ID.7). • Interpret the structure of expressions (A.SSE.1). • Interpret the parameters in a linear or exponential function in terms of a context (F.LE.5).

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (F.LE.1-3)

Instructional Strategies

Note: The model curriculum is not a collection of lessons nor a full curriculum; it does not suggest pace, sequence, or amount of time spent on topics. It provides information about a topic related to the standards including ideas for examples, strategies for teaching, common misconceptions, and possible connections between topics. In addition, the emphasis of the model curriculum is on instruction not on assessment; therefore, the examples in the Instructional Strategies section are written for instructional purposes not necessarily as assessment items. Examples may need to be rewritten to accommodate the needs of each individual classroom.

This cluster is recommended to be taught in conjunction with F.BF.1-2 and F.IF.1-3. The main difference between F.BF.1-2 and F.IF.1-3 is that in F.IF.1-3 the emphasis is on sequences being functions with integer domains, whereas F.BF.1-2 focuses on writing functions recursively and explicitly. The focus of the cluster F.LE.1-3 is comparing the different types of functions including those formed by sequences and using the appropriate type to solve problems.

Standards for Mathematical Practice

This cluster focuses on but is not limited to the following practices:

MP.1 *Make sense of problems and persevere in solving them.*

MP.2 *Reason abstractly and quantitatively.*

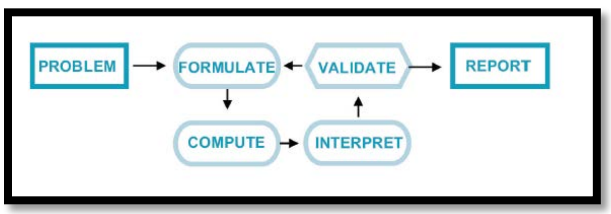
MP.4 *Model with mathematics.*

MP.7 *Look for and make use of structure.*

MP.8 *Look for and express regularity in repeated reasoning.*

MODELING

This cluster is included in the modeling standards. Geometric models that students can build are especially helpful to see patterns. See page 13 for more information about modeling.



COMPARING LINEAR AND EXPONENTIAL GROWTH

The phrase "increasing exponentially" in everyday language means "really fast." In mathematics, increasing exponentially means increasing using the model $y = ab^x$ (where $a > 0$ and $b > 1$), but it may be increasing incredibly slowly ($b = 1.01$). Because a graph of an exponential function eventually curves up, it will eventually have output values greater than a linear or quadratic (or polynomial) function. To understand this, students need to compare two graphs to see where the two graphs intersect. Then they will see that function behavior for values of x close to 0 is different than large (positive) values of x. Note: If $y = ab^x$ with $a > 0, 0 < b < 1$, then we have exponential decay.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (F.LE.1-3)

EXAMPLE

- **a.** Is it better to be paid a penny on the first day, and then double that amount each day thereafter for a month, or is it better to be paid $100 a day for the month?
- **b.** If we add a third option to get $1 times the square of the day number, which of the three options would take? Explain.

Discussion: Have students make both a tabular representation and a graph of the situation before writing an equation. (An alternate introductory lesson could be on the fable "One Grain of Rice" by Demi. See Instructional Tools/Resources section for more resources on "One Grain of Rice.")

Explore simple linear and exponential functions by engaging in hands-on experiments. For example, students can measure the diameters and corresponding circumferences of several circles and discover that a function that relates the diameter to the circumference is a linear function with a first common difference. Then they can explore the value of an investment for an account that will double in value every 12 years and see that it is an exponential function with a base of 2.

Use a graphing calculator or computer program to compare tabular and graphic representations of exponential and polynomial functions to show how the *y*- (output) values of the exponential function eventually exceed those of quadratic functions. A simple example would be to compare the graphs (and tables) of the functions $y = x^2$ and $y = 2^x$ to find that the *y*-values are greater for the exponential function when $x > 4$.

Students may also incorrectly believe that the end behavior of all functions depends on the situation and not on the fact that exponential function values will eventually get larger than those of any other polynomial function. Provide situations where students can discover this concept.

EXAMPLE

Suppose you wanted to join a gym that charges a $80 initiation fee, and then quotes you that it will cost you $230 for 6 months. Three price functions are given, all of which meet the quoted price, where *k* is the time in months and *P(k)* is the total cost. Which is the best model for you and why. Which is the best model for the gym?

Plan A: $P(k) = 80 + \frac{115}{3}k$

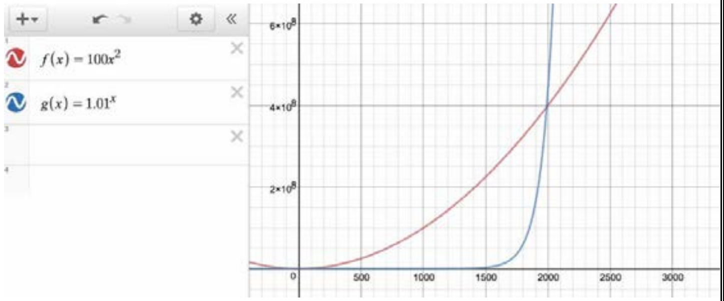
Plan B: $P(k) = 80 + \frac{97}{3}k + k^2$

Plan C: $P(k) = 80(1.25327)^k$

Discussion: Students should discover that the *y*-intercept of each graph represents the initiation fee. If you only needed a gym membership for less than 6 months and there is no cancellation fee, then Plan C would be best for the consumer. In contrast after 6 months, Plan C is significantly better for the gym. The best plan for the consumer who wants to be a member for 6 months or longer is Plan A.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (F.LE.1-3)

Think about comparing *f(x)* = $(1.01)^x$ to *g(x)*=$100x^2$. For these pairs, in a standard graphing window, it will not look as if *f(x)* will ever exceed *g(x).* With some strategic work, the graphing window can be adjusted to see that the exponential function *g(x)* will eventually exceed the quadratic function *f(x).* In this example, notice that for *x*-values between 1 and 10, the quadratic function appears much larger than the exponential function, but for *x*-values near 2,000, the exponential function becomes much larger than the quadratic (see the graph to the right). Students should be given the opportunity to compare the graphs of various functions by zooming out carefully and strategically, possibly by different horizontal and vertical ratios, in order to see what happens to the graphs of the functions for very large values of *x*. This example may be too hard for a first example, but it might be appropriate for a culminating task.



COMPARING LINEAR, QUADRATIC, AND EXPONENTIAL MODELS

Linear Function

x	f(x)	
-2	-9	
		+0.5
-1	-8.5	
		+0.5
0	-8	
		+0.5
1	-7.5	
		+0.5
2	-7	

Quadratic Function

x	f(x)		
-2	-6		
		+1	
-1	-5		+2
		+3	
0	-2		+2
		+5	
1	3		+2
		+7	
2	10		

Exponential Function

x	f(x)	
-2	$\frac{1}{2}$	
		×4
-1	2	
		×4
0	8	
		×4
1	32	
		×4
2	128	

Compare tabular representations of a variety of functions to show that over equal *x*-intervals linear functions have a constant first difference (equal differences over equal *x*-intervals), while exponential functions do not (instead function values grow by equal factors over equal *x*-intervals). Also, quadratic functions have a constant second difference over equal *x*-intervals. Have students explore these concepts instead of just telling them. Require them to explain why these patterns hold true and justify their thinking.

Students may believe that all functions have a constant first difference and need to explore in order to realize that, for example, a quadratic function will have equal constant second differences in a table. In addition some students may believe that every function with a constant rate of change is linear. For example, exponential functions have a constant multiplicative (percent) rate of change.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (F.LE.1-3)

EXAMPLE

Give each student 3 tables, and tell them to create an equation for each type of function. Then complete the table.

Linear:___________ **Exponential:**___________ **Quadratic:**___________

x	y	Patterns in y

x	y	Patterns in y

x	y	Patterns in y

a. What pattern did you notice about the linear functions?
b. What pattern did you notice about the exponential functions?
c. What pattern did you notice about the quadratic functions?
d. Share your equations with 5 other people in the class. How were the observations about your patterns similar or different than your classmates? Explain.
e. Will the patterns that you found, always hold true? Explain and justify you thinking.

Apply linear and exponential functions to real-world situations. For example, a person earning $10 per hour experiences a constant rate of change in salary given the number of hours worked, while the number of bacteria on a dish that doubles every hour will have equal factors over equal intervals.

CONSTRUCT ARITHMETIC AND GEOMETRIC SEQUENCES

Provide examples of arithmetic and geometric sequences in graphical, verbal, or tabular forms, and have students generate formulas and equations that describe the patterns. Explicit and recursive representations of functions could be constructed by analyzing the representations of linear and exponential functions.

Make connection to S.ID.6-7 with respect to exponential growth. For example have students create a simulation to model the exponential growth of cancer cells.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (F.LE.1-3)

Instructional Tools/Resources

These tools and resources may be helpful for instruction. Before use, they should be evaluated at the local level to determine their appropriateness for instruction.

Manipulatives/Technology

- Desmos is a free graphing calculator that is available to students as website or an app.
- Geogebra is a free graphing calculator that is available to students as website.
- Wolframalpha is dynamic computing tool.
- Visual Patterns is a website that shows pictures of linear, exponential, and quadratic patterns.
- Patterns Posters for Algebra 1 from Finding Ways by Fawn Nguyen is a blog about a class activity where students are given patterns and they have to make posters from them. She is the creator of the visual patterns link above.

One Grain of Rice and The King's Chessboard

Both are children's stories that can be used to teach exponential growth.

- *One Grain of Rice: A Mathematica Folktale* by Demi is a children's book about a rajah who takes his people's rice very year until a wise girl develops a clever plan using exponential growth.
- One Grain of Rice in an NCTM Illuminations lesson on exponential growth. *NCTM now requires a membership to view their lessons.*
- One Grain of Rice is another lesson from the Jim Wilson's University of Georgia's webpage using Demi's story and a spreadsheet.
- One Grain of Rice: Exponential Growth is a YouTube video on the story.
- *The Kings Chessboard,* by David Birch and Devis Grebu is a children's book about exponential growth as a wise man who refuses a king's reward for a favor instead takes a payment of rice.
- The Legend of a Chessboard: Teaser is a YouTube video that puts the quantity of rice in the context of different places such as a chessboard, a room, cities, and the country of Switzerland.
- The Legend of a Chessboard is a YouTube video based on the story.

Exponential Growth

- Exponential Models: Rhinos and M&M's is lesson from PBS that uses paperfolding, M&M's, and Rhinos to show exponential growth and decay.
- Exponential Growth & Decay (Ashby) is a lesson by Achieve the Core that uses several different representations to demonstrate exponential growth and decay.
- An Intro to Exponential Growth and Decay is a Desmos activity that also models exponential growth and decay using pennies and M&M's.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (F.LE.1-3)

Exponential Growth, continued

- Growth and Decay is a Desmos activity that uses March Madness brackets to illustrate exponential growth and decay.
- Overrun by Skeeters-Exponential Growth and Skeeter Populations and Exponential Growth are lessons from Annenberg Learner where students model functions that represent exponential growth about skeeters (mosquitoes).
- Predicting your Financial Future is an NCTM Illumination's lesson about compound interest. *NCTM now requires a membership to view their lessons.*
- Fry's Bank is a 3-Act Math Task by Dan Meyer that introduces exponential growth.
- Pixel Pattern is a 3-Act Math Task by Dan Meyer that explores patterns.
- Identifying Exponential Functions is a task by Illustrative Mathematics that introduces exponential functions by experimenting with the parameters of the function.
- Two Points Determine an Exponential Function I and Two Points Determine an Exponential Function II are tasks by Illustrative Mathematics where students have to find the values of a and b in an exponential function given two points.

