Ohio's Model Curriculum | Mathematics

with Instructional Supports

Geometry Course

Ohio | Department of Education

Mathematics Model Curriculum
with Instructional Supports
Geometry Course

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Introduction

PURPOSE OF THE MODEL CURRICULUM

Just as the standards are required by Ohio Revised Code, so is the development of the model curriculum for those standards. Throughout the development of the standards (2016-17) and the model curriculum (2017-18), the Ohio Department of Education (ODE) has involved educators from around the state at all levels, Pre-K–16. The model curriculum reflects best practices and the expertise of Ohio educators, but it is not a complete a curriculum nor is it mandated for use. The purpose of Ohio's model curriculum is to provide clarity to the standards, a foundation for aligned assessments, and guidelines to assist educators in implementing the standards. The model curriculum is not a collection of lessons nor a full curriculum; it does not suggest pace, sequence, or amount of time spent on topics. It provides information about a topic related to the standards including ideas for examples, strategies for teaching, and possible connections between topics in addition to highlighting some misconceptions.

COMPONENTS OF THE MODEL CURRICULUM

The model curriculum contains two sections: Expectations for Learning and Content Elaborations.

Expectations for Learning: This section begins with an introductory paragraph describing the cluster's position in the respective learning progression, including previous learning and future learning. Following are three subsections: Essential Understandings, Mathematical Thinking, and Instructional Focus.

- **Essential Understandings** are the important concepts students should develop. When students have internalized these conceptual understandings, application and transfer of learning results.
- **Mathematical Thinking** statements describe the mental processes and practices important to the cluster.
- **Instructional Focus** statements are key skills and procedures students should know and demonstrate.

Together these three subsections guide the choice of lessons and formative assessments and ultimately set the parameters for aligned state assessments.

Content Elaborations: This section provides further clarification of the standards, links the critical areas of focus, and connects related standards within a grade or course.

Introduction, continued

COMPONENTS OF INSTRUCTIONAL SUPPORTS

The Instructional Supports section contains the **Instructional Strategies** and **Instructional Tools/Resources** sections which are designed to be fluid and improving over time, through additional research and input from the field. The **Instructional Strategies** are descriptions of effective and promising strategies for engaging students in observation, exploration, and problem solving targeted to the concepts and skills in the cluster of standards. Descriptions of common misconceptions as well as strategies for avoiding or overcoming them and ideas for adapting instructions to meet the needs of all students are threaded throughout. The **Instruction Tools/Resources** are links to relevant research, tools, and technology.

There are several icons that help identify various tips in the Instructional Strategies section:

= a common misconception

= a technology tip

= a career connection

TIP! = a general tip which may include diverse learner or English language learner tips.

Standards for Mathematical Practice—Geometry

The Standards for Mathematical Practice describe the skills that mathematics educators should seek to develop in their students. The descriptions of the mathematical practices in this document provide examples of how student performance will change and grow as students engage with and master new and more advanced mathematical ideas across the grade levels.

MP.1 Make sense of problems and persevere in solving them.

Students construct accurate diagrams of geometry problems to help make sense of them. They organize their work so that others can follow their reasoning, e.g., in proofs.

MP.2 Reason abstractly and quantitatively.

Students understand that the coordinate plane can be used to represent geometric shapes and transformations, and therefore they connect their understanding of number and algebra to geometry.

MP.3 Construct viable arguments and critique the reasoning of others.

Students use formal and informal proofs to verify, prove, and justify geometric theorems with respect to congruence and similarity. These proofs can include paragraph proofs, flow charts, coordinate proofs, two-column proofs, diagrams without words, indirect proofs, or the use of dynamic software.

MP.4 Model with mathematics.

Students apply their new mathematical understanding to real-world problems. They learn how transformational geometry and basic trigonometric functions can be used to model the physical world.

MP.5 Use appropriate tools strategically.

Students make use of visual tools for representing geometry, such as simple patty paper, transparencies, or dynamic geometric software.

MP.6 Attend to precision.

Students develop and use precise definitions of geometric terms. They verify that a particular shape has specific properties and justify the categorization of the shape, e.g., a rhombus versus a quadrilateral.

MP.7 Look for and make use of structure.

Students construct triangles in quadrilaterals or other shapes and use congruence criteria of triangles to justify results about those shapes.

MP.8 Look for and express regularity in repeated reasoning.

Students explore rotations, reflections, and translations, noticing that some attributes of shapes (e.g., parallelism, congruency, orientation) remain the same. They develop properties of transformations by generalizing these observations.

Modeling

Modeling links classroom mathematics and statistics to everyday life, work, and decision-making. Modeling is the process of choosing and using appropriate mathematics and statistics to analyze empirical situations, to understand them better, and to improve decisions. Quantities and their relationships in physical, economic, public policy, social, and everyday situations can be modeled using mathematical and statistical methods. When making mathematical models, technology is valuable for varying assumptions, exploring consequences, and comparing predictions with data.

A model can be very simple, such as writing total cost as a product of unit price and number bought, or using a geometric shape to describe a physical object like a coin. Even such simple models involve making choices. It is up to us whether to model a coin as a three-dimensional cylinder, or whether a two-dimensional disk works well enough for our purposes. Other situations—modeling a delivery route, a production schedule, or a comparison of loan amortizations—need more elaborate models that use other tools from the mathematical sciences. Real-world situations are not organized and labeled for analysis; formulating tractable models, representing such models, and analyzing them is appropriately a creative process. Like every such process, this depends on acquired expertise as well as creativity.

Some examples of such situations might include the following:

- Estimating how much water and food is needed for emergency relief in a devastated city of 3 million people, and how it might be distributed.
- Planning a table tennis tournament for 7 players at a club with 4 tables, where each player plays against each other player.
- Designing the layout of the stalls in a school fair so as to raise as much money as possible.
- Analyzing the stopping distance for a car.
- Modeling a savings account balance, bacterial colony growth, or investment growth.
- Engaging in critical path analysis, e.g., applied to turnaround of an aircraft at an airport.
- Analyzing risk in situations such as extreme sports, pandemics, and terrorism.
- Relating population statistics to individual predictions.

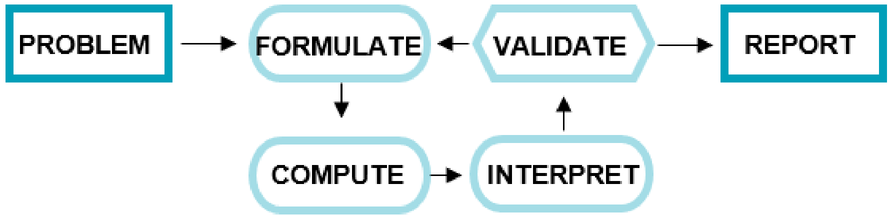
In situations like these, the models devised depend on a number of factors: How precise an answer do we want or need? What aspects of the situation do we most need to understand, control, or optimize? What resources of time and tools do we have? The range of models that we can create and analyze is also constrained by the limitations of our mathematical, statistical, and technical skills, and our ability to recognize significant variables and relationships among them. Diagrams of various kinds, spreadsheets and other technology, and algebra are powerful tools for understanding and solving problems drawn from different types of real-world situations.

Continued on next page

Modeling, continued

One of the insights provided by mathematical modeling is that essentially the same mathematical or statistical structure can sometimes model seemingly different situations. Models can also shed light on the mathematical structures themselves, for example, as when a model of bacterial growth makes more vivid the explosive growth of the exponential function.

The basic modeling cycle is summarized in the diagram. It involves (1) identifying variables in the situation and selecting those that represent essential features, (2) formulating a model by creating and selecting geometric, graphical, tabular, algebraic, or statistical representations that describe relationships between the variables, (3) analyzing and performing operations on these relationships to draw conclusions, (4) interpreting the results of the mathematics in terms of the original situation, (5) validating the conclusions by comparing them with the situation, and then either improving the model or, if it is acceptable, (6) reporting on the conclusions and the reasoning behind them. Choices, assumptions, and approximations are present throughout this cycle.



In descriptive modeling, a model simply describes the phenomena or summarizes them in a compact form. Graphs of observations are a familiar descriptive model—for example, graphs of global temperature and atmospheric CO_2 over time.
Analytic modeling seeks to explain data on the basis of deeper theoretical ideas, albeit with parameters that are empirically based; for example, exponential growth of bacterial colonies (until cut-off mechanisms such as pollution or starvation intervene) follows from a constant reproduction rate. Functions are an important tool for analyzing such problems. Graphing utilities, spreadsheets, computer algebra systems, and dynamic geometric software are powerful tools that can be used to model purely mathematical phenomena, e.g., the behavior of polynomials as well as physical phenomena.

Modeling Standards

Modeling is best interpreted not as a collection of isolated topics but rather in relation to other standards. Making mathematical models is a Standard for Mathematical Practice, and specific modeling standards appear throughout the high school standards indicated by a star symbol (★).

Mathematics Model Curriculum

with Instructional Supports

Geometry Course

STANDARDS	MODEL CURRICULUM (G.CO.1-5)
Geometry **CONGRUENCE** **Experiment with transformations in the plane.** **G.CO.1** Know precise definitions of ray, angle, circle, perpendicular line, parallel line, and line segment, based on the undefined notions of point, line, distance along a line, and arc length. **G.CO.2** Represent transformations in the plane using, e.g., transparencies and geometry software; describe transformations as functions that take points in the plane as inputs and give other points as outputs. Compare transformations that preserve distance and angle to those that do not, e.g., translation versus horizontal stretch. **G.CO.3** Identify the symmetries of a figure, which are the rotations and reflections that carry it onto itself. **a.** Identify figures that have line symmetry; draw and use lines of symmetry to analyze properties of shapes. **b.** Identify figures that have rotational symmetry; determine the angle of rotation, and use rotational symmetry to analyze properties of shapes. *Continued on next page*	**Expectations for Learning** In middle school, students first learn about the basic rigid motions (translations, rotations, and reflections) and verify their properties experimentally. In this cluster, students formalize the notion of a transformation as a function from the plane to itself. Building on their hands-on work, students develop mathematical definitions of the basic rigid motions. These definitions serve as a logical basis for the theorems that students prove in Geometry. An important step in high school is to perform appropriate transformations and give precise descriptions of sequences of basic rigid motions that carry one figure onto another. Transformations provide language to be precise about symmetry; this is the first-time students have encountered formal symmetry. The student understanding of this cluster begins at van Hiele Level 1 (Analysis) and moves to Level 2 (Informal Deduction/Abstraction). **ESSENTIAL UNDERSTANDINGS** • A transformation is a function from the plane to itself; input and output values are points, not numbers. • Rigid motions are transformations that preserve distance and angle. • Some transformations preserve distance and angle measures, and some do not. • In order to perform a translation, a distance and a direction is required. • A rotation requires a center and an angle. • A reflection requires a line. • The symmetries of a figure are the transformations that carry the figure onto itself. *Continued on next page*

STANDARDS	MODEL CURRICULUM (G.CO.1-5)
G.CO.4 Develop definitions of rotations, reflections, and translations in terms of angles, circles, perpendicular lines, parallel lines, and line segments. **G.CO.5** Given a geometric figure and a rotation, reflection, or translation, draw the transformed figure using items such as graph paper, tracing paper, or geometry software. Specify a sequence of transformations that will carry a given figure onto another.	**Expectations for Learning, continued** **MATHEMATICAL THINKING** • Use accurate and precise mathematical vocabulary and symbolic notations. • Make connections between concepts, terms, and properties within the grade level and with previous grade levels. • Recognize, apply, and justify mathematical concepts, terms, and their properties. **INSTRUCTIONAL FOCUS** • Know precise definitions of basic terms: ray, angle, circle, perpendicular line, parallel line, and line segment. • Develop and use appropriate geometric notation. • Formalize definitions of basic rigid motions (translations, rotations, and reflections). • Perform and identify transformations using a variety of tools. • Identify the symmetries shown in a figure (rotational and line symmetries). **Content Elaborations** **OHIO'S HIGH SCHOOL CRITICAL AREAS OF FOCUS** • Geometry, Number 2, page 4-5 **CONNECTIONS ACROSS STANDARDS** • Understand congruence in terms of rigid motion (G.CO.6-8). • Prove and apply geometric theorems (G.CO.9). • Make formal geometric constructions (G.CO.12). • Justify the slope criteria for parallel and perpendicular lines (G.GPE.5). • Understand similarity in terms of similarity transformations (G.SRT.1-2).

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.CO.1-5)

Instructional Strategies

Note: The model curriculum is not a collection of lessons nor a full curriculum; it does not suggest pace, sequence, or amount of time spent on topics. It provides information about a topic related to the standards including ideas for examples, strategies for teaching, common misconceptions, and possible connections between topics. In addition, the emphasis of the model curriculum is on instruction not on assessment; therefore, the examples in the Instructional Strategies section are written for instructional purposes not necessarily as assessment items. Examples may need to be rewritten to accommodate the needs of each individual classroom.

Standards for Mathematical Practice

This cluster focuses on but is not limited to the following practices:

***MP.1** Make sense of problems and persevere in solving them.*

***MP.4** Model with mathematics.*

***MP.5** Use appropriate tools strategically.*

***MP.6** Attend to precision.*

There are many approaches to teaching the concepts of congruence and similarity. The standards define congruence and similarity in terms of transformations. This allows the concepts to be grounded in hands-on experiences. In Grade 8, students should have used many manipulatives to *explore* geometric concepts and definitions. Now in high school, students will use transformations to *prove* geometric concepts and experience definitions. Rigid motions will be used to lay the foundation of proving theorems.

Effective teaching requires students to use a variety of geometric tools, such as graph paper, transparencies, tracing paper, dynamic geometric software, straightedge, compass, and protractor to obtain images of a given figure under specified transformations both on and off the coordinate plane. Different tools lead to different understandings. For example, transparencies are especially useful because two copies of the plane can be seen (the paper and the transparency). The original piece of paper can be easily seen as the domain and the transparency can be seen as the range. The preservation of distance and angle can also be clearly seen using a transparency since the transparency is not torn, stretched, or distorted. Emphasize to students that a transformation acts on the entire plane mapping each point to another corresponding point.

VAN HIELE CONNECTION

In Geometry students are expected to move from Level 1(Analysis) to Level 2 (Informal Deduction/Abstraction). In van Hiele Level 1 students use coordinates to examine transformations and recognize, use, and describe properties of transformations. Whereas van Hiele Level 2 can be characterized by the student doing some or all of the following:

- analyzing relationships within and between transformations;
- algebraically expressing transformations: reflections about the *x*- and *y*-axes, and rotations of 45°, 90°, and 180° about the origin; *Note: In high school, students should be given an opportunity to experience rotations around any point (not just the origin) and onto any angle, not just multiples of 90. Also, reflections are not limited to the x-and y-axes but could be about any line.*
- understanding and justifying how to use perpendicular bisectors of corresponding points to locate the turn center of a rotation;
- understanding how the composition of two reflections can be a translation or a rotation; and/or
- following a rationale if shown proof.

See the van Hiele pdf on ODE's website for more information about van Hiele levels.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.CO.1-5)

DEFINING GEOMETRIC TERMS

One of the major differences between geometric concepts in middle school and geometry concepts in high school is the usage of precise language and exact definitions. Standard G.CO.1 fits nicely alongside the cluster G.CO.12-13 where students make geometric constructions. Remind students that although they may be familiar with many geometric concepts, those taught in high school require greater precision with respect to definitions.

TIP! Many students have trouble distinguishing between undefined terms, definitions, properties, axioms, formulas etc. Helping students understand the difference between them will aid in the use of precise language.

Explain to students that multiple definitions can exist for a given object. For example, both a rectangle with all equal sides and a rhombus with four right angles are definitions that can be used to define a square. A correct definition will encompass all examples of squares and exclude all examples of non-squares.

Precise Definitions Based Upon Undefined Notions

Some definitions in mathematics are precisely defined, while others are built upon undefinable concepts (notions) such as a point, line, plane, distance along a line, betweenness, space, and arc length etc. They are concepts (although representations can be drawn) that only exist as an idea or mental image. For example, a point is a location in space. It has no dimension (length, area, or volume). Although, to convey the idea of a point, it is typically drawn (thus giving it a perceived dimension). Undefinable concepts are the building blocks needed for creating precise definitions.

Students incorrectly think that all a definition can include all properties that one knows about the geometric object. To address this misconception, have students engage in the process of defining definitions. Students should come to the conclusion that a good definition states what something is and attend to what makes it important. It should include just enough information, but not too much information as precision is important. Students also need to understand that a definition is reversible, so it is helpful to have students write their definitions as biconditional statements. Another helpful exercise is to have students analyze imprecise definitions (either from the teacher or from their peers) and revise them to make them more precise. (Nirode, 2019)

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.CO.1-5)

EXAMPLE

Comparing Dots and Points

- How do you draw a point?
- How big should the representation of the point be?
- What is the biggest dot you could use to represent a point?
- What is the smallest dot you could use to represent a point?
- What are the advantages and disadvantages to using different size dots to represent a point?
- What do you feel is the ideal sized dot used to represent a point? Explain.

Discussion: Students often confuse a point with a dot. A point is a location in space. It has no dimensions such as length, width, volume, or area. Although, it is usually represented by a dot, it is not the dot as dots have dimensions. Push students towards abstracting a point by comparing different dots that students make to represent a point. Discuss how the bigger the dot, the less precise it is as far as determining location. However, if a dot is drawn too small, it becomes hard to read.

Some geometric assumptions are—

- Through any two distinct points there is exactly one line that contains both.
- Through at least three noncollinear points, there is exactly one plane that contains all three.
- There exists a real number distance for every pair of points. The distance from A to B is equal to the distance from B to A. If the distance between the two points is 0, then the points coincide. Otherwise the distance must be greater than 0.
- Every line may have a coordinate number line associated with it.

See EngageNY, Geometry, Module 1, Topic G, [Lesson 33: Review of the Assumptions] for a list of all geometric assumptions.

TIP! Challenge students to find and correct imprecise definitions in resources such as websites, worksheets, textbooks, etc.

Students come to high school with less precise definitions. Push them to extend their preconceived definitions to more precise definitions. For example, many students enter high school defining parallel lines as lines that do not intersect. Push them to formalize their understandings of parallel lines by using the Parallel Line Postulate: For a line and a point, not on the line, there is exactly one line parallel to the line through the point which would be coplanar.

In addition, students come to high school with the notion that perpendicular lines intersect at 90° angles and that parallel lines have the same slope. These are not incorrect; however, at the high school level, they should build toward the following definitions: Two lines are perpendicular if the four angles formed by their intersection are congruent, and two lines in the same plane are parallel if they have no points in common.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.CO.1-5)

Students may have difficulty interpreting expressions that contain symbols such as ⊥ or ∥. Students often incorrectly interpret these symbols as operators instead of as an indicator of a relationship, whereas still others have difficulty expressing the meaning of the symbol. It may be helpful to use written words when introducing notation along with the symbol "line m is perpendicular to line n ($m \perp n$), so students associate the symbols with the words. Eventually, students can be weaned off the written expression.

A good geometric definition—

- names the term being defined;
- classifies the term and differentiates it from others in a similar class;
- uses precise language;
- stands against counterexamples;
- uses only previously defined terms;
- uses proper symbolism; and
- is reversible.

Give each student in the class a term to define. As they present their definition, encourage their classmates to challenge their definitions using counterexamples. Stress the idea that every single word in the definition matters and cannot be disregarded.

EXAMPLE

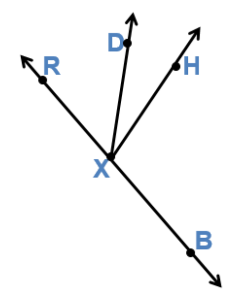
Precise Definitions

Carol gets the word "supplementary" to define.

Discussion:

- For example, Carol defines supplementary angles as "Supplementary angles are angles that add up to 180°." Mike draws the picture on the right and asks, "Is this what you mean?"
- Carol then revises her definition based on Mike's challenge: "Supplementary angles are **two** angles that add up to 180°."

Since $m\angle RXD + m\angle DXH + m\angle HXB = 180°$, therefore they are supplementary angles.



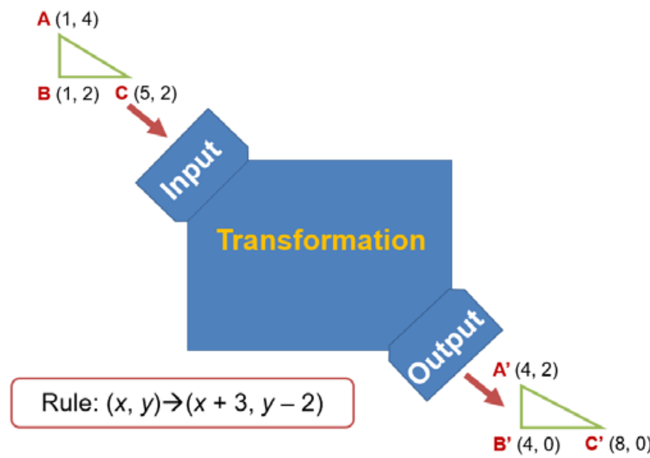
Review vocabulary associated with transformations (e.g. point, line, segment, angle, circle, polygon, parallelogram, perpendicular, rotation, reflection, translation, parallel, arc length, and ray). Be sure that students are aware of the differences between defined terms, undefined terms, properties, formulas etc.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.CO.1-5)

Betweenness is the quality or state of being between two others in an ordered mathematical set. This allows us to state if on the line a point X is between two points A and B, the length of the line segment AX plus the length of the line segment XB will equal the length of the line segment AB, which is presented as Segment Addition Postulate. Once betweenness has been established, it can be used as a foundation for Segment Addition Postulate.

TRANSFORMATIONS AS FUNCTIONS

A transformation, T, is a function that assigns to each point P of the plane a unique point P'such that $P' = T(P)$. A rigid motion R is a transformation that maps any pair of distinct points P and Q of the plane onto a pair of distinct points P' and Q' of the plane, so that $P' = R(P)$, $Q' = R(Q)$ and $\overline{PQ} = \overline{P'Q'}$. *Note: Throughout the Model Curriculum different notations are used such a function notation and prime notation etc. As each districts' resources are different, it is up to each district to determine the notation that students need to use. It may be helpful to show students a variety of notations, so they can be mathematically literate when seeing different notations in different resources/courses/schools.*



Connect geometric transformations to function transformations that students learn in Algebra standards. Make the transition from transformations as physical motions to the concept of a function that takes all points in the plane as inputs and give other points as outputs. The correspondence between the original points and their final corresponding points determines the transformation. A function machine may be useful to illustrate the similarities and differences. Point out that the rule maps all points on the figure as well as the plane, not just the vertices.

TIP! In an algebraic function, the *x*-coordinate is the independent variable and the *y*-coordinate is the dependent variable. However, in a geometric transformation function all the ordered pairs (that include both the *x*- and *y*-coordinates) that make up the preimage are considered the independent variable (of which there are an infinite amount) and all the ordered pairs that make up the image (of which there are an infinite amount) are considered the dependent variable. This is a shift for many students.

Preimage ABC (x, y)	Image A'B'C' (x + 3, y – 2)
(1, 4)	(4, 2)
(1, 2)	(4, 0)
(5, 2)	(8, 0)
(1, 2.25)	(4, 0.25)
(1.00001, 2)	(4.00001, 0)
(4.84, 2)	(7.84, 0)
(1, 3.9999)	(4, 1.9999)

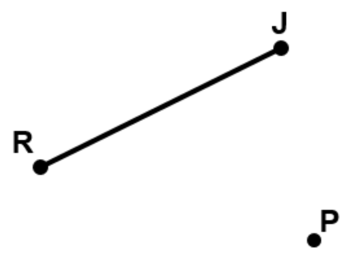
Teachers can help students connect input/preimage to domain and output/image to range. Draw attention to the fact that the image consists of all points that lie on the figure, not just the vertices.

Making constructions using a straightedge and a compass can highlight functional aspects of transformations.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.CO.1-5)

EXAMPLE

Highlighting Functional Aspects of a Translation



a. Given line segment $\overline{RJ}$and a point P that does not lie on $\overline{RJ}$, construct a line through P that is parallel to $\overline{RJ}$, using a compass and straight edge.
b. Mark off the distance RJ along the new parallel line containing P and label it $R'J'$.
c. Describe why $\overline{R'J'}$ is a translation of $\overline{RJ}$.
d. How are $\overline{RJ}$ and $\overline{R'J'}$ related to the concept of a function?

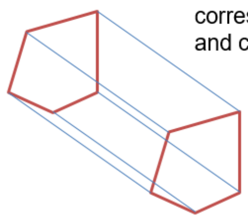
DEFINITIONS OF RIGID MOTIONS

A rigid motion is a function that keeps distances unchanged. Using hands-on experiences in Grade 8, students should come to high school Geometry with at least three assumptions about rigid motions (translations, reflections, rotations):

1. They map lines to lines, rays to rays, and segments to segments.
2. They preserve distance.
3. They preserve angle measure.

Provide both individual and small-group activities, allowing adequate time for students to explore and verify conjectures about rigid motions and develop precise definitions of rotations, reflections, and translations both on and off the coordinate plane. Students should come to the realization that translations move points a specified distance along parallel lines; rotations move points along a circular arc with a specified center and angle, and reflections move points across a specified line of reflection so that the line of reflection is the perpendicular bisector of each line segment connecting corresponding pre-image and image points.

Students should visualize rigid motions in terms of these definitions not just coordinate rules. For example, they could use perpendicular lines and distance to identify a reflection. Direct students to pay careful attention to properties of figures that are preserved. For example, as the result of a translation, a line segment is both parallel and congruent to its corresponding line segment in its pre-image.

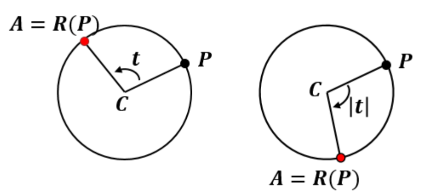


In this two-dimensional translation, all blue lines connecting the corresponding vertices are parallel and congruent.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.CO.1-5)

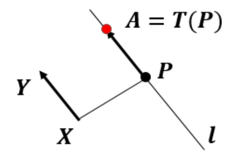
When students use coordinate notation such as (*x*, *y*)→(–*x*, *y*), they may incorrectly believe that –*x* is always negative instead of understanding that it indicates the opposite sign of the *x-coordinate*. See Model Curriculum 7.NS.1-3 for more common errors with negative numbers.

Definition of Rotation



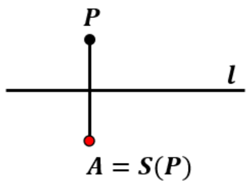
The rotation R around the Point C through the angle t is a rigid motion takes a point P to the point $A = R(P)$ as follows. If $P = C$, then $R(P) = C$. If $P \neq C$ and $t \geq 0$, then A is on the circle with center C and radius $|CP|$ so that $\angle PCA = t°$ and A is counterclockwise from P. If $t < 0$, we rotate clockwise by $|t|°$.

Definition of Translation



The translation T along the directed line segment $\overrightarrow{XY}$ is a rigid motion takes the point P to the point $A = T(P)$ as follows: The line l passing through P and parallel to line through X and Y. Point A is the point on l so that the direction from P to A is the same as the direction from X to Y and so that $|PA| = |XY|$.

Definition of Reflection



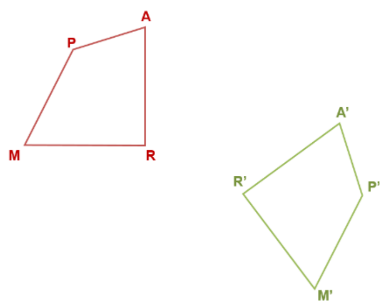
The reflection S across the line l is a rigid motion that takes each point on l to itself, and takes any other point P to the point $A = S(P)$ which is such that l is the perpendicular bisector of the line segment PA.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.CO.1-5)

EXAMPLE

Lines of Reflection

Draw a line of reflection for the figure PARM and its reflected image P'A'R'M'.



Discussion: A task like this reinforces that the line of reflection is equidistant from corresponding vertices of the two figures. To find the line of reflection students should connect the vertices of the original figure and its image, and then find the perpendicular bisector of the connecting lines using a compass and straight edge. Once students have practice finding the perpendicular bisector without coordinates, they can move toward using coordinates and the midpoint formula. Explain to students that coordinates are used to represent rigid motions as functions that map points in the plane to other points in the plane. See 8.G.1-4 Model Curriculum for a similar example using the coordinate plane.

In Grade 8 students should have used transparencies or tracing paper to have figures translate onto one another. Some students may need to review these concepts with transparencies. Emphasize that the plane (transparency or tracing paper) is moving and not the figure. In Grade 8, students described translations in terms of horizontal and vertical movements. Move students toward an informal understanding of a vector that is represented by a directed line segment. A vector, where one of its points designated as the starting point and the other as the ending point, shows the direction and the amount of translations (magnitude) of a plane.

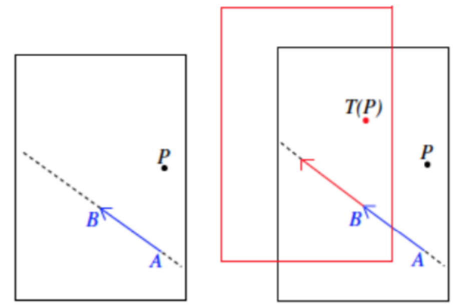
INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.CO.1-5)

EXAMPLE

Translating a Point

Use a transparency or tracing paper to perform a translation of a point.

- **a.** Draw point P and a dotted line AB on a piece of paper representing a plane. (Remember a plane extends infinitely in all directions.) Draw vector $\overrightarrow{AB}$ on top of the dotted line AB.
- **b.** Place a transparency or tracing paper over the piece of original diagram from part **a**. Trace P and the vector $\overrightarrow{AB}$ in another color such as red.
- **c.** Slide the transparency along line AB moving in the direction from A to B noting the distance from A to B fixed. The translation, T, moves the given point P to its image $T(P)$ denoted by the red dot.
- **d.** Repeat the process using geometric figures in place of P and different vectors.
- **e.** Draw attention to the fact that $Translation_{\overrightarrow{AB}}$ is different than $Translation_{\overrightarrow{BA}}$ since the direction is different.



From EngageNY Grade 8, Module 2, Lesson 2.

Discussion: Connect translating along a vector to the Pythagorean Theorem and the Distance Formula. Previously in Grade 8 a student might translate a figure 4 units to the right and 3 units up. This is the same as moving 5 units at an approximate 36.87° angle. Once students have an understanding of how a vector is a transformation of a plane that maps each point on the plane to its image, have students discover that the segment with endpoints P and $T(P)$ is parallel and congruent to vector $\overrightarrow{AB}$. Once they are able to make these connections to the particular segments, help them to realize that these properties hold true for all translations.

Provide real-world examples of rigid motions using precise definitions (e.g. a Ferris wheels for a rotation and arc length; mirrors for a reflection; moving vehicles for a translation).

In a rotation the center point is fixed, and the rest of the plane moves the same number of degrees along the arc of the circle. Students need to make a connection between a rotation, as a rigid motion, and the circular movement with the center at point C that rotates any point P in a way that $\overline{CP} \cong \overline{CP'}$ that are the radii of a circle. In high school, students should be given an opportunity to experience rotations around any point (not just the origin) and onto at any angle, not just a multiple of 90.

Rotations should be addressed again when looking at arc length and sector area. For example, rotating a point on a circle creates an arc or rotating a radius creates a sector.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.CO.1-5)

Translations and rotations preserve orientation (Orientation is a geometric notion that in two-dimensions allows one to say when a figure goes around clockwise or counterclockwise, and in three-dimensions when a figure is left-handed or right-handed. This is different than the common usage of orientation); reflections do not. Sometimes students want to incorrectly say that a rotation changes the orientation of the shape because the shape has been turned to create a shape that looks different. This is not an orientation change because the vertices can still be named in a clockwise fashion. However, in a reflection, the order of vertices is reversed, thus the orientation is not preserved.

Some students may struggle to use the correct center of rotation if they have memorized the rule $(x, y) \rightarrow (y, -x)$. However, this rule only holds true for 90° clockwise rotations when the center of rotation is the origin. Provide students with problems that include a variety of centers of rotation and using angles other than multiples of 90°.

TRANSFORMATIONS THAT PRESERVE DISTANCE AND THOSE THAT DO NOT

Focus attention on the attributes (e.g. distances or angle measures) of a geometric figure that remain constant under various transformations. Also explore transformations that do (dilations) and do not (horizontal or vertical stretch) preserve angle measure and/or length. For example, use control points on electronic pictures and work with stretching and cropping images.

It may be useful to connect the idea that a translation is a sequence (a composition) of two reflections over parallel lines and a rotation is the composite of two reflections over intersecting lines.

TIP!

Connect transformations, rigid motions, and symmetries to works of art. M.C. Escher has examples of translations, reflections, and/or rotations in some of his pieces such as "Two Birds," "Fish," "Clown," "Lizard," or "Three Birds." Examples of his work can be found at http://www.mcescher.com/gallery/switzerland-belgium/

Discuss the concept of symmetry and how it connects to the symmetry of graphs of functions and shapes of data displays.

TIP!

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.CO.1-5)

EXAMPLE

Finding the Center or Rotation

Find the center of rotation of the image using a compass and straightedge, transparency, folding, or another method. Explain how you found it.

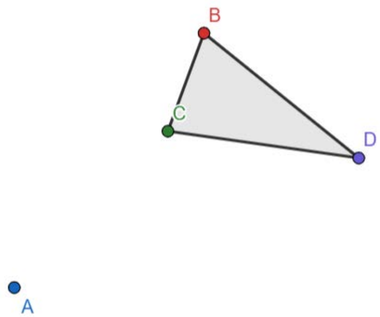
REPRESENTING AND DRAWING TRANSFORMATIONS

Remember a basic rigid motion F is a rule so that, for each point P of the plane, F assigns (or moves) a point $F(P)$ to P where the distance between the two points are preserved. Emphasize the importance of understanding a transformation as the correspondence between initial and final points, rather than the physical motion. Be precise with respect to the descriptions of sequences of rigid motions. This differentiates high school Geometry from Grade 8. For rotations students must state the center and angle of rotation. For reflections student must state the line of reflection. For a translation, students must state the direction and the distance.

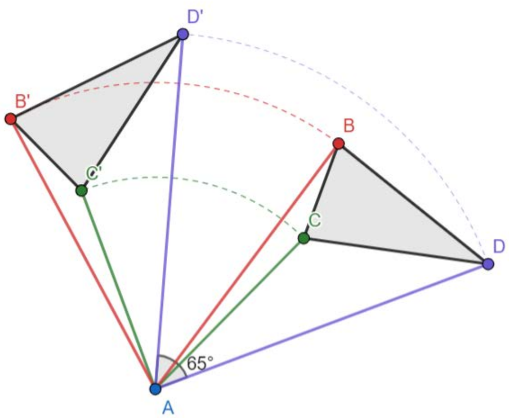
EXAMPLE

Rotating a Figure

Rotate triangle BCD 65° counterclockwise around point A.



Discussion: Using a straight edge, create a line segment containing points A and D. Using a protractor measure 65° counterclockwise and make a mark. Sketch a light line that goes through point A and the new mark. Then use a compass to make an arc centered at A and containing point D. The intersection point of the sketched 65° line and the arc containing D is D'. Repeat this process for points B and C. Finish by connecting the vertices of the triangle.

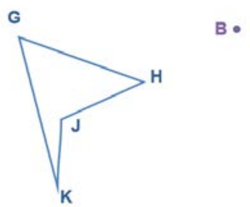


INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.CO.1-5)

EXAMPLE

Rotating a Figure

Rotate quadrilateral $GHJK$ 125° clockwise around point B.



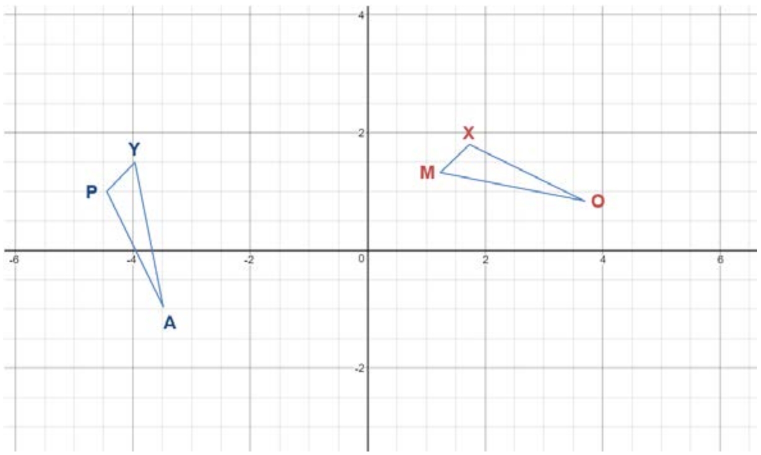
Discussion: Draw students' attention to the fact that 125° clockwise rotation is the same as a –235° rotation.

Provide students with a pre-image and a final image, and ask them to describe the steps required to generate the final image both on and off the coordinate plane. Show examples with more than one answer (e.g. a reflection might result in the same image as a translation).

EXAMPLE

Describing Steps for a Transformation

Describe the steps required to transform the image $\triangle YAP$ onto $\triangle MOX$. Be precise in your description.

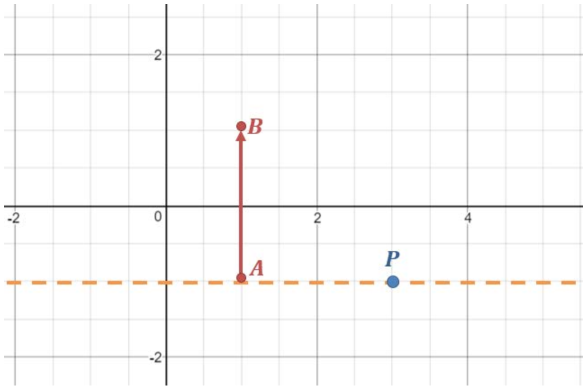


Discussion: One method that could be used to transform triangle YAP to triangle MOX is a 90° rotation around point A (–3.5, –1), and then a reflection across *x* = –1 and then a translation (*x*, *y*)→(*x* + 1, *y* + 2). Students may use a variety of methods. To indicate corresponding points, students may wish to use function notation.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.CO.1-5)

EXAMPLE

Analyzing a Transformation



a. Does it matter if you translate point P along $\overrightarrow{AB}$ and then reflect it across $y = -1$ or if you first reflect point P across $y = -1$ and then translate it along the vector AB? Explain.

b. Are sequences of translations commutative? Explain.

Work backwards to determine a sequence of transformations that will map one figure onto another of the same size and shape both on and off the coordinate plane.

SYMMETRY

A symmetry of a figure is a basic rigid motion that maps the figure back onto itself. Students in Ohio no longer have any explicit learning standards on symmetry until this course, so students may or may not have an understanding of the concept.

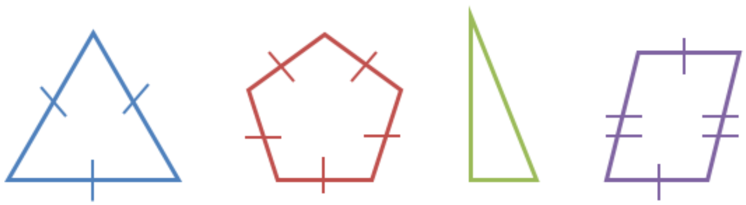
Analyze various figures (e.g. regular polygons, folk art designs, or product logos) to determine which rotations and reflections carry (map) the figure onto itself. These transformations are the "symmetries" of the figure. Use symmetry in proofs of properties of geometric figures. For example, use symmetry to prove that the diagonals of a rectangle are congruent. Include finding the line of symmetry on the coordinate plane (for a rectangle, for example). Include point symmetry as a subset of rotational symmetry. Remind students that if the figure has no symmetry other than 360° rotation, the figure has no rotational symmetry.

Transparencies work well to represent the symmetries of a figure because they keep the original and transposed figures on separate planes. Geometry software, although very useful in many ways in the classroom, requires students to imagine this concept, whereas transparencies are more concrete.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.CO.1-5)

EXAMPLE

Lines of Symmetry



a. Use a compass and straightedge to draw a line of symmetry onto each object. Sketch in any remaining lines of symmetry.
b. Justify that your lines are truly lines of symmetry.
c. How do you know that you found all the lines of symmetry?

TIP! To help diagnose misconceptions from student assessments, try to grade all student responses item by item instead of test by test. For example, grade all the answers to question 1 together and then diagnose any common misconceptions, and then do the same thing for question 2 and then question 3, etc. Afterwards, write discussion questions addressing misconceptions that a significant number of your students have. These questions can be either from a slightly different context or contrasting two different solution methods. After the class solves the discussion questions, hand back the assessments and have them figure out which test question it matches.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.CO.1-5)

Instructional Tools/Resources

These tools and resources may be helpful for instruction. Before use, they should be evaluated at the local level to determine their appropriateness for instruction.

Manipulatives/Technology

- Tracing paper (patty paper)
- Transparencies
- Graph paper
- Ruler
- Protractor
- Compass
- Mira
- Computer dynamic geometric software (such as Geometer's Sketchpad®, Desmos, Cabri®, or GeoGebra®).
- Graphing calculators and other handheld technology such as TI-Nspire™.
- *Flatland: A Romance of Many Dimensions* by Edwin Abbott.

Analyzing Definitions

- Defining Parallel Lines by Illustrative Mathematics is a task that has students examine and analyze three different definitions for parallel lines.
- Defining Perpendicular Lines by Illustrative Mathematics is a task that has students examine and analyze three different definitions for perpendicular lines.
- Practice: Geometric Definitions by Khan Academy has students analyze three definitions and match teacher's critiques to the definitions.
- Defining Rotations by Illustrative Mathematics is a task that encourages students to be precise in their definitions.
- Defining Reflections by Illustrative Mathematics is a task that has students compare and contrast different definitions.

Connecting Functions in Geometry to Algebra

- Connect Geometry and Algebra Through Functions by Technologically Embodied Geometric Functions is a webpage with interactive activities connecting *Flatland* with Geometry and Algebraic Functions.
- Transformations as Functions by MathBitsNotebook is an informational page about transformations as functions.

Comparing Functions that Preserve Distances to Those that Do Not

- Horizontal Stretch of the Plane by Illustrative Mathematics is a task that has students compare transformations of those that preserve distance and angles and those that do not.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.CO.1-5)

Analyzing Rigid Motions

- Seven Circles II by Illustrative Mathematics is an instructional task that has students analyze rigid motions.
- Symmetries of a Quadrilateral I and Symmetries of a Quadrilateral II by Illustrative Mathematics is a task that has students examine rigid motions in the context of a quadrilateral.

Representing and Drawing Rigid Motions

- Showing a Triangle Congruence: the General Case and Showing Triangle Congruence: A Particular Case by Illustrative Mathematics are tasks that have students compare different transformations to obtain an image from an pre-image.
- Transformation Golf: Rigid Motions has students use transformations to play a round of golf.

Curriculum and Lessons from Other Sources

- EngageNY, Grade 8, Module 2, Topic A, Lesson 2: Definition of Translation and Three Basic Properties and Lesson 3: Translating Lines are lessons that pertain to this cluster.
- EngageNY, Geometry, Module 1, Topic A, Lesson 1: Construct an Equilateral Triangle is a lesson that pertains to this cluster.
- EngageNY, Geometry, Module 1, Topic C, Lesson 12: Transformations—The Next Level, Lesson 13: Rotations, Lesson 14: Reflections, Lesson 15: Rotations, Reflections, and Symmetry, Lesson 16: Translations, Lesson 17: Characterize Points on a Perpendicular Bisector are lessons that pertain to this cluster.
- Georgia Standards of Excellence Curriculum Frameworks, Geometry, Unit 1: Transformations in the Coordinate Plane has many tasks that pertain to this cluster.
- Mathematics Vision Project, Geometry, Module 1: Transformations and Symmetry has many tasks that pertain to this cluster.
- Illustrative Mathematics, Geometry, Unit 1, Lesson 10: Rigid Transformations, Lesson 11: Defining Reflections, Lesson 12: Defining Translations, Lesson 13: Incorporating Rotations, Lesson 14: Defining Rotations, Lesson 15: Symmetry, Lesson 16: More Symmetry, Lesson 17: Working with Rigid Transformations, and Lesson 18: Practicing Point by Point Transformation are lessons that pertain to this cluster.
- Illustrative Mathematics, Geometry, Unit 6, Lesson 1: Rigid Transformations in a Plane, Lesson 2: Transformations as Functions, and Lesson 3: Types of Transformations are lessons that pertain to this cluster.

General Resources

- Arizona 7-12 Progression on Geometry is an informational document for teachers. This cluster is addressed on pages 13-15.
- High School Coherence Map by UnboundED is a map of the relationships among high school mathematics standards. Please note: This resource is aligned to the Common Core not Ohio's Learning Standards. Although the tool is useful, it needs to be checked with respect to Ohio's Learning Standards.
- van Hiele Model of Geometric Thinking is a pdf created by ODE that summarizes the van Hiele levels.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.CO.1-5)

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STANDARDS

Geometry

CONGRUENCE

Understand congruence in terms of rigid motions.

G.CO.6 Use geometric descriptions of rigid motions[G] to transform figures and to predict the effect of a given rigid motion on a given figure; given two figures, use the definition of congruence in terms of rigid motions to decide if they are congruent[G].

G.CO.7 Use the definition of congruence in terms of rigid motions to show that two triangles are congruent if and only if corresponding pairs of sides and corresponding pairs of angles are congruent.

G.CO.8 Explain how the criteria for triangle congruence (ASA, SAS, and SSS) follow from the definition of congruence in terms of rigid motions.

MODEL CURRICULUM (G.CO.6-8)

Expectations for Learning

In middle school, students understand congruence through a sequence of basic rigid motions (reflections, rotations, and translations). In this cluster, students will build on this knowledge to prove that two figures are congruent if there is a sequence of rigid motions carrying one onto the other. The triangle congruence criteria can then be established using the definition of congruence in terms of rigid motions. This is the time when students are first exposed to the criteria for triangle congruence; students should know and be able to use AAS, ASA, SAS, and SSS and understand that the criteria follow from rigid motions.

The student understanding of this cluster begins at van Hiele Level 1 (Analysis) and moves to Level 2 (Informal Deduction/Abstraction).

ESSENTIAL UNDERSTANDINGS

- Two figures are defined to be congruent if one can be mapped onto the other by rigid motions.

MATHEMATICAL THINKING

- Explain mathematical thinking.
- Recognize, apply, and justify mathematical concepts, terms, and their properties.
- Represent concepts symbolically.
- Use formal and informal reasoning.
- Use accurate and precise mathematical vocabulary.

INSTRUCTIONAL FOCUS

- Use rigid transformations to determine if the figures are congruent
- Given congruent triangles, describe the rigid transformations that map one triangle onto the other.
- Establish the criteria for triangle congruence (AAS, ASA, SAS, and SSS) in terms of rigid motions.
- Know and be able to use triangle congruence (AAS, ASA, SAS, and SSS) in solving problems.

Continued on next page

STANDARDS	MODEL CURRICULUM (G.CO.6-8)
	Content Elaborations **OHIO'S HIGH SCHOOL CRITICAL AREAS OF FOCUS** • Geometry, Number 2, pages 4-5 **CONNECTIONS ACROSS STANDARDS** • Experiment with transformations in the plane (G.CO.1-5). • Prove and apply theorems about triangles (G.CO.10). • Prove and apply theorems about parallelograms (G.CO.11). • Use coordinates to prove simple geometric theorems algebraically (G.GPE.4). • Prove and apply theorems involving similarity (G.SRT.5).

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.CO.6-8)

Instructional Strategies

Note: The model curriculum is not a collection of lessons nor a full curriculum; it does not suggest pace, sequence, or amount of time spent on topics. It provides information about a topic related to the standards including ideas for examples, strategies for teaching, common misconceptions, and possible connections between topics. In addition, the emphasis of the model curriculum is on instruction not on assessment; therefore, the examples in the Instructional Strategies section are written for instructional purposes not necessarily as assessment items. Examples may need to be rewritten to accommodate the needs of each individual classroom.

Standards for Mathematical Practice

This cluster focuses on but is not limited to the following practices:

***MP.3** Construct viable arguments and critique the reasoning of others.*

***MP.4** Model with mathematics.*

***MP.5** Use appropriate tools strategically.*

***MP.6** Attend to precision.*

VAN HIELE CONNECTION

In Geometry students are expected to move from Level 1(Analysis) to Level 2 (Informal Deduction/Abstraction). In van Hiele Level 1 students use coordinates to examine transformations and recognize, use, and describe properties of transformations. Whereas van Hiele Level 2 can be characterized by the student doing some or all of the following:

- analyzing relationships within and between transformations;
- algebraically expressing transformations: reflections about the *x*- and *y*-axes, and rotations of 45°, 90°, and 180° about the origin; *Note: In high school, students should be given an opportunity to experience rotations around any point (not just the origin) and onto any angle, not just multiples of 90. Also, reflections are not limited to the x-and y-axes but could be about any line.*
- understanding and justifying how to use perpendicular bisectors of corresponding points to locate the turn center of a rotation;
- understanding how the composition of two reflections can be a translation or a rotation; and/or
- following a rationale if shown proof.

See the van Hiele pdf on ODE's website for more information about van Hiele levels.

CONGRUENCE AND RIGID MOTIONS

In Grade 8 students experimented with congruence. They should have noticed that when rigid motions are performed, size and shape of figures are preserved. They may have even defined congruence as "same size and shape." Although, this idea helps create an intuitive understanding of congruence, high school students need to use a more precise definition. Students should develop the relationship between transformations and congruency and understand that two figures are defined to be congruent if and only if there is a sequence of rigid motions that maps one onto the other.

Students should identify rigid motions (translations, reflections, and/or rotations) that map one figure onto another to determine if two figures are congruent. Allow adequate time and provide hands-on activities for students to visually and physically explore the relationship between rigid motions and congruence. The use of graph paper, tracing paper, and/or dynamic geometric software to obtain images of a given figure under specified rigid motions should be used to achieve this experience.

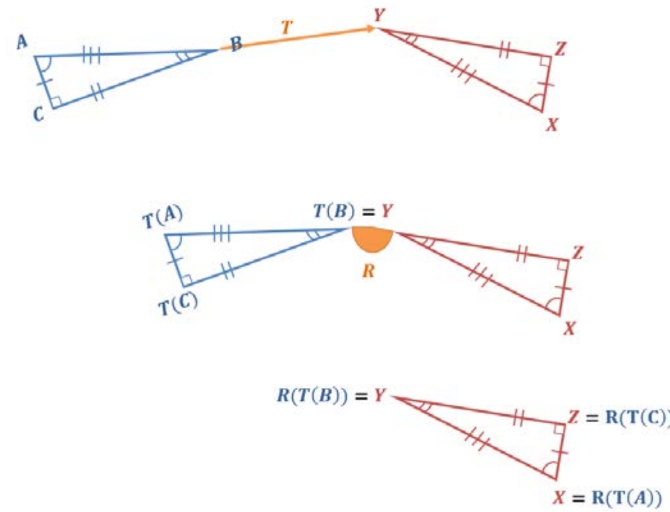
INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.CO.6-8)

Avoid circular reasoning in establishing the definition of congruency. Define congruence in terms of rigid motions before referring to proof statements about corresponding parts.

EXAMPLE

Congruency of Rigid Motions

Prove that $\triangle ABC$ and triangle $\triangle XYZ$ are congruent given that the corresponding side lengths and corresponding angle measures are equal.



Discussion: A student may translate $\triangle ABC$ along line segment BY, so that $T(B) = Y$. Then, apply a rotation, R, clockwise about point Y through the angle $\angle T(C)YX$. Because rigid motions preserve angles and sides lengths, the line segment through points Y and Z coincides with the line segment through points $R(T(B))$ and $R(T(C))$ and the line segment that goes through points Y and X coincides with the line segment that goes through $(T(B))$ and $R(T(A))$ and the line segment that goes through points X and Z coincides with the line segment that goes through points $(T(A))$ and $R(T(C))$. Since all three points coincide, there is a sequence of rigid motions that maps $\triangle ABC$ onto $\triangle XYZ$, and therefore the triangles are congruent. *Note: A point $T(C)$ is the image of point C after it was translated along line segment BY. A point $R(T(C))$ is the image of point C after it was first translated along line segment BY and then rotated about point Y on $\angle T(C)YX$. The same logic applies to other points on the figure.*

Work backwards – given two figures that are congruent, find a sequence of rigid motions that will map one onto the other and prove it.

Some students may incorrectly believe that all transformations, including dilations, are rigid motions. Provide opportunities for students to create counterexamples of this misconception, including dilations and vertical and horizontal stretch.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.CO.6-8)

Some students may incorrectly believe that any two figures that have the same area are the result of rigid transformation(s). Students should recognize that the preservation of area does not guarantee the preservation of side lengths and angle measures.

TIP!

A symmetry is a rigid motion that carries a figure to itself. It is nothing other than a congruence of an object with itself. When a figure has many rigid motions that map a figure onto itself, it is because the symmetries in the objects are being compared.

Students may incorrectly believe that two angles cannot be congruent if the rays forming the angles have different lengths or different directions. Show students two congruent angles, one with longer rays than the other. Have students measure the angles to determine congruency. Then discuss that although one appears larger than the other, they are really the same angle measure since rays continue forever.

CORRESPONDING PARTS OF CONGRUENT FIGURES ARE CONGRUENT

Although there is a relationship between correspondence and congruence, correspondences do not necessarily imply congruency. Correspondence is simply matching. Since any two parts of figures can be compared or matched, not all correspondences imply congruency. The two corresponding figures may or may not have the same measures (angles and side lengths), and they may or may not be able to be mapped using rigid motions, for example similar triangles. An example of this happens in construction when parts of old building have settled and no longer have square corners, yet windows or doors (with square corners) still need to be replaced with new objects. In this case workers need to adjust the matching parts to make the corresponding (not congruent) parts fit. However, the opposite is true: if there is congruence between two figures there is a correspondence between their congruent parts. This is because an image created by a rigid motion produces a one-to-one correspondence between its parts (points, angle measures, and side lengths) because the two figures can be mapped onto each other. Therefore, corresponding parts of congruent figures are congruent.

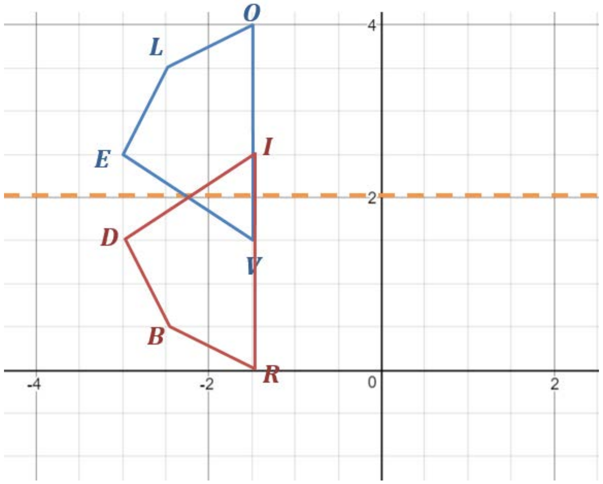
INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.CO.6-8)

EXAMPLE

Exploring Correspondence and Congruence

The figure LOVE (shown in blue) has been mapped to the second figure (shown in red) by a reflection across $y = 2$.

- **a.** Identify the corresponding vertices, sides, and angles.
- **b.** State the congruency between the two figures.
- **c.** Explain why you can state that the two figures are congruent.



Corresponding Vertices	Corresponding Segments	Corresponding Angles	Congruent Figures
			Explanation of Congruency:

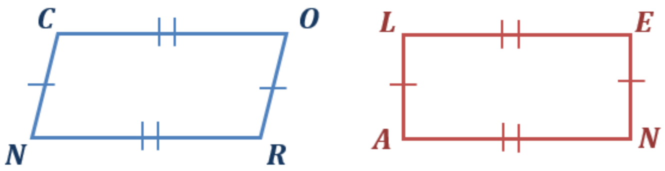
Discussion: Emphasize to students that the two figures are congruent because they can be mapped onto each other using a rigid motion—not just because their corresponding parts can be located. Other situations, such as similar figures, also have corresponding parts, but are not congruent.

TIP! Some students may believe that it is not important to list corresponding vertices of congruent figures in order; however, it is useful to stress the value of listing corresponding vertices in the same order so that corresponding sides and angles can be easily identified and that included sides or angles are apparent.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.CO.6-8)

EXAMPLE

Exploring Correspondence and Congruence



a. List all the corresponding parts between the two figures so that sides of the first quadrilateral are congruent to the sides of the second quadrilateral.

b. Are the two figures congruent? Explain.

Discussion: The figures are not congruent. The definition of rigid motions does not apply, because not all the distances between pairs of corresponding points remain equal. Therefore, the corresponding angles are not congruent. The parts will not coincide if quadrilateral CORN is mapped onto quadrilateral LEAN as required by rigid motions.

Corresponding Parts of Congruent Triangles are Congruent (CPCTC)

Special attention is given to the corresponding parts of congruent triangles that are congruent usually written as CPCTC. Students should describe rigid motion(s) (translations, reflections and/or rotations) that map one triangle onto another to determine if two triangles are congruent. Compare a given triangle and its image to verify that corresponding sides and corresponding angles are congruent.

Criteria for Triangle Congruence

Since basic rigid motions preserve segments, lines, and angles, congruent figures must have all pairs of corresponding equal sides and all pairs of corresponding equal angles. The opposite (converse) is also true: If a figure has all corresponding congruent pairs of sides and all corresponding congruent pairs of angles, the figures are congruent. The same logic applies to triangles; however instead of needing six conditions to prove congruency, there are some shortcuts using only three conditions.

Build on previous learning of transformations and congruency to develop a formal criterion for proving the congruency of triangles. This is the time when students are first exposed to the criteria for triangle congruence; students should know and be able to use ASA, SAS, and SSS and understand that the criteria follow from rigid motions. See the EngageNY, Geometry, Module 1, Topic D lessons listed in the Instructional Resources section for guidance on establishing congruency of triangles.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.CO.6-8)

Students should construct pairs of triangles that satisfy the ASA, SAS, or SSS congruence criteria, and use rigid motions to verify that they satisfy the definition of congruent figures. Investigate rigid motions and congruence both algebraically (using coordinates) and logically (using proofs).

Students should be able to use ASA, SSS, and SAS in formal and informal proofs. Traditionally, AAS and HL are included in the list of congruence theorems. These are special cases of triangle congruence, for example HL is a specific case of SAS for right triangles (via Pythagorean theorem) and AAS is a specific case of ASA (via Third Angle Theorem).

Some students may incorrectly believe that combinations such as SSA or AAA are also a congruence criterion for triangles. Provide opportunities to expose students to counterexamples to confront this misconception.

Some students may incorrectly believe that SSS, ASA, and SAS apply to all figures, not just triangles. Provide counterexamples to help students.

Instructional Tools/Resources

These tools and resources may be helpful for instruction. Before use, they should be evaluated at the local level to determine their appropriateness for instruction.

Manipulatives/Technology

- Tracing paper (patty paper)
- Transparencies
- Graph paper
- Ruler
- Protractor
- Compass
- Mira
- Computer dynamic geometric software (such as Geometer's Sketchpad®, Desmos, Cabri®, or GeoGebra®).
- Graphing calculators and other handheld technology such as TI-Nspire™.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.CO.6-8)

Triangle Congruency

- Evaluating Conditions for Congruency by Mathematics Assessment Project has students analyze the truth about conjectures of congruency.
- Why Does SSS Work? by Illustrative Mathematics is a task that has students explore SSS criteria by performing different rigid motions. There is also an attached GeoGebra file to be used with this activity.
- Why Does ASA Work? by Illustrative Mathematics is a task that has students explore ASA criteria.
- Why Does SAS Work? by Illustrative Mathematics is a task that has students explore SAS criteria.
- Side-Angle-Side Congruence by Basic Rigid Motions is a video that explains a proof H. Wu's "Teaching Geometry According to the Common Core Standards" publication.
- When Does SSA Work to Determine Triangle Congruence? by Illustrative Mathematics is a task that has students explore congruency using different criteria to determine congruence.
- SSS Congruence Criterion by Illustrative Mathematics is a task that has students establish SSS congruence using rigid motions.

Curriculum and Lessons from Other Sources

- EngageNY, Geometry, Module 1, Topic C, Lesson 19: Construct and Apply a Sequence of Rigid Motions, Lesson 20: Applications of Congruence in Terms of Rigid Motions, Lesson 21: Correspondence and Transformations are lessons that pertain to this cluster.
- EngageNY, Geometry, Module 1, Topic D, Lesson 22: Congruence Criteria for Triangles—SAS, Lesson 23: Base Angles of Isosceles Triangles, Lesson 24: Congruence Criteria for Triangles—ASA and SSS, Lesson 25: Congruence Criteria for Triangles—AAS and HL are lessons that pertain to this cluster.
- Georgia Standards of Excellence Curriculum Frameworks, Geometry, Unit 2: Similarity, Congruence, and Proofs. This cluster is addressed on pages 25-38.
- Mathematics Vision Project, Geometry, Module 2: Congruence, Construction, and Proof has many tasks that pertain to this cluster.
- Illustrative Mathematics, Geometry, Unit 2: Congruence, Lesson 1: Congruent Parts, Part 1, Lesson 2: Congruent Parts, Part 2, Lesson 3: Congruent Triangles, Part 1, Lesson 4: Congruent Triangles, Part 2, Lesson 5: Points, Segments, and Zigzags, Lesson 6: Side-Angle-Side Congruence, Lesson 7: Angle-Side Angle Triangle Congruence, Lesson 9: Side-Side-Side Triangle Congruence, Lesson 11: Side-Side-Angle (Sometimes) Congruence are lessons that pertain to this cluster.
- Illustrative Mathematics, Geometry, Unit 3, Lesson 10: Other Conditions for Triangle Similarity is a lesson that pertains to this cluster.

General Resources

- Arizona 7-12 Progression on Geometry is an informational document for teachers. This cluster is addressed on pages 13-15.
- High School Coherence Map by UnboundED is a map of the relationships among high school mathematics standards. Please note: This resource is aligned to the Common Core not Ohio's Learning Standards. Although the tool is useful, it needs to be checked with respect to Ohio's Learning Standards.
- van Hiele Model of Geometric Thinking is a pdf created by ODE that summarizes the van Hiele levels.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.CO.6-8)

References

- Common Core Standards Writing Team. (2016, March 24). *Progressions for the Common Core State Standards in Mathematics (draft). Grades 7–8, HS, Geometry*. Tucson, AZ: Institute for Mathematics and Education, University of Arizona.
- Eggleton, P. (October 2001). Triangles a la fettuccine: A hands-on approach to triangle-congruence theorems. *Mathematics Teacher*, *94(7),* 534-537.
- Piatek-Jimenz, K. (February 2008). Building intuitive arguments for the triangle congruence conditions. *Mathematics Teacher*, *101(6),* 463-466.
- Wu, H. (2013). Teaching geometry in Grade 8 and high school according to Common Core Standards. Retrieved from https://math.berkeley.edu/~wu/CCSS-Geometry_1.pdf
- Wu, H. (2013). Teaching geometry according the Common Core Standards. Retrieved from https://math.berkeley.edu/~wu/Progressions_Geometry.pdf

STANDARDS	MODEL CURRICULUM (G.CO.9-11)
Geometry **CONGRUENCE** **Prove geometric theorems both formally and informally using a variety of methods.** **G.CO.9** Prove and apply theorems about lines and angles. *Theorems include but are not restricted to the following: vertical angles are congruent; when a transversal crosses parallel lines, alternate interior angles are congruent and corresponding angles are congruent; points on a perpendicular bisector of a line segment are exactly those equidistant from the segment's endpoints.* **G.CO.10** Prove and apply theorems about triangles. *Theorems include but are not restricted to the following: measures of interior angles of a triangle sum to 180°; base angles of isosceles triangles are congruent; the segment joining midpoints of two sides of a triangle is parallel to the third side and half the length; the medians of a triangle meet at a point.* *Continued on next page*	**Expectations for Learning** In middle school, students informally define and apply the relationships of lines, angles, triangles, and parallelograms. For this cluster, students now develop conjectures and construct valid proofs about lines, angles, triangles, and parallelograms. They should begin with informal proof and work toward formal proof using a variety of methods including coordinate-based methods. Also, students should apply these relationships to real-world settings and to proofs. The student understanding of this cluster begins at van Hiele Level 2 (Informal Deduction/Abstractions) and moves to Level 3 (Deduction). **ESSENTIAL UNDERSTANDINGS** • The process of proof can vary from informal to formal reasoning. • A proof is a deductive argument that explains why a claim must be true. • Proof can rely on formal and informal language; there are many ways to justify a claim, not all of which rely on technical vocabulary. • Students should demonstrate a knowledge of the content listed in the standards and be able to apply those concepts in various problem-solving settings. **MATHEMATICAL THINKING** • Explain mathematical thinking. • Recognize, apply, and justify mathematical concepts, terms, and their properties. • Represent concepts symbolically. • Use formal and informal reasoning. • Use accurate and precise mathematical vocabulary. • Plan a solution pathway. • Make and analyze mathematical conjectures. • Solve real-world and mathematical problems accurately. • Create a drawing and add components as appropriate. *Continued on next page*

STANDARDS	MODEL CURRICULUM (G.CO.9-11)
G.CO.11 Prove and apply theorems about parallelograms. *Theorems include but are not restricted to the following: opposite sides are congruent, opposite angles are congruent, the diagonals of a parallelogram bisect each other, and conversely, rectangles are parallelograms with congruent diagonals.*	**Expectations for Learning, continued** **INSTRUCTIONAL FOCUS** • Form conjectures about geometric relationships and examine their validity, providing evidence to support or refute the claim. • Using previously established facts about lines, angles, triangles, and parallelograms, construct a valid argument for why a conjecture is true or not true. • Solve problems involving lines, angles, triangles, and parallelograms by applying theorems. **Content Elaborations** **OHIO'S HIGH SCHOOL CRITICAL AREAS OF FOCUS** • Geometry, Number 2, page 4-5 **CONNECTIONS ACROSS STANDARDS** • Experiment with transformations in the plane (G.CO.1, 3, 4). • Understand congruence in terms of rigid transformations (G.CO.6-8). • Use coordinates to prove simple geometric theorems algebraically (G.GPE.4-5). • Prove and apply theorems involving similarity (G.SRT.4,5).

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.CO.9-11)

Instructional Strategies

Note: The model curriculum is not a collection of lessons nor a full curriculum; it does not suggest pace, sequence, or amount of time spent on topics. It provides information about a topic related to the standards including ideas for examples, strategies for teaching, common misconceptions, and possible connections between topics. In addition, the emphasis of the model curriculum is on instruction not on assessment; therefore, the examples in the Instructional Strategies section are written for instructional purposes not necessarily as assessment items. Examples may need to be rewritten to accommodate the needs of each individual classroom.

Standards for Mathematical Practice
This cluster focuses on but is not limited to the following practices:
MP.3 *Construct viable arguments and critique the reasoning of others.*
MP.4 *Model with mathematics.*
MP.5 *Use appropriate tools strategically.*
MP.6 *Attend to precision.*

A proof is an argument that demonstrates that a statement is always true. A collection of examples is not a proof unless the examples exhaust all possible cases.

The purposes of proof are to verify and explain results, promote discovery, and to communicate and formalize truth. Although proof does not end in the knowledge of the truth but goes further to explain the "why" something is true. The idea is to communicate truth (of mathematics or other disciplines) to skeptics. A student will continue to use proof, verification, and justification throughout not only his/her math career, but throughout life.

Proofs are beneficial forms of communication when they convince others *that* a statement must be true. Proofs are especially useful when they explain *why* the statement must be true. Proofs are usually organized into steps, each of which follows from previous steps and from accepted theorems, definitions, and assumptions. Proofs can vary in level of formality and amount of detail. Sometimes, for example, a brief sketch of a proof is more informative than a formal proof that attends precisely to every detail. At other times, it might be important to check every step very carefully.

Proofs can be presented in various formats, including paragraphs, flow charts, or two columns. In geometry, proofs can be based on transformations, triangle congruence, or coordinates. Geometry proofs can make use of algebra; algebra proofs can make use of geometry. This sort of diversity is worth encouraging, for understanding is more stable and flexible when it is informed by multiple perspectives.

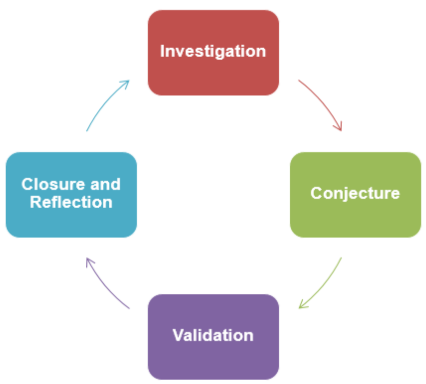
When aiming to prove a statement, it helps to reframe it in the form, "If A then B." For example,

Statement	"If A then B" form
The sum of two odd numbers is even.	If two numbers are odd, then their sum is even.
Opposite sides of a parallelogram are congruent.	If a figure is a parallelogram then its opposite sides are congruent.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.CO.9-11)

In this form, A is called the hypothesis, and B is called the conclusion. The idea is, "Whenever (hypothesis) A is true, it follows that (conclusion) B must also true." Proofs may consist of one or more statements. Each justified statement follows from the previous one(s). Each conclusion may be supported either by a particular definition, a particular postulate, a particular theorem, particular property, etc.

Process of Proving



Uncertainty drives proof and makes it meaningful. Avoid asking students to prove a routine theorem they already know to be true for it can be demotivating. Instead provide them with tasks that need proofs, so they can participate in the process of proving as they have a reason to provide a proof. Give students an opportunity to investigate figures. Have them list things that they are most likely true, most likely false, or things that tend to be true or false. Use their observations as a launching pad into proving different theorems. Point out that since nothing has yet to be proven, students are just making conjectures based on their observations. Challenge students when they give imprecise conjectures or definitions. (This cluster should reemphasize the precise use of definitions in G.CO.1.) It may be more useful to highlight proofs that promote understanding.

A conjecture is an educated guess or opinion. To tell whether a conjecture is true or false, students should usually start by examining instances. For conjectures about geometric figures, this means that drawings are made and explored. If even one counterexample is found, the conjecture is not true. If a counterexample is not found, there is evidence that the conjecture is true. Still, for a conjecture to be accepted as true for all cases, it must be proved.