Comparing Linear, Exponential and/or Quadratic Growth

- Piles of Paper is a CPalms activity where students fold paper to demonstrate linear and exponential growth.
- Rainforest Deforestation-Problem or Myth? is a lesson by NCTM Illuminations that allows students to explore deforestation as a exponential function. Students will use first or second differences to determine whether data models are linear, quadratic, or exponential. *NCTM now requires a membership to view their lessons.*
- National Debt and Wars is a lesson by NCTM Illuminations where student collect information about the National Debt, plot the data by decade, and decide whether an exponential curve is a good fit. *NCTM now requires a membership to view their lessons.*
- Shrinking Candles, Running Waters, Folding Boxes is a lesson by NCTM Illuminations that has students determine which function type best fits the data. Skip the "Weather, It's a Function" section as it is above grade-level. *NCTM now requires a membership to view their lessons.*
- Birthday Gifts and Turtle Problem is a Mathematics Design Collaborative lesson from the State of Georgia Department of Education that explores the rates of changes of linear functions versus exponential functions.
- Representing Linear and Exponential Growth by Mathematics Assessment Project is a lesson that has students interpret exponential and linear functions.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (F.LE.1-3)

Curriculum and Lessons from Other Sources

- EngageNY Algebra 1, Module 3, Topic A, Lesson 5: The Power of Exponential Growth, Lesson 6: Exponential Growth—U.S. Population and World Population, Lesson 7: Exponential Decay are lessons that pertain to this cluster.
- EngageNY Algebra 1, Module 3, Topic D, Lesson 21: Comparing Linear and Exponential Models Again, Lesson 22: Modeling an Invasive Species Population, Lesson 23: Newton's Law of Cooling are lessons that pertain to this cluster.
- The Georgia Standards of Excellence Curriculum Frameworks for Algebra 1, Unit 5: Comparing and Contrasting Functions compares and contrasts functions. There are many tasks in this document that align with this cluster.
- The Mathematics Vision Project Secondary Math 1, Module 2: Linear and Exponential Functions has many task that align with this cluster.
- A lesson on Exponential Modeling developed by the Virginia Department of Education that uses a graphing calculator. It has the following activities: Who Wants to be a Millionare?, Paper Folding, M&M Decay, Decaying Dice Game, Population Growth, and Baseball Players' Salaries.

General Resources

- Arizona High School Progression on Functions is an informational document for teachers. This cluster is addressed on pages 16-17.
- Arizona High School Progression on Modeling is an informational document for teachers. This cluster is addressed on page 5.
- High School Coherence Map by UnboundED is a map of the relationships among high school mathematics standards. Please note: This resource is aligned to the Common Core not Ohio's Learning Standards. Although the tool is useful, it needs to be checked with respect to Ohio's Learning Standards.

References

- Chan, H. (May 2015). How do they grow? *Mathematics Teaching in the Middle School*, *20(9),* 548-555.
- Common Core Standards Writing Team. (2013, March 1). *Progressions for the Common Core State Standards in Mathematics (draft). Grade 8, High School, Functions.* Tucson, AZ: Institute for Mathematics and Education, University of Arizona.
- Common Core Standards Writing Team. (2013, July 4). *Progressions for the Common Core State Standards in Mathematics (draft). High School, Modeling*. Tucson, AZ: Institute for Mathematics and Education, University of Arizona.
- Kara, M., Eames, C., Miller, A., & Chieu, A. (May 2015). Staircases, towers, and castles. *Mathematics Teacher*, *108(9),* 663-670.
- Kirwan, J. (April 2017). Using visualization to generalize on quadratic patterning tasks. *Mathematics Teacher*, *110(8),* 588-593.
- Yarma, C. & Sampson, J. (October 2001). Just say "charge it!" *Mathematics Teacher*, *94(7),* 558-564.

STANDARDS	MODEL CURRICULUM (F.LE.5)
Functions **LINEAR, QUADRATIC, AND EXPONENTIAL MODELS** **Interpret expressions for functions in terms of the situation they model.** **F.LE.5** Interpret the parameters in a linear or exponential function in terms of a context. ★	**Expectations for Learning** This standard does not present new expectations for student learning. Rather, it emphasizes important habits to complement F.LE.1-3. In this cluster, students connect their understanding of the defining characteristics of linear functions (initial value and rate of change) to the defining characteristics of exponential functions (initial value and growth rate/growth factor) and by interpreting them in the context of a real-world problem. **ESSENTIAL UNDERSTANDINGS** • Linear functions have a constant additive change. • Exponential functions have a constant multiplicative change. • Linear and exponential functions both have initial values. • To highlight the constant growth/decay rate, r, often expressed as a percentage, exponential functions can be written in the form, $f(n) = a(1 + r)^n$. • To highlight the growth/decay factor, b, exponential functions can be written in the form, $f(n) = a(b)^n$. • An arithmetic sequence is a linear function, and a geometric sequence is an exponential function. **MATHEMATICAL THINKING** • Connect mathematical relationships to contextual scenarios. • Use accurate mathematical vocabulary to describe mathematical reasoning. • Attend to meaning of quantities. • Make connections between concepts, terms, and properties within the grade level and with previous grade levels. **INSTRUCTIONAL FOCUS** • For linear functions (arithmetic sequences), focus on the constant rate of change across the tables, graphs, contexts, and the explicit and recursive representations. • For exponential functions (geometric sequences), focus on the constant growth/decay rate (or factor) across the tables, graphs, contexts, and the explicit and recursive representations. Continued on next page

<table>
<tr><th>STANDARDS</th><th>MODEL CURRICULUM (F.LE.5)</th></tr>
<tr><td>F.LE.5, continued</td><td>
<h2>Content Elaborations</h2>
<h3>OHIO'S HIGH SCHOOL CRITICAL AREAS OF FOCUS</h3>
Algebra 1, Number 2, pages 5-7
<h3>CONNECTIONS ACROSS STANDARDS</h3>

Build a function that models a relationship between two quantities (F.BF.1a, 2).
Interpret functions that arise in applications in terms of the context (F.IF.4-5).
Analyze functions using different representations (F.IF.7a, e).
Summarize, represent, and interpret data on two categorical and quantitative variables (S.ID.6c).
Interpret linear models (S.ID.7).
Interpret the structure of expressions (A.SSE.1).

</td></tr>
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INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (F.LE.5)

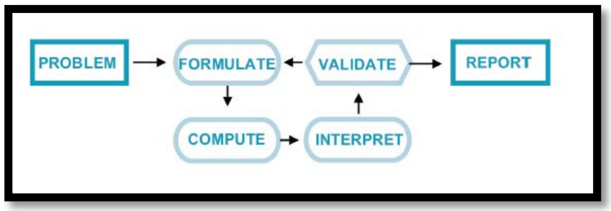
Instructional Strategies

Note: The model curriculum is not a collection of lessons nor a full curriculum; it does not suggest pace, sequence, or amount of time spent on topics. It provides information about a topic related to the standards including ideas for examples, strategies for teaching, common misconceptions, and possible connections between topics. In addition, the emphasis of the model curriculum is on instruction not on assessment; therefore, the examples in the Instructional Strategies section are written for instructional purposes not necessarily as assessment items. Examples may need to be rewritten to accommodate the needs of each individual classroom.

Standards for Mathematical Practice
This cluster focuses on but is not limited to the following practices:
MP.2 *Reason abstractly and quantitatively.*
MP.7 *Look for and make use of structure.*

MODELING

This cluster is included in the modeling standards. Geometric models that students can build are especially helpful to see patterns. See page 13 for more information about modeling.



INTERPRETING PARAMETERS

Emphasis should be put on using units to understand problems. Students should recognize the meaning of the parameters of a function. Draw attention to the units of the parameters and connect them to the context.

Use real-world contexts to help students understand how the parameters of linear and exponential functions depend on the context. For example, a plumber who charges \$50 for a house call and \$85 per hour would be expressed as the function $f(x) = 85x + 50$, and if the rate were raised to \$90 per hour, the function would become $f(x) = 90x + 50$. On the other hand, an equation of $f(x) = 8{,}000(1.04)^x$ could model the rising population of a city with 8,000 residents when the annual growth rate is 4%. Students can examine what would happen to the population over 25 years if the rate were 6% instead of 4% or the effect on the equation and function of the city's population were 12,000 instead of 8,000.

Students may incorrectly believe that the first term in a linear equation is always the rate of change. However, in the equation $y = 10 + 2x$, 10 is the constant value (or y-intercept) not the rate of change.

TIP!

Students may want to multiply the initial value by the base before raising the base to its exponential value. However, $3 \cdot 2^4$ is not equivalent to $(3 \cdot 2)^4$. Review the properties of exponents using expanded form.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (F.LE.5)

EXAMPLE

Illegal Fish

A fisherman illegally introduces some fish into a lake, and they quickly breed. The growth of the population of this new species (within a period of a few years) is modeled by $P(x) = 5b^x$ where x is the time in weeks following the introduction and b is a positive unknown base.

a. Exactly how many fish did the fisherman release into the lake?
b. Find b if you know the lake contains 33 fish after eight weeks. Show step-by-step work. What's the percent growth rate by week?
c. Instead, now suppose that $P(x) = 5b^x$ and $b = 2$. What is the weekly percent growth rate in this case? What does this mean in every-day language?

Task from Illustrative Mathematics. For solutions and discussion, see https://www.illustrativemathematics.org/content-standards/tasks/579.

Provide students with opportunities to research raw data on the internet (such as increases in gasoline consumption in China over x number of years) and graph and make generalizations about trends in growth, determining whether the growth is linear or exponential. Working in pairs or small groups, students can be given different parameters of a function to manipulate and compare the results to draw conclusions about the effects of the changes.

Graphs and tables can be used to examine the behaviors of functions as parameters are changed, including the comparison of two functions. For example, what would happen to a population if it grew by 500 people per year in contrast to the population rising an average of 8% per year over the course of 10 years?

EXAMPLE

a. John deposited \$400 in the bank at 2.25% simple interest rate for 6 months. Write an equation modeling the situation. How long will it take him to make \$500 in interest?
b. Jasmine deposited \$400 in the bank at 2.25% rate compounded every 6 months. Write an equation modeling the situation. How long will it take her to earn \$500 in interest?
c. Compare the parameters of the equations in both situations.

Note: The coefficients and constant, a, b, and c are the parameters of the equation $\boldsymbol{h}(\boldsymbol{t}) = \boldsymbol{a}\boldsymbol{t}^{\mathbf{2}} + \boldsymbol{b}\boldsymbol{t} + \boldsymbol{c}$. In physics, the constants a, b, and c have definite meanings. In projectile problems, one does not have the luxury of choosing friendly numbers. In the equation, a is $\frac{1}{2}$ of the gravitational constant, g. Since g is 9.8 m/s^2 or 32 ft/s^2, a is –4.9 or –16 depending on the choice of units. In the equation, b, represents the initial upward speed, and c is the initial height above the ground.

Taken from EngageNY Lesson 2: Graphs of Quadratics.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (F.LE.5)

Instructional Tools/Resources

These tools and resources may be helpful for instruction. Before use, they should be evaluated at the local level to determine their appropriateness for instruction.

Manipulatives/Technology

- Graphing calculators or computer software that generates graphs and tables of functions
- Web sites and other sources that provide raw data, such as the cost of products over time, population changes, etc.
- Desmos is a free graphing calculator that is available to students as website or an app.
- Geogebra is a free graphing calculator that is available to students as website.
- Wolframalpha is dynamic computing tool.

Interpreting Parameters

- Illegal Fish is a task from Illustrative Mathematics where students interpret the relevant parameters of exponential growth in a real-world context.
- Taxi is a task from Illustrative Mathematics where students interpret the parameters of a linear equation.
- U.S. Population 1982-1998 is a modeling task from Illustrative Mathematics using U.S. Census data. Students are required of make predictions using a linear model without using an equation.
- DDT-cay is a task from Illustrative Mathematics that allows students to encounter negative exponents in a contextual situation.
- Avi & Benita's Repair Shop is a Desmos activity focusing on parameters of linear and exponential functions in a real-world context.

Curriculum and Lessons from Other Sources

- EngageNY Algebra 1, Module 1, Topic A, Lesson 2: Graphs of Quadratic Functions is a lesson that pertains to this cluster.
- EngageNY Algebra 1, Module 3, Topic D, Lesson 21: Comparing Linear and Exponential Models Again, Lesson 22: Modeling an Invasive Species Population, Lesson 23: Newton's Law of Cooling, Lesson 24: Piecewise and Step Functions in Context are lessons that pertain to this cluster.
- The Georgia Standards of Excellence Curriculum Frameworks for Algebra 1, Unit 5: Comparing and Contrasting Functions compares and contrasts functions. This cluster is addressed on pages 61-69.
- The Mathematics Vision Project Secondary Math 1, Module 2: Linear and Exponential Functions has many tasks that align with this cluster.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (F.LE.5)

General Resources

- Arizona High School Progression on Functions is an informational document for teachers**.** This cluster is addressed on page 17.
- Arizona High School Progression on Modeling is an informational document for teachers. This cluster is addressed on page 12.
- High School Coherence Map by UnboundED is a map of the relationships among high school mathematics standards. Please note: This resource is aligned to the Common Core not Ohio's Learning Standards. Although the tool is useful, it needs to be checked with respect to Ohio's Learning Standards.

References

- Common Core Standards Writing Team. (2013, March 1). *Progressions for the Common Core State Standards in Mathematics (draft). Grade 8, High School, Functions.* Tucson, AZ: Institute for Mathematics and Education, University of Arizona.
- Common Core Standards Writing Team. (2013, July 4). *Progressions for the Common Core State Standards in Mathematics (draft). High School, Modeling*. Tucson, AZ: Institute for Mathematics and Education, University of Arizona.

STANDARDS	MODEL CURRICULUM (S.ID.1-3)
Statistics and Probability **INTERPRETING CATEGORICAL AND QUANTITATIVE DATA** **Summarize, represent, and interpret data on a single count or measurement variable.** **S.ID.1** Represent data with plots on the real number line (dot plots[G], histograms, and box plots) in the context of real-world applications using the GAISE model. ★ **S.ID.2** In the context of real-world applications by using the GAISE model, use statistics appropriate to the shape of the data distribution to compare center (median and mean) and spread (mean absolute deviation[G], interquartile range[G], and standard deviation) of two or more different data sets. ★ **S.ID.3** In the context of real-world applications by using the GAISE model, interpret differences in shape, center, and spread in the context of the data sets, accounting for possible effects of extreme data points (outliers). ★	**Expectations for Learning** In middle school students learn about the framework of the GAISE model of statistical problem solving. It consists of four steps: Formulating Questions; Collecting Data; Analyzing Data; and Interpreting Results. Students integrate this model whenever they use statistical reasoning. This process will continue throughout high school as students deepen their statistical reasoning skills. Middle school students create dot plots, histograms, and box plots and draw informal comparisons between two populations using graphs. They also summarize data sets using mean absolute deviation. In Algebra 1/Math 1 students should use and expand their learning to more sophisticated problems and by comparing single or multiple data sets through graphical representations. Standard deviation is a new concept for students, and it builds upon their previous understanding of mean absolute deviation (MAD). In Algebra 2/Math 3**,** students then extend their knowledge of mean and standard deviation from Algebra 1/Math 1 to normal distributions. **<u>The GAISE Model</u>** Students will use the GAISE Model framework for statistical problem solving in all courses. The GAISE Model should not be taught in isolation. Students are building on the framework that was developed in middle school. As students progress through the courses, the learning will move toward a greater level of precision and complexity. Students in middle school start at Level A and move toward Level B. As students progress from Level A to Levels B and C, the learning becomes less teacher-driven and more student-driven. In this cluster students are at Level B moving towards Level C, and Steps 1 and 2 continue to be emphasized with added depth on Steps 3 and 4. "Understanding the statistical concepts of GAISE model Level B enables a student to grow in appreciation that data analysis is an investigative process consisting of formulating their own questions; collecting appropriate data through various sources; analyzing data through graphs and simple summary measures; and interpreting results with an eye toward inference to a population based on a sample" (Guideline for Assessment and Instruction in Statistics Education (GAISE) Report, 2007, page 58). *Continued on next page*

STANDARDS	MODEL CURRICULUM (S.ID.1-3)
S.ID.1-3, continued	**Expectations for Learning, continued** **The GAISE Model, continued** **Step 1: Formulate the Question** • Students should pose their own statistical question of interest (Level C). • Students are starting to form questions that allow for generalizations of a population (Level B-C). **Step 2: Collect Data** • Students should begin to use random selection or random assignment (Level B). **Step 3: Analyze Data** • Students measure variability within a single group using MAD, IQR, and/or standard deviation (Level B). • Students compare measures of center and spread between groups using displays and values (Level B). • Students describe potential sources of error (Level B). • Students understand and use particular properties of distributions as tools of analysis moving toward using global characteristics of distributions (Level B-C). **Step 4: Interpret Results** • Students acknowledge that looking beyond the data is feasible by interpreting differences in shape, center, and spread (Level B). • Students determine if a sample is representative of a population and start to move toward generalization (Level B-C). • Students note the difference between two groups with different conditions (Level B). *Continued on next page*

STANDARDS	MODEL CURRICULUM (S.ID.1-3)

S.ID.1-3, continued	**Expectations for Learning, continued**
	ESSENTIAL UNDERSTANDINGS
	• Univariate quantitative data can be represented using dot plots, box plots, and histograms.
	• Mean and median are approximately equal for symmetric distributions, but tend to be different for nonsymmetric distributions.
	• Standard deviation is a measure of variation from the mean (spread).
	• Extreme values (outliers) have an effect on the shape, center, and spread of a distribution.
	  o The median and interquartile range are appropriate measures of center and spread if the distribution is extremely skewed or has outliers.
	  o The mean and standard deviation are appropriate measures of center and spread if the distribution is not skewed and has no extreme outliers.
	
	MATHEMATICAL THINKING
	• Use accurate and precise mathematical vocabulary.
	• Construct formal and informal arguments to verify claims and justify conclusions.
	• Solve real-world and statistical problems.
	• Use appropriate tools to display and analyze data.
	Continued on next page

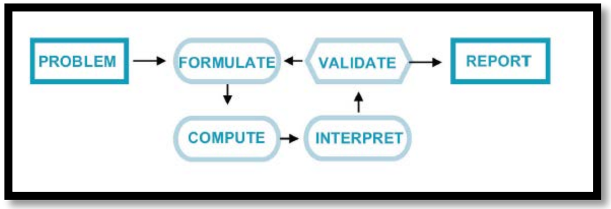
STANDARDS	MODEL CURRICULUM (S.ID.1-3)
S.ID.1-3, continued	**Expectations for Learning, continued** **INSTRUCTIONAL FOCUS** • Compare the mean to the median of the same data set and relate them to the shape of the distribution (symmetric, skewed). • Develop the formula for and a conceptual understanding of standard deviation by building on the conceptual understanding and formula of mean absolute deviation. • Compare two or more distributions based upon their means and standard deviations. • Explain how outliers affect the mean, the median, and standard deviation. • Given two or more data sets or graphs, do the following: o Compare the shape (symmetric, skewed, uniform). o Compare the spread (greater than, less than, equal). o Compare the centers (mean, median). • Interpret the mean, standard deviation, outliers, as well as differences and similarities between two or more sets of data within a context. **Content Elaborations** **OHIO'S HIGH SCHOOL CRITICAL AREAS OF FOCUS** • Algebra 1, Number 3, page 8 **THE GAISE MODEL** • GAISE Model, pages 14 – 15 o Focus of the cluster for Algebra 1/Math 1 is Level B, pages 37-60 **CONNECTIONS ACROSS STANDARDS** • Summarize, represent, and interpret data on two categorical and quantitative variables (S.ID.5-6).

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (S.ID.1-3)

Instructional Strategies

Note: The model curriculum is not a collection of lessons nor a full curriculum; it does not suggest pace, sequence, or amount of time spent on topics. It provides information about a topic related to the standards including ideas for examples, strategies for teaching, common misconceptions, and possible connections between topics. In addition, the emphasis of the model curriculum is on instruction not on assessment; therefore, the examples in the Instructional Strategies section are written for instructional purposes not necessarily as assessment items. Examples may need to be rewritten to accommodate the needs of each individual classroom.

S.ID.1-3 are modeling standards. See page 13 for more information about modeling.



Standards for Mathematical Practice

This cluster focuses on but is not limited to the following practices:

MP.1 *Make sense of problems and persevere in solving them.*
MP.2 *Reason abstractly and quantitatively.*
MP.3 *Construct viable arguments and critique the reasoning of others.*
MP.4 *Model with mathematics.*
MP.5 *Use appropriate tools strategically.*
MP.6 *Attend to precision.*
MP.8 *Look for and express the regularity in repeated reasoning.*

THE GAISE MODEL

Opportunities should be provided for students to work through the statistical process outlined in the GAISE model that students learned in middle school. The process consists of four steps: formulating a question that can be answered by data; designing and implementing a plan that collects appropriate data; analyzing the data by graphical and/or numerical methods; and interpreting the analysis in the context of the original question.

Step 1: Formulating Questions

By high school students should be posing their own statistical questions, designing their own data collection procedure, and conducting their own analysis that includes generalizations from a sample to a population. Teachers should provide access to real-world datasets which are readily available online. Use problems that are of interest to the students. The richer the question forumulated, the more interesting the process.

EXAMPLE

- Are males and females more involved in after-school activities?
- To what extent has the opiod crisis affected my community?
- How often do teens text?
- How many minutes do high schoolers spend on social media?

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (S.ID.1-3)

Step 2: Collecting Data

Although this domain addresses both categorical and quantitative data, standards S.ID.1-4 and S.ID.6 are focused on only quantitative data. Note that S.ID.5 addresses analysis for two categorical variables on the same individual. It may be helpful to contrast categorical (using bar graphs, two-way frequency tables, pie graphs, and/or the central tendency of mode) and quantitative data.

In 7th grade (7.SP.1) students developed an informal idea of sampling. By high school students should be able to differentiate between a population, a census, and a sample. When collecting data, students should begin to informally discuss the importance of random selection in terms of "fairness" especially if they have not done so in middle school. In Algebra 2 students will gain a more in-depth understanding of random sampling (S.IC.1). In this cluster even though students will not be able to calculate true randomness, they can develop the basic idea that generalizations about a population are only valid if the sample truly represents the population. This can be done by discussing biases such as only sampling one's friends. This should allow them to take the first step toward a generalization about a population. Using the probability concepts that students learned in Grade 7, students should be able to have a discussion about the basic understanding of the role of probability in random selection.