Students tend not to write conjectures as conditional statements, and therefore have difficulty understanding what they are supposed to be proving in a particular figure. Given a figure, have students practice writing their own conjectures, and then practice rewriting them in conditional statements using the words if and then. (Nirode, 2019)

Students may incorrectly think that a conjecture is true because it worked in all examples that were explored, when there may be some types of examples that were not explored, and therefore the conjecture may be false. Teachers should challenge students to find another method (instead of using examples) that is valid.

When conjectures are not written in biconditional statements, students oftentimes confuse the hypothesis with the conclusion. They have difficulty selecting from the conjecture what they are assuming to be true and what they are trying to prove and applying to particular figures. Once students rewrite the conjectures, then have them identify the hypothesis and the conclusion. (Nirode, 2019)

Students may incorrectly think that they only need to provide an example or two to make a conclusion, but a proof is needed because it covers all possible examples.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.CO.9-11)

VAN HIELE CONNECTION

In Geometry students are expected to move from Level 2 (Informal Deduction/Abstraction) to Level 3 (Deduction). Whereas in Level 2 students use informal arguments, Level 3 requires a more formal approach to thinking.

Level 3 can be characterized by the student doing some or all of the following:

- constructing proofs and understanding the necessity of proofs;
- understanding the role and relationships between of axioms, theorems, definitions, and postulates;
- differentiating between necessary and sufficient conditions; and/or
- making logical conclusions.

PROOFS

Here is general information about proof instruction:

- The main concept that should be gleaned from these standards is that students need to be able to explain reasons for their thinking and why/how something is true, much like in the ELA Writing Standards where *evidence* must be included for any claim.
- Geometry is visual and should be taught in ways that leverage this aspect. Sketching, drawing and constructing figures and relationships between geometric objects should be central to any geometric study and certainly to proof. (Niven, 1987)
- Exploring the history of geometry and real-world applications may help students develop conceptual understandings before they begin to use formal proof.
- Teachers should vary the level of formality that is appropriate for the content, and the reader of the proof. However, every level of formality includes students' ability to formally/informally reference the appropriate source—a definition, a property, a law, an axiom, a theorem, etc. Make sure that the level of formality does not distract from the main idea of the proof.
- Proving methods could include but are not limited to the following: deductive, inductive, two-column, paragraph, flow chart, visual, and/or counterexample.
- Using dynamic geometric software allows students to make conjectures that can, in turn, be formally proven. For example, students might notice that the base angles of an isosceles triangle always appear to be congruent when manipulating triangles on the computer screen. This could lead to a more formal discussion of why this occurs every single time.
- The emphasis should be on a progression toward proof and not on formal proof. Students need to be able to come up with their own conjectures and then provide mathematically sound justification for the conjectures' validity. Ultimately students should construct a complete argument, in a variety of formats, to move from given information to a conclusion.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.CO.9-11)

- Direct instruction is not the best way to introduce formal proof. Instead the focus should "be on justifying ideas while helping students build the visual and empirical foundation for higher levels of geometric thought, we can lead students to appreciate the need for formal proof." (Battista and Clements, 1995)
- A complete argument should include a marked diagram, when applicable. For example, without a marked diagram a student may mistake congruent triangles by AAS instead of ASA.
- Students continue to use precise language and relevant vocabulary to justify steps in their work to construct viable arguments that defend their method of solution.
- All statements need to be examined on their own merit—conditionals and their converse statements are not always both true. For the times when both conditionals and their converses are true, biconditional statements can be written.
- The concepts of inverse and contrapositive are not emphasized but can be explored.
- A valuable activity is to have students critique each other's work. A good resource with potential problems on page 7 (labeled p. 5): Introduction to Proof.
- Coordinate proof is not that main focus of this cluster, but it can be used. Notice that coordinate proof is addressed in G.GPE.4 and 5.

TIP!

Students may think that justifications are not needed for statements they view as "obvious." Emphasis should be placed on providing students with examples contradicting the "obvious" or giving reasoning for every claim to create a complete argument.

Students often struggle in a proof situation that requires the converse of a theorem in the proof.

TIP!

Student oftentimes incorrectly think they can draw conclusions from a diagram based on what "looks" true rather then what is stated in the given and appropriate definitions. To address this misconception, teach students to explicitly draw conclusions *before* teaching proof. For example, give students a diagram and have them state two things they can conclude from the diagram and two things they cannot conclude from the diagram. (Nirode, 2019)

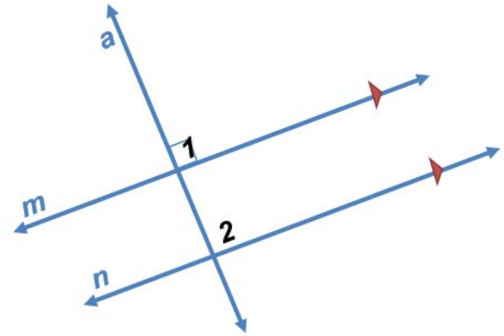
Proofs about Lines and Angles and their Applications

- Problem situations should involve algebraic relationships with and without labeled diagrams.
- Some of the theorems listed in this cluster (e.g. the ones about alternate interior angles and the angle sum of a triangle) are logically equivalent to the Euclidean Parallel Postulate and should be acknowledged.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.CO.9-11)

EXAMPLE

If you are given that $a \perp m$ and $m \parallel n$, what kind of conjectures about lines and angles can you draw from the given information?



Proof: A student may want to prove that if a transversal is perpendicular to one of the lines included in the set of parallel lines, then it is perpendicular to the rest of the parallel lines. Angle 1 is 90° because it is formed by two perpendicular lines. Since $\angle 1$ and $\angle 2$ are formed by parallel lines cut by a transversal, they are corresponding angles, and by the Corresponding Angles Criteria they are congruent. Therefore $\angle 2$ is also 90° and line $a \perp n$. Therefore, in a plane, if a transversal is perpendicular to one of the lines included in the set of parallel lines, then it is perpendicular to the rest of the parallel lines.

Students may incorrectly try to apply the Transitive Property by writing statements with mixed symbols such as if $a \perp m$ and $m \perp n$, then $a \parallel n$. Explain to students that the symbols must be the same in all three statements in order to apply the Transitive Property.

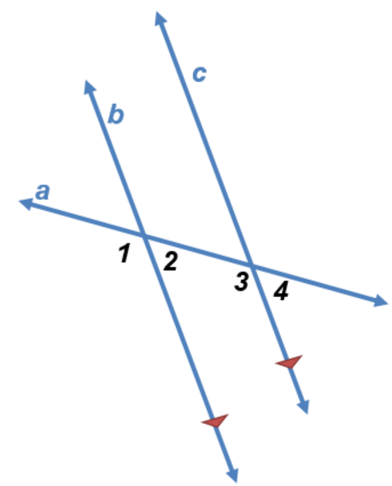
EXAMPLE

Prove that if two parallel lines are cut by a transversal, then the consecutive exterior angles are supplementary.

Given: Lines b and c are parallel.

Statement	Reason
Lines *b* and *c* are parallel and cut by a transversal.	Given
$\angle 1$ and $\angle 2$ are supplementary.	Two angles that form a linear pair are supplementary.
$\angle 3$ and $\angle 4$ are supplementary.	Two angles that form a linear pair are supplementary.
$\angle 1 \cong \angle 3$	If two $\parallel$ lines are cut by a transversal, then corresponding angles are $\cong$.
$\angle 1$ and $\angle 4$ are supplementary.	Substitution

Statement	Reason
Lines *b* and *c* are parallel and cut by a transversal.	Given
$\angle 1$ and $\angle 2$ are supplementary.	Two angles that form a linear pair are supplementary.
$\angle 2 \cong \angle 4$	If two $\parallel$ lines are cut by a transversal, then corresponding angles are $\cong$.
$\angle 1$ and $\angle 4$ are supplementary.	Substitution



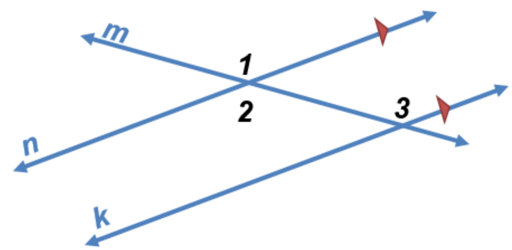
Prove: Angle 1 and Angle 4 are supplementary angles.

Discussion: There are generally several acceptable ways to prove a statement. As students become more advanced in their mathematical thinking, push them toward more efficient proofs.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.CO.9-11)

EXAMPLE

Assuming that the Vertical Angles Conjecture (If two angles are vertical, they are equal in measure.) is true and that the Corresponding Angles Conjecture (If two parallel lines are cut by a transversal, then their corresponding angles are congruent.) is true. Then write a proof showing the Alternate Interior Angles conjectures (If two parallel lines are cut by a transversal, then their alternate interior angles are congruent.) is also true.



Proof: We can draw a pair of vertical angles and label them $\angle 1$ and $\angle 2$. Then we can draw line *k* so that it is parallel to line *n*. Angle $\angle 3$ would be a corresponding angle to $\angle 1$ and an alternate interior angle to $\angle 2$. Since $\angle 2 \cong \angle 1$ (vertical angles) and $\angle 1 \cong \angle 3$ (corresponding angles), then $\angle 2 \cong \angle 3$ (alternate interior angles) is true because of the Transitive Property.

Discussion: This is an example of a paragraph proof. After doing several proofs about pairs of angles formed by parallel lines and a transversal, it is important for students to understand that any pair of congruent angles such as alternate interior angles (not only corresponding angles are congruent) can be used as the given to prove that the other pair of angles (e.g., alternate exterior angles) is also congruent.

Students may incorrectly think that the converse of a statement is always true. For example, if the statement is "Parallel lines do not intersect.", then the converse would be "Lines that do not intersect are parallel.", which is not always true.

Proofs about Triangles and their Applications

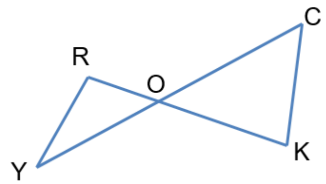
When being introduced to proofs, students can begin with labeled diagrams to draw conclusions. Then require students to label/mark their own diagrams to draw conclusions. Students should move towards writing a complete argument, in a variety of formats, to move from given information to a conclusion, including drawing and marking a diagram.

After students are familiar with the transformational definition of congruence in terms of rigid motions, have them prove that the base angles of an isosceles triangle are congruent.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.CO.9-11)

Students often may incorrectly think they cannot make any assumptions about diagrams. To correct this misconception, have students be explicit about what assumptions they can and cannot make about diagrams. For example, students can make assumptions about relative location, collinearity, and betweenness, but they cannot make assumptions about measurements or congruence. (Nirode, 2019)

What can you assume about the diagram?



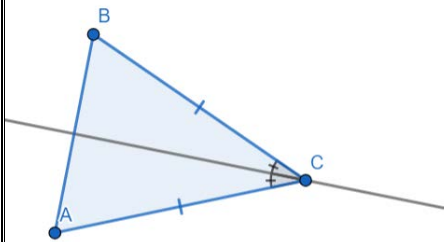
Taken from Addressing Misconceptions in Secondary Geometry Proofs, 2019, Cirillo and Hummer.

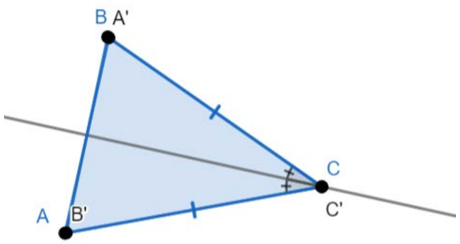
Can **Assume**	***Cannot*** **Assume**
If any two lines intersect, they intersect at a point.	If lines look parallel or perpendicular, they may not be.
If points look collinear, they are.	If angles or segments look congruent, they may not be.
Relative location of points—if a point looks like it is between two other points or to the right or to the left of a point, it is.	Relative size of segments and angles—if one angle (or line segment) looks bigger than another, it may not be.
If angles look adjacent, they are.	If angles look like right angles, they may not be.
If angles look like vertical angles, they really are.	

EXAMPLE

Base Angles of an Isosceles Triangle

Prove that the base angles of an isosceles triangle are congruent.





Discussion: Students may also use transformations to prove theorems. This can be done using transparencies, tracing paper, or dynamic geometric software. *Note: In order to use congruence in an explanation or a proof, the rigid motion(s) need to be specified.* By the definition of an isosceles triangle, the two sides of the triangle are congruent. A student can create an angle bisector through the vertex of the isosceles triangle creating two congruent angles. A student may choose to use the angle bisector as a line of reflection, reflecting the entire triangle over the angle bisector (so A' maps to B and B' maps to A). Because reflections preserve angle measure and side length, the two congruent sides are taken to each other and maps the triangle on to itself. Since the triangle is mapped onto itself, it means that the base angles also map onto each other, so they must be congruent.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.CO.9-11)

After proving a generalization, students may still incorrectly believe that exceptions to the generalization might exist. Ask students to find an example that counters the generalization and see if this example exists.

EXAMPLE

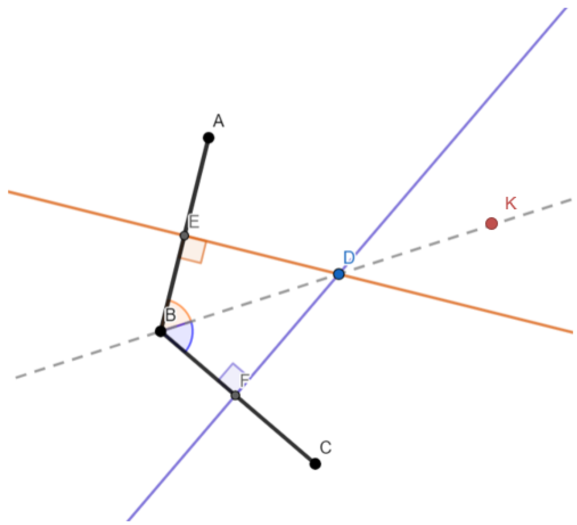
Angle Bisectors

Prove that if a point is on the angle bisector, then it is equidistant from the sides of the angle.

Discussion:

Method 1

Using geometric software such as GeoGebra, students can construct an angle bisector. Then they can construct a perpendicular line between the point on the angle bisector to each side of the angle. The advantage of using geometric software is that students can manipulate the angle and point on the angle bisector to show how the angle bisector and the perpendicular lines remain even when the angle size is adjusted or the point on the angle bisector moves. Then students could use triangle congruence to prove that the point on the angle bisector is equidistant from the sides.



- $\overline{BD} \cong \overline{BD}$ because it's the same line segment
- $\triangle EBD \cong \triangle FBD$ because of AAS
- $\overline{ED} \cong \overline{DF}$ because of CPCTC

Therefore, a point on an angle bisector is equidistant from the sides of its angle.

Method 2

To prove that a point on the angle bisector is equidistant from the sides of angle, the definition of the distance between the point and a line should be recalled: a distance between a point and a line is the length of a perpendicular line segment that goes from that point toward the line. Select any point D on the angle bisector $\overline{BK}$. Draw two perpendicular lines from point D toward the sides of the angle BAC. The lengths of the line segment $\overline{DE}$ and $\overline{DF}$ respectively represent the distances from a point D to the sides $\overline{BA}$ and $\overline{BC}$. Two right triangles EBD and FBD are congruent by AAS (Angle BED is congruent to angle BFD since they are both right angles. Angle EBD is congruent to angle FBD by the definition of the angles bisector. $\overline{BD} \cong \overline{BD}$ by the Reflexive Property). Since sides $\overline{ED}$ and $\overline{DF}$ are congruent by CPCTC, their lengths are equal. Since point D was selected randomly, it is proved that any point on the angle bisector is equidistant from the sides of the angle.

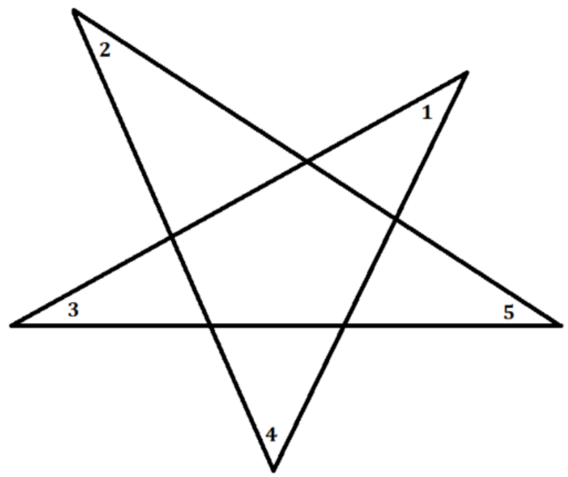
INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.CO.9-11)

EXAMPLE

Exploring a Proof Without Words

Part 1

What is the sum of the 5 vertex angles?



Part 2

- **a.** How was it determined that one of the angles equals $m\angle 2 + m\angle 4$ and the other equals $m\angle 1 + m\angle 3$?
- **b.** What does the dotted line going upward from angle 5 represent (paying attention to the labeled angles)?
- **c.** Would the ideas represented by the image be applicable to 5-pointed stars of any shape?

Discussion: Students do not always need to use words for proofs. Proofs without words are a viable method for proving theorems and conjectures at this level. Although a proof without words can be useful as a context for discussion, a proof without words can also stand alone. Students should explore how to explain what is represented and possibly add their own written descriptions. (Some conjectures/theorems lend themselves more to this strategy than others.) For part **a.** students can utilize the fact the sum of the angles in a triangle = 180° with various triangles within the star. For part **b.** students realized that the dotted line is drawn in and is parallel to the line segment adjacent to angle 1 and angle 4 so that students make the connection to the properties of parallel lines intersected by a transversal. Part **c.** shows how this image can be a "proof" instead of just an example.

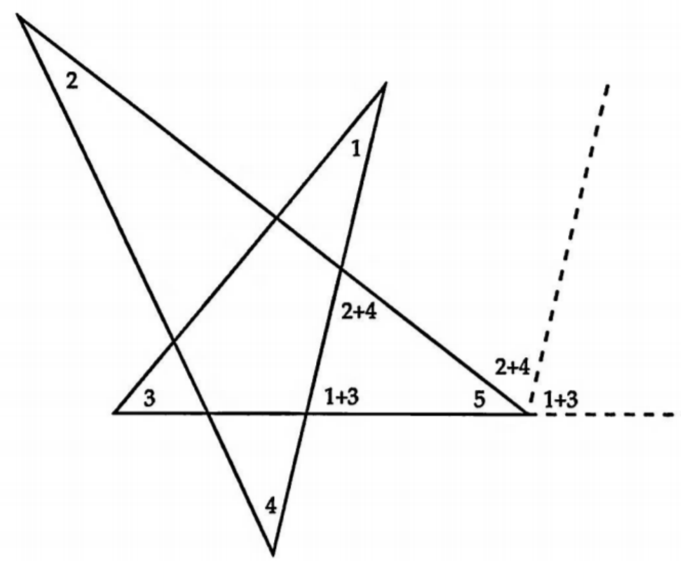


Image taken from Fouad Nakhli's [Proof Without Words] published by The Mathematical Association

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.CO.9-11)

TIP! Explicitly modeling mathematical thinking in classroom discussion helps students reason through proofs on their own.

Proofs about Parallelograms and their Applications

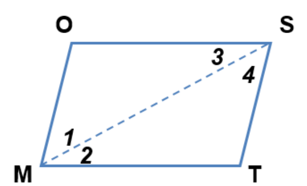
Students will combine the definition of a parallelogram with triangle congruence criteria and previously proved theorems to yield properties of parallelograms assumed in earlier grades. A variety of proof formats can be used here, including those based in symmetrical and transformational arguments.

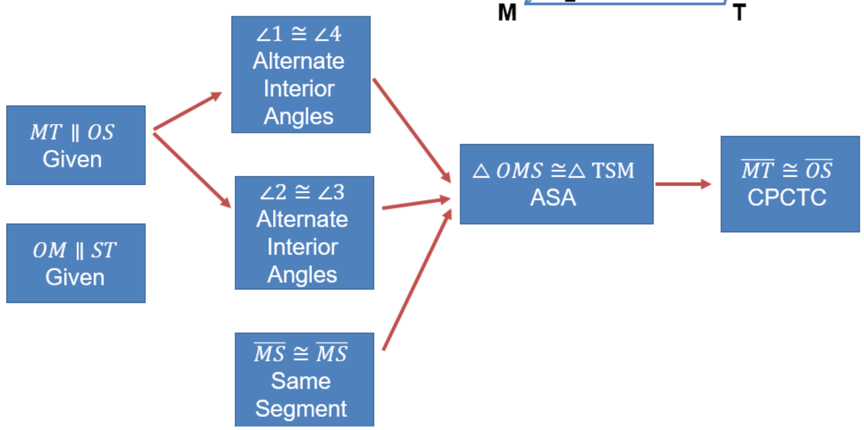
EXAMPLE

Proof about a Parallelogram

Given: $MT \parallel OS$
$OM \parallel ST$

Prove: $\overline{MT} \cong \overline{OS}$





Discussion: This proof could be given after students are asked what they know about parallelograms. Some students may say that opposite sides are congruent. Then students could be challenged about how they know that its true (besides the teacher told them). This example shows a flow chart proof. A flow chart could be scaffolded for struggling learners in many ways. One way is by giving them the boxes and having them put them in order. Another way could be to give students the boxes with the statements and have them come up with the rationales or give students the rationales and have them come up with the statements. Another way could be by only giving students some of the boxes. The goal should be for students to prove the statement independently without any scaffolding, but it may take time and experience.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.CO.9-11)

TIP!

Have some students present their proofs and other students critique the proofs. This promotes communication, critical thinking, and mathematical reasoning.

Indirect Proof

Explain to students that sometimes it is easier to disprove a conjecture than to prove it. This is called an indirect proof. Practice in developing statements that lead to contradiction helps bridge students to indirect proof.

Students may incorrectly think that more than one counterexample is necessary for disproving a statement. It would be valuable to explore multiple counterexamples, but teachers should emphasize that only one is necessary.

TIP!

Although not required in the standards, some students may benefit from exploring rules of inference. Truth tables may help students develop logical reasoning and see how conditional statements apply under different circumstances.

Instructional Tools/Resources

These tools and resources may be helpful for instruction. Before use, they should be evaluated at the local level to determine their appropriateness for instruction.

Manipulatives/Technology

- Computer dynamic geometric software (such as Geometer's Sketchpad®, Desmos, Cabri®, or GeoGebra®).
- Graphing calculators and other handheld technology such as TI-Nspire™.
- Tracing paper (patty paper)
- Transparencies
- Graph paper
- Ruler
- Protractor
- Compass
- Proofs Without Words: Exercises in Visual Thinking

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.CO.9-11)

Proofs about Lines and Angles

- Points Equidistant from Two Points in the Plane by Illustrative Mathematics is a task where students prove that the perpendicular is exactly the set of points equidistant to the endpoints of the segment.
- Congruent Angles Made by Parallel Lines and a Transverse by Illustrative Mathematics is a task where students prove congruence of vertical angles.

Proofs about Triangles

- Circles in Triangles by Mathematics Assessment Project has students apply theorems for circles inscribed in triangles.
- Classifying Triangles by Illustrative Mathematics is a task where students synthesize their knowledge of triangles.
- Midpoints of Triangle Sides by Illustrative Mathematics is a task where students use similarity transformations to relate two triangles.
- Congruent Angles in Isosceles Triangles by Illustrative Mathematics is a task where students establish that the base angles in an isosceles triangle are congruent.
- Sum of Angles in a Triangle by Illustrative Mathematics is a task where students provide an argument for the sum of the angles in a triangle at the high school level.

Proofs about Quadrilaterals

- Quadrilaterals is a Performance Assessment Task by Inside Mathematics where students use geometric properties to solve a problem.
- Midpoints of the Sides of a Parallelogram by Illustrative Mathematics is a task where students use previously known facts about parallelograms to prove new facts.
- Is This a Parallelogram? by Illustrative Mathematics is a task where students develop an alternative characterization of a parallelogram in terms of congruence of opposite sides.

Curriculum and Lessons from Other Sources

- EngageNY, Geometry, Module 1, Topic C, Lesson 18: Looking More Carefully at Parallel Lines is a lesson that pertains to this cluster.
- EngageNY, Geometry, Module 1, Topic D, Lesson 26: Triangle Congruency Proofs, Lesson 27: Triangle Congruency Proofs are lessons that pertain to this cluster.
- Georgia Standards of Excellence Curriculum Frameworks, Geometry, Unit 2: Similarity, Congruence and Proof has many tasks that pertain to this cluster.
- EngageNY, Geometry, Module 1, Topic E, Lesson 28: Properties of Parallelograms, Lesson 29: Special Lines in Triangles, Lesson 30: Special Lines in Triangles are lessons that pertain to this cluster.
- EngageNY, Geometry, Module 1, Topic G, Lesson 33: Review of the Assumptions, Lesson 34: Review of the Assumptions, are lessons that pertain to this cluster.
- Mathematics Vision Project, Geometry, Module 2: Congruence, Construction, and Proof has many tasks that pertain to this cluster.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.CO.9-11)

Curriculum and Lessons from Other Sources, continued

- Illustrative Mathematics, Geometry, Unit 1, Lesson 19: Evidence, Angles, and Proof, Lesson 20: Transformations, Transversals, and Proof, Lesson 21: One Hundred and Eighty are lessons that pertain to this cluster.
- Illustrative Mathematics, Geometry, Unit 2, Lesson 8: The Perpendicular Bisector Theorem, Lesson 10: Practicing Proofs, Lesson 12: Proofs about Quadrilaterals, Lesson 13: Proofs about Parallelograms, Lesson 14: Bisect It, Lesson 15: Congruence for Quadrilaterals are lessons that pertain to this cluster.
- Illustrative Mathematics, Geometry, Unit 7, Lesson 6: A Special Point is a lesson that pertains to this cluster.

General Resources

- Arizona 7-12 Progression on Geometry is an informational document for teachers. This cluster is addressed on pages 15-16.
- High School Coherence Map by UnboundED is a map of the relationships among high school mathematics standards. Please note: This resource is aligned to the Common Core not Ohio's Learning Standards. Although the tool is useful, it needs to be checked with respect to Ohio's Learning Standards.
- van Hiele Model of Geometric Thinking is a pdf created by ODE that summarizes the van Hiele levels.

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STANDARDS	MODEL CURRICULUM (G.CO.12-13)
Geometry **CONGRUENCE** **Make geometric constructions.** **G.CO.12** Make formal geometric constructions with a variety of tools and methods (compass and straightedge, string, reflective devices, paper folding, dynamic geometric software, etc.). *Copying a segment; copying an angle; bisecting a segment; bisecting an angle; constructing perpendicular lines, including the perpendicular bisector of a line segment; and constructing a line parallel to a given line through a point not on the line.* **G.CO.13** Construct an equilateral triangle, a square, and a regular hexagon inscribed in a circle.	**Expectations for Learning** In elementary and middle school, students learn to use measurement tools to informally draw geometric shapes with given conditions. In this cluster, students make formal and precise constructions using a variety of tools, and they understand the geometric relationships upon which the constructions are based. The student understanding of this cluster begins at van Hiele Level 1 (Analysis) and moves to Level 2 (Informal Deduction/Abstraction). **ESSENTIAL UNDERSTANDINGS** • Construction is a process of reasoning that does not use a scale and does not use measurement. • Simple constructions can be used to develop an understanding of mathematical relationships. **MATHEMATICAL THINKING** • Make sound decisions about using tools. • Strategically use technology to deepen understanding. • Plan a pathway to complete constructions. • Determine accuracy of results. • Create a drawing and add components as appropriate. **INSTRUCTIONAL FOCUS** • Distinguish between a rough sketch, a careful drawing with measurements, and a construction with compass and straightedge. • Use a variety of geometric tools to make precise constructions. *Continued on next page*

STANDARDS	MODEL CURRICULUM (G.CO.12-13)
	Content Elaborations **OHIO'S HIGH SCHOOL CRITICAL AREAS OF FOCUS** • Geometry, Number 2, pages 4-5 **CONNECTIONS ACROSS STANDARDS** • Experiment with transformations in the plane (G.CO.1, 5). • Understand and apply theorems about circles (G.C.3, (+)4). • Prove and apply geometric theorems (G.CO.9-11). • Prove and apply theorems involving similarity (G.SRT.4-5).

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.CO.12-13)

Instructional Strategies

Note: The model curriculum is not a collection of lessons nor a full curriculum; it does not suggest pace, sequence, or amount of time spent on topics. It provides information about a topic related to the standards including ideas for examples, strategies for teaching, common misconceptions, and possible connections between topics. In addition, the emphasis of the model curriculum is on instruction not on assessment; therefore, the examples in the Instructional Strategies section are written for instructional purposes not necessarily as assessment items. Examples may need to be rewritten to accommodate the needs of each individual classroom.

Standards for Mathematical Practice

This cluster focuses on but is not limited to the following practices:

MP.1 *Make sense of problems and persevere in solving them.*

MP.2 *Reason abstractly and quantitatively.*

MP.4 *Model with mathematics.*

MP.5 *Use appropriate tools strategically.*

MP.6 *Attend to precision.*

VAN HIELE CONNECTION

In Geometry students are expected to move from Level 1(Analysis) to Level 2 (Informal Deduction/Abstraction). In van Hiele Level 1 students may list all the properties of a shape, but they may not see the relationships between properties. Whereas van Hiele Level 2 can be characterized by the student doing some or all of the following:

- noticing relationships within and between properties and figures;
- forming meaningful definitions;
- justifying reasoning using informal arguments;
- drawing logical maps and diagrams; and/or
- not yet understanding formal deduction.

This cluster when combined with G.CO.9-11 will also move into Level 3 (Deduction) where students start to construct proofs and understand the necessity of proofs.

See the van Hiele pdf on ODE's website for more information about van Hiele levels.

MAKING FORMAL CONSTRUCTIONS

Mathematicians have been making constructions using a compass and straight edges without markings since ancient Greece. This is because handheld measurement tools, such as protractor or ruler, result in approximations while constructions, supported by formal Geometry, are theoretically perfect. Constructions also allow students to have a kinesthetic learning experience and make proofs relevant as they have to justify the correctness of their construction. This cluster should be taught in conjunction with other clusters such as G.CO.1-5 and G.CO.9-11.

It is more important that students understand why the construction works instead of manually performing the construction by memorizing steps. For students who are struggling because of dexterity problems, geometric software may be useful.

Some students may believe that a construction is the same as a sketch or drawing. Emphasize the need for precision and accuracy when doing constructions.

TIP!

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.CO.12-13)

Provide opportunities for students to practice executing basic constructions. Here is a list of some important constructions. Classroom instruction should include but not be limited to these constructions:

- Reproduce a line segment on a ray with a specified endpoint.
- Construct an equilateral triangle on a given side.
- Reproduce and angle with one side specified.
- Construct a line perpendicular to a given line from a given point.
- Construct the perpendicular bisector of a line segment.
- Construct an angle bisector.
- Construct a line parallel to a given line through a given point.
- Divide a given line segment into any number of equal segments.
- Construct an equilateral triangle inscribed in a square.
- Construct a regular hexagon inscribed in a circle.
- Draw tangents to a circle from a point outside the circle.
- Construct the sum, difference, product, and quotient of two given positive numbers.
- Construct the square root of a positive number.

(List taken from Wu, 2013, Teaching Geometry in Grade 8 and High School According to the Common Core Standards, page 145)

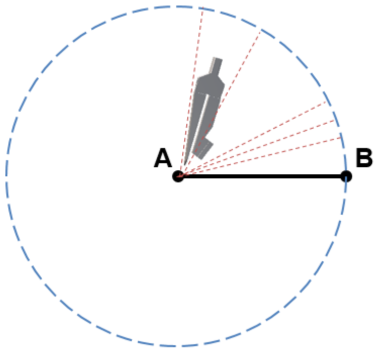
Students should not only construct figures, but they should be able to justify or prove why they are correct. Oftentimes the students will utilize the congruence theorems.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.CO.12-13)

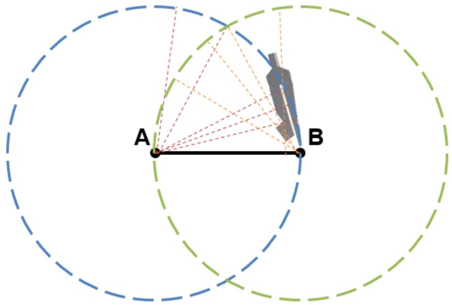
EXAMPLE

Constructing an Equilateral Triangle

Construct an equilateral triangle using $\overline{AB}$ as one of its sides.

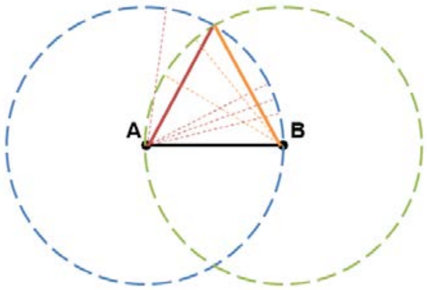


Discussion: Ask students what they know about equilateral triangles. They should recall that an equilateral triangle has all equal sides and angles. They may also know that all angles have a measure of 60°. Prepare students to connect their future constructions to the circles. Ask students what they know about circles. They should say that all radii in a circle have the same length. Constructing a circle may be helpful for constructing an equilateral triangle using two distinct radii of the circle as two out of the three congruent sides of an equilateral triangles is the initial thought. A student may start with their compass point on point A and extend it to point B to make the circle with radius $\overline{AB}$, but now the question must be asked "Which radius will be part of the equilateral triangle?" Although, some students may try to eyeball one that is 60°, emphasize that it must be exact. This may also be good place to give students the opportunity to recall the formal definition of an equilateral triangle and a circle.



Ask students how they could get another side of the triangle. Hopefully a student will come up with the idea of drawing another circle using the other point on the line segment (in the diagram point B.) Again, have students explain that all radii in a circle are congruent. Have students decide which radii should be used to make an equilateral triangle. They should come to the conclusion that the radii needed are those that intersect each other and also the intersection point of the circle. This is because the point of intersection of the two circles along with points A and B will become the vertices of the equilateral triangle.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.CO.12-13)



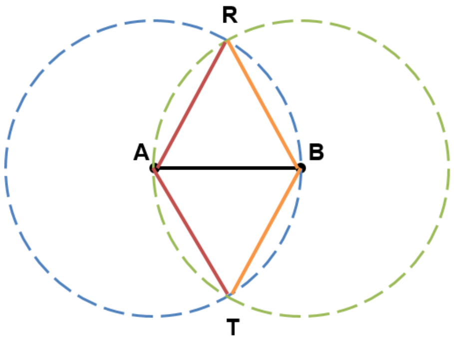
To reinforce the idea, ask students how they can prove that the triangle they constructed is in fact an equilateral triangle. They should state that it is an equilateral triangle because all the sides of the triangle are radii of congruent circles. Therefore, all three side lengths are congruent, and the triangle is an equilateral triangle.

EXAMPLE

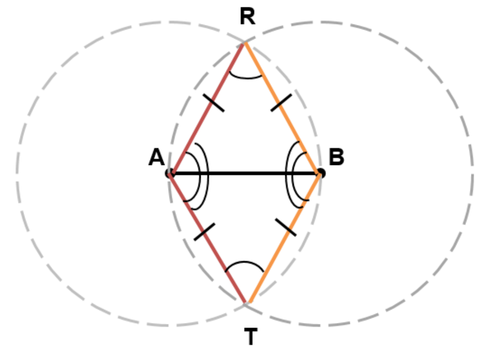
Making Observations and Justifying Thinking in Relation to Constructions

Construct the perpendicular bisector of a line segment.