Step 3: Analyzing Data

Students should focus on using technology to create graphs for S.ID.1 since creating graphs by hand was one of the focuses in middle school. (See 6.SP. 4 and 7.SP3 Model Curriculum for scaffolding ideas.) In addition, students should not only become fluent in creating graphical displays but also on knowing how to choose the most appropriate graph given the data set. They should use initial graphical displays (for single or more than one group) as a first step during the exploratory stage of their analysis. Then the graphical displays could be used to prompt deeper and more meaningful questions.

Students may think that a bar graph and a histogram are the same. A bar graph is appropriate when the horizontal axis has categories and the vertical axis is labeled by either frequency (e.g., book titles on the horizontal axis and the number of students who like the respective books on the vertical axis) or a measurement of some numerical variable (e.g., days of the week on the horizontal axis and the median length of root growth of radish seeds on the vertical axis). A histogram has units of measurement of a numerical variable on the horizontal axis such as ages with intervals of equal length.

Provide students with multiple data sets that will generate different types of distributions. Give students guiding questions to discuss in order to help them determine why the distribution looks the way that it does and consider the possible implications for the distribution. They should be discussing shape, center, and spread in context. Although in middle school students had practice in describing shape, center, and spread, students should now become more fluent in these descriptions so that it becomes a habit of mind.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (S.ID.1-3)

This exploratory analysis should also include the comparisons of mean and median and how they are affected by the data and the distribution. The best measures of center and spread to describe data sets without outliers are the mean and standard deviation, whereas median and interquartile range are better measures for data sets with outliers. Note on outliers: Outliers should always be included in the analysis. They can be discussed to verify the validity of the data point such as "Why was it different?" or "Could it have been a collection error?"

Students may believe that the lengths of the intervals of a boxplot (min, Q1), (Q1, Q2), (Q2, Q3), (Q3, max) are related to the number of subjects in each interval. Students should understand that each interval theoretically contains (approximately) one-fourth (25%) of the total number of subjects. Sketching an accompanying dot plot or histogram or constructing a live boxplot may help in alleviating this misconception.

Misleading Graphs

Students should be given misleading or distorted graphs such as dot plots, histograms, and box plots to analyze. They can be used to stress the importance of labeling and proper scaling. Some examples of things that can be misleading are as follows:

- non-evenly spaced scales;
- evenly spaced scales for uneven intervals;
- non-proportional area or volume changes for a picture;
- loaded or bias labels;
- using more dimensions than variables in the plotted data;
- histograms or bar graphs not starting at 0 (truncated graph);
- the size of the intervals in histograms;
- the bars on a histogram may not be uniform;
- a superfluous third dimension is used; and/or
- area of the box in a box and whiskers plot.

One site that has misleading graphs is 31 Misleading Graphs and Statistics by Dr. Marcel B. Finan from Arkansas Tech University.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (S.ID.1-3)

Standard Deviation and Mean Absolute Deviation
Introduce the formula of standard deviation by reviewing the previously learned MAD (mean absolute deviation). The MAD is very intuitive and gives a solid foundation for developing the more complicated standard deviation measure.

In addition to a discussion about the shape of the distribution, students should be exploring standard deviation:

- What it means in the context of the situation?
- How the standard deviation changes as the data changes?

To introduce standard deviation, it may be helpful to calculate the standard deviation by hand for small sets of data with a focus on understanding the concept and its relationship to MAD but then transition to using technology for larger sets of data. The focus should be on the conceptual understanding in lieu of the calculations.

Step 4: Interpreting Data
Informally observing the extent to which two or more appropriate graphs overlap begins the discussion of drawing inferential conclusions.

TIP! When making two or more graphs, students should be sure to use the same scale for all graphs so that comparisons can be made.

Students should explore the differences between variations and errors. Draw attention to naturally occurring errors that happen in the classroom and discuss the impact of the errors on the results. These notions can be used to explain outliers, clusters, and gaps. Understanding variability is vital for developing data sense.

VARIABILITY
Variability is everywhere in statistics. Data vary; samples vary; distributions vary; and variation occurs both within and across samples and distributions.

Initially when given a data set, students calculate the mean or median to analyze the data set. However, relying exclusively on using a measure of center masks important features. However students who attend to variability are much more likely to predict an interval of outcomes rather than a single number.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (S.ID.1-3)

Students may reason additively, proportionally, or distributively about variability. Additive reasoners focus on frequencies rather than relative frequencies. Proportional reasoners are able to make connections between sample proportions and population proportions. Distributional reasoners are able to use both center and spread to reason about a problem. Distributional reasoners like proportional reasoners, are able to make connections between the sample and population proportions, but they are also able to explicitly mention variation about the expected value. It takes time and experience for students to gain distributive reasoning. It is difficult for students to think about center, shape, and spread simultaneously (Shaughessy, 2007).

Students in high school should start to realize there is not only variability within a group but also variability between groups. Measures of spread such as mean absolute deviations (MAD) and standard deviation are ways students in Algebra 1 can express measures of spread. Students should be able to compare data values to a measure of center and quantify how different the data are from the measure of center. Informally discuss the benefits of imposing randomness into sampling procedures.

Instructional Tools/Resources

These tools and resources may be helpful for instruction. Before use, they should be evaluated at the local level to determine their appropriateness for instruction.

Manipulatives/Technology

- Census at School allows students to complete a brief online survey, analyze their class census results, and compare their class with random samples for students in the United States or other countries.
- A Little Stats: Adventures in Teaching Statistics created by Amy Hogan is a list of free internet sources for real data and datasets for public use.
- Rossmanchance.com is a website by Allan Rossman and Beth Chance that has a link to many applets that could be useful for statistics and probability.
- Statistics Calculator: Box Plot is an applet that generates a box plot.
- Visual Understanding Educational Apps for Statistics
- StatKey by Lock, Lock, Lock, Lock, and Lock is an applet for representing statistics
- Box Plot by Shodor is an applet that creates a box plot.
- Graphing calculators
- Printed media (e.g., almanacs, newspapers, professional reports)
- Desmos is a free graphing calculator that is available to students as website or an app.
- Geogebra is a free graphing calculator that is available to students as website.
- Wolframalpha is dynamic computing tool.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (S.ID.1-3)

Box Plots

- Are Female Hurricanes Deadlier than Male Hurricanes? is a lesson by Mary Richardson from the Statistics Education Web (STEW) published in June 2014. Students will apply the GAISE model and use box plots to compare two data sets.
- Colors Challenge by Reischman is a lesson from the Statistics Education Web (STEW) published June 2013. In this lesson students collect data using the GAISE model to investigate whether or not the ability to name a color of ink is more difficult when the word written in the same ink is the name of a color; students will design an experiment and use box plots.
- Representing Data with Box Plots is a lesson from Mathematics Assessment Project where students interpret box plots and create box plots from frequency graphs.
- Saga of Survival by Richardson and Rogness is a lesson from the Statistics Education Web published July 2013. In this lesson students use demographic data from the Donner Party tragedy using box plots and two-way frequency tables.
- Did I Trap the Median? is a lesson by Parks, Steinwachs, Diaz, and Molinaro from the Statistics Education Web (STEW) published in June 2014. In this lesson students use the GAISE model to collect data about the median foot size of the class. They use box plots to evaluate the data.
- Commuting to Work is a unit by the United States Census Bureau where students use measures of central tendency and box plots to represent the number of people who bike to work.
- Speed Trap is a task by Illustrative Mathematics where students construct boxplots and use them to compare distributions.
- Haircut Costs is an introductory task by Illustrative Mathematics that introduces group comparisons using boxplots.
- Now You See It, Now You Don't: Using See It to Compare Stacked Dotplots to Boxplots by Guzman-Alvarez, Smith, Molinaro, and Diaz is a lesson from the Statistics Education Web (STEW) published May 2014. In this lesson students use the GAISE model collect data by measuring the height of their right-hand reach using dot plots and box plots.
- BMI Calculations is a CTE lesson from Achieve the Core where students evaluate BMI measurements using statistics such as a 5-Number summary and box plots.
- Differences in Earning Across Sex and Educational Attainment: Comparing Box Plots is a unit by the United States Census Bureau where students use data and boxplots to compare the earnings of men and women.
- Glued to the Tube or Hooked to Books? is a lesson from the Melt Institute where students compare the time watching TV and the time reading books by using a boxplot.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (S.ID.1-3)

Histograms

- Census in Counties-Describing and Comparing Histograms to Understand American Life is a unit by the United States Census Bureau that uses data on employment, technology, and transportation to create histograms.
- Math Class is a lesson from Georgia Standards of Excellence Curriculum Framework's Algebra 1 course could be used to formatively assess students' ability to summarize, represent, and interpret data on a single count or measurement variable using histograms. This lesson can be found on pages 20-24.
- Georgia Standards of Excellence Curriculum Framework, Algebra 1, Unit 6: Describing Data, The Basketball Star is a lesson where students use dot plots, histograms, and box plots to compare the shape of two data sets. This lesson can be found on pages 25-32.
- Describing and Comparing Data Distributions is a unit by the United States Census Bureau where students use data on the government organization, spending, and populations at different levels (city, county, state) to compare and contrast the distributions using histograms and boxplots.
- Census in Counties—Describing and Comparing Histograms to Understand American Life is a lesson by the United States Census Bureau where students use histograms to analyze life in America.

Frequency Graphs

- Representing Data with Frequency Graphs by Mathematics Assessment Project has students use frequency graphs to indeify a range of measures and make sense of the data in a real-world context.

Mean Absolute Deviation (MAD)

- Measuring Variability in a Data Set is a task by Illustrative Mathematics that compares MAD to standard deviation.
- Georgia Standards of Excellence Curriculum Framework, Algebra 1, Unit 6: Describing Data, If the Shoe Fits? is a lesson where students explore data from shoe prints to figure out who came on school grounds without permission. They will use measures of center, IQR, Box Plots, and MAD. It also incorporates scatterplots. This lesson can be found on pages 48-61.

Misleading Graphs

- The Most Misleading Charts of 2015, Fixed is a website by Quartz Media that has examples of misleading graphs.
- 31 Misleading Graphs and Statistics by Dr. Marcel B. Finan of Arkansas Tech University has many misleading graphs.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (S.ID.1-3)

Curriculum and Lessons from Other Sources

- EngageNY, Algebra 1, Module 2, Topic A, Lesson 1: Distributions and their Shapes, Lesson 2: Describing the Center of a Distribution, Lesson 3: Estimating Centers and Interpreting the Mean as a Balance Point are lessons that pertain to this cluster.
- EngageNY, Algebra 1, Module 2, Topic B, Lesson 4: Summarizing Deviations from the Mean, Lesson 5: Measuring Variability for Symmetrical Distributions, Lesson 6: Interpreting the Standard Deviation, Lesson 7: Measuring Variability for Skewed Distributions (Interquartile Range), Lesson 8: Comparing Distributions are lessons that pertain to this cluster.
- Mathematics Vision Project, Secondary Math 1, Module 9: Modeling Data is a unit on statistics. Tasks 9.1 and 9.2 are applicable to this cluster.
- Math Statistics in Schools are lessons provided by the United States Census Bureau that use real-life data.