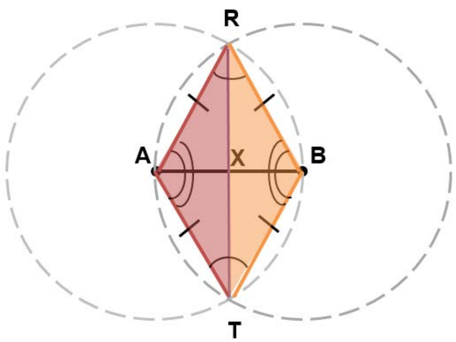
Discussion: Before having students construct, use what students learned from the previous example about equilateral triangles. Label where the circles intersect R and T, and draw line segments connecting R and T to A and B forming equilateral triangles. Instead of telling students what to prove, ask students what they are able to prove. Tell students to share everything they know to be true about the picture, justifying their thinking. (*Note: Letting students direct the instruction may result in many other theorems being proved before arriving at constructing the perpendicular bisector.)* Although your goal may be to have students construct the perpendicular bisector of a line segment, let them form their own goals. The ultimate goal is to let students make observations and justify their thinking to think mathematically about constructions. *Note: Students may skip all the proof steps in this example if they have already proved previously in class that the diagonals of a rhombus are perpendicular bisectors.*

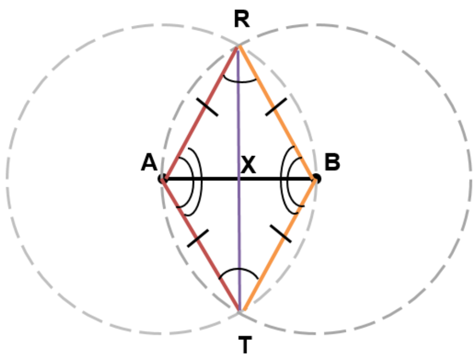


INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.CO.12-13)

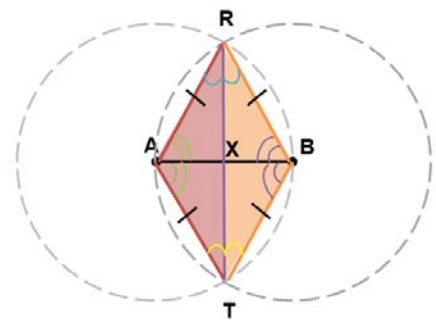


- Students should realize that $\triangle ARB \cong \triangle ATB$ because they are both equilateral triangles whose side lengths are radii of two congruent circles.
- They should eventually come to the conclusion that, by definition, quadrilateral $ARBT$ is a rhombus because sides $\overline{AR}, \overline{RB}, \overline{BT},$ and $\overline{AT}$ are congruent since the side lengths are radii of two congruent circles.
- Since $\triangle ARB \cong \triangle ATB$, students should realize that $\angle ARB \cong \angle ATB$ since they are both angles in an equilateral triangle.
- Students know that $\angle RAB \cong \angle BAT \cong \angle RBA \cong \angle ABT$ because they are angles in two congruent equilateral triangles. Then they can see that $\angle RAT \cong \angle RBT$ because $\angle RAB + \angle BAT \cong \angle RBA + \angle ABT$ due to the Substitution Property and the Angle Addition Postulate. **Therefore, they have just proved that opposite angle in a rhombus are congruent.**





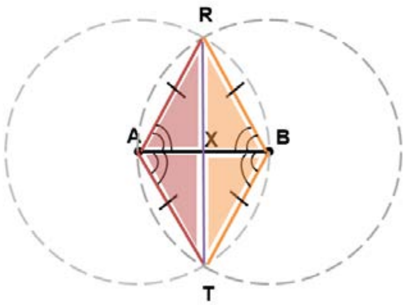
- Now draw a line segment connecting point R and point T, and ask students to share everything they now know about the picture to be true and justify their thinking.
- They may say that $\triangle ART \cong \triangle BRT$ by SAS.

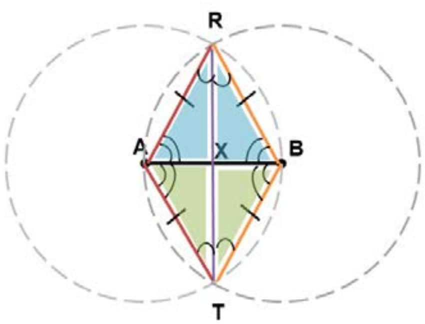


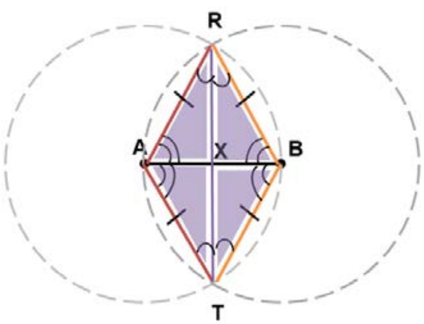
- Since $\triangle ART \cong \triangle BRT$, then $\angle BRX \cong \angle ARX$ and $\angle ATX \cong \angle BTX$ by CPCTC, which means $\overline{RT}$ bisects $\angle ARB$ and $\angle BTA$.
- Since $\angle RAX \cong \angle TAX$ and $\angle RBX \cong \angle TBX$ because they are all angles in congruent equilateral triangle, they should come to the conclusion that $\overline{AB}$ bisects $\angle RAT$ and $\angle RBT$ which means the students just proved **that the diagonals of a rhombus bisect its angles.**

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.CO.12-13)

- Since $\overline{AX} \cong \overline{AX}$ and $\overline{XB} \cong \overline{XB}$ due to the Reflexive Property $\angle RAX \cong \angle TAX$ and $\angle RBX \cong \angle TBX$ because they are all angles in congruent equilateral triangle, they should also see that $\triangle RAX \cong \triangle TAX$ and $\triangle RBX \cong \triangle TBX$ by SAS.

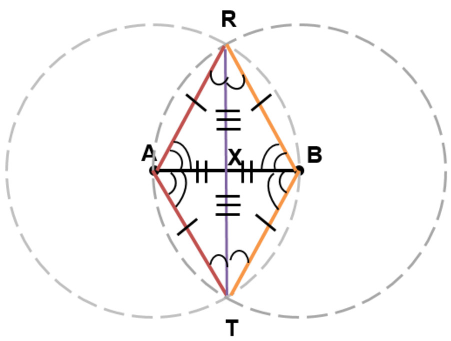






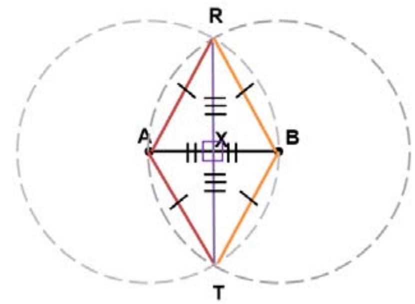
- Students can also say $\triangle RAX \cong \triangle RBX$ and $\triangle TAX \cong \triangle TBX$ by ASA, so by the Substitution Property $\triangle RAX \cong \triangle RBX \cong \triangle TAX \cong \triangle TBX$.

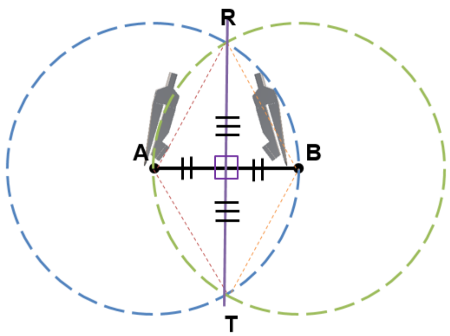
- This means that $\overline{AX} \cong \overline{XB}$ and $\overline{RX} \cong \overline{XT}$ by CPCTC. Therefore X is the midpoint of line segments $\overline{AB}$ and $\overline{RT}$, and $\overline{RT}$ bisects $\overline{AB}$ at point X and $\overline{AB}$ bisects $\overline{RT}$ at point X, so students just proved that the **diagonals of a rhombus bisect each other.**



INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.CO.12-13)

Students should be able to reason that the measure of each angle in an equilateral triangle equals 60° so $\angle RAX, \angle TAX, \angle TBX, \angle ATB,$ and $\angle ARB = 60°$, and since $\overline{RT}$ bisects $\angle ATB$ and $\angle ARB$ angles $\angle ARX, \angle BRX, \angle ATX,$ and $\angle BTX = 30°$. Since all angles in a right triangle must equal 180°, $\angle AXR, \angle BXR, \angle AXT,$ and $\angle BXT = 90°$ because $180° – 60° – 30° = 90°$, therefore students just proved **that diagonals of a rhombus are perpendicular bisectors of each other**.





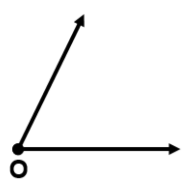
Now bring students back to the original question: Construct the perpendicular bisector of a line segment. Ask students if they can see that $\overline{RT}$ is the perpendicular bisector of $\overline{AB}$ and vice versa. Point out that the endpoints of the two-line segments of $\overline{AB}$ and $\overline{RT}$ become vertices of the rhombus. So, by constructing two equilateral triangles with the common side $\overline{AB},$ they constructed a rhombus whose diagonals are perpendicular segments and angle bisectors. Therefore, they can conclude that if they construct a rhombus, they should also get a perpendicular bisector. Now, have them construct a rhombus using a compass and straight edge. If they get stuck, prompt them to start with constructing an equilateral triangle.

TIP! Draw attention to the fact that in many constructions there are congruent triangles that often make the construction work. For example, applying the Perpendicular Bisector Theorem (If a point is on the perpendicular bisector of a segments, then it is equidistant from the segment's endpoints.) helps create congruent triangles.

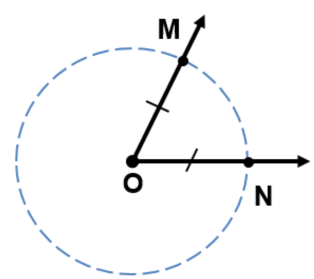
EXAMPLE

Making Observations and Justifying Thinking in Relation to Constructions

Construct the bisector of an angle.

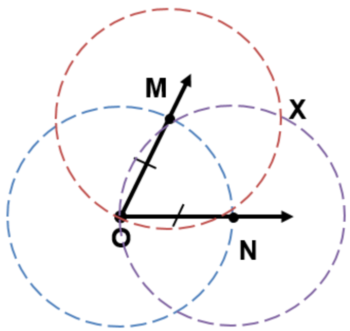


Discussion: Have students start with what they know. If they construct a circle with the center at the vertex O, then $\overline{OM} \cong \overline{ON}$ since they are both radii of the same circle. Ask students questions such as "What shapes have equal angles?" and then discuss which shapes with equal angles could be useful.

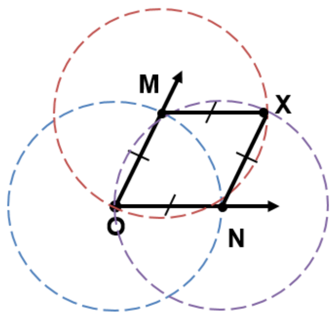


INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.CO.12-13)

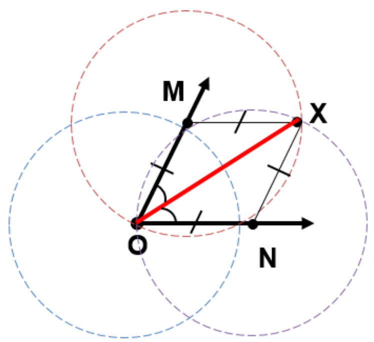
If students keep their compass the same length as $\overline{OM}$ and create two new circles with the center at M and N, students will see the new circles intersect at point X.



Students should also observe that $\overline{OM} \cong \overline{ON} \cong \overline{MX} \cong \overline{XN}$ since they are all radii of congruent circles, and quadrilateral $OMXN$ is a rhombus.



In the previous example students discovered that the diagonal of the rhombus was proved to be also an angle bisector. Now they can conclude that drawing the segment $\overline{XO}$ will bisect $\angle MON$.



Have students use different tools to make constructions. For instance, challenge students to perform the same construction using a variety of tools, for example, a compass or string. Have students compare dynamic geometric software commands to sequences of compass-and-straightedge steps.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.CO.12-13)

Students often incorrectly think that bisectors divide triangles in half or act as lines of symmetry. Misconceptions around bisectors can be difficult to correct. They have a difficult time drawing valid conclusions about angle bisectors, line segment bisectors, and perpendicular bisectors. When confronted with a bisector ask students, "What is being bisected?" or "What type of bisector is this?" Then reinforce the definition for the appropriate type of bisector. Formally teaching about and distinguishing between bisectors before teaching students to prove may be more effective. Draw attention to the fact that a perpendicular bisector is a special type of line segment bisector. (Nirode, 2019).

Instructional Tools/Resources

These tools and resources may be helpful for instruction. Before use, they should be evaluated at the local level to determine their appropriateness for instruction.

Manipulatives/Technology

- Compass
- Straightedge/Ruler
- String
- Origami paper
- Tracing paper
- Protractor
- Mira
- Computer dynamic geometric software (such as Geometer's Sketchpad®, Desmos, Cabri®, or GeoGebra®).
- Graphing calculators and other handheld technology such as TI-Nspire™.

Constructions

- Students will construct perpendicular bisectors while designing a delivery route for a local pizza shop, as described in Dividing a Town into Pizza Delivery Regions by NCTM Illuminations. Lead a class discussion where students will identify application of these skills across various career fields (e.g., landscaping, agriculture, construction, architecture, logistics). Students will apply the information to their plan for education and training through high school and beyond. *NCTM now requires a membership to view their lessons.*
- Constructions and Concurrencies by MathBitsNotebook.com is a webpage that contains many links to pages that explain constructions.
- Construction by Math Open Reference is a webpage that has links for tutorials on constructions. The students can follow through step-by-step or watch the entire way through independently.
- Geometric Constructions by GeoGebra is an interactive course on geometric constructions.
- Desmos Geometry allows students to make geometric constructions on a computer.
- Create My Constructions by Andrew Stadel is a Desmos activity that contains 6 challenges that help students and teachers become more familiar with the Desmos tool.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.CO.12-13)

Curriculum and Lessons from Other Sources

- EngageNY, Geometry, Module 1, Topic A, Lesson 1: Construct an Equilateral Triangle, Lesson 2: Construct an Equilateral Triangle, Lesson 3: Copy and Bisect and Angle, Lesson 4: Construct a Perpendicular Bisector, Lesson 5: Points of Concurrencies are lessons that pertain to this cluster.
- Georgia Standards of Excellence Curriculum Frameworks, Geometry, Unit 2: Similarity, Congruence, and Proof has lessons that pertain to this cluster. Lessons on construction start on page 77.
- Mathematics Vision Project, Geometry, Module 2: Congruence, Construction, and Proof has several lessons that pertain to this cluster.
- Illustrative Mathematics, Geometry, Unit 1: Constructions and Rigid Transformations, Lesson 1: Build It, Lesson 2: Constructing Patterns, Lesson 3: Constructing Techniques 1: Perpendicular Bisectors, Lesson 4: Construction Techniques 2: Equilateral Triangles, Lesson 5: Construction Techniques 3: Perpendicular Lines and Angle Bisectors, Lesson 6: Construction Techniques 4: Parallel and Perpendicular Lines, Lesson 7: Construction Techniques 5: Squares, Lesson 8: using Technology for Constructions, Lesson 9: Speedy Delivery, Lesson 22: Now What Can You Build? are lessons that pertain to this cluster.

General Resources

- Arizona 7-12 Progression on Geometry is an informational document for teachers. This cluster is addressed on page 16.
- High School Coherence Map by UnboundED is a map of the relationships among high school mathematics standards. Please note: This resource is aligned to the Common Core not Ohio's Learning Standards. Although the tool is useful, it needs to be checked with respect to Ohio's Learning Standards.
- van Hiele Model of Geometric Thinking is a pdf created by ODE that summarizes the van Hiele levels.

References

- Common Core Standards Writing Team. (2016, March 24). *Progressions for the Common Core State Standards in Mathematics (draft). Grades 7–8, HS, Geometry*. Tucson, AZ: Institute for Mathematics and Education, University of Arizona.
- Nirode, W. (April 2019). Going over the test. *Mathematics Teacher*, 112, (6), 422-425.
- Prasad, P. (November 2016). Leveraging interactive software to prompt discussion. *Mathematics Teaching in Middle School*, 22*(4),* 226-232.
- Wu, H. (2013). Teaching geometry according the Common Cores Standards. Retrieved from https://math.berkeley.edu/~wu/Progressions_Geometry.pdf
- Wu, H. (2013). Teaching geometry in Grade 8 and high school according the Common Cores Standards. Retrieved from https://math.berkeley.edu/~wu/CCSS-Geometry_1.pdf

STANDARDS	MODEL CURRICULUM (G.CO.14)
## Geometry **CONGRUENCE** **Classify and analyze geometric figures.** **G.CO.14** Classify two-dimensional figures in a hierarchy based on properties	## Expectations for Learning In elementary school, students learn to classify two-dimensional figures based on their properties. In middle school, students focus on drawing quadrilaterals and triangles with given conditions. Now in high school, they learn to analyze and relate categories of two-dimensional shapes explicitly based on their properties. Based on analysis of properties, students create hierarchies for two-dimensional figures. The student understanding of this cluster begins at van Hiele Level 1 (Analysis) and moves to Level 2 (Informal Deduction/Abstraction). **ESSENTIAL UNDERSTANDINGS** • There is a distinction between the definition of a figure and its properties, e.g., side lengths, angles, parallel/perpendicular sides, diagonals, symmetry. • Figures may be categorized in different ways based on their properties. **MATHEMATICAL THINKING** • Use accurate mathematical vocabulary to describe geometric relationships. • Make connections between terms and properties. • Recognize, apply, and justify mathematical concepts, terms, and their properties. • Generalize concepts based on patterns. • Use formal reasoning. *Continued on next page*

STANDARDS	MODEL CURRICULUM (G.CO.14)
	Expectations for Learning, continued **INSTRUCTIONAL FOCUS** • Explain the difference between the definition of a figure and its properties. • Know precise definitions of special polygons, e.g., rhombus, parallelogram, rectangle, square, kite, trapezoid, isosceles trapezoid, equilateral triangle, isosceles triangle, and regular polygon. • Compare and contrast definitions of quadrilaterals, including both definitions of trapezoids. • Know and apply properties of special polygons and use them to classify figures. • Explain the relationships among special quadrilaterals. • Explain the relationships among special triangles. • Create hierarchies in order to represent the relationship between pairs of figures and among several figures. **Content Elaborations** **OHIO'S HIGH SCHOOL CRITICAL AREAS OF FOCUS** • Geometry, Number 2, pages 4-5 **CONNECTIONS ACROSS STANDARDS** • Prove and apply theorems about quadrilaterals and triangles (G.CO.10-11). • Prove theorems algebraically about quadrilaterals and triangles using coordinates (G.GPE.4). • Justify the slope criteria for parallel and perpendicular lines (G.GPE.5). • Know precise definitions (G.CO.1).

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.CO.14)

Instructional Strategies

Note: The model curriculum is not a collection of lessons nor a full curriculum; it does not suggest pace, sequence, or amount of time spent on topics. It provides information about a topic related to the standards including ideas for examples, strategies for teaching, common misconceptions, and possible connections between topics. In addition, the emphasis of the model curriculum is on instruction not on assessment; therefore, the examples in the Instructional Strategies section are written for instructional purposes not necessarily as assessment items. Examples may need to be rewritten to accommodate the needs of each individual classroom.

VAN HIELE CONNECTION

In Geometry students are expected to move from Level 1(Analysis) to Level 2 (Informal Deduction/Abstraction). In van Hiele Level 1 students may list all the properties of a shape, but may not see the relationships between properties. Whereas van Hiele Level 2 can be characterized by the student doing some or all of the following:

- noticing relationships within and between properties and figures;
- forming meaningful definitions;
- justifying reasoning using informal arguments;
- drawing logical maps and diagrams; and/or
- not yet understanding formal deduction.

Standards for Mathematical Practice

This cluster focuses on but is not limited to the following practices:

MP.1 *Make sense of problems and persevere in solving.*

MP.3 *Construct a viable argument and critique the reasoning of others.*

MP.4 *Model with mathematics.*

MP.6 *Attend to precision.*

MP.7 *Look for and make use of structure.*

MP.8 *Look for and express regularity in repeated reasoning.*

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.CO.14)

See the van Hiele pdf on ODE's website for more information about van Hiele levels.

HIERARCHIES

This standard has moved from Grade 5 to high school, for research has shown that students are not ready to classify shapes into a hierarchy until high school. Hierarchies could include tree diagrams, Venn diagrams, or other conceptual maps (whatever that could be) and could be organized based on different properties of figures (diagonals, sides, angles, symmetry) and/or using different figures (quadrilaterals, triangles, polygons). Before students make a hierarchy of geometric figures, it may be useful to introduce hierarchies using non-mathematical ideas or by making cross curricular connections.

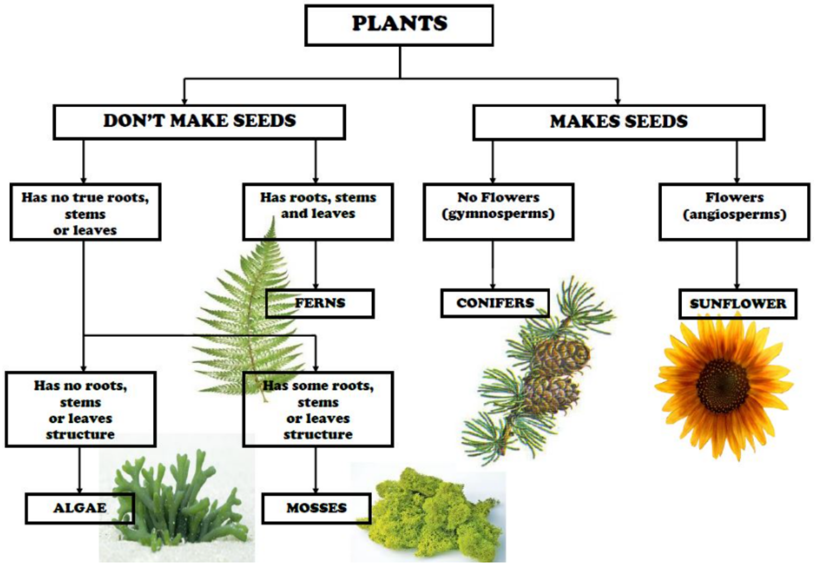


Image taken from Plant Classification Chart posted on Iman's Home-School page found at http://imanshomeschool.blogspot.com/2015/05/plant-classification-chart.html

EXAMPLE

Classifying Objects

Think of a real-life situation that you could classify. Create a flow chart and a Venn Diagram to illustrate your classification.

Discussion: Students can use cross-curricular situations in science or social studies to create classifications or even find creative ways to classify animals or students in their classroom or school.

Before showing students a quadrilateral hierarchy, challenge students to create their own quadrilateral hierarchy using a flow chart or Venn diagram. It may be helpful to give groups of students cards with pictures of shapes on each card and have students sort them by using their own established criteria. See Polygon Capture in the Instructional Tools and Resources for a worksheet containing quadrilaterals that can be cut out and sorted.

Students may incorrectly think that orientation of a figure effects the classification. Show several examples of figures with different orientations.

In order to classify shapes, students need to understand the conditions that define a regular, equilateral, or equiangular figures, especially triangles and quadrilaterals. Although, the quadrilateral hierarchy is the focus of this cluster, a hierarchy can be used to organize other polygons. Extend classification of shapes to more than 4 sides, when it is appropriate for the students.

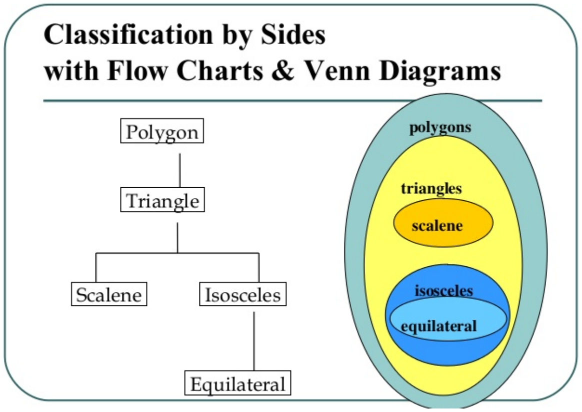
INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.CO.14)

Special polygons include but are not limited to rhombus, parallelogram, rectangle, square, kite, trapezoid, isosceles trapezoid, equilateral triangle, isosceles triangle, and regular polygons.

TIP! Compare and contrast different definitions of quadrilaterals. This is an opportunity to include both definitions of a trapezoid or a kite. Students can explore how these different definitions affect a potential hierarchy.

Have students become fluid in converting a hierarchy presented in a flow chart to a Venn diagram and vice versa.

While students come with experience about properties such as side lengths and angles in polygons, especially triangles and quadrilaterals, this may be their first formal experience with properties of diagonals. Students should investigate the relationships of diagonals in terms of being perpendicular, bisecting each other, and being congruent. See cluster G.CO.12-13 for an example of students proving properties of the diagonals of a rhombus.



Taken from a SlideShare titled [Classify Polygons] from Dreams4school published on Dec 25, 2014.

Students may incorrectly think that diagonals must be slanted. Show several examples of polygons with vertical and/or horizontal diagonals.

Students need to be able to, given specific properties, determine what figure is being described. For example, questions like "I have perpendicular diagonals and four congruent sides. What shape am I?" could be useful. Also, one could cover up part of a figure and ask, "What shape could I be?"

When learning about geometric concepts in high school, it is important that students are able to reason logically about properties of figures. Questions involving replies such as *Sometimes, Always, Never* are appropriate.

EXAMPLE

Reason Logically About Properties of Figures

Use *Sometimes*, *Always*, or *Never* to answer the question. Then explain why.

- Is a square a rectangle?
- Is a rectangle a square?
- Is a rhombus a rectangle?
- Is a rectangle a rhombus?

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.CO.14)

Instructional Tools/Resources

These tools and resources may be helpful for instruction. Before use, they should be evaluated at the local level to determine their appropriateness for instruction.

Materials/Manipulatives

- Tracing paper (patty paper)
- Transparencies
- Graph paper
- Ruler
- Protractor
- Compass
- Mira
- Computer dynamic geometric software (such as Geometer's Sketchpad®, Desmos, Cabri®, or GeoGebra®).
- Graphing calculators and other handheld technology such as TI-Nspire™.

Classifying Quadrilaterals

- Polygon Capture by NCTM Illuminations is a game that could be used to motivate students to examine relationships among geometric properties. *NCTM now requires a membership to view their lessons.*
- What is a Trapezoid? (Part 2) is a task by Illustrative Mathematics that has students compare different definitions of trapezoids and relate them to a Venn diagram.
- What Do These Shapes Have in Common? is a task by Illustrative Mathematics that has students classify shapes based on their properties.
- Bridges in Mathematics, Grade 3 Supplement, Set C4 Geometry: Quadrilaterals is a compilation of activities that relate to the sorting of quadrilaterals.
- Quadrilateral Properties and Relationships Lesson Plan includes an "Who Am I?" activity and a quadrilateral sorting activity using geometric software.
- Don Steward's Complete the Quadrilateral by Fawn Nguyen published on February 8, 2013 is an activity where students use quadrilateral definitions to create quadrilaterals on dot paper.
- Parallelograms and Translations by Illustrative Mathematics is a task where students apply the definition of a parallelogram in the context of a geometric construction.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.CO.14)

Diagonals

- Geometry: Exploring Diagonals of Quadrilaterals by Texas Instruments is a lesson where students understand and identify quadrilaterals based solely on their diagonals.
- Diagonals of Quadrilaterals by gschettini is a GeoGebra activity where students explore the diagonals of quadrilaterals.
- Investigate Quadrilateral Diagonals by David Petro is a Desmos investigation where students explore diagonals and use them to identify a quadrilateral.
- EngageNY, Grade 5, Module 5, Topic D, Lesson 16: Draw Trapezoids to Clarify Their Attributes, and Define Trapezoids Based on Those Attributes, Lesson 17: Draw Parallelograms to Clarify their Attributes, and Define Parallelograms Based on those Attributes, Lesson 18: Draw Rectangles and Rhombuses to Clarify their Attributes, and Define Rectangles and Rhombuses Based on Those Attributes, Lesson 19: Draw Kites and Squares to Clarify their Attributes, and Define Kites and Squares Based on those Attributes, Lesson 20: Classify Two-Dimensional Figures in a Hierarchy Based on Properties, Lesson 21: Draw and Identify Varied Two-Dimensional Figures from Given Attributes are lessons that pertain to this cluster. Some of the earlier lessons in this module need to be adjusted to meet the rigor of high school Geometry, but they include some useful resources.
- Georgia Standards of Excellence Curriculum Frameworks, Fifth Grade, Unit 5: 2D Figures has many lessons that pertain to this cluster.

General Resources

- Arizona K-6 Progression on Geometry is an informational document for teachers. This cluster is addressed on pages 17-18.
- van Hiele Model of Geometric Thinking is a pdf created by ODE that summarizes the van Hiele levels.

References

- Common Core Standards Writing Team. (2013, September 19). *Progressions for the Common Core State Standards in Mathematics (draft). Grades K–5*, Geometry. Tucson, AZ: Institute for Mathematics and Education, University of Arizona.
- McGraw, R. (March 2017). Quadrilaterals: Diagonals and area. *Mathematics Teacher*, *110(7),* 498-504.
- Szydlik, J., Parrott, A., & Belnap, J. (March 2016). Conversations to transform Geometry class. *Mathematics Teacher*, *109(7),* 507-513.

STANDARDS	MODEL CURRICULUM (G.SRT.1-3)
Geometry **SIMILARITY, RIGHT TRIANGLES, AND TRIGONOMETRY** **Understand similarity in terms of similarity transformations.** **G.SRT.1** Verify experimentally the properties of dilations[G] given by a center and a scale factor: **a.** A dilation takes a line not passing through the center of the dilation to a parallel line and leaves a line passing through the center unchanged. **b.** The dilation of a line segment is longer or shorter in the ratio given by the scale factor. **G.SRT.2** Given two figures, use the definition of similarity in terms of similarity transformations[G] to decide if they are similar; explain using similarity transformations the meaning of similarity for triangles as the equality of all corresponding pairs of angles and the proportionality of all corresponding pairs of sides. **G.SRT.3** Use the properties of similarity transformations to establish the AA criterion for two triangles to be similar.	**Expectations for Learning** The standards in this cluster make more precise the informal notion of "same shape, different size." In middle school, students represent proportional relationships within and between similar figures; create scale drawings; describe the effect of dilations on two-dimensional figures; and understand similarity transformations as a sequence of basic rigid motions and dilations. In this cluster, students verify the properties (given center and scale factor) of dilations and use those properties to establish the AA criterion for triangles. They also explore the relationships among corresponding parts of similar figures. The student understanding of this cluster begins at van Hiele Level 1 (Analysis) and moves to Level 2 (Informal Deduction/Abstraction). **ESSENTIAL UNDERSTANDINGS** • A dilation requires a center and a scale factor. • A similarity transformation often requires a sequence of basic rigid motions, in addition to a dilation. • A scale factor is a ratio corresponding lengths between figures. • A similarity transformation with a scale factor of 1 is a special case, which is a congruence transformation. • While the definition of similarity applies to polygons, it also applies to non-polygonal shapes, e.g., circles, parabolas, etc. • The AA criterion is equivalent to the AAA criterion because the angle sum in a triangle is 180 degrees. • The AA criterion and the AAA criterion apply only to triangles. **MATHEMATICAL THINKING** • Use accurate mathematical vocabulary to represent geometric relationships. • Use formal reasoning with symbolic representation. • Determine reasonableness of results. • Recognize, apply, and justify mathematical concepts, terms, and their properties. • Make connections between terms and properties. *Continued on next page*

STANDARDS	MODEL CURRICULUM (G.SRT.1-3)
	Expectations for Learning, continued **INSTRUCTIONAL FOCUS** • Use basic rigid motions and dilations to map similar figures onto one another. • Given a figure, carry out a similarity transformation, and then verify its properties. • Know the precise definitions of dilation and similarity. • Identify center and scale factor of a dilation. • Determine the scale factor. • Establish that triangles with two pairs of corresponding congruent angles are similar. **Content Elaborations** **OHIO'S HIGH SCHOOL CRITICAL AREAS OF FOCUS** • Geometry, Number 3, page 6 **CONNECTIONS ACROSS STANDARDS** • Prove theorems involving similarity (G.SRT.4-5). • Solve problems involving right triangles trigonometry (G.SRT.6-8). • Apply the understanding that all circles are similar (G.C.1). • Represent transformations in the plane (G.CO.2). • Understand the relationships between lengths, areas, and volumes in similar figures (G.GMD.6). • Use coordinates to prove simple geometric theorems algebraically (G.GPE.6). • Apply geometric concepts in modeling situations (G.MG.1-3).

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.SRT.1-3)

Instructional Strategies

Note: The model curriculum is not a collection of lessons nor a full curriculum; it does not suggest pace, sequence, or amount of time spent on topics. It provides information about a topic related to the standards including ideas for examples, strategies for teaching, common misconceptions, and possible connections between topics. In addition, the emphasis of the model curriculum is on instruction not on assessment; therefore, the examples in the Instructional Strategies section are written for instructional purposes not necessarily as assessment items. Examples may need to be rewritten to accommodate the needs of each individual classroom.

The concept of similarity builds on the concept of congruence. This is one of the reasons for teaching congruence prior to teaching similarity.

It may be helpful to teach this cluster in conjunction with G.SRT.4-5 and G.GMD.5-6 especially with respect to the Side Splitter Theorem and the Fundamental Theorem of Similarity.

Standards for Mathematical Practice

This cluster focuses on but is not limited to the following practices:

MP.3 *Construct viable arguments and critique the reasoning of others.*

MP.4 *Model with mathematics.*

MP.5 *Use appropriate tools strategically.*

MP.7 *Look for and make use of structure.*

MP.8 *Look for and express regularity in repeated reasoning.*

VAN HIELE CONNECTION

In Geometry students are expected to move from Level 1(Analysis) to Level 2 (Informal Deduction/Abstraction). In van Hiele Level 1 students use coordinates to examine transformations and recognize, use, and describe properties of transformations. Whereas van Hiele Level 2 can be characterized by the student doing some or all of the following:

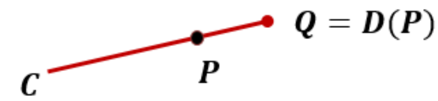
- analyzing relationships within and between transformations;
- algebraically expressing transformations: reflections about the *x*- and *y*-axes, and rotations of 45°, 90°, and 180° about the origin; *Note: In high school, students should be given an opportunity to experience rotations around any point (not just the origin) and onto any angle, not just multiples of 90. Also, reflections are not limited to the x-and y-axes but could be about any line.*
- understanding and justifying how to use perpendicular bisectors of corresponding points to locate the turn center of a rotation;
- understanding how the composition of two reflections can be a translation or a rotation; and/or
- following a rationale if shown proof.

See the van Hiele pdf on ODE's website for more information about van Hiele levels.