General Resources

- Arizona's High School Progression on Statistics and Probability is an informational document for teachers. This cluster is addressed on pages 2-4.
- Arizona's High School Progression on Modeling is an informational document for teachers. This cluster is addressed on pages 2 and 10.
- High School Coherence Map by UnboundED is a map of the relationships among high school mathematics standards. Please note: This resource is aligned to the Common Core not Ohio's Learning Standards. Although the tool is useful, it needs to be checked with respect to Ohio's Learning Standards.
- LOCUS is an NSF Funded project focused on developing assessments of statistical understanding. Teachers must create an account to access the assessments.
- K-12 Statistics Education Resources is a collection of websites put together by the American Statistical Association for teachers.
- A Sequence of Activities for Developing Statistical Concepts by Christine Franklin & Gary Kader is an article published in *The Statistics Teacher Network*, Number 68 ,Winter 2006. It has an overview of the GAISE model and its levels, and it includes activities at each level.
- Statistics Teacher is an online journal published by the American Statistical Association (ASA) and NCTM. It includes articles, lesson plans, and professional development opportunities.
- Significance is a magazine that demonstrates the practical use of statistics and shows how statistics benefits society.
- Chance is a magazine about statistics for anyone who is interested in using data to advance science, education, and society.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (S.ID.1-3)

Research

- Burrill, G., Franklin, C., Godbold, L., & Young. (2003). *Navigating through Data Analysis in Grades 9-12*. Reston, VA: National Council of Teachers of Mathematics.
- Common Core Standards Writing Team. (2016, March 24). *Progressions for the Common Core State Standards in Mathematics (draft). High School Statistics and Probability*. Tucson, AZ: Institute for Mathematics and Education, University of Arizona.
- Common Core Standards Writing Team. (2013, July 4). *Progressions for the Common Core State Standards in Mathematics (draft). High School, Modeling*. Tucson, AZ: Institute for Mathematics and Education, University of Arizona.
- Franklin, C., Kader, G., Mewborn, D, Moreno, J., Peck, R., Perry, M., & Scheaffer, R. (2007). *Guidelines for Assessment and Instruction in Statistics Education (GAISE) Report*. Alexandria, VA: American Statistical Association.
- Shaughnessy, J. (2007). Research on statistics learning and reasoning. In F. K. Lester (Ed.), *Second Handbook of Research on Mathematics Teaching and Learning* (pp. 957-1010). Charlotte, NC: Information Age Publishing.

STANDARDS	MODEL CURRICULUM (S.ID.5-6)

Statistics and Probability
INTERPRETING CATEGORICAL AND QUANTITATIVE DATA
Summarize, represent, and interpret data on two categorical and quantitative variables.
S.ID.5 Summarize categorical data for two categories in two-way frequency tables. Interpret relative frequencies in the context of the data (including joint, marginal, and conditional relative frequencies). Recognize possible associations and trends in the data. ★
S.ID.6 Represent data on two quantitative variables on a scatter plot, and describe how the variables are related. ★
c. Fit a linear function for a scatterplot that suggests a linear association. (A1, M1)
	Expectations for Learning
	For this cluster, the GAISE Model framework continues to be used: Formulating Questions; Collecting Data; Analyzing Data; and Interpreting Results. In the middle grades, students visually approximate a linear model and informally judge its goodness of fit. In Algebra 1/Math 1, students extend this knowledge to find the equation of a linear model, with and without technology. They will also use more precise language to describe the relationship between variables. In Algebra 2/Math 3, concepts extend to quadratic and exponential functions as well as working with residuals.
	
	The learning at this level is at the developmental Level B. See pages 62-63 for more information on Level B.
	
	ESSENTIAL UNDERSTANDINGS
	Note: Students should be able to talk sensibly about the meanings of joint, marginal, and conditional frequencies within a context but should not be held responsible for precise usage of this vocabulary.
	• Row totals and column totals constitute the marginal frequencies.
	• Individual table entries represent joint frequencies.
	• A relative frequency is found by dividing the frequency count by the total number of observations for a whole set or subset.
	    o A marginal relative frequency is calculated by dividing the row (or column) total by the table total.
	    o A joint relative frequency is calculated by dividing the table entry by the table total.
	    o A conditional relative frequency is calculated by restricting to one row or one column of the table.
	• Relative frequencies are useful in considering association between two categorical variables.
	• A linear function can be used as a model for a linear association of two quantitative variables.
	Continued on next page

STANDARDS	MODEL CURRICULUM (S.ID.5-6)
S.ID.5-6, continued	**Expectations for Learning, continued** **MATHEMATICAL THINKING** • Use accurate and precise mathematical vocabulary. • Construct formal and informal arguments to verify claims and justify conclusions. • Solve real-world and statistical problems. • Use appropriate tools to display and analyze data. • Accurately make computations using data. • Determine reasonableness of predictions. **INSTRUCTIONAL FOCUS** **Categorical Data** • Calculate and interpret, within a context, joint, marginal, and conditional relative frequencies. • Recognize possible relationships (trends) in the context of the data by using percentages from two-way frequency tables. **Quantitative Data** • Describe, within a context, how variables are related in a linear relationship using scatter plots. • Calculate and interpret, within a context, the slope and *y*-intercept of a linear model, given a set of data or graph, with or without technology. *Continued on next page*

STANDARDS	MODEL CURRICULUM (S.ID.5-6)
S.ID.5-6, continued	**Content Elaborations** **OHIO'S HIGH SCHOOL CRITICAL AREAS OF FOCUS** • Algebra 1, Number 3, page 8 **THE GAISE MODEL** • GAISE Model, pages 14 – 15 o Focus of this cluster for Algebra 1/Math 1 is Level B moving toward Level C, pages 37-60 **CONNECTIONS ACROSS STANDARDS** • Interpret linear models (S.ID.7-8). • Build a function that models a relationship between two quantities (F.BF.1). • Distinguish between situations that can be modeled with linear and exponential functions (F.LE.1). • Create equations that describe numbers or relationships (A.CED.2).

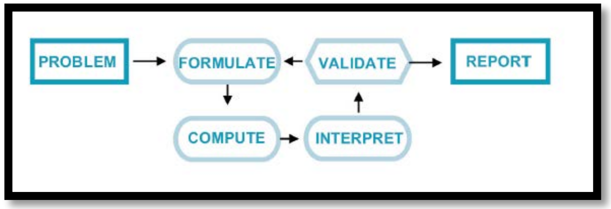
INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (S.ID.5-6)

Instructional Strategies

Note: The model curriculum is not a collection of lessons nor a full curriculum; it does not suggest pace, sequence, or amount of time spent on topics. It provides information about a topic related to the standards including ideas for examples, strategies for teaching, common misconceptions, and possible connections between topics. In addition, the emphasis of the model curriculum is on instruction not on assessment; therefore, the examples in the Instructional Strategies section are written for instructional purposes not necessarily as assessment items. Examples may need to be rewritten to accommodate the needs of each individual classroom.

Standards for Mathematical Practice
This cluster focuses on but is not limited to the following practices:
MP.4 *Model with mathematics.*
MP.6 *Attend to precision.*
MP.7 *Look for and make use of structure.*

S.ID.5-6 are modeling standards. See page 13 for more information about modeling.



This cluster builds upon what students learned in Grade 8 regarding scatter plots and two-way tables. It is also a continuation of statistical thinking using the GAISE model that was introduced in S.ID.1 – 4. Students should continue to formulate their own statistical question, collect data, analyze data (with two-way tables and scatterplots), and then interpret data (which continues in the next cluster for scatterplots).

In this cluster, the focus is on two categorical or two quantitative variables that are being measured on the same subject. Help students clearly distinguish between categorical and quantitative variables by providing multiple examples of each type.

CATEGORICAL BIVARIATE DATA

The focus is on statistical thinking within a given data set using relative frequencies. The standard S.ID.5 is almost identical to 8.SP.4 except now in high school students delve deeper into concepts as they transition from Level A to Level B. See 8th Grade Model Curriculum 8.SP.1-4 for scaffolding ideas. In the categorical case, begin with two categories for each variable and represent them in a two-way table.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (S.ID.5-6)

EXAMPLE

Step 1: Formulate Questions

Students came up with the question: Is there a relationship between gender and whether a student has a job in high school?

TIP! Use more open-ended types of question to promote a higher level of statistical reasoning.

Step 2: Collect Data

Based on the question in Step 1, students may decide that a survey can be given to Mr. Johnson's two Algebra 1 classes and that they can use a table to organize their data. A discussion should occur about whether the two Algebra 1 classes can really represent the population of the school and if not discuss how the survey should be adjusted. Some students may point out that many Algebra 1 students are only Freshman, so it may be better to do a survey of a 9th, a 10th, an 11th, and a 12th grade English class. Since most honors students took Algebra 1 in middle school, students may wonder whether it is more likely that honors students have jobs compared to non-honors students. This could lead to a discussion about how variability not only occurs within a group, but can also occur between groups. Students should be encouraged to explore these different types of possibilities in tandem with the class discussion time permitting.

A discussion also needs to take place about clearly defining the event: What constitutes a job? Does babysitting count as a job?

Discuss how using discrete questions rather than open-ended questions is preferable when creating a survey in order to make it easier to analyze data. Discuss how loaded words or biased language could change the outcome of survey. A fun exploration could be to have the class design two similar surveys using different word choices and explore the differences in the results.

	Job	No Job
Male	30	35
Female	45	25

TIP! When creating surveys, students may be stuck on the idea of "fairness" instead of the idea of random selection. For example they may want to pick two boys and two girls from every class, so every student has a chance of being surveyed. Another example could be that they may think it would be better to have an optional survey so that any student can choose to participate not realizing that self-selection may lead to bias. Instead students should realize that an accurate sample does not mean that every student needs to be surveyed.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (S.ID.5-6)

Two-way Frequency Tables vs Two-way Relative Frequency Tables

Information can be organized into Two-Way Frequency Tables or Two-Way Relative Frequencies. Students need to be exposed to both types of tables. Whereas frequency tables display counts, relative frequency tables display the data in ratios, decimals, or percents. Notice that the total in relative frequency table (bottom right-hand corner) is 1 or 100%. Students should have been exposed to both types of tables in 8th grade, but since High School students should be at GAISE Level B, the emphasis is on proportional reasoning and the two-way relative frequency table in order to interpret data in terms of fractions or percents.

Note: In high school, rarely will students be given the column and row for total (marginal frequency) but rather they will be expected to calculate this information independently. Students should be able to talk sensibly about the meanings of joint, marginal, and conditional frequencies within a context but should not be held responsible for precise usage of this vocabulary.

EXAMPLE, CONTINUED

Two-way Frequency Table

	Job	No Job	TOTAL
Male	30	35	
Female	45	25	
TOTAL			

Two-way Relative Frequency Table

	Job	No Job	TOTAL
Male	$^{30}/_{135} \approx$ 22%	$^{35}/_{135} \approx$ 26%	
Female	$^{45}/_{135} \approx$ 33%	$^{25}/_{135} \approx$ 19%	
TOTAL			$^{135}/_{135} =$ 100%

TIP!

Some students will have more success with two-way frequency tables as opposed to two-way relative frequency tables. Depending on the data, some students may have more success changing a two-way relative frequency table into a two-way frequency table where the total is 100 or 1,000. Although this method is not always as precise, it is acceptable for Algebra 1 students. See article: https://opinionator.blogs.nytimes.com/2010/04/25/chances-are/ for more information.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (S.ID.5-6)

Step 3: Analyze Data

Joint and Marginal Relative Frequency

Joint frequency is where the two variables "join" such as job and male. These can be found in the body of the table.

Row totals and column totals constitute the marginal frequencies. These are found in the margins of the table. Marginal frequencies can also be found by adding across columns or rows.