VERIFYING PROPERTIES OF DILATIONS EXPERIMENTALLY

Students build upon the informal idea of dilation that they learned in Grade 8. Now in high school students use a more formal definition of a dilation as a function on a plane. A dilation is a rule that expands or contracts the plane about a center. The center of dilation is a fixed point in the plane about which all distances from the center are expanded or contracted. A scale factor informs the magnitude or the amount of expansion or contraction. A dilation must name a center and a scale factor. *Note: Throughout the Model Curriculum different notations are used such a function notation and prime notation etc. As each districts' resources are different, it is up to each district to determine the notation that students need to use. It may be helpful to show students a variety of notations, so they can be mathematically literate when seeing different notations in different resources/courses/schools.*

Definition of Dilation



The dilation D with center C and positive scale factor r, leaves C unchanged and takes every point P to the point $Q = D(P)$ on the ray CP whose distance from C is $r|CP|$ so that $|CQ| = r|CP|$.

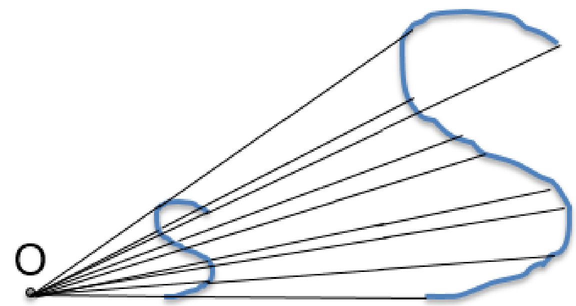
Students should recognize that—

- A scale factor between 0 and 1 $(0 < r < 1)$ pulls (contracts) every point on the plane proportionally closer to the center;
- A scale factor greater than 1 $(r > 1)$ pushes (expands) every point on the plane proportionally the same amount away from the center; and
- A scale factor of 1 leaves every points' position unchanged $(r = 1)$.

TIP! Emphasize to students that the entire plane dilates not just the image as the center of dilation remains fixed. The image not only increases (or decreases) in size but the image is pushed farther away from (or pulled closer to) the pre-image because the plane has changed.

Introduce dilation by discussing a topic such as "How do we triple the size of a wiggly curve?"

After discussion, lead students toward assigning an arbitrary point, O, on the plane, and pushing every point on the squiggly line three times as far away from O. Explain to students that this is a dilation. A dilation pushes out (expands) or pulls in (contracts) every point in the plane as well as the figure from its center of dilation proportionally. This can be easily modeled by pushing in or pulling out a light source that displays an image such as an overhead projector or a flashlight. In this case the center of dilation is O, however the center of dilation can be located anywhere on the plane.

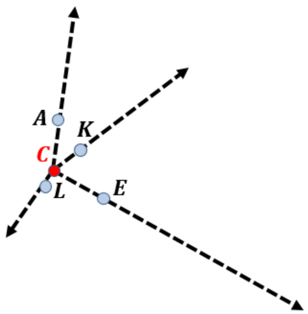


See Model Curriculum Grade 7.G.1-3 and Grade 8.G.1-5 for scaffolding ideas.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.SRT.1-3)

TIP! Use rubber bands to explore dilations. Tie two rubber bands together with a knot in between. Given a figure and a point of dilation, hold the end of the rubber band at the dilation point. With a writing utensil at the far end, make the knot trace the original figure. This should create a dilated image of the original figure. See the YouTube video called Rubber Band Dilation, by Robin Betcher for an illustration of the process.

EXAMPLE



a. Using a ruler or compass map the points $L, A, K,$ and E using a dilation about point C where the scale factor is $k = 3$.

b. Write four equations to describe how the distance between the original point and point C relates to the distance between its image and point C.

EXAMPLE

Draw a dilation image of polygon $DEFGH$.

a. Pick a center of dilation, and label it C.

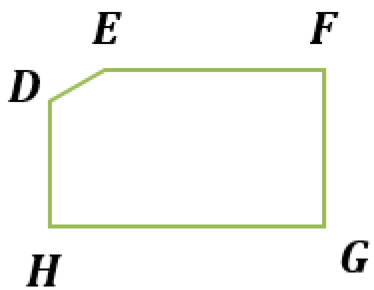
b. Draw a ray from your center of dilation through each vertex on the figure.

c. Use a compass or ruler to map the points D', E', F', G', and H' using a dilation by a scale factor of $\frac{1}{3}$.

d. Then connect the corresponding images of the original vertices creating line segments.

e. How does your dilated figure compare to a scale drawing?

f. How does your dilated figure compare to your classmates?



Have students focus on the properties of dilations. In Grade 8 students explored the basic properties of dilations. Now students need to explain *why* the properties of dilations are true by using reasoned arguments. Since, in general, a dilation changes the distances between two fixed points, dilations do not preserve congruence (except when the scale factor is 1); they do, however, preserve similarity.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.SRT.1-3)

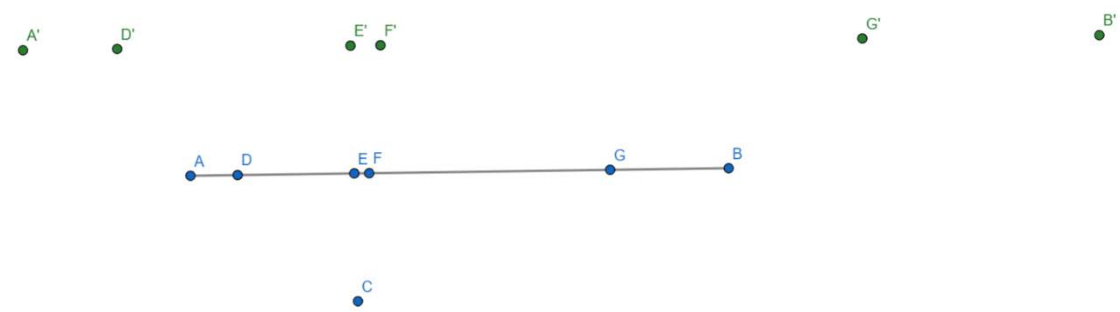
Dilations—

- Map lines to lines, rays to rays, and segments to segments;
- Change a distance by a factor of *r*, where *r* is the scale factor of the dilations;
- Map every line passing through the center of dilation to itself, and map every line not passing through the center of dilations to a parallel line; and
- Preserve angle measure, betweenness, and collinearity. (Distance is not preserved.)

Note: Discussion of the Fundamental Theorem of Similarity with respect to dilations and parallel lines will be discussed in G.GMD.5-6.

EXAMPLE

Use colored pencils, ruler, and a compass or dynamic geometric software to perform experiments exploring the properties of dilations.



Part 1: Collinearity, Betweenness and Distance

Draw 6 collinear points (not equally spaced). Then draw point C which is not collinear, and let C be the center of dilation with a scale factor of 2.

a. Find the images of the 6 points.
b. Compare the relationships between corresponding points. Does distance appear to be preserved? Give rationale to support your answer.
c. Make a conjecture based on your discovery for part **b**.
d. Does collinearity appear to be preserved? Explain.
e. Does betweenness appear to be preserved? Support your answer with reasons.
f. Does the distance between a pre-image point and its image appear to be the same for all points? What is it about a dilation that impacts the distance in this way?

Discussion: Students should come to the conclusion that distance between the points is not preserved. For example, the distance between point A and point D in not the same as the distance between point A' and point D'. Students should see that the collinearity is preserved since the image points also make a straight line. Betweenness is also preserved. For example, just as point D is between A and E, point D' is between point A' and E'. The distance between point A and point A' is not the same as the distance between point D and point D'.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.SRT.1-3)

Part 2: Segments

Draw a segment $\overline{AB}$ with a length of at least 3 inches. Draw a point C, which is not on $\overleftrightarrow{AB}$ and let C be the center of dilation with a scale factor of $\frac{2}{3}$. Use as many points as necessary to convince you what the image of $\overline{AB}$ looks like.

- **a.** Describe the resulting figure. What relationship exists between the distance from $A' = D(A)$ and $B' = D(B)$ to the distance of AB.
- **b.** Is the distance between $D(A)$ and $D(B)$ equal to AB?
- **c.** Does the dilation preserve either betweenness or collinearity? Support your response.
- **d.** Predict what you think the image of $\overrightarrow{AB}$ or $\overleftrightarrow{AB}$ would look like after a dilation?
- **e.** What is the easiest way for you to determine $D(\overline{AB})$?
- **f.** Does $D(\overline{AB})$ intersect $\overline{AB}$? Explain.

Discussion: In Part 2, ask the question, "How can we be sure that the dilation maps all the points between A and B to the image points between A' and B'? Students may need to use the coordinate plane to make the argument, but then ask them how they would know that its true without a coordinate plane. Students should also come to the conclusion that $D(\overline{AB})$ and $\overline{AB}$ are parallel.

Part 3: Angles

Draw an obtuse angle $\angle ABX$. Let O be a point in the interior of $\angle ABX$ and let D be a dilation with center O and scale factor of $\frac{1}{3}$. Find the images of A, B, X and as many other points as necessary to convince you what the image of $\angle ABX$ looks like.

- **a.** What kind of figure is $D(\angle ABX)$?
- **b.** Does the dilation preserve angle measure? Explain
- **c.** How are $\overrightarrow{BX}$ and $D(\overrightarrow{BX})$ related?
- **d.** True or False? If $D(B) = B'$ and $D(A) = A'$, then $\frac{|A\prime B\prime|}{|AB|} = k$, where k is the scale factor. Explain.
- **e.** What is the minimum number of points of $\angle ABX$ needed to find $D(\angle ABX)$? Explain.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.SRT.1-3)

Part 4: One Triangle, Two Transformations

Draw a triangle with side lengths of 2 inches, 1 inch, and 1.5 inches, and label it $\triangle PQR$. Draw point A in the interior of $\triangle PQR$ and point E in exterior of $\triangle PQR$. Let D be the dilation of $\triangle PQR$ with point A as the center of dilation and a scale factor of $\frac{3}{2}$. Then let d be the dilation of with point E as the center of dilation and a scale factor of $\frac{3}{2}$.

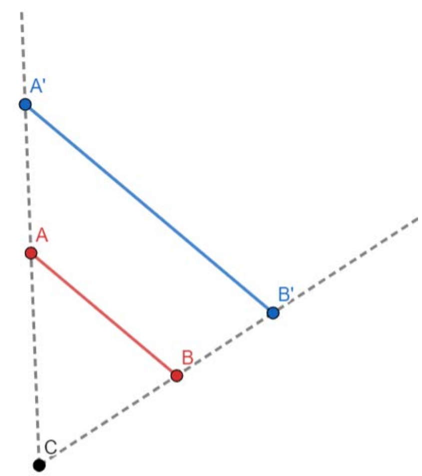
- **a.** What is the relationship between $D(\triangle PQR)$ and $d(\triangle PQR)$?
- **b.** Compare the corresponding angles of $\triangle PQR$ and $D(\triangle PQR)$. What is the relationship between them?
- **c.** Compare the corresponding angles of $\triangle PQR$ and $d(\triangle PQR)$. What is the relationship between them?
- **d.** What is the minimum number of points needed to create the image of the triangle under either one of these dilations? Explain.

Part 5: Parallels and Perpendiculars

Draw a rectangle $ABCE$ with vertices (2,2), (6,4), (6,2), and (2,4). Draw its image under a dilation with a center of (0,0) and a scale factor of 2.5.

- **a.** What are the coordinates of the vertices of the image?
- **b.** Does the image seem to be a rectangle? Explain.
- **c.** Does the dilation preserve parallelism?
- **d.** Does the dilation preserve perpendicularity?
- **e.** Does the dilation preserve orientation?

Example take from Coxford and Usiskin, 1971, *Geometry: A Transformational Approach.*



According to the standards, students need to verify that under a dilation, a line (or line segment) is parallel to its image. This could be done using geometric software or using coordinate geometry. Depending on the sequencing of concepts, more advanced students could prove this theorem assuming that in the classroom, students start with the postulate that dilations preserve angle measure.

EXAMPLE

Given $\overline{AB}$ has been dilated about point C. Prove that $\overline{AB} \parallel \overline{A'B'}$.

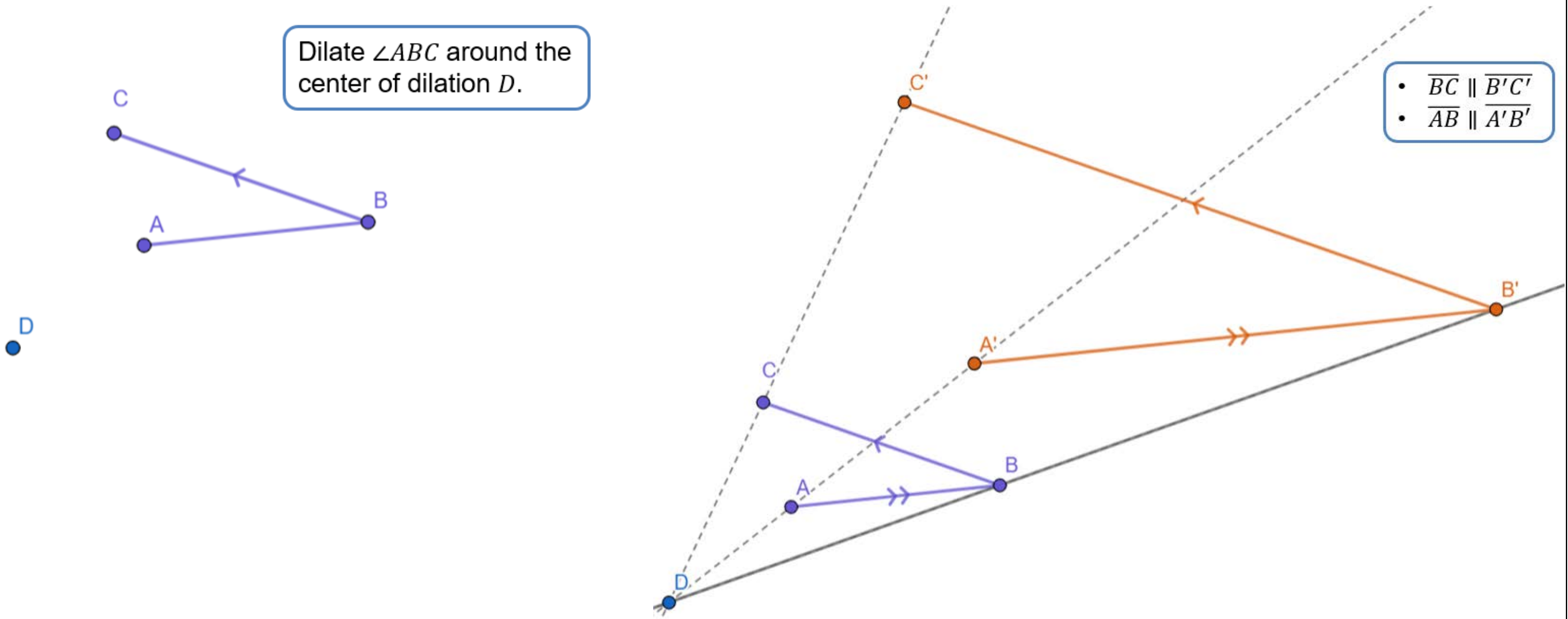
Discussion: Under the dilation with the center C, the image of point C is C, the image of point B is B', points C, B, and B' are collinear and $m\angle ABC = m\angle A'B'C$ because dilations preserve angle measure. Since angles ABC and $A'B'C$ are also corresponding angles that are formed by line segments $\overline{AB}$, $\overline{A'B'}$ and the transversal ray CB', line segments $\overline{CB}$ and $\overline{C'B'}$ must be parallel.

In contrast if the sequence of instruction has students start with the postulate that under a dilation, a line is parallel to its image (verified experimentally in G.SRT.1.b), they can prove the dilations preserve angle measure.

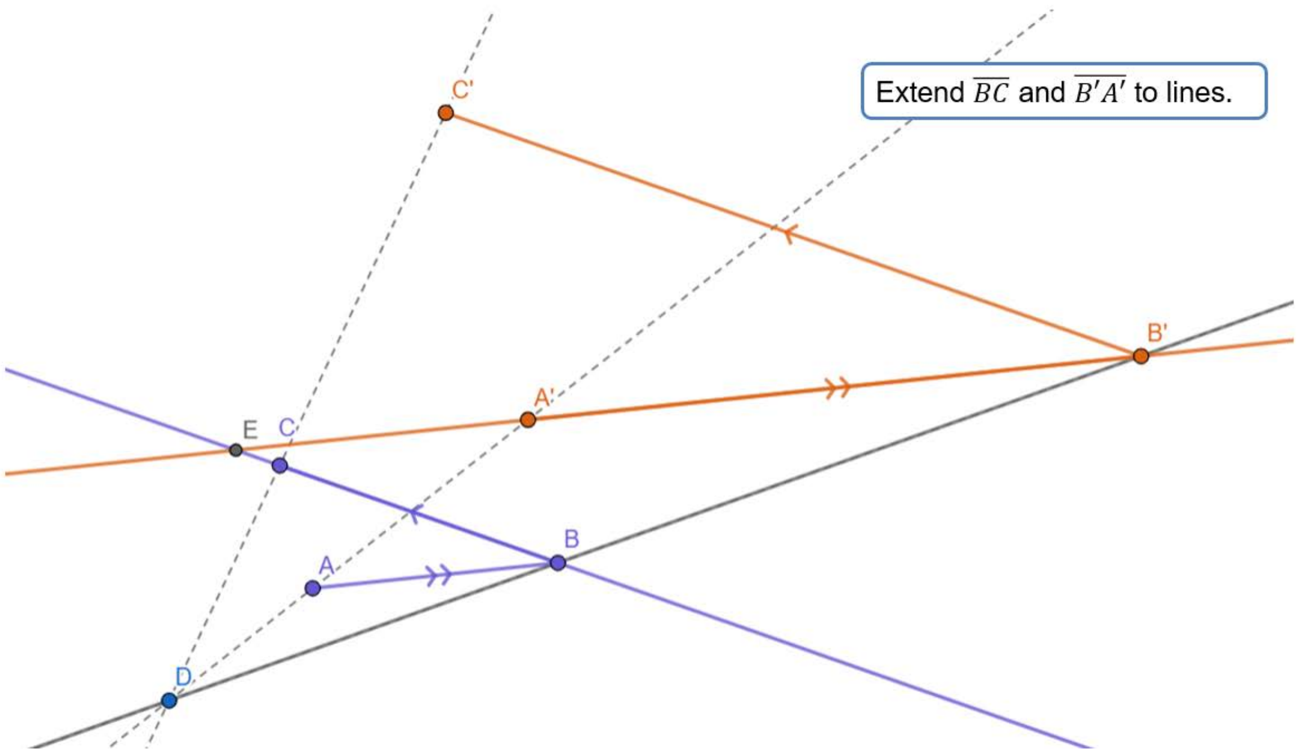
INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.SRT.1-3)

EXAMPLE

Prove that dilating $\angle ABC$ preserves angle measure after undergoing a dilation.

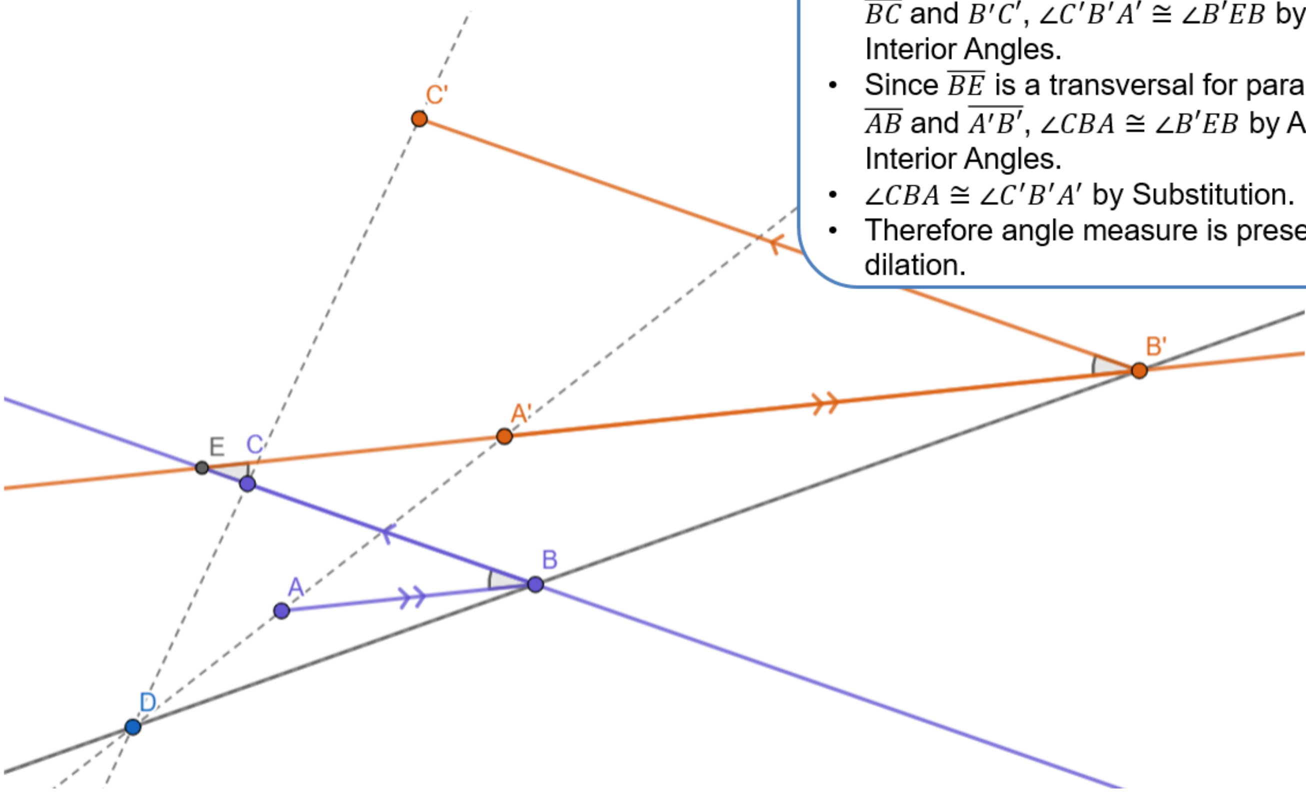


INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.SRT.1-3)



INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.SRT.1-3)

- Since $\overline{B'E}$ is a transversal for parallel lines $\overline{BC}$ and $\overline{B'C'}$, $\angle C'B'A' \cong \angle B'EB$ by Alternative Interior Angles.
- Since $\overline{BE}$ is a transversal for parallel lines $\overline{AB}$ and $\overline{A'B'}$, $\angle CBA \cong \angle B'EB$ by Alternative Interior Angles.
- $\angle CBA \cong \angle C'B'A'$ by Substitution.
- Therefore angle measure is preserved under a dilation.



Use graph paper and a ruler or dynamic geometric software to obtain images of a given figure under dilations having specified centers and scale factors. Carefully observe the images of lines passing through the center of dilation and those not passing through the center, respectively. A line segment passing through the center of dilation will simply be shortened or elongated but will lie on the same line, while the dilation of a line segment that does not pass through the center will be parallel to the original segment (this is intended as a clarification of Standard G.SRT.1a).

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.SRT.1-3)

EXAMPLE

Part 1

Dilate each of the following figures around the center of dilation X and a scale factor of 2.

Figure 1

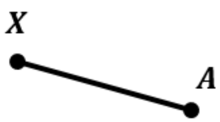


Figure 2

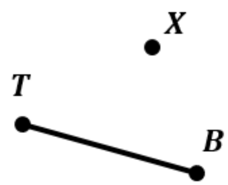


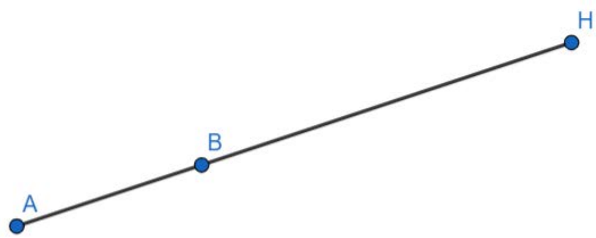
Figure 3

a. What is the same about all three dilations?
b. What is different about all three dilations?

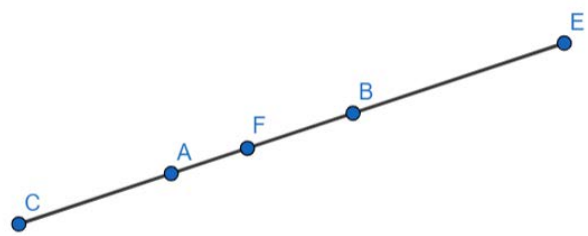
Part 2

Name the center of dilation.

a. If there is a dilation that maps $\overline{AB}$ to $\overline{AH}$, what is the center of dilation? Using the center of dilation that you found, how would the dilation of $\overline{AB}$ change if $\overline{AB}$ were $\overleftrightarrow{AB}$ or $\overrightarrow{AB}$?

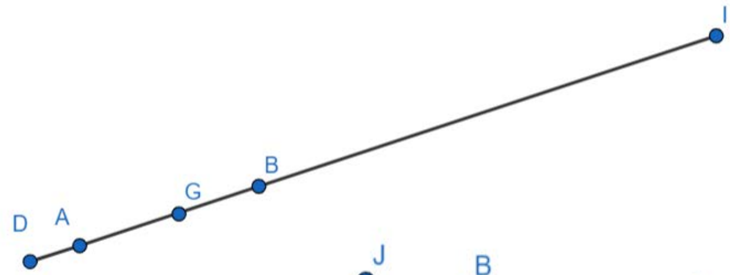


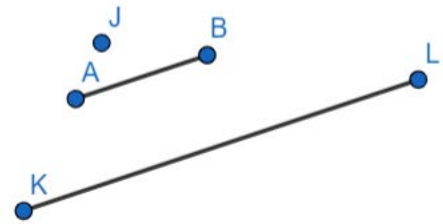
b. If there is a dilation that maps $\overline{AB}$ to $\overline{CE}$, what is the center of dilation? Using the center of dilation that you found, how would the dilation of $\overline{AB}$ change if $\overline{AB}$ were $\overleftrightarrow{AB}$ or $\overrightarrow{AB}$?



INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.SRT.1-3)

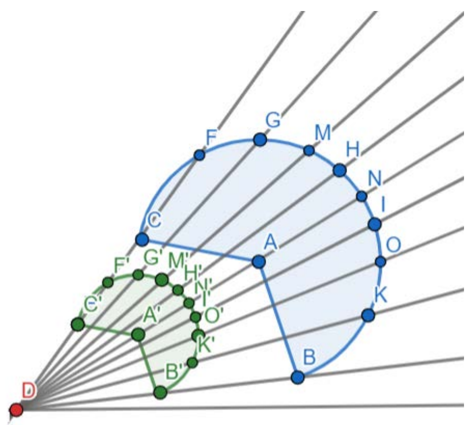
c. If there is a dilation that maps $\overline{AB}$ to $\overline{GI}$, what is the center of dilation? Using the center of dilation that you found, how would the dilation of $\overline{AB}$ change if $\overline{AB}$ were $\overleftrightarrow{AB}$ or $\overrightarrow{AB}$?





d. If there is a dilation that maps $\overline{AB}$ to $\overline{KL}$, what is the center of dilation? Using the center of dilation that you found, how would the dilation of $\overline{AB}$ change if $\overline{AB}$ were $\overleftrightarrow{AB}$ or $\overrightarrow{AB}$?

Discussion: Part 2 of this example is intended to be a progression for students to understand that the center of dilation can be an endpoint of the segment, in the middle of the segment, or outside of a line segment of which it is dilated.



Introduce students to examples of dilations that include curved lines to show that dilations are not limited to polygons. This will reinforce the idea that all the points in the pre-image undergo a dilation not just the vertices.

Connect dilations with perimeter and area in G.GMD.6. Include examples where orientation is changed.

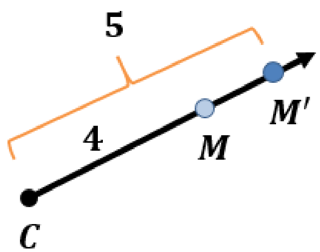
Students do not need to prove G-SRT.1b but specific cases may be appropriate for advanced students for example when the scale factor is 2.

Scale Factor

The scale factor is the ratio of corresponding lengths in an image to the corresponding lengths in the preimage. In case of circles, it is the ratio between two corresponding arc lengths or two radii. Give students scale factors in various forms: fractions, decimals, and percents, as oftentimes in real-life scale factor is expressed in various forms. For example, a photocopier typically uses percents and other computer software uses decimals.

Scale factor (or magnitude), usually represented by *k* or *r*, can be any factor except 0; however, in high school Geometry, *k* > 0.

EXAMPLE



Part 1

a. How far is M away from C?
b. How far is M' away from C?
c. How many times as far is M' away from C compared to M?
d. How many times as far is M away from C compared to M'?
e. Fill in the blank $M'C =$ ___MC to make the statement true.
f. Fill in the blank ___$M'C = MC$ to make the statement true.

Part 2

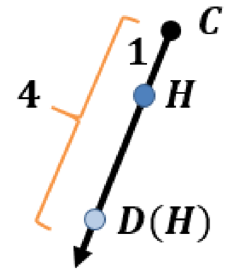
Use the results from **Part 1e**. to fill in the blank: Draw a picture showing all the points that are _____ times as far from C as M is.

Discussion: This example lends itself to a discussion of scale factor in relation to distance. Students can see that M' is $\frac{5}{4}$ times away from C compared to M and that M is $\frac{4}{5}$ times away from C compared to M . Draw attention to the Multiplicative Inverse Property and its relationship to the coefficients. Students should write the following equations: $M'C = \frac{5}{4}MC$ and $\frac{4}{5}M'C = MC$. Writing comparison equations is difficult for some students. Have discussion about why the two equations are equivalent. See Model Curriculum Instructional Supports A.CED.1-3 for more information about writing comparison equations. In Part 2, students come to understand that there is an infinite amount of points that can be drawn and that they in fact make a circle.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.SRT.1-3)

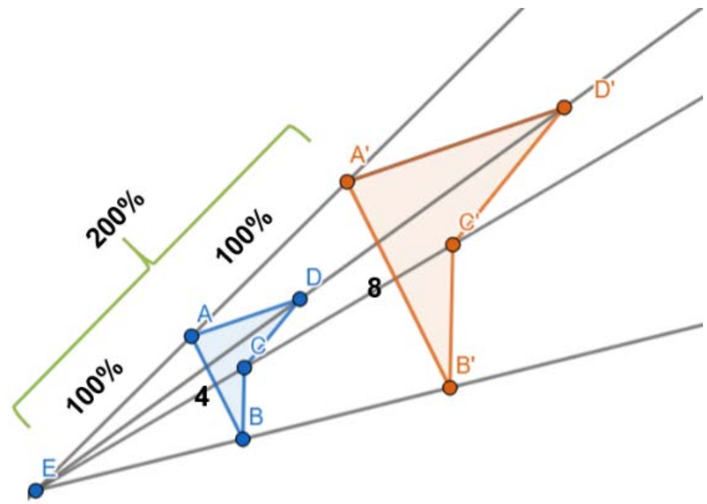
EXAMPLE

If C is center of dilation, find the scale factor for the given point H and its image $D(H)$.



Some students may incorrectly think that when a scale factor is applied to a figure, the scale factor also affects the size of the angles. In actuality, the scale factor only affects the side lengths of a figure, the angles stay the same regardless of the scale factor.

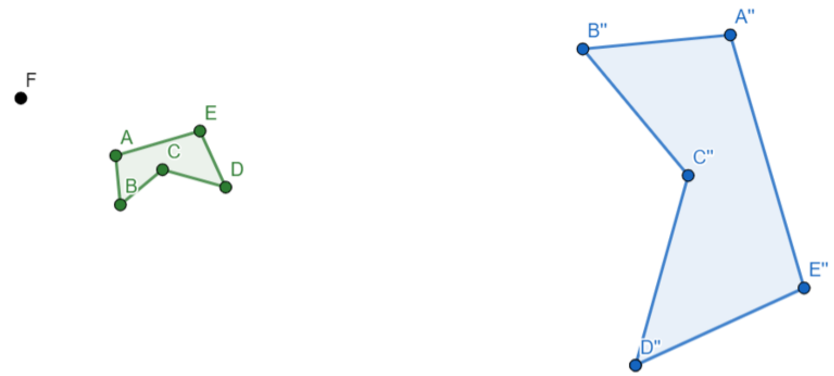
Students may incorrectly think that scale factor is the distance added onto the original distance as they focus on the ratios of the amounts of increase in the length of the corresponding sides rather than the ratio of the lengths of the sides. Draw attention to the lengths of the sides. For example, in the image to the right, students may think that the image has scale factor of 100% but rather it has a 100% increase where the scale factor is 200% because the image is twice as far from the center of dilation. Direct students who struggle with this concept to focus on the side lengths of the figure. They should notice that $\overline{A'B'}$ is twice as long as $\overline{AB}$.



SIMILARITY TRANSFORMATIONS

There are two ways to look at similarity. The traditional definition of similarity occurs between two figures if corresponding angles are congruent and corresponding line segments are proportional; however, this definition only applies to polygons. Another definition defines similarity in terms of transformations which is called a similarity transformation. Two figures are similar if there is a series of transformations including a dilation that maps one figure onto the other. Similarity cannot be defined by dilations alone because otherwise, for example, dilated figures that are also rotated would not be similar; therefore, congruence must be introduced. Therefore, two figures are similar if one figure is congruent to the dilation of the other. Once similarity is defined in terms of transformations and congruence, the traditional definition of similarity (and its converse) becomes a consequence of the definition instead of the definition itself. The benefit of using the transformational definition of similarity is that it can extend to all figures such as circles, ellipses, or even open figures and is not limited to polygons.

Similar figures that have been rotated and dilated.



TIP! Some students find it tempting to use the word dilation instead of similarity transformation. A dilation has a fixed center and scale factor. Many similarity transformations also require basic rigid motions in addition to the dilation.

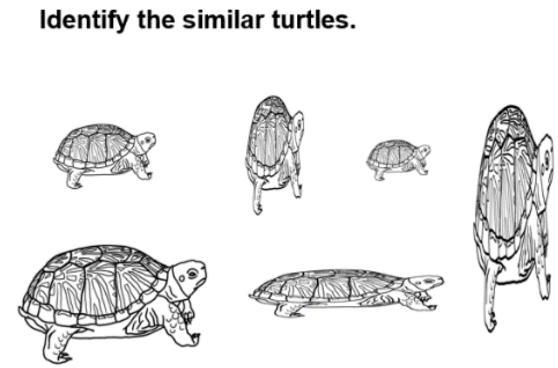
Every rigid motion is also a similarity transformation, since technically congruence is a special case of similarity. If the image and pre-image are congruent figures then there is a dilation with a scale factor of 1. Therefore, if two figures are congruent, they are also similar. When a scale factor equal to 1 is applied to a figure, it is called an identity transformation because each point coincides with its image, and the image is identical to the preimage. Figures also have inverse transformations that return a dilated point back to itself. Inverse transformations "undo" each other. The inverse transformation of a dilation D with center C and scale factor r, would be a dilation D with center C and a scale factor of $\frac{1}{r}$.

GeoGebra has an applet titled Dilation and Inverse by Alfred Estberg that allows students to explore inverse dilations.

This cluster could be taught in parallel with G.C.1: Proving that all circles are similar.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.SRT.1-3)

Use technology such as Microsoft Word, PowerPoint, Smartboard etc. to show how stretching or shrinking a figure in one direction does not maintain similarity but stretching or shrinking a figure in all directions does. Allow adequate time and hands-on activities for students to explore dilations visually and physically.