The ratio of the joint or marginal frequencies to the total number of subjects define relative frequencies (and percentages), respectively.

Note: Whereas in Grade 8 students were often given the marginal frequencies, in high school students are expected calculate the marginal frequencies by themselves.

EXAMPLE, CONTINUED

Discuss with students what kind of information could be found using two-tables. Discuss the pros and cons of a two-way frequency table vs a two-way relative frequency table. Strive for fluency in both, but push students to use the two-way relative frequency table.

	Job	No Job	TOTAL
Male	$\frac{30}{135}$	$\frac{35}{135}$	$\frac{65}{135}$
Female	$\frac{45}{135}$	$\frac{25}{135}$	$\frac{70}{135}$
TOTAL	$\frac{75}{135}$	$\frac{60}{135}$	$\frac{135}{135}$

Joint Frequencies

Marginal Frequencies

Now that students have the data, discuss what types of questions can be asked to help them answer the initial question. Have students discuss the benefit of calculating relative frequency over frequency.

- What is the frequency that a male will have a job?
- What is the relative frequency that a male will have a job?
- If a student is selected at random, what is the probability that he or she would not have a job?

Point out that students can calculate relative frequency from a two-way frequency table, or they can work directly from a two-way *relative* frequency table.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (S.ID.5-6)

Conditional Frequency

Conditional relative frequencies are determined by focusing on a specific row or column of the table. They are particularly useful in determining any associations between the two variables. Discuss the different types of conditional frequency questions students could ask.

- Using a frequency table, what is the likelihood that a student is male assuming the he has a job? $\frac{30}{75} = \frac{2}{5}$
- Using a relative frequency table, what is the likelihood that a student is male assuming has a job? $\left(\frac{30}{135}\right) \div \left(\frac{75}{135}\right) = \frac{2}{5}$.

	Job	No Job	TOTAL
Male	30	35	65
Female	45	25	70
TOTAL	75	60	135

	Job	No Job	TOTAL
Male	$^{30}/_{135} \approx$ 22%	$^{35}/_{135} \approx$ 26%	$^{65}/_{135} \approx$ 48%
Female	$^{45}/_{135} \approx$ 33%	$^{25}/_{135} \approx$ 19%	$^{70}/_{135} \approx$ 52%
TOTAL	$^{75}/_{135} \approx$ 55%	$^{60}/_{135} \approx$ 45%	$^{135}/_{135} =$ 100%

Again discuss the benefits and drawbacks of calculating conditional relative frequencies using the two different tables.

Discuss how these two statements are different:

- What is the relative frequency that a female student has a job?
- What is the relative frequency that a student who has a job is female?

Note: Although the above example is a simple example for a 2 by 2 frequency table, in high school students should gain experience and become fluent with larger tables (2 by 3, 3 by 4, etc.)

Direct students to use percentages when making claims about data presented in two-way tables, because frequencies can be misleading.

TIP!

Students need practice calculating conditional frequencies with and without the word "given" being used. Have students practice writing conditional frequencies in different ways.

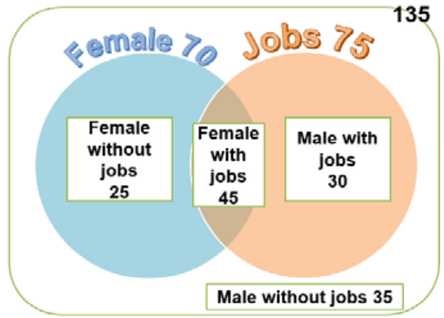
Students should also be able to convert between different data representations such as converting from a two-way table to a Venn diagram.

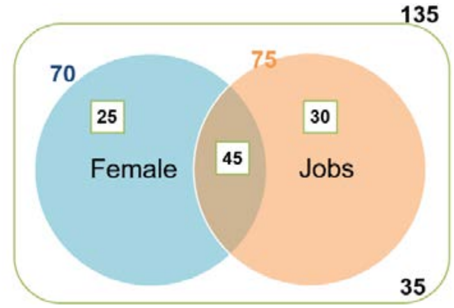
INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (S.ID.5-6)

EXAMPLE

Michelle prefers using Venn diagrams over two-way frequency tables. Convert the data from the two-way frequency table to a Venn diagram to help Michelle visualize whether a female has a job. Student can move from fully labeled Venn diagrams to those where less labels are given.

	Job	No Job	TOTAL
Male	30	35	65
Female	45	25	70
TOTAL	75	60	135





Step 4: Interpret Data

Students should gain additional practice in writing about statistics by validating conclusions and justifying arguments (see GAISE Steps 3 and 4 Level C). Questioning could be focused around associations and trends in the data. From our previous example a question could be "Is there an association between gender and whether or not an individual has a job? Explain."

Association

Association is a relationship between two types of quantities so that one event is dependent on the other. Have students redefine their question to address associations such as—

- Is there an association between gender and job status?
- Does gender predict whether a high school student will have a job?

In two-way frequency tables, two categorical variables are associated if the row conditional relative frequencies (or column relative frequencies are different from the rows (or columns) of the table. Discuss with students the pairs of conditional probability questions that could be asked to determine this.

- What is the conditional relative frequency that a male has a job? (30/65 = 0.46)
- What is the conditional relative frequency that a female has a job? (45/70 = 0.64)

Since 0.46 and 0.64 are not close, gender and job status are related.

Or

- What is the conditional relative frequency that someone who has a job is male? (30/75 = 0.40)
- What is the conditional relative frequency that someone who does not have a job is male? (35/60 = 0.58)

Since 0.40 and 0.58 are not close, gender and job statues are related.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (S.ID.5-6)

The greater the difference between the conditional relative frequencies, the greater the association. Although students will discuss correlation vs causation more formally in Algebra 2, a discussion about how an association does not necessarily mean a cause-and-effect relationship should occur. *Note: Association and independence will be formalized in the conditional probability standard S.CP.4 in the Geometry course.*

Agreement-Disagreement Ratio

Students in Grade 8 learn how to find association between two quantitative variables using the Quadrant Count Ratio (QCR). A comparable method is called the Agreement-Disagreement-Ratio (ADR) which can be employed for categorical data in a 2 by 2 table for two Yes-No variables. It is calculated by taking the sum of the agreements minus the sum of the disagreements divided by the total or

ADR $= \frac{(\boldsymbol{a}+\boldsymbol{d})-(\boldsymbol{b}+\boldsymbol{c})}{\boldsymbol{T}}$

Agreement / Disagreement

Question 2

Question 1	**Yes**	**No**	**Row Total**
Yes	a	b	$r_1 = a + b$
No	c	d	$r_2 = c + d$
Column Total	$c_1 = a + c$	$c_2 = b + d$	$T = a + b + c + d$

EXAMPLE

A yes or no survey was given to 144 students. They were asked whether they like soccer and whether they like basketball. The purpose was to see if there is an association between liking the two sports. Find and interpret the ADR based on the results shown in the table.

Discussion: Since the ADR $= \frac{(\mathbf{50}+\mathbf{18})-(\mathbf{35}+\mathbf{41})}{\mathbf{144}} = \frac{\mathbf{68}-\mathbf{76}}{\mathbf{144}} \approx -\mathbf{0.056}$ there is an extremely weak negative association between those who like basketball and those who like soccer.

Agreement / Disagreement

Do you like soccer?

Do you like basketball?	Yes	No	Row Total
Yes	50	35	
No	41	18	
Column Total			

A connection could be made between frequency tables and scatterplots to connect QCR and ADR. See GAISE Report page 96 for more information.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (S.ID.5-6)

ASSOCIATION

Students should investigate problems with more emphasis placed on associations among two or more variables. They should begin to quantify their conclusions by questions such as "How strong or weak is the association?" or "Does the strength of the data association allow us to make any useful predictions?" Students should also start to be able to distinguish between "association" and "cause and effect" when discussing the relationships between variables.

Phi Coefficient (Extension)

Another association that can be used in a 2 by 2 frequency table is the phi coefficient which is comparable to Pearson's correlation coefficient for quantitative data. Whereas the QCR and the ADR are additive in nature calculating "how many?" data values are in each quadrant or cell, Pearson's and Phi's coefficient are multiplicative in nature as they calculate "how far?" the point in each quadrant are from the center point. The Phi coefficient can be found as follows:

Phi $(\boldsymbol{\phi}) = \dfrac{\boldsymbol{ad - bc}}{\sqrt{\boldsymbol{r_1 r_2 c_1 c_2}}}$

QUANTITATIVE BIVARIATE DATA

Continue to use the GAISE model in exploring quantitative bivariate data. Emphasize using mathematical models to capture key elements of the relationship between two variables. Use real-world data and make connections to science. This cluster emphasizes the first two steps of the GAISE model: formulating questions and collecting data. The next two steps will be emphasized in the next cluster (S.ID.7-8).

EXAMPLE

Step 1: Formulate Questions

How strong is the association between the hours spent doing homework and the hours spent playing video games?

Step 2: Collect Data

Use the Census at School Random Sampler, plot the data of the hours spent doing homework and the hours spent playing videogames. Have an informal discussion about sample size and a random sample.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (S.ID.5-6)

In standard S.ID.7, the focus is on two quantitative variables being measured on the same subject. The paired data should be listed and then displayed in a scatterplot. What is being predicted (the dependent variable) is plotted on the vertical axis; the predictor variable (independent variable) is on the horizontal axis. If *time* is one of the variables, it usually is defined as the independent variable, so it typically is plotted on the horizontal axis.

In the numerical or quantitative case, display the paired data in a scatterplot. Quickly review creating the scatterplot and informally fitting the function by hand, but move on to more complex examples that require the use of technology.

Mean—Mean Line Method

To informally fit a linear function to data by hand, students can use the *Mean—Mean Line Method* to gain a conceptual understanding of the regression line.

- Order data from least to greatest based on the *x*-coordinates.
- Divide the data into a "lower half" and an "upper half." (Discard the median point if there is an odd number of data points)
- Determine the mean of the *x*-coordinates of both the "upper half" and the "lower half" of the data set.
- Determine the mean of the *y*-coordinates of both the "upper half" and the "lower half" of the data set.
- Have students write the equation of the line using the two points.
- Have students compare their regression line to one generated by technology such as a graphing calculator.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (S.ID.5-6)

EXAMPLE

An ice cream store collected data to compare the amount of sales to the daily temperature to see if there is an association.

- **a.** Graph the data on a scatterplot and fit a function to the data set.
- **b.** Order the data from least to greatest based on *x*-coordinates. (See table to the right.)
- **c.** Calculate the means of the upper and lower half of the data set.

Temp (F°)	Sales ($)	
32	250	
36	325	
38	300	
38	330	Upper Half
40	350	
42	345	
45	400	
49	415	Discarded
52	500	
56	525	
63	550	
65	525	Lower Half
66	560	
69	600	
72	650	

Lower Half (7pts)	Upper Half (7pts)
Mean Temp = 38.7	Mean Temp = 63.3
Mean Sales = 328.6	Mean Sales = 558.6

- **d.** Write an equation of a line using the two points (38.7, 328.6) and (63.3, 558.6) and then fit that line to the data.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{558.6 - 328.6}{63.3 - 38.7} = \frac{230}{24.6} \approx 9.35$$

$$y - y_1 = m(x - x_1)$$
$$y - 328.6 = 9.35(x - 38.7)$$
$$y - 328.6 = 9.35x - 361.845$$
$$y = 9.35x - 33.245$$

- **e.** Use technology to graph and compare the mean-mean line of fit you calculated with the actual line of best fit: $y = 9.05x - 18.41$**.** When you graph both lines on the scatterplot, they should be very similar.
- **f.** Using both equations, predict the sales when the temperature is 80°F. How do your values differ?
- **g.** Using both equations predict what temperature it would have to be in order to generate $540 in sales. How do your values differ?