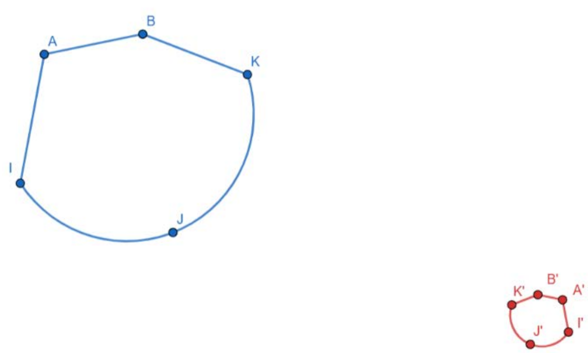
Some students may incorrectly think that a stretch in one-dimension results in similarity. Show examples of figures such as turtles that are stretched in only one dimension compared to those that are stretched in both dimensions.

Use graph paper and rulers or dynamic geometric software to obtain the image of a given figure under a series of transformations: a dilation followed by a sequence of rigid motions (or rigid motions followed by dilation). Similar Shapes & Transformations by Khan Academy illustrates this process. Students should experience similarity transformations with and without coordinates. Only using coordinates is too restrictive and causes students to create misconceptions.

Given two figures, students could translate, rotate, and/or reflect an image to line up a corresponding angle of the image and preimage and then dilate to show similarity.

EXAMPLE

Perform a similarity transformation(s) to show that $ABKJI$ is similar to $A'B'K'J'I'$.



Discussion: Students may find it helpful to line up a corresponding angle and then dilate to show similarity.

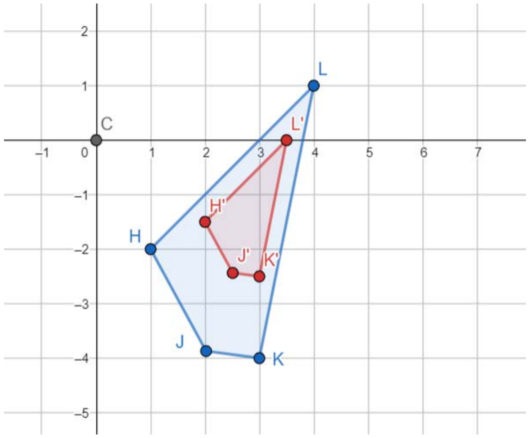
INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.SRT.1-3)

TIP! Students can use shadow puppets to explore properties of similar figures. They can also connect constructing hexagons and similarity. See the References section for NCTM articles for lessons surrounding these concepts.

Work backwards – given two similar figures that are related by dilation, determine the center of dilation and scale factor. Given two similar figures that are related by a dilation followed by a sequence of rigid motions, determine the parameters of the dilation and rigid motions that will map one onto the other.

EXAMPLE

Name the specific transformations that will map $H'J'K'L'$ to $HJKL$.



Discussion: Make sure students mention both a dilation and a translation. The dilation should include the center of dilation and the scale factor for the dilation. The translation should state the distance and direction.

Students may not realize that similarities preserve shape of the figure, but not size. Angle measures stay the same, but side lengths change by a constant scale factor.

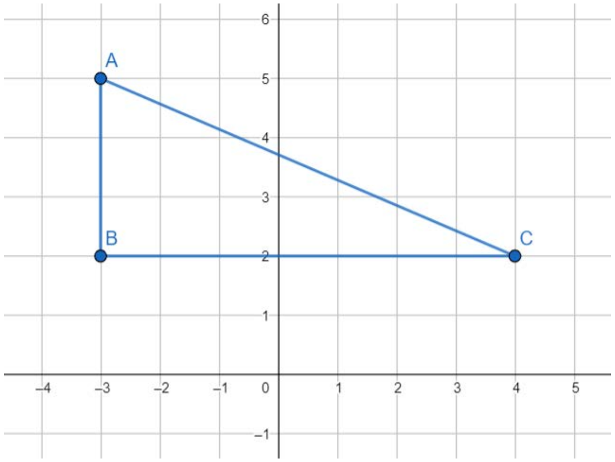
The Reflexive, Symmetric, and Transitive Properties also hold true for similarity. Have students form informal arguments about why these properties hold true for similarity.

Similar Triangles

Similar triangles should be thought of as an extension of other similar figures. Students should use similarity transformations to show that in similar triangles corresponding angles are congruent and corresponding sides are proportional. Students can measure the corresponding angles and sides of the original figure and its image to verify that the corresponding angles remain unchanged and the corresponding sides are proportionally stretched or shrunk by the same scale factor.

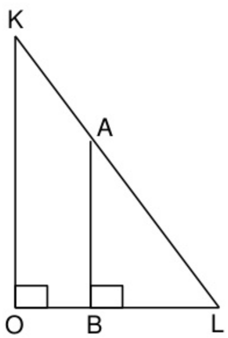
EXAMPLE

a. Using a series of transformations, create two similar triangles.
b. Using your two triangles, make a conjecture about corresponding parts of similar triangles? Justify your reasoning.
c. Compare your conjecture in part **b.** to several of your classmates' examples of similar triangles. Does your conjecture in part **b.** hold true? Explain.
d. Does your statement in part **b.** hold true for any pair of similar triangles? Explain.
e. Would your conjecture hold true for figures other than triangles? Explain.



EXAMPLE

Determine if $\triangle KOL \sim \triangle ABL$. Justify your steps.



Discussion: Given two triangles with a pair of congruent angles, students translate, rotate, and/or reflect one of the triangles in order to map one angle onto its corresponding angle so that the opposite sides from this angle are parallel. Then a dilation completes the map of one triangle onto the other.

Some students often do not list the vertices of similar triangles in order. However, the order of corresponding vertices is especially important for similar triangles so that proportional sides can be correctly identified.

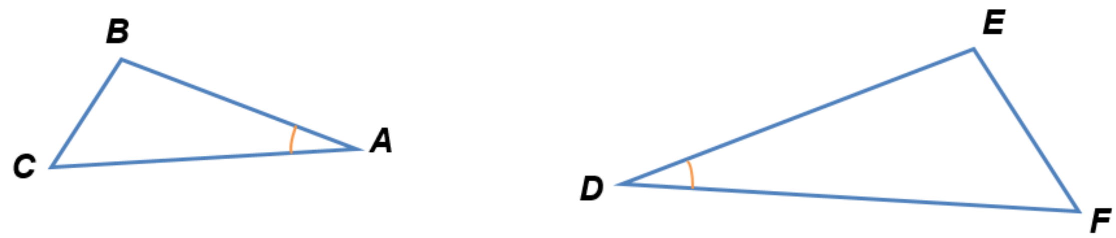
INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.SRT.1-3)

EXAMPLE

Part 1

If $\angle A \cong \angle D$, determine if two given triangles, $\triangle ABC$ and $\triangle DEF$, are similar using transformations. Justify your steps.

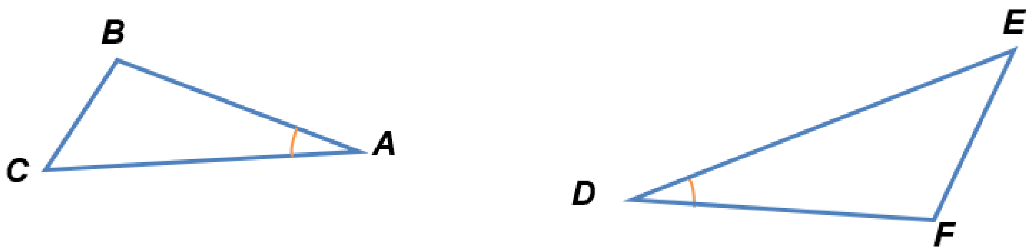
Discussion: Given that $\angle A \cong \angle D$, using patty paper, tracing paper, or geometric software, students should discover that a reflection followed by a translation will map $\angle A$ onto $\angle D$. Then a dilation of $\triangle ABC$ will map onto $\triangle DEF$. Then students can see $\angle B \cong \angle E$ and $\angle C \cong \angle F$, so the triangles are similar.



Part 2

If $\angle A \cong \angle D$, determine if two given triangles, $\triangle ABC$ and $\triangle DEF$, are similar using transformations. Justify your steps.

Discussion: Given that $\angle A \cong \angle D$, using patty paper, tracing paper, or geometric software, students should discover that a reflection followed by a translation will map $\angle A$ onto $\angle D$. However, a dilation of $\triangle ABC$ will not map onto $\triangle DEF$ as student can see $m\angle B \neq m\angle E$ and $m\angle C \neq m\angle F$, so the triangles are not similar. This would be a good opportunity to discuss why A-Similarity Criteria only works with right triangles.



INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.SRT.1-3)

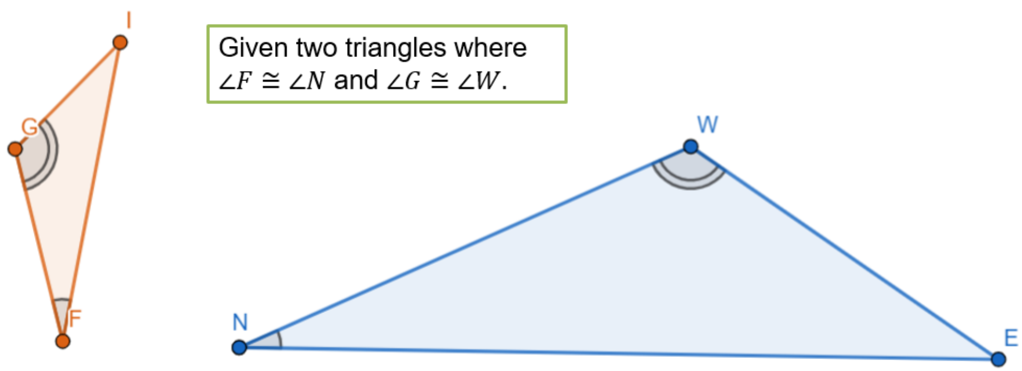
TIP! Optical range-finding golf scopes or similar hand-made devices could be used to illustrate similar triangles.

Angle-Angle (AA) Criterion

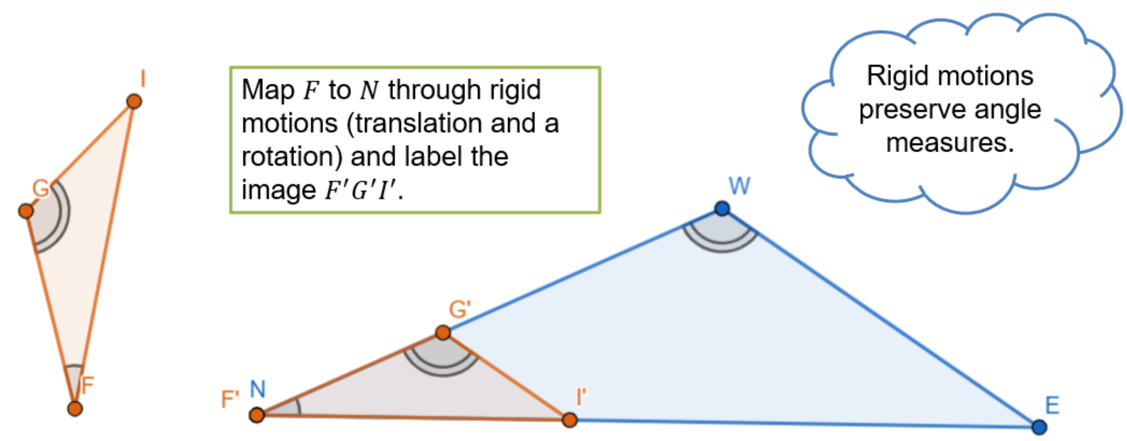
Prove AA similarity criteria for triangle, given two triangles that have two pairs of corresponding angles by using rigid motions to map a vertex of one triangle onto the corresponding vertex of the other in such a way that their corresponding sides are aligned. Then show that dilation will complete the mapping of one triangle onto the other.

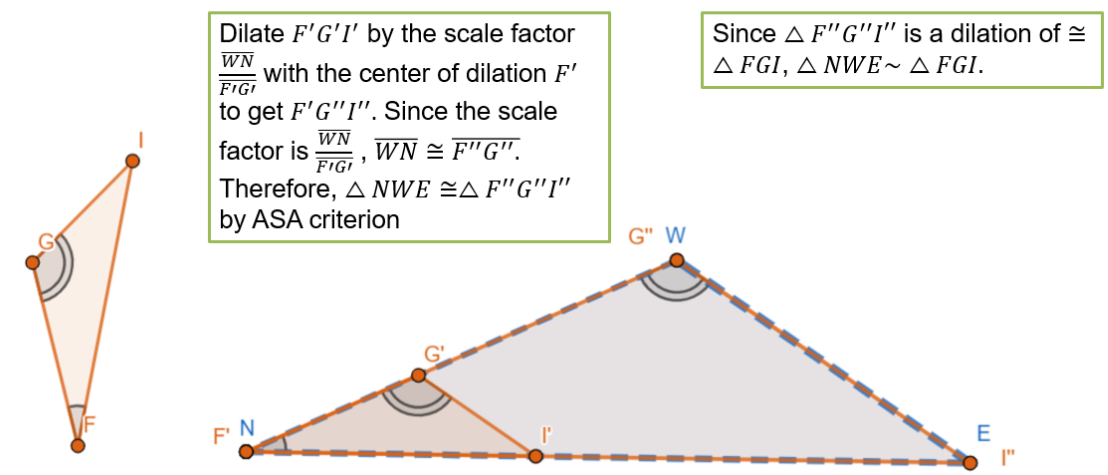
EXAMPLE

Prove two triangles are similar using only two pairs of corresponding congruent angles (AA-Similarity Criteria.)



INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.SRT.1-3)





INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.SRT.1-3)

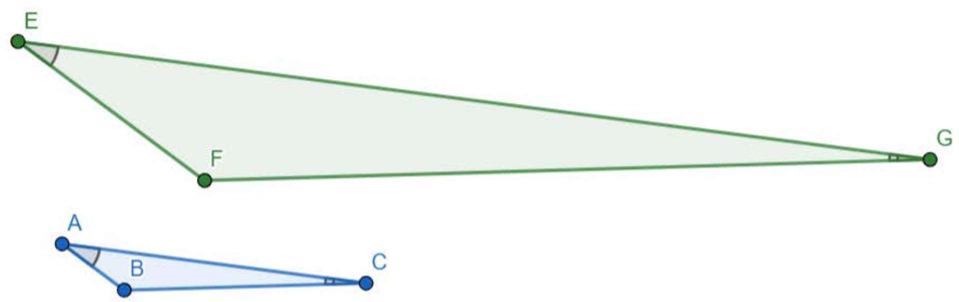
Using the theorem that the sum of the angles is a triangle is 180°, verify that the AA-Similarity Theorem is equivalent to the AAA-Similarity Criteria.

EXAMPLE

Prove that only two angles are needed to prove similarity.

Given: $\angle A \cong \angle E$ and $\angle C \cong \angle G$.

Prove: $\triangle ABC \sim \triangle EFG$.



Discussion:

- When all corresponding angles in a triangle are congruent, the triangles are similar.
- $\angle A + \angle B + \angle C = 180°$ because the sums of the angles in a triangle = $180°$.
- $\angle E + \angle F + \angle G = 180°$ because the sums of the angles in a triangle = $180°$.
- $\angle A + \angle F + \angle C = 180°$ by substitution.
- $\angle A + \angle B + \angle C = \angle A + \angle F + \angle C$ by substitution.
- $\angle B = \angle F$.
- Therefore if 2 pairs of corresponding angles are congruent, then the third angle must also be congruent. By default, if AA-Similarity Criteria has been met, then AAA-Similarity Criteria has also been met and $\triangle ABC \sim \triangle EFG$. Therefore, only 2 angles are needed to prove that two triangles are similar.

Note: This example presupposes that students know the theorem that if all corresponding angles in a triangle are congruent, then the figures are similar.

Students should investigate the SAS- and SSS-Similarity Criteria for triangles.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.SRT.1-3)

Instructional Tools/Resources

Manipulatives/Technology

- Dot paper
- Patty paper
- Graph paper
- Rulers
- Protractors
- Pantograph
- Photocopy machine
- Computer dynamic geometric software (Geometer's Sketchpad®, Desmos, Cabri®, or GeoGebra®).
- Web-based applets that demonstrate dilations, such as those at the National Library of Virtual Manipulatives.

Dilations

- Dilating a Line by Illustrative Mathematics is a task that gives students the opportunity to verify the properties of dilations experimentally.
- Working with Dilations by Caleb Rothe is a beginning Desmos activity that introduces students to dilations.
- Pantry Dilations by Mathycathy is a Desmos activity that uses dilated scaled images on items found in a pantry.
- The Shadow Knows Dilations (and Transformations) by Ivan Cheng is a Desmos activity that explores dilations in the context of shadows.
- Dilations by Victor Bartodej is a webpage that has many GeoGebra dilations activities.
- Enlarging Figures by Dilations by Shasta County Office of Education is a student handout that has students use rubber bands to explore dilations.
- Embracing Transformational Geometry in CCSS-Mathematics by Jim Short has several explorations on dilations.
- Dilations by Hand2Mind has students explore dilations through Geoboards.
- Key Visualizations: Geometry by The Mathematics Common Core Toolbox has an applet that pushes students to explore dilations of lines and circles.
- Exploring Dilations of the Plane by Michael Andrejkovics is a GeoGebra applet designed to help you visualize how dilations affect both the object in the plane and the plane itself. The applet also allows the center of dilation to be anywhere on the plane.
- Dilations around a Point and their Dilation Factor by Juan Castaneda from Online Math Tutor has three interactive dilations applets.

Similar Figures

- Similar Quadrilaterals is task by Illustrative Mathematics that has students explore how the AA criterion for triangles is to similar criteria for quadrilaterals.
- Episode 1: Similarity—Project Mathematics by Caltech is a YouTube video that explores similarity.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.SRT.1-3)

Similar Triangles

- Are They Similar? by Illustrative Mathematics is a task where students are given two triangles that appear to be similar but whose similarity cannot be proven without further information.
- Similar Triangles by Illustrative Mathematics is a task that examines the similarity of triangles from the viewpoint of proportional sides.
- Congruent and Similar Triangles by Illustrative Mathematics is a task where students understand similarity as a natural extension of congruence.

AA-Similarity Criteria Theorem

- AA Similarity Theorem by Tim Brzenzinski is a GeoGebra applet that explores AA-Similarity Criteria.
- AA Similarity Exploration by BIM is a GeoGebra exploration about AA-Similarity Criteria.

Curriculum and Lessons from Other Sources

- EngageNY, Geometry, Module 2, Topic A, Lesson 1: Scale Drawings, Lesson 2: Making Scale Drawing Using the Ratio Method, Lesson 3: Making Scale Drawings Using the Parallel Method are lessons that pertain to this cluster.
- EngageNY, Geometry, Module 2, Topic B, Lesson 6: Dilations as Transformations of the Plane, Lesson 7: How Do Dilations Map Segments?, Lesson 8: How Do Dilations Map Lines, Rays, and Circles?, Lesson 9: How Do Dilations Map Angles?, Lesson 10: Dividing the King's Foot into 12 Equal Pieces, Lesson 11: Dilations from Different Centers are lessons that pertain to this cluster.
- EngageNY, Geometry, Module 2, Topic C, Lesson 12: What are Similarity Transformations, and Why Do We Need Them?, Lesson 13: Properties of Similarity Transformations, Lesson 14: Similarity, Lesson 15: The Angle-Angle (AA) Criterion for two Triangles to Be Similar, Lesson 16: Between-Figure and Within-Figure Ratios, Lesson 17: The Side-Angle-Side (SAS) and Side-Side=Side (SSS) Criteria for Two Triangles to Be Similar are lessons the pertain to this cluster.
- Mathematics Vision Project, Geometry, Module 4: Similarity and Right Triangle Trigonometry has several tasks that pertain to this cluster.
- Georgia Standards of Excellence Curriculum Framework, Geometry, Unit 2: Similarity, Congruence, and Proofs has many tasks that pertain to this cluster.
- Illustrative Mathematics, Geometry, Unit 3: Similarity, Lesson 1: Scale Drawings, Lesson 3: Measuring Dilations, Lesson 4: Dilating Lines and Angles, Lesson 6: Connecting Similarity and Transformations, Lesson 7: Reasoning about Similarity with Transformations, Lesson 8: Are They All Similar, and Lesson 9: Conditions for Triangle Similarity are lessons that pertain to this cluster.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.SRT.1-3)

General Resources

- Arizona 7-12 Progression on Geometry is an informational resource for teachers. This cluster is addressed on pages 16-17.
- High School Coherence Map by UnboundED is a map of the relationships among high school mathematics standards. Please note: This resource is aligned to the Common Core not Ohio's Learning Standards. Although the tool is useful, it needs to be checked with respect to Ohio's Learning Standards.
- van Hiele Model of Geometric Thinking is a pdf created by ODE that summarizes the van Hiele levels.

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STANDARDS	MODEL CURRICULUM (G.SRT.4-5)
Geometry **SIMILARITY, RIGHT TRIANGLES, AND TRIGONOMETRY** **Prove and apply theorems both formally and informally involving similarity using a variety of methods.** **G.SRT.4** Prove and apply theorems about triangles. *Theorems include but are not restricted to the following: a line parallel to one side of a triangle divides the other two proportionally, and conversely; the Pythagorean Theorem proved using triangle similarity.* **G.SRT.5** Use congruence and similarity criteria for triangles to solve problems and to justify relationships in geometric figures that can be decomposed into triangles.	**Expectations for Learning** In middle school, students draw, construct, and describe geometric figures; use informal arguments to establish facts about similar triangles; and explain a proof of the Pythagorean Theorem and its converse. In this cluster, students prove theorems and solve problems involving similarity of triangles. They will also solve problems by applying these theorems to geometric figures that can be decomposed into triangles. The student understanding of this cluster begins at van Hiele Level 2 (Informal Deduction/Abstraction) and moves to Level 3 (Deduction). **ESSENTIAL UNDERSTANDINGS** • The altitude to the hypotenuse divides a right triangle into two triangles that are similar to the original triangle. • A line parallel to the side of a triangle makes similar triangles and divides the other two side lengths proportionally. • Two right triangles are similar if they have another congruent angle. • Polygons can be divided into congruent and/or similar triangles. **MATHEMATICAL THINKING** • Use accurate mathematical vocabulary to represent geometric relationships. • Recognize, apply, and justify mathematical concepts, terms, and their properties. • Use formal reasoning with symbolic representation. • Make conjectures. • Plan a solution pathway. • Justify relationships in geometric figures. • Determine reasonableness of results. • Create a drawing and add components as appropriate. *Continued on next page*

STANDARDS	MODEL CURRICULUM (G.SRT.4-5)
	Expectations for Learning, continued **INSTRUCTIONAL FOCUS** • Form conjectures and construct a valid argument for why the conjecture is true or not true, both formally and informally. • Recognize when polygons are divided into congruent and/or similar triangles. • Justify relationships in geometric figures that can be decomposed into triangles. • Solve problems using triangle congruence and similarity. **Content Elaborations** **OHIO'S HIGH SCHOOL CRITICAL AREAS OF FOCUS** • Geometry, Number 3, page 6 **CONNECTIONS ACROSS STANDARDS** • Understand similarity (G.SRT.1-3). • Define trigonometric ratios, and solve problems involving right triangles (G.SRT.6-8). • Understand the relationships between lengths, areas, and volumes in similar figures (G.GMD.5-6). • Use coordinate geometry (G.GPE.4). • Use coordinates to prove simple geometric theorems algebraically (G.GPE.6). • Represent transformations in the plane (G.CO.2). • Prove and apply geometric theorems (G.CO.9-10). • Make geometric constructions (G.CO.12-13). • Find arc lengths and areas of sectors of circles (G.C.5). • Apply geometric concepts in modeling situations (G.MG.2-3).

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.SRT.4-5)

Instructional Strategies

Note: The model curriculum is not a collection of lessons nor a full curriculum; it does not suggest pace, sequence, or amount of time spent on topics. It provides information about a topic related to the standards including ideas for examples, strategies for teaching, common misconceptions, and possible connections between topics. In addition, the emphasis of the model curriculum is on instruction not on assessment; therefore, the examples in the Instructional Strategies section are written for instructional purposes not necessarily as assessment items. Examples may need to be rewritten to accommodate the needs of each individual classroom.

Standards for Mathematical Practice
This cluster focuses on but is not limited to the following practices:
***MP.3** Construct viable arguments and critique the reasoning of others.*
***MP.5** Use appropriate tools strategically.*
***MP.6** Attend to precision.*
***MP.7** Look for and make use of structure.*

This cluster focuses on providing a logical argument that demonstrates the truth about theorems involving triangles. A proof is typically a series of justifications presented either formally or informally. They may use different proving methods such as deductive, inductive, two-column, paragraph, flow chart, visual, counter-examples, and others. The Pythagorean Theorem and its converse are proved and applied in Grade 8. In high school, another proof, based on similar triangles, is presented.

Note: The alternate interior angle theorem and its converse are established informally in Grade 8 and proved formally in high school. Depending on the sequencing of the course, students may not know theorems regarding parallel lines cut by a transversal.

The Fundamental Theorem of Similarity will be addressed in G.GMD.5-6.

VAN HIELE CONNECTION

In Geometry students are expected to move from Level 2 (Informal Deduction/Abstraction) to Level 3 (Deduction). Whereas in Level 2 students use informal arguments, Level 3 requires a more formal approach to thinking.

Level 3 can be characterized by the student doing some or all of the following:

- constructing proofs and understanding the necessity of proofs;
- understanding the role and relationships between of axioms, theorems, definitions, and postulates;
- differentiating between necessary and sufficient conditions; and/or
- making logical conclusions.

PROVING AND APPLYING THEOREMS ABOUT TRIANGLES

Here is some general information about proof instruction:

- The main concept that should be gleaned from these standards is that students need to be able to explain reasons for their thinking and why/how something is true, much like in the ELA Writing Standards where *evidence* must be included for any claim.
- Teachers should vary the level of formality that is appropriate for the content, and the reader of the proof. However, every level of formality includes students' ability to formally/informally reference the appropriate source—a definition, a property, a law, an axiom, a theorem, etc. Make sure that the level of formality does not distract from the main idea of the proof.
- The emphasis should be on a progression toward proof and not on formal proof. Students need to be able to come up with their own conjectures and then provide mathematically sound justification for the conjectures' validity. Ultimately students should construct a complete argument, in a variety of formats, to move from given information to a conclusion.
- Direct instruction is not the best way to introduce formal proof. Instead the focus should "be on justifying ideas while helping students build the visual and empirical foundation for higher levels of geometric thought, we can lead students to appreciate the need for formal proof." (Battista and Clements, 1995)
- Students continue to use precise language and relevant vocabulary to justify steps in their work to construct viable arguments that defend their method of solution.
- All statements need to be examined on their own merit—conditionals and their converse statements are not always both true. For the times when both conditionals and their converses are true, biconditional statements can be written.
- Proof methods could include but are not limited to the following: deductive, inductive, two-column, paragraph, flow chart, visual, and/or counterexample.
- The concepts of inverse and contrapositive are not emphasized but can be explored as an extension.

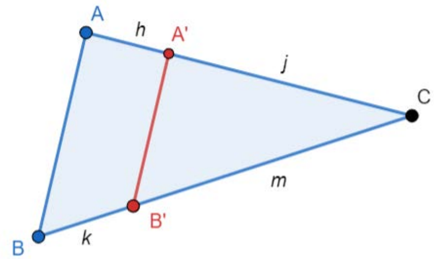
Triangle Side-Splitter Theorem

Another important application of similarity is the Triangle Side-Splitter Theorem: If a line is parallel to one side of a triangle and intersects the other two sides in distinct points, it splits these sides into proportional segments. Another way to phrase it is that a line segment splits two sides of triangle proportionally if and only if it is parallel to the third side. Students should also be able to prove its converse.

EXAMPLE

Given: $\overline{A'B'}$ is a dilation of $\overline{AB}$ with scale factor r and a point C as the center of dilation. The side lengths are as labeled.

Prove: $\frac{h}{j} = \frac{k}{m}$.



Method 1

This approach uses the definition of dilation (scale factor).

According the definition of a dilation, $|CA'| = r|CA|$ and $|CB'| = r|CB|$.

Therefore, $j = r(j+h)$ and $m = r(m+k)$, so

$r = \frac{j}{j+h}$ and $r = \frac{m}{m+k}$

$\frac{m}{m+k} = \frac{j}{j+h}$ by substitution

$m(j+h) = j(m+k)$

$mj + mh = jm + jk$

$mh = jk$

$\frac{mh}{mj} = \frac{jk}{mj}$

$\frac{h}{j} = \frac{k}{m}$

Method 2

This approach uses a property of dilation (parallel lines). Since $\overline{A'B'}$ is a dilation of $\overline{AB}$, $\overline{A'B'} \parallel \overline{AB}$ because a dilation takes a line to a parallel line if its center is not on the line, then $\angle CA'B' \cong \angle CAB$ and $CB'A' \cong \angle CBA$ by corresponding angles formed by a parallel lines and a transversal. Thus, $\Delta CA'B' \sim \Delta CAB$ by AA-Similarity Criteria. Since corresponding sides of similar figures are proportional,

$\frac{j+h}{j} = \frac{m+k}{m}$, so

$m(j+h) = j(m+k)$

$mj + mh = jm + jk$

$mh = jk$

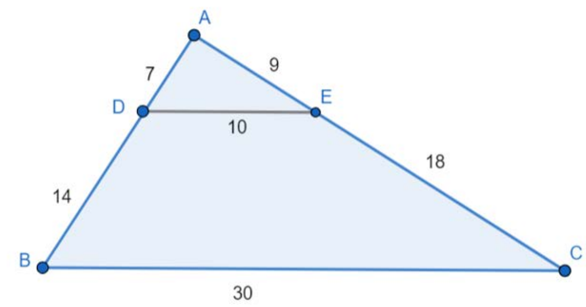
$\frac{mh}{mj} = \frac{jk}{mj}$

$\frac{h}{j} = \frac{k}{m}$

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.SRT.4-5)

TIP!

Using the AA-Similarity Criteria to prove the Triangle Side-Splitter Theorem uses circular reasoning. More advanced students can confront this idea by using the formula for the area of a triangle to explain the Side-Splitter Theorem. See Parallels in Geometry by Snapp and Findell pages 91-99 and EngageNY, Geometry, Module 2, Topic A, Lesson 4: Comparing the Ratio Method with the Parallel Method for more information.



When a line parallel to one side of a triangle divides the other two proportionally, students will sometimes misapply the Triangle Side-Splitter Theorem to situations with similar triangles. It is important to give students problems where they not only find missing parts as in the Triangle Side-Splitter Theorem but also give situations where one of the missing measures is one of the parallel sides. Sometimes students incorrectly say $part_1$:$part_2$ (sides divided proportionally) = $whole_1$:$whole_2$ (parallel side lengths), instead of correctly applying the Triangle Side-SplitterTheorem by stating $\frac{1st\ part\ of\ side\ 1}{2nd\ part\ of\ side\ 1} = \frac{1st\ part\ of\ side\ 2}{2nd\ part\ of\ side\ 2}$ or $\frac{1st\ part\ of\ side\ 1}{1st\ part\ of\ side\ 2} = \frac{2nd\ part\ of\ side\ 1}{2nd\ part\ of\ side\ 2}$ and differentiating between similar triangles by stating $\frac{1st\ side\ of\ litte\ triangle}{1st\ side\ of\ big\ triangles} = \frac{2nd\ side\ of\ litte\ triangle}{2nd\ side\ of\ big\ triangles}$ or $\frac{1st\ side\ of\ litte\ triangle}{2nd\ side\ of\ little\ triangles} = \frac{1st\ side\ of\ big\ triangle}{2nd\ side\ of\ big\ triangles}$. Decomposing the diagram into two similar triangles may help students recognize corresponding parts.

$\frac{Part_1}{Part_2} \neq \frac{Whole_1}{Whole_2}$, so $\frac{7}{14} \neq \frac{10}{30}$

Instead of using the Side-Splitter Theorem

$\frac{Part_1\ of\ \overline{AB}}{Part_2\ of\ \overline{AB}} = \frac{Part_1\ of \overline{AC}}{Part_2\ of\ \overline{AC}}$ such as $\frac{7}{14} = \frac{9}{18}$ or

$\frac{Part_1\ of\ \overline{AB}}{Part_1\ of\ \overline{AC}} = \frac{Part_2\ of \overline{AB}}{Part_2\ of\ \overline{AC}}$ such as $\frac{7}{9} = \frac{14}{18}$

Compared to using Corresponding Parts of Similar Triangles are Proportional (CPSTP)

$\frac{little_1}{little_2} = \frac{big_1}{big_2}$ such as $\frac{7}{10} = \frac{21}{30}$ or

$\frac{little_1}{big_1} = \frac{little_2}{big_2}$ such as $\frac{7}{21} = \frac{10}{30}$

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.SRT.4-5)

EXAMPLE

Given:

Figure 1: $\overline{BC} \parallel \overline{DF}$

Figure 2: $\overline{OK} \parallel \overline{PE}$

a. Find x in Figure 1.

b. Find y in Figure 2.

c.

- Rihanna said that she solved both problems the same way and got $x = 6$ for Figure 1 and $y = \frac{12}{5}$ for Figure 2.
- Marcus said he solved them both the same way (but differently from Rihanna) and got $x = 6$ for Figure 1 and $y = \frac{32}{5}$ for Figure 2.
- Lamar said that he did them different ways, but also got $x = 6$ for Figure 1 and $y = \frac{12}{5}$ for Figure 2.

Figure 1

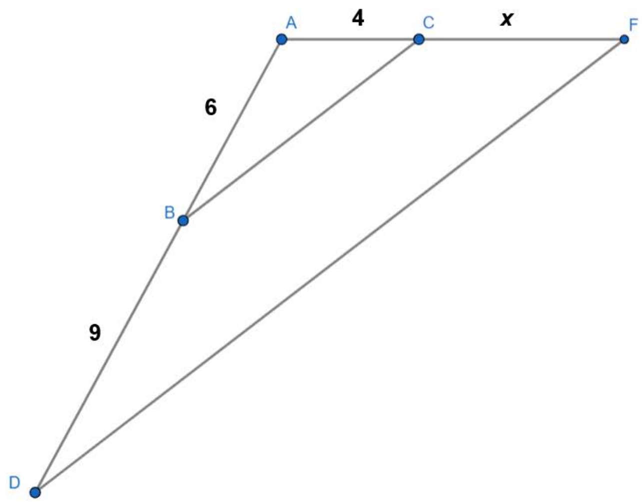
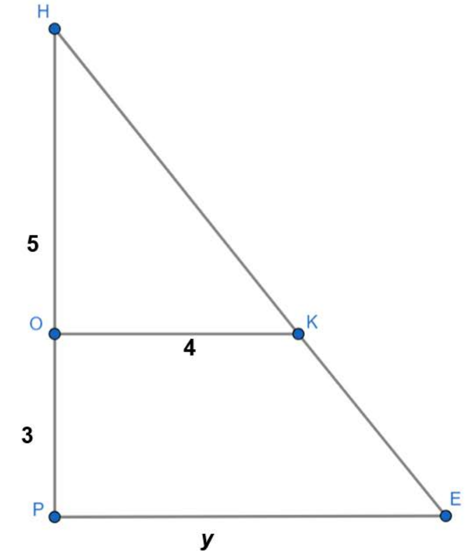


Figure 2



How did each person get their result? Who did the problem correctly?

Midsegment (or Midpoint-Connector) Theorem

Students should recognize that the Midsegment Theorem "The segment joining midpoints of two sides of a triangle is parallel to the third side and half the length." is a special case of the Triangle Side-Splitter Theorem. The Midsegment Theorem can be proved using SAS-Similarity Criteria for Triangles and the Alternate Interior Angle Theorem.

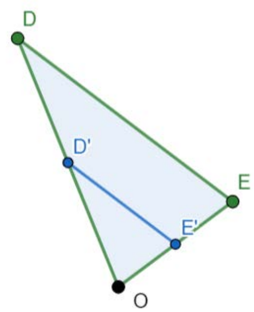
An optional application of the Midsegment Theorem is to prove that the figure formed by joining consecutive midpoints of sides of an arbitrary quadrilateral is a parallelogram. (This result is known as the Midpoint Quadrilateral Theorem or Varignon's Theorem.)

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.SRT.4-5)

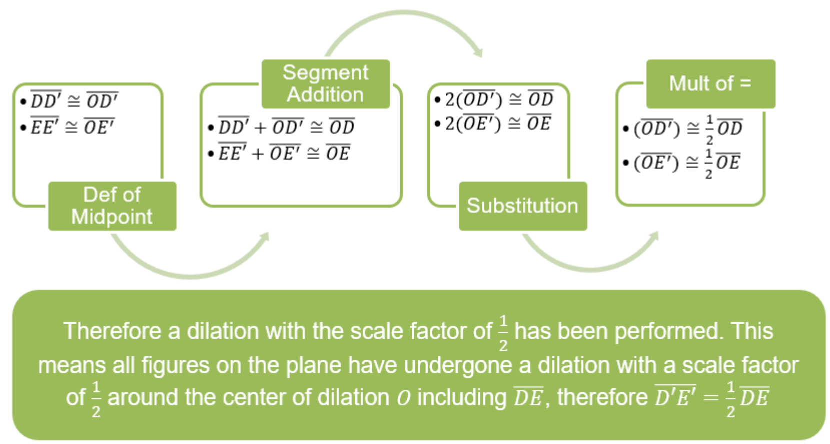
EXAMPLE

Given: $D'$is the midpoint of $\overline{OD}$, and $E'$is the midpoint of $\overline{OE}$.

Prove: $\overline{D'E'} = \frac{1}{2}\overline{DE}$



Discussion:

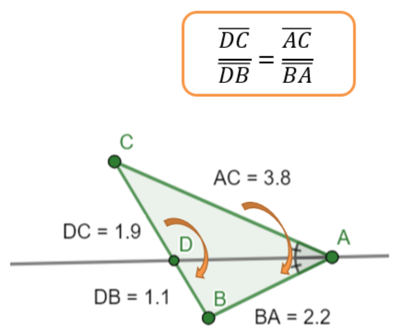


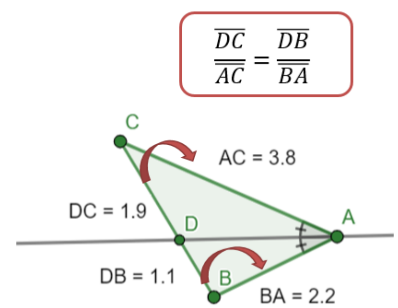
Note: This theorem could be approached from the perspective of similar triangles or could also be tied into G.GPE.4 as a coordinate proof.

Midsegment Theorem by Brad Findell is an interactive proof that explores the Midsegment Theorem: The segment joining the midpoints of two sides of a triangle is parallel to and half the length of the third side.

Angle Bisector Theorem

The Angle Bisector Theorem states the angle bisector of the interior angle of a triangle, divides the opposite side into two segments that are proportional to the other two sides of the triangle. It rests on the Angle Bisector Conjecture stating that any point on the bisector of an angle is equidistant from its sides (See G.CO.9-11).





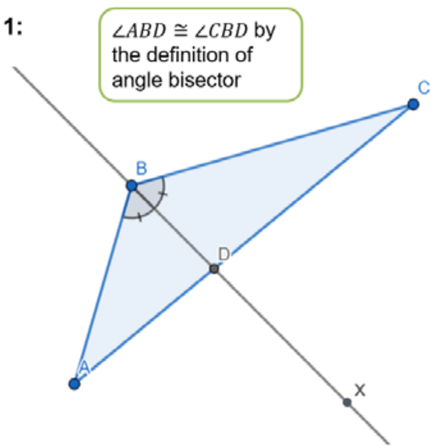
INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.SRT.4-5)

EXAMPLE

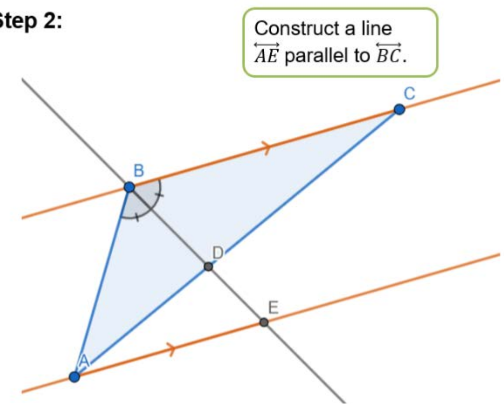
Given: $\overleftrightarrow{BX}$ bisects $\angle ABC$ in $\triangle ABC$

Prove: $\frac{\overline{DC}}{\overline{DA}} = \frac{\overline{BC}}{\overline{AB}}$ or $\frac{\overline{DC}}{\overline{BC}} = \frac{\overline{AD}}{\overline{AB}}$

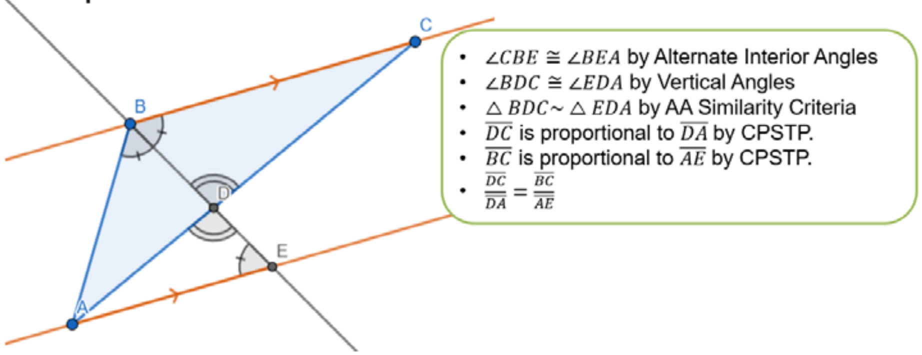
Step 1:



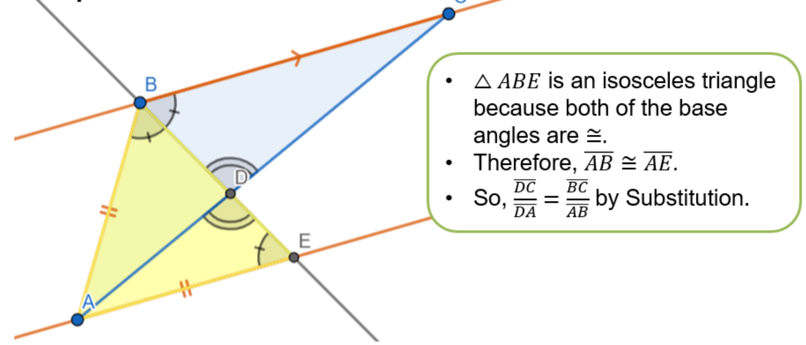
Step 2:



Step 3:



Step 4:

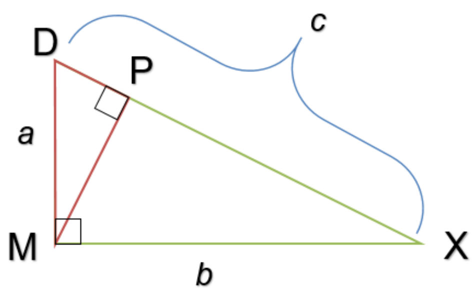


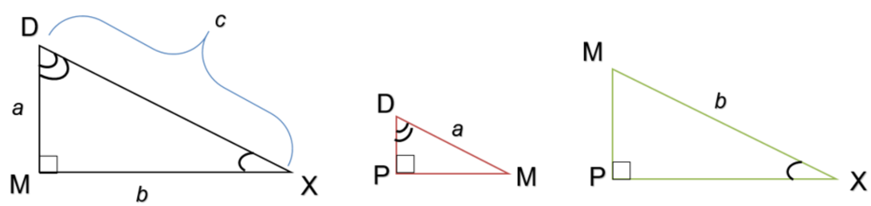
Pythagorean Theorem

It is important for students to understand an altitude drawn to the hypotenuse of a right triangle creates two triangles that are similar to the original triangle. Use physical models such as cardboard cutouts to illustrate that the altitude to the hypotenuse divides a right triangle into two triangles that are similar to the original triangle. Then use AA-Similarity Criteria to prove this theorem. This is used as a proof of the Pythagorean Theorem and a basis of Trigonometry. *Note: The geometric mean is discussed later in this section.*

EXAMPLE

Prove that if an altitude drawn to the hypotenuse of a right triangle creates two triangles, then the new right triangles are similar to the original triangle.





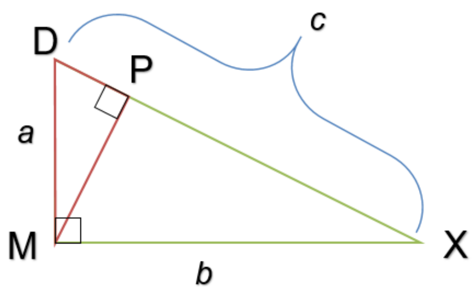
$\triangle MDX \sim \triangle PDM$
and
$\triangle MDX \sim \triangle PMX$
by AA similarity criteria

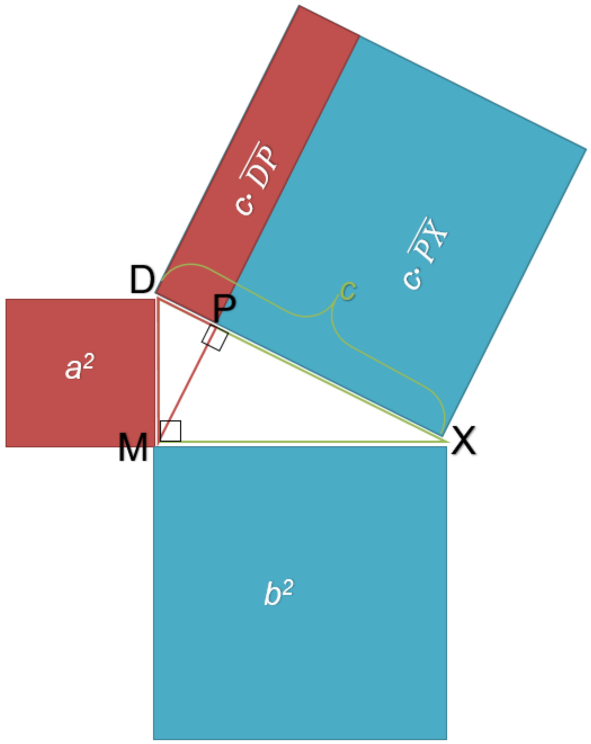
Discussion: Then, use this result to establish the Pythagorean relationship among the sides of a right triangle ($a^2 + b^2 = c^2$), and thus obtain an algebraic proof of the Pythagorean Theorem. See the next example.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.SRT.4-5)

EXAMPLE

Prove that $a^2 + b^2 = c^2$ for a right triangle using concepts of similarity.





Discussion:

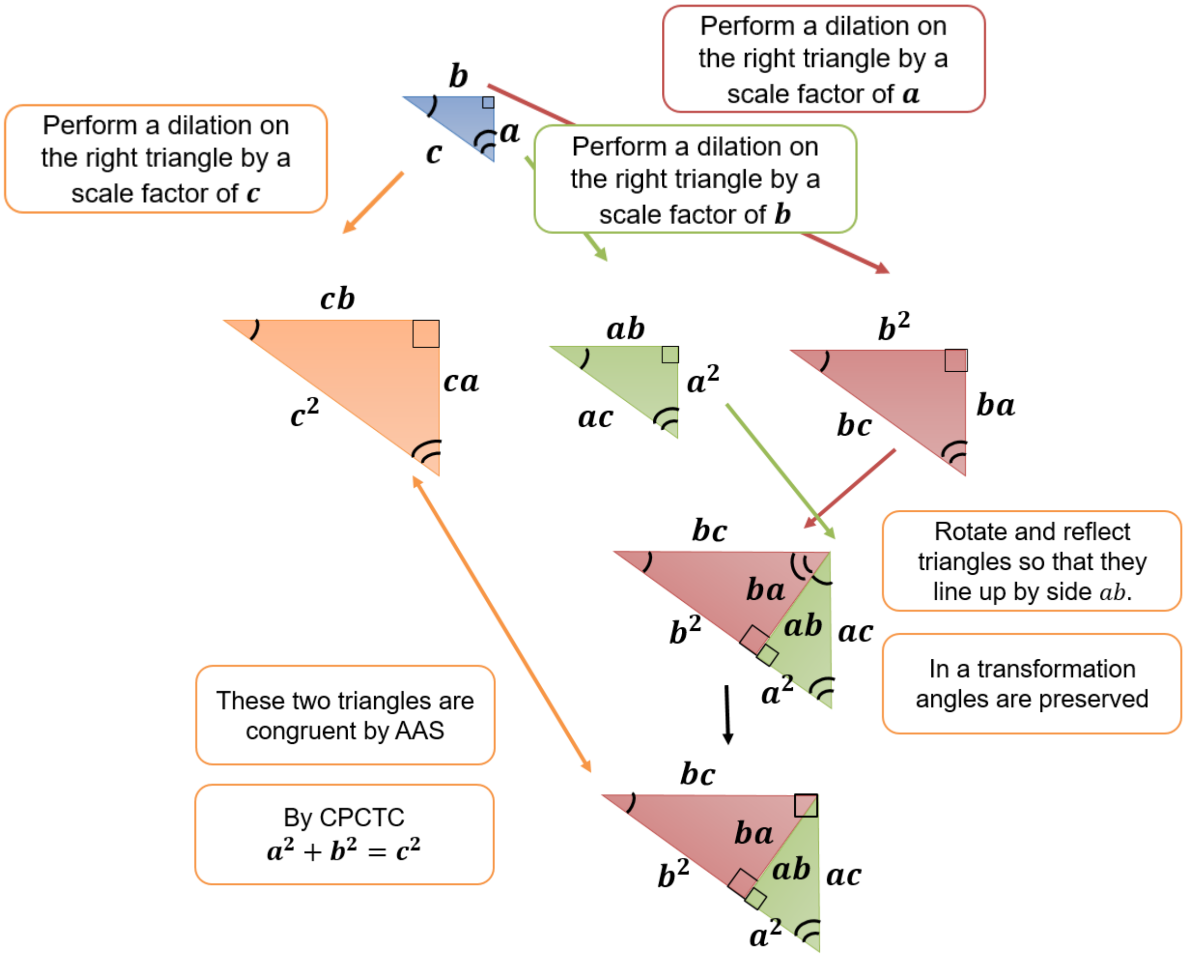
- By AA-Similarity Criteria students should see that $\Delta MDX \sim \Delta PDM$ and $\Delta PMX \sim \Delta MDX$ by AA-Similarity Criteria (See pervious example).
- Since corresponding sides are proportional in similar triangles $\frac{\overline{DP}}{\overline{DM}} = \frac{\overline{DM}}{\overline{DX}}$ and $\frac{\overline{XM}}{\overline{DX}} = \frac{\overline{PX}}{\overline{XM}}$.
- Then the equations could be rewritten as $\overline{DM}^2 = \overline{DP} \cdot \overline{DX}$ and $\overline{XM}^2 = \overline{PX} \cdot \overline{DX}$.
- Combine the two equations: $\overline{DM}^2 + \overline{XM}^2 = \overline{DP} \cdot \overline{DX} + \overline{PX} \cdot \overline{DX}$.
- Factor out $\overline{DX}$ to get $\overline{DM}^2 + \overline{XM}^2 = \overline{DX}\,(\overline{DP} + \overline{PX})$.
- Since $\overline{DX}$ is $\overline{DP} + \overline{PX}$, the equation can be rewritten as $\overline{DM}^2 + \overline{XM}^2 = \overline{DX}^2$.
- Then by substituting lengths $a, b,$ and c back into the equation we get $a^2 + b^2 = c^2$ which is the Pythagorean Theorem for triangle ΔMDX.

Pythagorean Theorem Proof for Similar Right Triangles is a video by bikes4fish explaining the proof a little more-in depth.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.SRT.4-5)

EXAMPLE

Prove the Pythagorean Theorem using transformations.

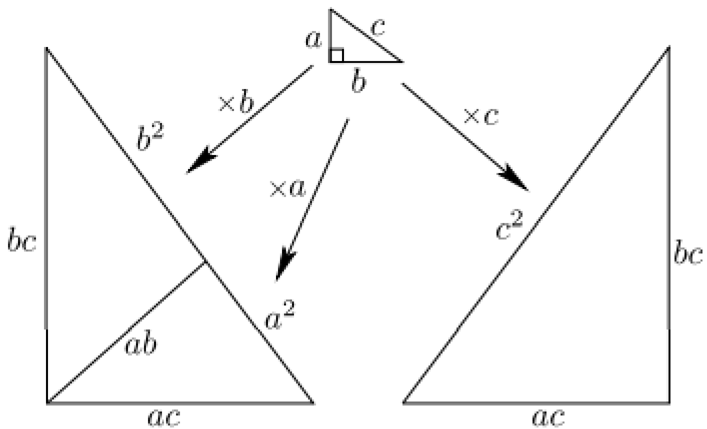


INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.SRT.4-5)

Similar Right Triangles by Brad Findell is an interactive proof where students can explore proving triangles created by dropping a perpendicular line to the hypotenuse are also similar.

EXAMPLE (EXTENSION)

To challenge students, give them the image below as a "Proof without Words." Have them explain how it proves the Pythagorean Theorem. *Note: To be an extension, it should be in place of the previous example.*



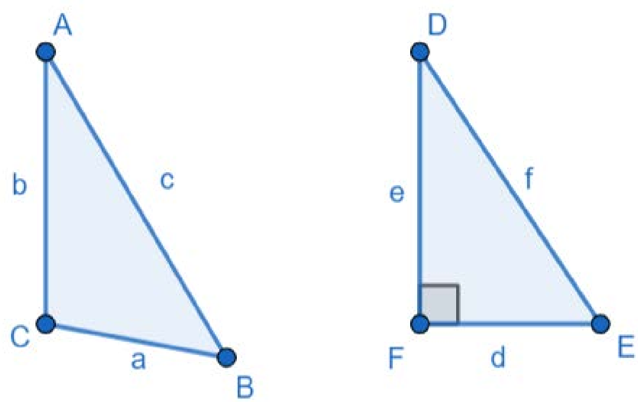
Taken from Snapp & Findell, 2016, Parallels in Geometry

Prove the converse of the Pythagorean Theorem, using the theorem itself as one step in the proof.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.SRT.4-5)

EXAMPLE

Prove that if $a^2 + b^2 = c^2$, then a triangle is a right triangle so $\angle C$ must be a right angle. To use a method of contradiction, assume that in $\triangle ABC, c^2 = a^2 + b^2$, and $\angle C$ not a right angle.



- Construct another triangle $\triangle DEF$ that is a right triangle where $\overline{FE} \cong \overline{CB}$ and $\overline{DF} \cong \overline{AC},$ and $FE = d$ and $DF = e$.
- Since $\triangle DEF$ is a right triangle, by the Pythagorean Theorem, $d^2 + e^2 = f^2$,
- Since $a = d$ and $b = e,$ then by substitution, $a^2 + b^2 = f^2$,
- Since $a^2 + b^2 = c^2$ and $a^2 + b^2 = f^2$, then by substitution $c^2 = f^2$ or $c = f,$
- Since $a = d$, $b = e$, and $c = f$, then by SSS, $\triangle ABC \cong \triangle DEF$,
- By CPCTC, $\angle C \cong \angle F$,
- By definition of congruence, then $m\angle C = m\angle F = 90°$,
- This means that $\angle C$ is a right angle and contradicts our assumption,
- Therefore for $a^2 + b^2 = c^2$ to be true, $\angle C$ must be a right angle and $\triangle ABC$ must be a right triangle.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.SRT.4-5)

TIP! Have students create a flipbook animation to demonstrate the Pythagorean Theorem or have students create a fractal tree using the Pythagorean Theorem. How to Draw Fractal Tree by Dearing Wang illustrate how to draw the tree.

TIP! Some students may confuse theorems and their converses such as the Alternate Interior Angle Theorem and its converse or the Pythagorean Theorem and its converse.

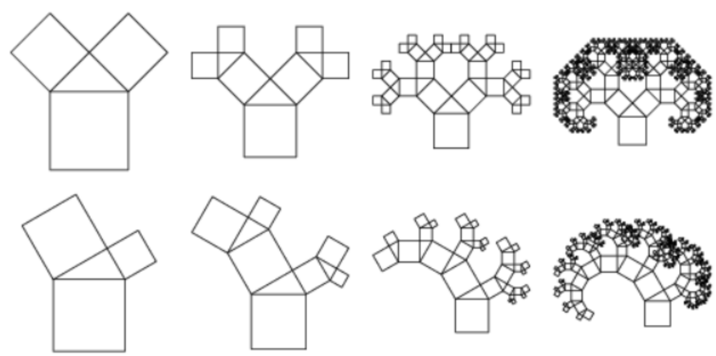


Image taken from http://mathworld.wolfram.com/PythagorasTree.html

Connecting Irrational Roots to the Pythagorean Theorem

Give students opportunities to conceptually understand irrational numbers. One way is for students to draw a square that is one unit by one unit and find the diagonal using the Pythagorean Theorem. The diagonal drawn has an irrational length of $\sqrt{2}$. Other irrational lengths can be found using the same strategy by finding diagonal lengths of rectangles with various side lengths. Students can also explore the Wheel of Theodorus to explore irrational numbers without using a number line. See the Instructional Resources/Tools for more ideas.

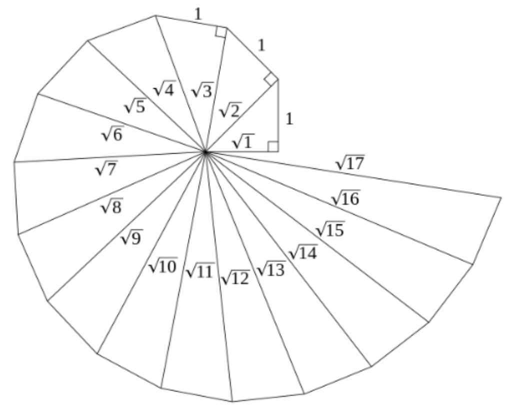


Image taken from https://en.wikipedia.org/wiki/Spiral_of_Theodorus

SOLVING PROBLEMS USING TRIANGLE CONGRUENCE AND SIMILARITY CRITERIA

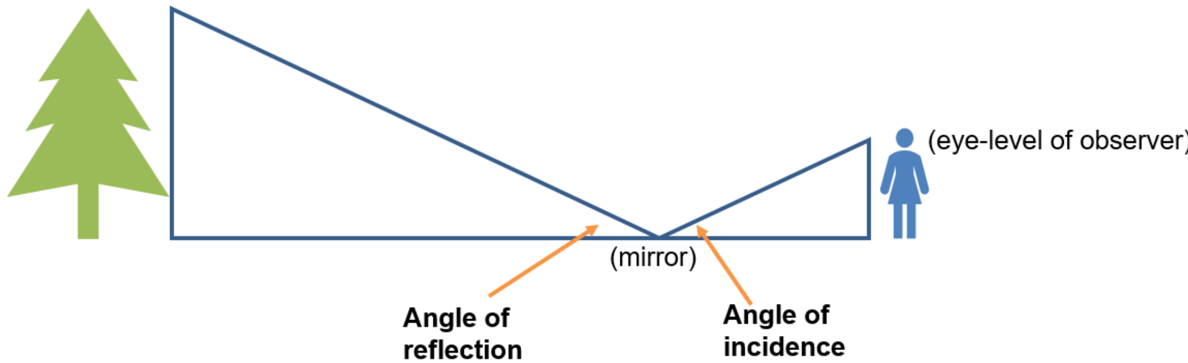
Students should apply their knowledge of congruence and similarity to solve mathematical and real-world problems surrounding triangles.

TIP! Students sometimes are dependent on given triangles presented with the same orientation instead of using corresponding parts for similarity. Give students shapes with different orientations to confront this issue.

EXAMPLE

Indirectly find the height of a nearby tree or telephone pole using a mirror and a meter stick.

Discussion: Students should apply their knowledge of similar triangles and the idea that the angle of incidence equals the angle of reflection to find the height of a nearby object.



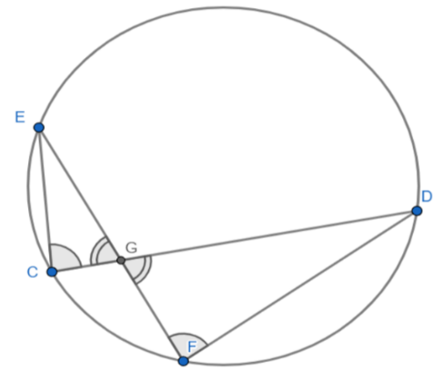
TIP! The concept of using angles to find images can be tied to Physics applications of light reflecting. [The Physics Classroom] has some information about images, light, reflections, and angles.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.SRT.4-5)

EXAMPLE

A tree has a shadow of 15 feet. Ray who is 6'1" has a shadow of 4'2" feet. How tall is the tree in feet?

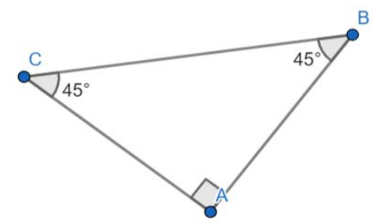
Students can also discover that triangles formed by angles inscribed in circles can be an application of similarity.

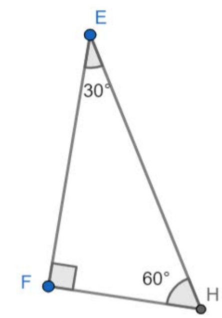


$\triangle GCE \sim \triangle GFD$ because
- $\angle EGC \cong \angle DGF$ by vertical angles, and
- $\angle ECG \cong \angle DFG$ are angles that intercept the same arc, so
- $\triangle GCE \sim \triangle GFD$ by AA criteria.

Special Right Triangles

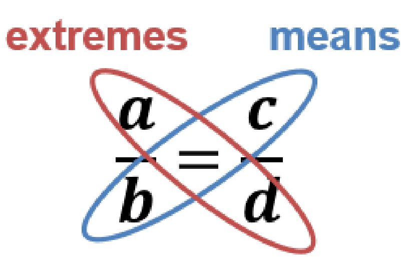
Students should explore special right triangles such as 30°-60°-90° triangles and 45°-45°-90° triangles.





Pythagorean Triples

Pythagorean triples should be taught as an application of similar triangles, for example a (a 6:8:10 right triangle is similar to a 3:4:5 right triangle). They can be used as an algebraic extension and an opportunity to explore patterns.



Geometric Mean

Traditionally middle school students have solved proportions using cross products also known as the Means-Extremes Property. The Means-Extreme Property states that in a true proportion, the product of the means equals the product of the extremes or "If $\frac{a}{b} = \frac{c}{d}$, then $ad = bc$." However, Ohio's Learning Standard for Mathematics in the middle grades now deemphasize this method because students at that level do not understand why it works, and therefore oftentimes misapply it. Instead they emphasize solving proportions using within and between relationships, common denominator method, the unit rate method, and graphing proportions, so students may not be familiar with solving proportions this way. See Model Curriculum 7.RP.1-3 for more information on solving proportions.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.SRT.4-5)

In a proportion the means (or the extremes) can be changed to form an equivalent true proportion or the reciprocals can be used to create an equivalent true proportion. These properties, although not formerly named, should have been explored in middle school. They can be proved through algebraic methods.

Means Exchange Property

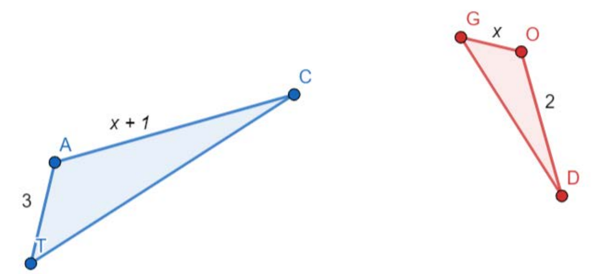
If $\frac{a}{b} = \frac{c}{d}$, then $\frac{a}{c} = \frac{b}{d}$.

Reciprocals Property

If $\frac{a}{b} = \frac{c}{d}$, then $\frac{b}{a} = \frac{d}{c}$.

EXAMPLE

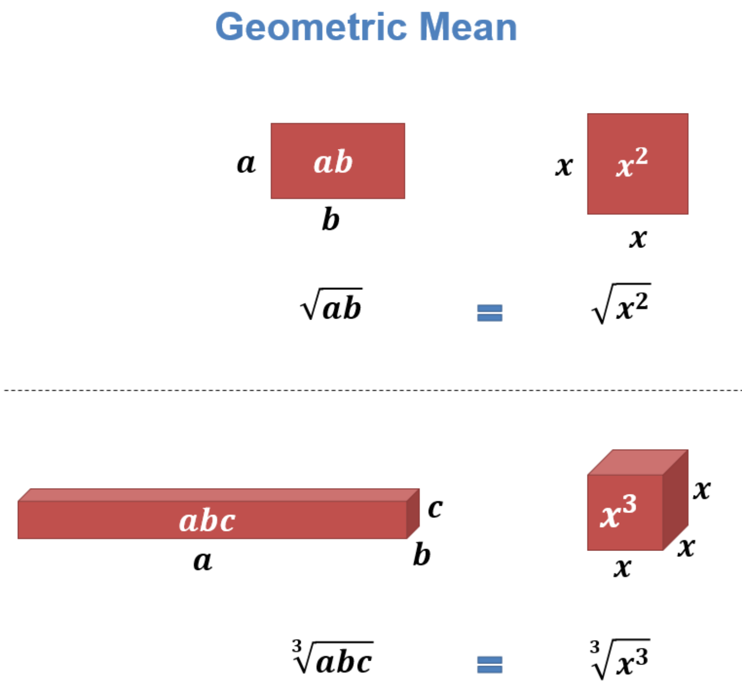
$\Delta CAT \sim \Delta DOG$. Find the missing side lengths of the similar triangles. Triangles are not drawn to scale.



When the means of a proportion are identical, the identical segment or number is called the geometric mean.

In $\frac{a}{m} = \frac{m}{d}$, m is the geometric mean

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.SRT.4-5)

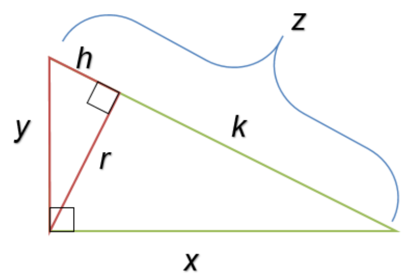


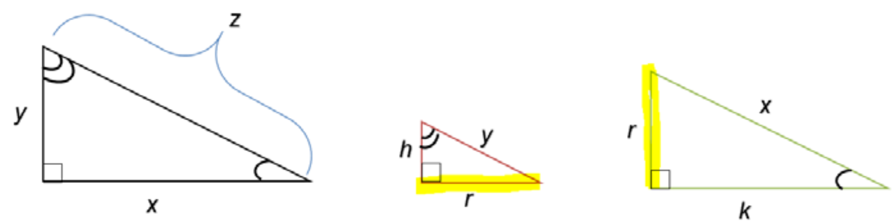
Although the geometric mean is often used in the contexts of proportions, it is not limited to proportional situations. It is a special type of mean where the factors are multiplied together, and then the root (corresponding to the number of factors) is taken. The nth root of the product for n numbers. It is useful when comparing things with different properties. Whereas the arithmetic mean sums all the values in a data set and then divides the sum by the number of terms, the geometric mean multiplies all the values in a data set, and takes the root corresponding to the number of terms. The geometric mean is useful when comparing values with different units. The concept of geometric mean should be connected to right triangles.

The geometric mean can also be viewed in light of geometric figures. The geometric mean of two numbers, a and b, is the length of one side of a square whose area is equal to the area of the rectangles with sides of lengths a and b. Similarly, the geometric mean of three numbers, a, b, and c is the length of one side of a cube whose volume is equal to the volume of the prism with side lengths, a, b, and c.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.SRT.4-5)

EXAMPLE





Part 1

- **a.** How many triangles are in the picture?
- **b.** Draw all the triangles you see in the same orientation keeping the labels.
- **c.** What is the relationship between the triangles? Explain.
- **d.** What is the same on the two smaller triangles? Explain.
- **e.** Why is this important?

Part 2

- **a.** Use the information from Part 1 to find the missing side in figures 1 and 2.
- **b.** Create a rule describing the relationship between *h*, *k*, and *r*.
- **c.** The rule you created in part **b.** illustrates the geometric mean. What other rules can you create using the side lengths of the three right triangles.

Figure 1:

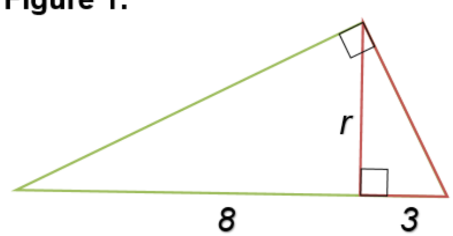
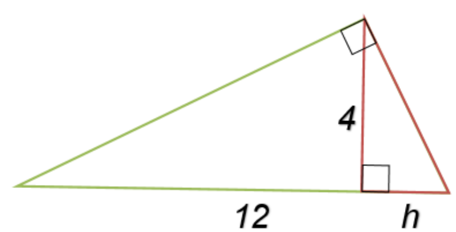


Figure 2:



Discussion: Students should notice that there are three right triangles in the diagram, and that they are all similar by AA-Triangle Similarity Criteria. They should be able to come up with the rule: The altitude to the hypotenuse is the geometric mean of the two segments into which it divides the hypotenuse or $r = \sqrt{hk}$. They should also be able to deduce that each leg is the geometric mean of the hypotenuse, and the segment of the hypotenuse adjacent to the leg or $x = \sqrt{zk}$ and $y = \sqrt{zh}$. This is the Right Triangle Altitude Theorem.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.SRT.4-5)

The geometric mean can also be connected to concepts in Algebra.

EXAMPLE

Ted wants to book a hotel and is looking at two different websites. The Cabana Inn is ranked 3.7 stars on the first website and 8.63 on the second site. The Holiday Hotel is ranked 4.2 on the first website and 8.12 on the second website. Which hotel has the best ratings?

Discussion: Although these examples are algebraic instead of geometric, they lay the foundation for understanding the geometric mean and help students make connections between algebraic and geometric concepts. The idea of geometric mean will be connected to right triangles. Since the Cabana Inn has a geometric mean of $\sqrt{3.7 \cdot 8.63} \approx 5.65$, and the Holiday Hotel has a geometric mean of $\sqrt{4.2 \cdot 8.12} \approx 5.84$, the Holiday Hotel has better overall rankings.

EXAMPLE

A stock grows by 15% one year, declines 25% the second year, and then grows by 35% the third year. Use the geometric mean to calculate the average growth rate (known as the compounded annual growth rate).