The focus of this cluster is not necessarily on writing equations of lines, which is covered in F.LE.1-2. (Although it could be an application of the concept.) Rather, students should be able to pick the best model by approximating the slope and *y*-intercept and be able to informally justify this choice in context. Students should progress to more complex data sets requiring the use of technology to generate the best linear model.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (S.ID.5-6)

Fitting functions to such data will help students avoid difficulties such as the interpretation of a slope when the scales are different, for example total SAT score versus grade-point average. Once students are comfortable with the same scale cases, introducing different scales in situations becomes less problematic.

Students may incorrectly believe that a 45-degree line in the scatterplot of two quantitative variables always indicates a slope of 1. However this is only the case when the two variables have the same scaling.

Although examples should be limited to linear models, include situations where data are not best represented by a linear model. Then have an informal discussion about how a linear model is not always the best fit for a data set. This will set the foundation for learning in Algebra 2 where students use quadratic and exponential function models in addition to linear models to make sense of scatterplots and data displays.

Misleading Graphs

Students should analyze misleading scatterplots. See cluster S.ID.1-3 for more information about misleading graphs.

Instructional Tools/Resources

These tools and resources may be helpful for instruction. Before use, they should be evaluated at the local level to determine their appropriateness for instruction.

Manipulatives/Technology

- Excel/Google Sheets
- Graphing calculators.
- Desmos is a free graphing calculator that is available to students as website or an app.
- Geogebra is a free graphing calculator that is available to students as website.
- Wolframalpha is dynamic computing tool.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (S.ID.5-6)

Two-Way Frequency Tables

- First Day Statistics Activity—Grouping Qualitative Data is a lesson by R.B. Campbell from Statistics Education Web (STEW) published in 2014 that explicitly aligns to the GAISE model.
- You Will Soon Analyze Categorical Data (Classifying Fortune Cookies Fortunes) is a lesson by Mary Richardson from Statistics Education Web (STEW) published in 2014 that explicitly aligns to the GAISE model. Students will construct two-way frequency tables and investigate their results using joint relative frequencies and marginal and conditional distributions.
- A Sweet Task by Fiedler, Huey, Jenkins, and Flinspach from Statistics Education Web (STEW) published in 2014 that explicitly aligns to the GAISE model. Students will create two-way frequency tables and interpret relative frequencies using M&M's.
- Saga of Survival (Using Data about the Donner Party to Illustrate Descriptive Statistics) by Richardson and Rogness from Statistics Education Web (STEW) published in 2013 that explicitly aligns to the GAISE model. Students explore survival rates of emigrants trapped in the Sierra Nevada winter of 1846-1847 using two-way frequency tables and box plots.
- Support for a Longer School Day? is a task by Illustrative Mathematics that provides students with an opportunity to calculate and interpret joint, marginal, and conditional relative frequencies.
- Musical Preferences is a task by Illustrative Mathematics that explores association using two-way frequency tables.
- The Case of the Careless Zookeeper by Malloure, Richardson, and Rogness from Statistics Education Web (STEW) published in 2013 that explicitly aligns to the GAISE model. This is an extension activity on two-way frequency tables that uses a chi-square test of independence.

Scatterplots

- Text Messaging Is Time Consuming! What Gives? by Gibson, McNelis, Bargagliotti, and Project-SET from Statistics Education Web (STEW) published in 2013 that explicitly aligns to the GAISE model. Students use the Census at School Data to create and interpret scatterplots.
- NFL Quarterback Salaries by Gibson, Bargagliotti, and Project-SET from Statistics Education Web (STEW) published in 2013 that explicitly aligns to the GAISE model. Students use scatterplots to determine which variables are the bet predictor of an NFL player's salary.
- How High Can You Jump? by Diann Reischmann from Statistics Education Web (STEW) published in 2012 that explicitly aligns to the GAISE model. Students use scatterplots and box plots to summarize the data.
- Devising a Measure: Correlation is a task by Mathematics Assessment Project involving students working with correlation involving a drive-in movie theater.
- EllipSeeIT: Visualizing Strength and Direction of Correlation is a lesson by Olvera, Dias, and Colleagues from the Statistics Education Web (STEW) published in February 2017 that explicitly aligns to the GAISE model. This lesson focuses on correlation as a way of measuring association. Students will then collect data about themselves to create a scatterplot.
- Correlation Guessing Game is an applet by Rossman/Chance where students can guess the correlation of various scatterplots.
- Analyzing Two Quantitative Variables in an applet by Rossman/Chance that allows students to analyze two quantitative variables on a scatterplot.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (S.ID.5-6)

Scatterplots, continued

- Applying Correlation Coefficients—Educational Attainment and Unemployment by the United States Census Bureau is a lesson that uses scatterplots to explore the relationship between educational level and unemployment.
- Educational Attainment and Marital Age by the United States Census Bureau is a lesson that explores the relationship between educational attainment and marriage age.

Curriculum and Lessons from Other Sources

- EngageNy's Algebra 1, Module 2: Topic C, Lesson 9: Summarizing Bivariate Categorical Data, Lesson 10: Summarizing Bivariate Categorical Data with Relative Frequencies, and Lesson 11: Conditional Relative Frequencies and Association are lessons involving categorical data and two-way frequency tables.
- EngageNy's Algebra 1, Module 2: Topic D, Lesson 12: Relationships Between Two Numerical Variables and Lesson 14: Modeling Relationships with a Line are lessons involving scatterplots and the least-squares regression line.
- Mathematics Vision Project, Algebra 1, Module 9: Modeling Data has several tasks that pertain to this cluster.
- Georgia Standards of Excellence Curriculum Framework, Algebra 1, Unit 6: Describing Data has many lessons that pertain to this cluster. This cluster is addressed on pages 39-106.
- Exploring Linear Relations by Burrill and Hopfensperger is a pdf version of a textbook from the Data-Driven Mathematics series published by Dale Seymour Publications. It has some lessons that pertain to this cluster. The student edition is found here.

General Resources

- Arizona's High School Progression on Statistics and Probability is an informational document for teachers. This cluster is addressed on pages 4-6.
- Arizona's High School Progression on Modeling is an informational document for teachers. Statistics and Probability is discussed on page 10.
- High School Coherence Map by UnboundED is a map of the relationships among high school mathematics standards. Please note: This resource is aligned to the Common Core not Ohio's Learning Standards. Although the tool is useful, it needs to be checked with respect to Ohio's Learning Standards.
- Statistics Teacher is an online journal published by the American Statistical Association (ASA) and NCTM. It includes articles, lesson plans, and professional development opportunities.
- Significance is a magazine that demonstrate the practical use of statistics and shows how statistics benefits society.
- Chance is a magazine about statistics for anyone who is interested in using data to advance science, education, and society.
- Levels of Conceptual Understanding in Statistics (LOCUS) is an NSF funded project that has assessment questions around statistical understanding.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (S.ID.5-6)

References

- Common Core Standards Writing Team. (2016, March 24). *Progressions for the Common Core State Standards in Mathematics (draft). Grades 6-8, Statistics and Probability*. Tucson, AZ: Institute for Mathematics and Education, University of Arizona.
- Common Core Standards Writing Team. (2013, July 4). *Progressions for the Common Core State Standards in Mathematics (draft). High School, Modeling*. Tucson, AZ: Institute for Mathematics and Education, University of Arizona.
- Franklin, C., Kader, G., Mewborn, D, Moreno, J., Peck, R., Perry, M., & Scheaffer, R. (2007). *Guidelines for Assessment and Instruction in Statistics Education (GAISE) Report*. Alexandria, VA: American Statistical Association.

STANDARDS	MODEL CURRICULUM (S.ID.7-8)

Statistics and Probability
INTERPRETING CATEGORICAL AND QUANTITATIVE DATA
Interpret linear models.
S.ID.7 Interpret the slope (rate of change) and the intercept (constant term) of a linear model in the context of the data. ★
S.ID.8 Compute (using technology) and interpret the correlation coefficient of a linear fit. ★
	Expectations for Learning
	In middle school, students interpret the slope and *y*-intercept of a linear model. In Algebra 1/Math 1, students build on this knowledge with more sophisticated problems. Since scales may vary, students require a deeper conceptual understanding of slope. They also need to recognize when the *y*-intercept is not always meaningful in the context of the data. This leads to the computation and interpretation of the correlation coefficient and its interpretation. In Algebra 2/Math 3, students are introduced to and explore the distinction between correlation and causation.
	
	The learning of standard S.ID.7 is at the developmental Level B. The learning of standard S.ID.8 is at developmental Level C. See pages 62-63 for more information on Level B, and see Algebra 2 Model Curriculum for more information on Level C.
	
	ESSENTIAL UNDERSTANDINGS
	• In a linear model, the slope represents the change in the predicted value for every one unit of increase in the independent (*x*) variable.
	• When appropriate, the *y*-intercept represents the predicted value of the dependent variable when $x = 0$.
	• In a linear model, the *y*-intercept may not always be appropriate for the context.
	• The correlation coefficient (*r*) is a measure of the strength of a linear association in the data. Correlation coefficients are between –1 and 1, inclusive.
	  o If *r* is close to 0, then there is a weak correlation.
	  o If *r* is close to 1, then there is a strong correlation with a positive slope.
	  o If r is close to –1, then there is a strong correlation with a negative slope.
	Continued on next page

STANDARDS	MODEL CURRICULUM (S.ID.7-8)
S.ID.7-8, continued	**Expectations for Learning, continued** **MATHEMATICAL THINKING** • Use accurate and precise mathematical vocabulary. • Construct formal and informal arguments to verify claims and justify conclusions. • Solve real-world and statistical problems. • Use appropriate tools to display and analyze data. • Determine reasonableness of predictions. **INSTRUCTIONAL FOCUS** • Given a linear model, interpret the slope and the *y*-intercept within a context. • Compute, with technology, and interpret correlation coefficient (*r*). **Content Elaborations** **OHIO'S HIGH SCHOOL CRITICAL AREAS OF FOCUS** • Algebra 1, Number 3, page 8 **THE GAISE MODEL** • GAISE Model, pages 14 – 15 o The focus of S.ID.7 is at Level B for Algebra 1/Math 1, pages 37-60 o The focus of S.ID.8 is at Level C for Algebra 1/Math 1, pages 61-85 **CONNECTIONS ACROSS STANDARDS** • Interpret the structure of functions (F.IF.4). • Construct linear models, and solve problems (F.LE.1-2). • Interpret expressions for functions in terms of context (F.LE.5). • Build a function that models a relationship between two quantities (F.BF.1). • Summarize, represent, and interpret data in two categories and quantitative variables (S.ID.6).

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (S.ID.7-8)

Instructional Strategies

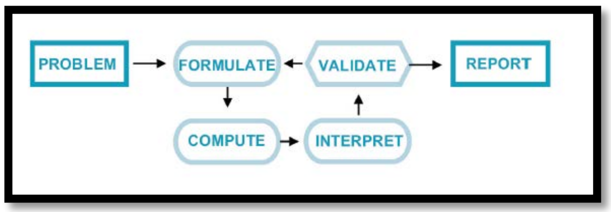
Note: The model curriculum is not a collection of lessons nor a full curriculum; it does not suggest pace, sequence, or amount of time spent on topics. It provides information about a topic related to the standards including ideas for examples, strategies for teaching, common misconceptions, and possible connections between topics. In addition, the emphasis of the model curriculum is on instruction not on assessment; therefore, the examples in the Instructional Strategies section are written for instructional purposes not necessarily as assessment items. Examples may need to be rewritten to accommodate the needs of each individual classroom.

Standards for Mathematical Practice
This cluster focuses on but is not limited to the following practices:
MP.4 *Model with mathematics.*
MP.6 *Attend to precision.*
MP.7 *Look for and make use of structure.*

As this cluster is a continuation of statistical thinking in both 8.SP.3 and S.ID.6, continue to use the GAISE model in exploring quantitative bivariate data. Emphasize using mathematical models to capture key elements of the relationship between two variables. Use real-world data and make connections to science. This cluster emphasizes the last two steps of the GAISE model: analyze data and interpret results. The first two steps were emphasized in the previous cluster (S.ID.6) with respect to quantitative data.

MODELING

S.ID.7-8 is a modeling standards. See page 13 for more information about modeling.



INTERPRETING THE SLOPE AND THE INTERCEPT

It will be helpful for students to write models in the context of the data. For example, when simply writing $y = 20x + 10$ to represent the height of a hot air balloon over time, students may struggle to make the connections between the variables and the model. An alternative strategy would be to write the model as follows:

$$\text{height (feet) of balloon} = 20\,\frac{feet}{minute}\cdot(x\ \text{minutes}) + 10\ \text{feet}$$

The interpretation of the slope is an increase of 20 feet for every additional minute. The interpretation of the y-intercept would be that the balloon is at a height of 10 feet above the ground when time is equal to 0.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (S.ID.7-8)

In some cases, the interpretation of the *y*-intercept (constant term) is not meaningful in the context of the data. This happens when the zero point on the horizontal axis is of considerable distance from the initial values of the horizontal variable, or in some case has no meaning at all. For example, if the horizontal axis measures student heights, it is impractical to reason that a person would have a height of zero. In Grade 8, students interpret the slope and *y*-intercept of a linear model, including situations when the *y*-intercept does not have an appropriate interpretation in context. Now since students have the vocabulary of "domain", they can explicitly state that when a *y*-intercept does not have an appropriate interpretation, 0 is excluded from the domain of the model, whereas in 8th grade they could not yet do so.

A visual rendering of slope makes no sense in most scatterplots. For example, students may believe that a 45-degree line in scatterplot of two numerical variables always indicates a slope of 1 which is the case only when the two variables have the same scaling. Because the scaling for many real-world situations varies greatly, students need to be given opportunities to compare graphs of differing scales. Asking students questions like "What would this graph look like with a different scale or using this scale?" is essential in addressing this misconception.

CORRELATION COEFFICIENT

The correlation coefficient, *r*, measures "tightness" of the data points to a line, and the strength of the linear relationship between the two variables. Computing the formula for *r* is long and cumbersome, so students need to find the correlation coefficient using technology. The focus is on interpreting the value, not on calculating the value.

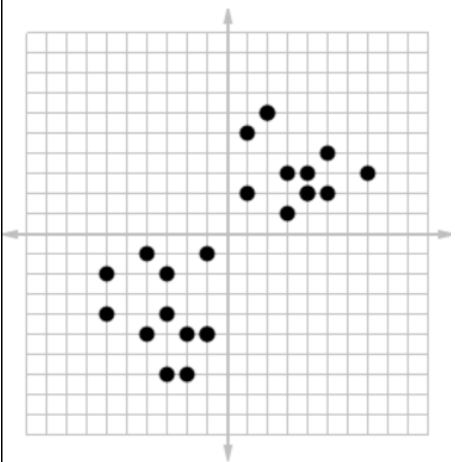
To make some sense of Pearson's r, correlation coefficient, students should recall their middle school experience with the Quadrant Count Ratio (QCR) as a measure of the relationship between two quantitative variables (See Grade 8 Model Curriculum 8.SP.1-4). The difference between QCR and Pearson's correlation coefficient is that the QCR is only based upon how many points are in each quadrant whereas Pearson's looks at how far each point is from the dividing line of each quadrant. To make the connection for students, the QCR should be reviewed and its limitations should be discussed. The primary weakness of the QCR is that all points have the same weight whether or not they are far or close to the line of best fit. Another drawback is that just because the QCR = ± 1, the relationship between the two variables is not necessarily perfectly linear. In addition, the QCR can even be ± 1 when it is not perfectly positive. Even though students are not required to calculate Pearson's coefficient, there should be some discussion in the classroom on how it developed. Resources such as The Evolution of Pearson's Correlation Coefficient/Exploring the Relationships Between Two Quantitative Variables by Gary Kader from the Science Education Resource Center and Carleton College and The Evolution of Pearson's Correlation Coefficient can help guide the discussion.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (S.ID.7-8)

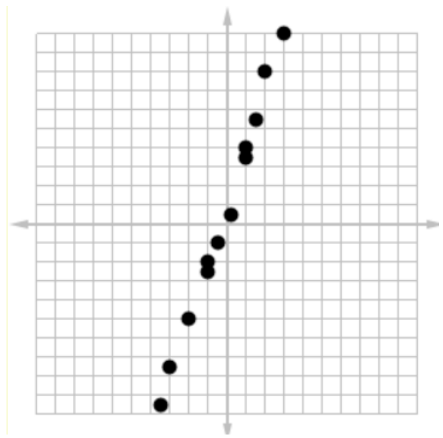
EXAMPLE

Have students discuss something similar to the following to graphs.

Graph A



Graph B



a. Calculate the QCR for both graphs. What do you notice?

b. However, is one graph more positively correlated than the other? Explain.

c. How could we adjust the QCR formula to give more weight to the points that are closer to the regression line?

Discussion: Although, it is unlikely that student's will come up with the formula for r, this activity will help them see the need for such a formula and develop an intuitive understanding of what exactly r is calculating. To extend the discussion, the development of Pearson's formula could be introduced as a discussion point, but students should be expected to do the actual calculations via technology.

Explain that in order for the correlation coefficient to be exactly 1 or –1, the data points must fall exactly on the line of best fit. The closer r is to ± 1, the stronger the correlation. The closer r is to 0, the weaker the correlation. However, to be noted getting a correlation coefficient of exactly 0 is extremely unlikely.

A huge benefit for using r squared (instead of r) is that the values of –1 and 1 separately do not need to be discussed separately. Instead, an r squared value close to 1 is a strong correlation, while an r squared value close to 0 is a weak correlation.

Students may incorrectly think that r must have units, and that the units relate to the context of the problem. Instead, the only way to relate r to the data is to use it to measure the strength and the direction of the relationship.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (S.ID.7-8)

UNDERSTANDING VARIABILITY

Students should explore the differences between variations and errors. Teachers can draw attention to naturally occurring errors that happen in the classroom and discuss the impact of the errors on the results. These notions can be used to explain outliers, clusters, and gaps. Understanding variability is vital for developing data sense (also in S.ID.1-3 and 8.SP.1-4).

At Level B students start to develop the ideas of covariability, induced variability, and sampling variability. See GAISE model pages 6 and 20.

ASSOCIATION

The correlation coefficient is a quantity that measure the strength and direction of an association. Students in 8th grade learn about the Quadrant Count Ratio (QCR) which allows students to have a conceptual understanding of correlation coefficient. Students need to have an informal understanding that an association present between two variables does not necessarily mean that there is a cause and effect relationship.

Instructional Tools/Resources

These tools and resources may be helpful for instruction. Before use, they should be evaluated at the local level to determine their appropriateness for instruction.

Manipulatives/Technology

- Excel/Google Sheets
- Graphing calculators
- Desmos is a free graphing calculator that is available to students as website or an app.
- Geogebra is a free graphing calculator that is available to students as website.
- Wolframalpha is dynamic computing tool.
- Correlation and Regression is a statistical applet that allows students to explores why the correlation and least-squares regression line changes as points are added or deleted from a scatterplot.

Interpreting Linear Models

- Coffee and Crime is a task by Illustrative Mathematics where students analyze bivariate quantitative data.
- Used Suburban Foresters II is a task by Illustrative Mathematics that emphasizes that regression lines always have a context by interpreting the slope and the intercept.
- Statistics: Does a Correlation Exist? is an activity by Texas Instruments to determine if a data set has a positive or negative correlation coefficients.
- Olympic Men's 100-meter Dash is a task by Illustrative Mathematics that has students describe the strength and direction of an association.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (S.ID.7-8)

Interpreting Linear Models, continued

- Text Messaging is Time Consuming! What Gives? is a lesson by Gibson, McNelis, and Bargagliotti from the Statistics Education Web (STEW) published in August 2013. It allows students to explore relationships between two variables using data from the Census at School project. They explore variability in slope, intercepts, and correlations.
- NFL Quarterback Salaries is a lesson by Gibson and Bargagliotti from the Statistics Education Web (STEW) published in July 3013. Students will set up a statistical question and interpret a linear regression equation and the correlation coefficient.
- Guessing Correlations in an applet by Stat.istics.net where students have to match correlations with the scatterplots.
- Spurious Correlations by Tyler Vigen is a website that shows unrelated correlations. Although truly understanding the difference between correlation and causation is not required until Algebra 2, an informal discussion about association and cause-and-effect could take place.
- From the NCTM Illuminations, Impact of a Superstar is a lesson that uses technology tools to plot data, identify lines of best fit, and detect outliers. Then, students compare the lines of best fit when one element is removed from a data set, and interpret the results. *NCTM now requires a membership to view their lessons.*
- iPhone 6s Opening Weekend Sales by Clifford Pate is a Desmos lesson paired with Will the New iPhone Sales Be Huge? from Yummy Math has students make predictions about the number of iPhone units sold.
- Line of Best Fit by Desmos has students visualize a line to fit a data set, then graph that line with sliders and make predictions.

Curriculum and Lessons from Other Sources

- Algebra 1, Module 9: Modeling Data is a unit by the Mathematics Vision Project. Tasks 9.5, 9.6, and 9.7 align to this cluster.
- EngageNY Algebra 1 Module 2, Topic D, Lesson 19 is a lesson about interpreting the correlation coefficient as a measure of strength.

General Resources

- Arizona's High School Progression on Statistics and Probability is an informational text for teachers. This cluster is addressed on pages 6-7.
- Arizona's High School Progression on Modeling is an informational text for teachers. This cluster is addressed on pages 2, 5, and 14.
- High School Coherence Map by UnboundED is a map of the relationships among high school mathematics standards. Please note: This resource is aligned to the Common Core not Ohio's Learning Standards. Although the tool is useful, it needs to be checked with respect to Ohio's Learning Standards.
- Statistics Teacher is an online journal published by the American Statistical Association (ASA) and NCTM. It includes articles, lesson plans, and professional development opportunities.
- Significance is a magazine that demonstrate the practical use of statistics and shows how statistics benefits society.
- Chance is a magazine about statistics for anyone who is interested in using data to advance science, education, and society.
- K-12 Statistics Education Resources is a webpage put together by the American Statistics Association to provide resources for K-12 educators.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (S.ID.7-8)

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- Common Core Standards Writing Team. (2013, July 4). *Progressions for the Common Core State Standards in Mathematics (draft). High School, Modeling*. Tucson, AZ: Institute for Mathematics and Education, University of Arizona.
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- Kader, G. & Franklin, C. (November 2008). The evolution of Pearson's Correlation Coefficient. *Mathematics Teacher*, *102(4),* 292-299.

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