Discussion: This can be calculated by $\sqrt[3]{(1+0.15)(1-0.25)(1+0.35)}-1$ which is approximately 0.052 or 5.2% annually.

Have students explore how a right triangle's altitude connects with the geometric mean.

EXAMPLE

Find the three missing terms in the geometric sequence 2, ___, ___, ___, 10.125…

Discussion: Students can find the 3rd term by finding the geometric mean of 2 and 10.125: $b=\sqrt{2 \cdot 10.125}=4.5$. Then, they can find the 2nd term by finding the geometric mean of 2 and 4.5: $a=\sqrt{2 \cdot 4.5}=3$. Finally, they can find the 4th term by finding the geometric mean of 4.5 and 10.125: $c=\sqrt{4.5 \cdot 10.125}=6.75$. If desired, they could also then take the concept a step further and find the common ratio of the sequence which is 1.5.

Decomposing Polygons into Triangles to Justify Relationships

Students should justify relationships in geometric figures that can be decomposed into triangles. For example, given a trapezoid, students can discover that the diagonals create two similar triangles.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.SRT.4-5)

Instructional Tools/Resources

These tools and resources may be helpful for instruction. Before use, they should be evaluated at the local level to determine their appropriateness for instruction.

Manipulatives/Technology

- Cardboard models of right triangles.
- Dynamic geometric software (Geometer's Sketchpad®, Desmos®, Cabri®, or GeoGebra®).
- Mirrors
- Tape Measures
- Protractors

Applying Theorems about Triangles

- Solving Problems with Circles and Triangles by Mathematics Assessment Project is a lesson where students solve problems by determining the lengths of the sides in right triangles.

Midpoints of a Sides of a Triangle

- Joining Two Midpoints of Sides of a Triangle by Illustrative Mathematics is a task where students solve a problem involving midpoints of sides of a triangle.

Pythagorean Theorem

- Pythagorean Theorem by Illustrative Mathematics is a task where students prove the Pythagorean Theorem using similar triangles.
- Pythagorean Theorem by Davis Associates, Inc. has an animated proof of the Pythagorean Theorem.
- Pythagorean Theorem by Cut the Knot has 18 approaches to the Pythagorean Theorem.
- Video: *The Theorem of Pythagoras* from Project MATHEMATICS!
- Lunar Rover by NASA is a lesson where students apply the Pythagorean Theorem to a situation involving a lunar rover.
- Wheel of Theodorus is also called the spiral of Theodorus, the Square Root Spiral, the Einstein Spiral, or the Pythagorean Spiral. Starting with a right triangle with legs the length of one, each succeeding right triangle is formed with one of the legs being one and the second leg being the hypotenuse of the preceding triangle.
  - Root Spiral of Theodorus is a YouTube video by bikes4fish that shows creating the Wheel of Theodorus.
  - Pythagorean Spiral Video by Jendar40 is a YouTube video that shows creating the Wheel of Theodorus.
  - Create the Wheel of Theodorus and Find its Pattern of Irrational Numbers by Using the Iterations of the Pythagorean Theorem is a lesson by LearnZillion.
  - Wheel of Theodorus by Bill Lombard is a GeoGebra animation that shows the Wheel of Theodorus.
  - Wheel of Theodorus Art Project Part 1 by Sara Scholes is a YouTube Video that shows examples of the Wheel of Theodorus in nature and shows you how to make it and examples of student work with a corresponding grading rubric. Wheel of Theodorus Calculation Chart by Sara Scholes is the second part of the first video. Students calculate the measures of each of the legs of the right triangles in the wheel to determine the patterns in the triangles.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.SRT.4-5)

Geometric Mean

- Geometric Mean and Right Triangles by Jeffery P. Smith is a GeoGebra applet where students can explore the geometric mean using a right triangle.
- Similar Right Triangle Side Lengths by Math Warehouse has a video, applet, and examples of using the geometric mean to solve problems involving similar right triangle.

Using Similarity Criteria to Solve Problems

- Bank Shot by Illustrative Mathematics is a task where students use similarity to solve a problem in the context of a pool table.
- Tangent Line to Two Circles by Illustrative Mathematics is a task where students use similarity to calculate a side length.
- Points from Directions by Illustrative Mathematics is a task where students use similarity and the Pythagorean Theorem to solve a problem using directions.
- Extensions, Bisections and Dissections in a Rectangle by Illustrative Mathematics is a task where students use an application of similar triangles.
- Folding a Square into Thirds by Illustrative Mathematics is a task where students apply knowledge about similar triangles to an origami construction.
- How Far Is the Horizon? by Illustrative Mathematics is a modeling task where students use similarity and the Pythagorean Theorem in the context of looking at a horizon in Alaska. Students will need to make reasonable assumptions as part of the modeling process and seek out information for themselves.
- Mirror, Mirror, on the Ground by CPalms is a lesson that applies similar triangles to situations that use the angle of incidence and angle of reflection.
- Similar Triangle Applications by Passy World of Mathematics is a blog that has many examples of similar triangle applications.
- Similar Triangle Applications by Rochester City School District is a worksheet that has many similar triangle application problems.
- Real Life Real World Activity: Forestry Similar Triangles and Trigonometry by Texas Instruments is a lesson where students solve forestry problems using similar triangles and trigonometry ratios.

Special Right Triangles

- Covering the Plane with Rep-Tiles is a lesson by NCTM Illuminations where students use rep-tiles (a geometric figure whose copies can fit together to form a larger similar figure) to create patterns. These can be used to illustrate isosceles right triangles, 30°-60°-90° triangles, equilateral triangles, parallelograms, trapezoids etc. *NCTM now requires a membership to view their lessons.*

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.SRT.4-5)

Curriculum and Lessons from Other Sources

- EngageNY, Geometry, Module 2, Topic A, Lesson 4: Comparing the Ratio Method with the Parallel Method is a lesson that pertains to this cluster.
- EngageNY, Geometry, Module 2, Topic C, Lesson 18: Similarity and the Angle Bisector Theorem, Lesson 19: Families of Parallel Lines and the Circumference of the Earth, Lesson 20: How Far Away Is the Moon? are lessons that pertain to this cluster.
- EngageNY, Geometry, Module 2, Topic D, Lesson 21: Special Relationships Within Right Triangles—Dividing into Two Similar Sub-Triangles and Lesson 24: Prove the Pythagorean Theorem Using Similarity are lessons that pertain to this cluster.
- Mathematics Vision Project, Geometry, Module 4: Similarity and Right Triangle Trigonometry has many tasks that pertain to this cluster.
- Georgia Standards of Excellence Curriculum Frameworks, Geometry, Unit 2: Similarity, Congruence, and Proofs has several lessons that pertain to this cluster.
- Georgia Standards of Excellence Curriculum Frameworks, Geometry, Unit 3: Right Triangle Trigonometry has several lessons that pertain to this cluster.
- Illustrative Mathematics, Geometry, Unit 3, Lesson 5: Splitting Triangle Sides with Dilation, Part 1, Lesson 6: Connecting Similarity and Transformations, Lesson 11: Splitting Triangle Sides with Dilation, Part 2, Lesson 12: Practice with Proportional Relationships, Lesson 13: Using the Pythagorean Theorem and Similarity, Lesson 14: Proving the Pythagorean Theorem, Lesson 15: Finding All the Unknown Values in Triangles, and Lesson 16: Bank Shot are lessons that pertain to this cluster.

General Resources

- Arizona 7-12 Progression on Geometry is an informational resource for teachers. This cluster is addressed on page 17.
- High School Coherence Map by UnboundED is a map of the relationships among high school mathematics standards. Please note: This resource is aligned to the Common Core not Ohio's Learning Standards. Although the tool is useful, it needs to be checked with respect to Ohio's Learning Standards.
- van Hiele Model of Geometric Thinking is a pdf created by ODE that summarizes the van Hiele levels.
- Not Sure About MCC9-12.GSRT.5 and Others is a blog entry in Bill McCallum's website Mathematical Musings that discusses the standard. Note: Bill McCallum was one of the writers of the Common Core.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.SRT.4-5)

References

- Common Core Standards Writing Team. (2016, March 24). *Progressions for the Common Core State Standards in Mathematics (draft). Grades 7–8, HS, Geometry*. Tucson, AZ: Institute for Mathematics and Education, University of Arizona.
- Snapp, B. & Findell, B. (2016). Parallels in Geometry. Creative Commons. Retrieved from: https://people.math.osu.edu/findell.2/ParallelsInGeometry2016.pdf
- Cirillo, M. & Hummer, J. (April 2019). Addressing misconception in secondary Geometry proof. *Mathematics Teacher*, *112, (6),* 410-417.
- Coxford, A. & Usiskin, Z. (1971). *Geometry: A Transformational Approach*. Laidlaw Brothers Publishers: Riverforest, IL.
- Coxford, A., Usiskin, Z., & Hirschorn, D. (1991). *The University of Chicago Mathematics Project: Geometry, Teacher's Edition.* Scott Foresman and Company, Glenview: IL
- Van De Walle, J., Karp, K., Bay-Williams, J. (2010). *Elementary and Middle School Mathematics* (7th ed.). Boston, MA: Pearson Education, Inc.

STANDARDS	MODEL CURRICULUM (G.SRT.6-8)
Geometry **SIMILARITY, RIGHT TRIANGLES, AND TRIGONOMETRY** **Define trigonometric ratios, and solve problems involving right triangles.** **G.SRT.6** Understand that by similarity, side ratios in right triangles are properties of the angles in the triangle, leading to definitions of trigonometric ratios for acute angles. **G.SRT.7** Explain and use the relationship between the sine and cosine of complementary angles. **G.SRT.8** Solve problems involving right triangles.★ **a.** Use trigonometric ratios and the Pythagorean Theorem to solve right triangles in applied problems if one of the two acute angles and a side length is given. (G, M2)	**Expectations for Learning** In middle school, students draw, construct, and describe geometric figures; use informal arguments to establish facts about similar triangles; and explain a proof of the Pythagorean Theorem and its converse. In this cluster, students use similarity to define trigonometric ratios and then solve problems using right triangles (excluding inverse trigonometric functions). The student understanding of this cluster aligns with van Hiele Level 2 (Informal Deduction/Abstraction). **ESSENTIAL UNDERSTANDINGS** • Because right triangles with the same acute angle are similar, within-figure ratios are equal. Three of these possible ratios are named sine, cosine, and tangent. • The sine of an acute angle is equal to the cosine of its complement and vice versa. • Given an angle and a side length of a right triangle, the triangle can be solved, which means finding the missing sides and angles. **MATHEMATICAL THINKING** • Use accurate mathematical vocabulary to represent geometric relationships. • Recognize, apply, and justify mathematical concepts, terms, and their properties. • Discern and use a pattern or structure. • Plan a solution pathway. • Justify relationships in geometric figures. • Determine reasonableness of results. • Create a drawing and add components as appropriate. • Use technology strategically to deepen understanding. • Solve routine and straightforward problems accurately. • Connect mathematical relationships to real-world encounters. *Continued on next page*

STANDARDS	MODEL CURRICULUM (G.SRT.6-8)
	Expectations for Learning, continued **INSTRUCTIONAL FOCUS** • Define trigonometric ratios for acute angles (sine, cosine, tangent). • Explain and apply the relationship between sine and cosine of complementary angles. • Solve problems involving right triangles (excluding inverses of trigonometric functions). • Use the Pythagorean Theorem to explore exact trigonometric ratios for 30, 45, and 60-degree angles (fluency not required). • Use triangle similarity criteria to define trigonometric ratios. • Given the sine, cosine, or tangent of an angle, find other trigonometric ratios in the triangle. • Solve mathematical and real-world problems given a side and an angle (or the sine, cosine, or tangent of an angle) of a right triangle. **Content Elaborations** **OHIO'S HIGH SCHOOL CRITICAL AREAS OF FOCUS** • Geometry, Number 3, page 6 **CONNECTIONS ACROSS STANDARDS** • Understand similarity (G.SRT.1-3). • Prove and apply theorems involving similarity (G.SRT.4-5). • Apply geometric concepts in modeling situations (G.MG.1-3). • Use coordinates to prove simple geometric theorems algebraically (G.GPE.4).

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.SRT.6-8)

Instructional Strategies

Note: The model curriculum is not a collection of lessons nor a full curriculum; it does not suggest pace, sequence, or amount of time spent on topics. It provides information about a topic related to the standards including ideas for examples, strategies for teaching, common misconceptions, and possible connections between topics. In addition, the emphasis of the model curriculum is on instruction not on assessment; therefore, the examples in the Instructional Strategies section are written for instructional purposes not necessarily as assessment items. Examples may need to be rewritten to accommodate the needs of each individual classroom.

In Geometry, students will only be finding missing *side lengths* of triangles or shapes that can be decomposed into triangles through trigonometry and Pythagorean Theorem. In Algebra 2, students will use trigonometric functions to find missing angles, because inverses of trigonometric functions will not be introduced until Algebra 2.

Standards for Mathematical Practice

This cluster focuses on but is not limited to the following practices:

MP.1 *Make sense of problems and persevere in solving them.*

MP.4 *Model with mathematics.*

MP.5 *Use appropriate tools strategically.*

MP.6 *Attend to precision.*

MP.7 *Look for and make use of structure.*

Trigonometry comes froom the Greek words *trigonon* meaning triangle and *-metria* meaning measurement. However, the word trigonometry was not always used when describing the concepts behind trigonometry. Originally the concepts behind trigonometry were used in relation to circles and spheres as the concepts were used primarily for studying astronomy. Tangents were treated seperately as they were originally used in relation to studying shadows. It took a long time for these concepts to be combined and then applied to right triangles without a direct connection to circles.

Right triangle trigonometry (a geometry topic) has implications when studying algebra and functions concepts in later math courses. For example, students learn that trigonometric ratios are functions of the size of an angle. Then, the trigonometric functions will be revisited in later courses after radian measure has been studied, and the Pythagorean Theorem will be used to show that $(\sin A)^2 + (\cos A)^2 = 1$.

Students should be proficient in using the trigonometric functions on a calculator.

Some students may incorrectly believe that right triangles must be oriented a particular way. When applying trigonometric ratios to situations, show examples of right triangles in different orientations to confront this misconception.

Note: The Geometry standards no longer include inverse trigonometric functions; however, some schools such as career technical schools may want to include this content.

VAN HIELE CONNECTION

In Geometry students are expected to move be at Level 2 (Informal Deduction/Abstraction) and move towards Level 3 (Deduction).
Level 2 can be characterized by the student doing some or all of the following:

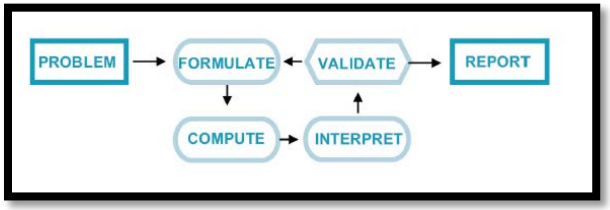
- noticing relationships within and between properties and figures;
- forming meaningful definitions;
- justifying reasoning using informal arguments;
- drawing logical maps and diagrams; and/or
- not yet understanding formal deduction.

Level 3 can be characterized by the student doing some or all of the following:

- constructing proofs and understanding the necessity of proofs;
- understanding the role and relationships between of axioms, theorems, definitions, and postulates;
- differentiating between necessary and sufficient conditions; and/or
- making logical conclusions.

See the van Hiele pdf on ODE's website for more information about van Hiele levels.

MODELING



This cluster is included in the modeling standards. Geometric models that students can build are especially helpful to see patterns. See page 10 for more information about modeling.

DEFINITION OF TRIGONOMETRIC RATIOS

Have students make their own diagrams of a right triangle with labels showing the trigonometric ratios. Make cutouts or drawings of right triangles or manipulate them on a computer screen using dynamic geometric software, and ask students to measure side lengths and compute side ratios. Observe that when two or more triangles satisfy the similarity criteria, corresponding side ratios are equal. Although students like mnemonics such as SOHCAHTOA, these are not a substitute for conceptual understanding. As an extension some students may investigate the reciprocals of sine, cosine, and tangent to discover cosecant, secant, and cotangent.

Some students incorrectly believe the trigonometric ratio is the answer when finding the measure of an angle or length of a side. Instead, the ratio shows the relationship between the sides' lengths and the measure of the specified angle in the right triangle. They are also functions of angles, which students will discover in Algebra 2.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.SRT.6-8)

EXAMPLE

a. Draw 5 different right triangles with an angle of $60°$. Label the $60°$ angle θ. Measure the lengths of the corresponding sides of each triangle. What do you notice?

b. Draw 5 different right triangles with an angle of $45°$. Label the $45°$ angle θ. Measure the lengths of the corresponding sides of each triangle. What do you notice?

c. Draw 5 different right triangles with an angle of $15°$. Label the $15°$ angle θ. Measure the lengths of the corresponding sides of each triangle. What do you notice?

d. Draw 5 different right triangles with an angle of $20°$. Label the $20°$ angle θ Measure the lengths of the corresponding sides of each triangle. What do you notice?

e. Given a specified angle measure for a right triangle, what do you know about all triangles that are similar to the given triangle? Explain.

Discussion: Students should understand that any acute angle in the right triangle has several fixed ratios (trigonometric ratios) related to that specified angle. This is true because based on AA-Similarity Criteria of Triangles (or A-Similarity Criteria for Right Triangles) the triangles are similar; therefore, sides are proportional. It may help some students to demonstrate this using dynamic geometric software.

TIP!

Right Triangle Similarity stems for similarity criteria of general triangles. For example, A-Similarity Criteria of Right Triangles stems from AA-Similarity Criteria of Triangles, since in a right triangle, one angle is already known; HL stems from SSS.

Students often confuse the sine or cosine of an angle as an angle measure instead of a ratio of side lengths for a particular angle.

Sine and Cosine

Although the emphasis of trigonometric study in Geometry is not on the unit circle, some informal workings with circles and the unit circle (where the radius is 1) can lay the foundation for the conceptual understanding of sine beyond simply memorizing the mnemonic SOHCAHTOA, which has no meaning for students. Students can apply concepts of similar triangles to compare any right triangle to a similar triangle with the hypotenuse of 1.

Sine comes from the word "half-chord" or "bowstring."
A sine is—

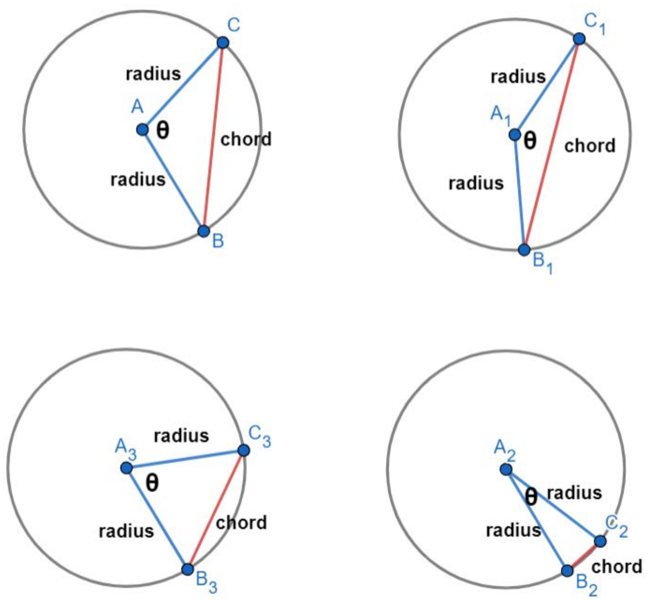
- A half-chord on a circle;
- Part of a right triangle when the radius (or hypotenuse) is 1; and
- A ratio comparing the lengths of 2 sides of a right triangle.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.SRT.6-8)

Avoid using the notation "$\sin x$" until more advanced courses since it can lead students to think that "$\sin x$" is short for "sin · x" and incorrectly divide out the variable $\frac{\sin x}{x} = sin$. Instead at this level, refer to the sine of an angle as $(\sin \angle A)$ or sine of an angle measure $(\sin \theta)$. *Note the parenthesis are used for clarity instead of representing a function.*

EXAMPLE

Note: The word "sine" will be used throughout this example in place of the abbreviation "$sin\ \theta$." The abbreviation will be introduced after students comprehend that $\sin \theta$ is the sine-to-radius ratio of a right triangle whose central angle is θ which can be described as $\frac{length\ of\ opposite\ side\ of\ \theta}{length\ of\ hypotenuse}$. The abbreviation of $sin\ \theta$ will be introduced at the end of this example.



Part 1

- **a.** Use geometric software or a compass and straight edge to construct several triangles inscribed in circles where two radii of the circle are the sides of the triangle.
- **b.** Label the sides of the triangle according to its relationship with the circle.
- **c.** What kind of angles are present in each of the triangles?
- **d.** What is different about each of the triangles? What is the same about each of the triangles?

Discussion: Students should be able to label the radius and chords of each circle and realize that they are also parts of the sides of a triangle. Students should also notice that all the triangles contain a central angle. They should realize that all the triangles are isosceles because the radii create two congruent sides, but the central angle measures may be different than the base angle measures in each triangle.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.SRT.6-8)

Part 2

a. Draw or create a circle and a chord.

b. Create a triangle by connecting the endpoints of the chord to the center of the circle using the radii of the circle. Make the length of the radii 1 unit.

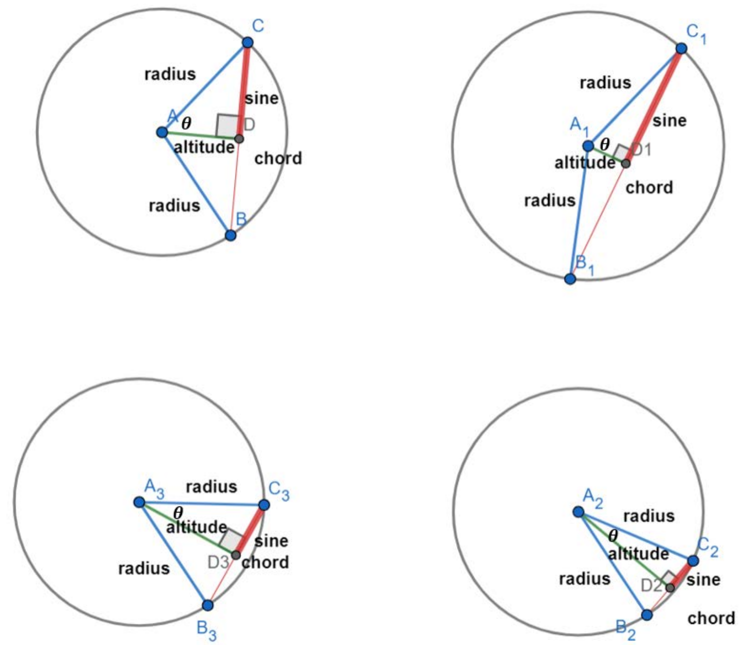
c. Draw an altitude from the center of the circle to the chord, and label the altitude.

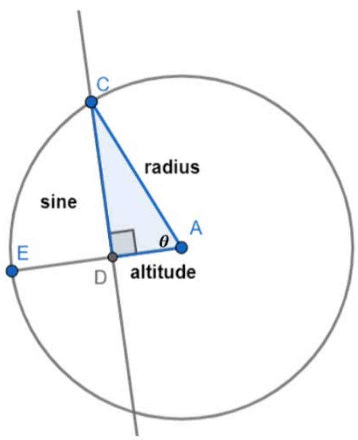
d. What kind(s) of triangles are formed? How do you know?

e. Sine is also known as a half-chord (or more accurately the sine of a central angle θ is half the chord of twice the angle or 2θ). Label sine and theta (θ) in each one of your triangles.

f. How are the θs and the central angles in Part 1 related?

g. Notice that any half-chord (sine) and the radius (which is the endpoint of the half-chord on the circle), and the altitude (from the center to the half-chord) form a right triangle. What part of the right triangle is the radius? (If using geometric software, drag the intersection point of the chord and the radius that is located on the circle to illustrate the concept.)





Discussion: Draw attention to the fact that the sine of the central angle θ is half the chord. When the chord is not cut in half, the central angle intercepting the chord is 2θ. Both θ and 2θ are central angles. Students should conclude that the radius is the hypotenuse of the right triangle. Using half-chords instead of chords forces the triangles to be right triangles.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.SRT.6-8)

Part 3

- **a.** Create a right triangle in a circle with the same central angle θ that you used in Part 2 where the radius (which is also the hypotenuse of your triangle) is 1. Label the half-chord sine and the radius 1 (Figure 1).
- **b.** Create another right triangle in a circle where the radius of your choosing is the hypotenuse of your triangle. Label the half-chord *x* and the radius *r* (Figure 2).
  - How does your circle compare to the circle in part **a.**? Explain.
  - How does your triangle compare to the triangle in part **a.**? Explain.
- **c.** Sine θ can be thought of a sine-to-radius ratio of a right triangle whose central angle is θ. Write a proportion comparing the sine-to-radius ratios of both triangles.
- **d.** Why is the half-chord only needed to express sine, when the radius is 1?
- **e.** Why do we need both the half-chord and the radius to express the ratio, when the radius $\neq 1$?
- **f.** Find sine θ for the triangle in Figure 3.
- **g.** How does that relate to a similar triangle inscribed in a circle with radius of 1? Explain. See Figure 4.

Figure 1

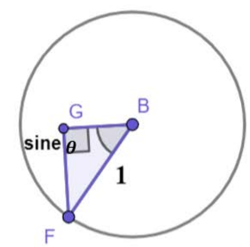


Figure 2

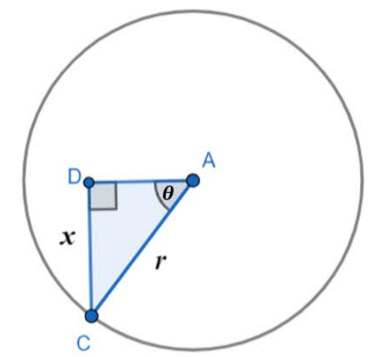
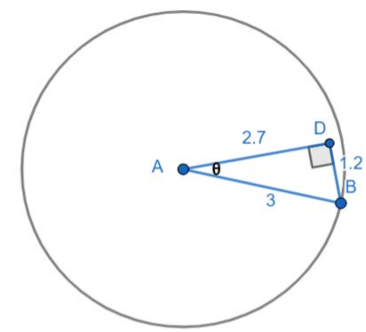
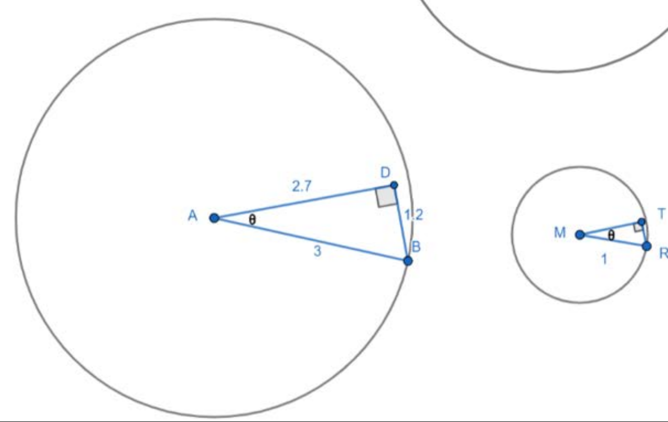


Figure 3



Discussion: Students should realize that both circles are similar since all circles are similar (G.C.1) and that the triangles are similar by AA-Triangle Similarity Criteria. Students should be able to write the proportion $sine\ of\ the\ central\ angle\ \theta = \frac{sine}{1} = \frac{x}{r}$, which can be rewritten as $sine\ \theta = \frac{x}{r}$. Since $\frac{sine}{1} = sine$, the sine of the central angle θ is the length of the half-chord when the radius equals 1. When the radius does not equal 1, the sine-to-radius ratio needs to be described by both the half-chord and the radius. However, when the ratio is simplified it describes the half-chord of a similar circle (and triangle) with a radius of 1. Students should recognize that the sine-to-radius ratio in Figure 3 equals the half-chord (sine) in Figure 4 because the radius is 1 and the figures are similar. Therefore, all sine ratios can be thought of as a description of the half-chord in a circle with a radius of 1.

Figure 4



INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.SRT.6-8)

Part 4

a. Consider two circles with the same radius and triangles formed by central angles and half chords (Figure 1).
b. What do you notice about both triangles? Explain.
c. What do you notice about the central angles θ and α? (Figure 2)
d. Write a statement describing the relationship between the altitude and the sine of both triangles.
e. A complementary angle's sine is referred to as the cosine. Therefore, what is another name for the altitude? Label cosine on your diagrams.
f. Fill in the blanks:
- $sine\ \theta = cosine$ ____ and $sine\ \alpha = cosine$ ____.
- $sine$ and $cosine$ of ___________ angles are _____.

g. In a right triangle with a hypotenuse (radius) of 1, what determines which side is the sine and which side is the cosine?

Discussion: Students should recognize both triangles are congruent because the radii are equal, and the two angles are congruent. They are transformations of one another (a rotation and a translation). The central angle of $\triangle ADC$ is the complement to the central angle of $\triangle JIH$. The reflection also causes the sines and the altitudes to interchange. $The\ sine\ of\ \triangle ADC = the\ altitude\ of\ \ \triangle JIH$ and vice versa (since the radius is 1). Students should note that the cosine is the altitude when the radius is 1. Therefore $the\ sine\ of\ \triangle ADC = the\ cosine\ of\ \ \triangle JIH$ or $sine\ \theta = cosine\ \alpha$ and $sine\ \alpha = cosine\ \theta$. Therefore, the sine and cosine of complementary angles are equal. The students should understand that position relative to the central angle determines which side of the triangle is sine and which is cosine. The sine is the half-chord, which is opposite from the central angle of the circle and the cosine is the altitude, which is adjacent to the central angle. Ask students how they would know which is which if the circle is removed. That discussion will help them transition to Part 5.

Figure 1:

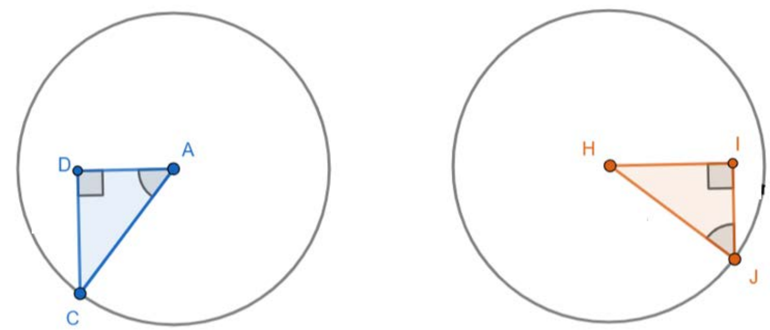


Figure 2:

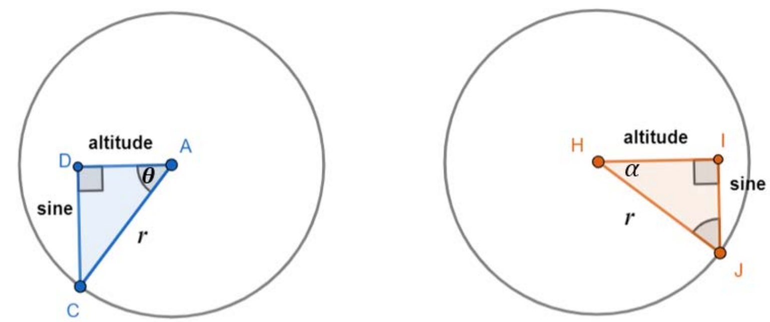
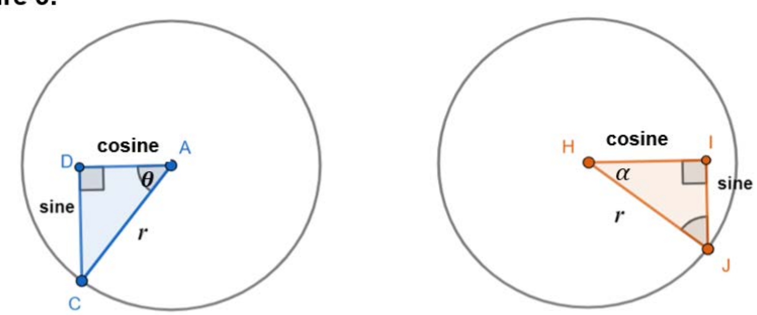


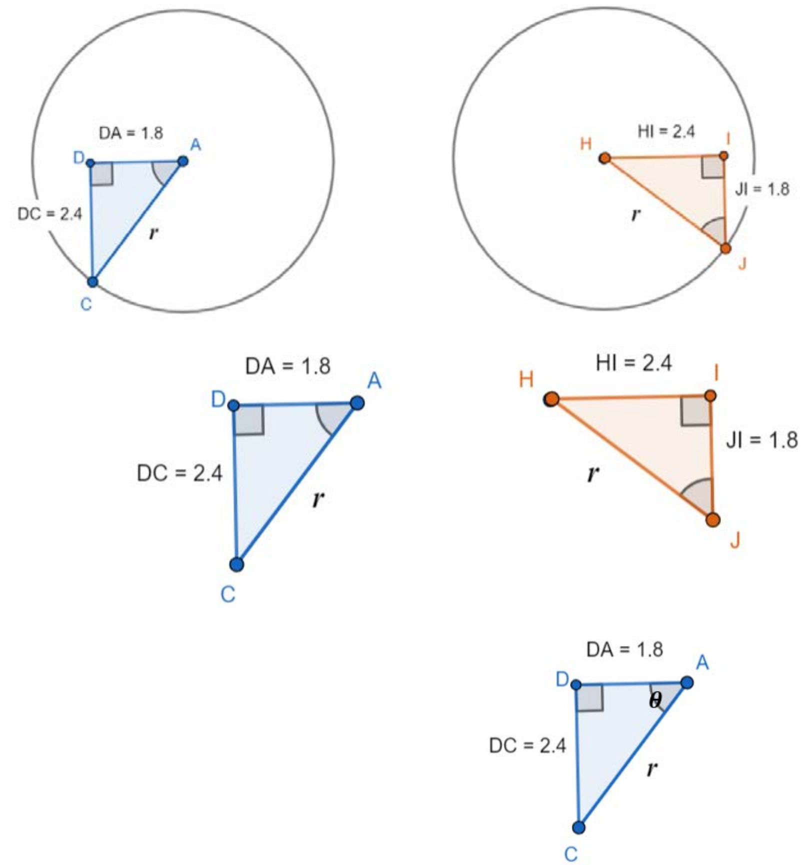
Figure 3:



INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.SRT.6-8)

Part 5

a. Now if you remove the circles, how do you determine which angle is the central angle and which side length is the half-chord (sine)?
b. Look back at the position of the sine (half-chord) in relation to the central angle in each of the triangles. What do you notice?
c. When the circle is removed, what is the name for the radius in relation to the rest of the right triangle?
d. Use words to describe the sine-to-radius ratio for the angle θ that can be consistently seen when right triangles are not part of the radius and half-chord inside a circle.
e. The cosine of an angle is the sine of its complementary angle. So what words can be used to describe the cosine of θ?
f. Write the sine and cosine ratio of the angle θ for the triangle in Figure 1.
g. What is the sine and cosine of the other acute angle in the triangle (Figure 2).



Discussion: Once the circles are removed it may be difficult to determine which angle is the central angle θ and which side length is the half-chord (sine) of the right triangle. As one can see either of the acute angles can be θ depending on the position of the triangle in the relationship to the circle. Therefore, when triangles are not inside circles, it makes more sense to talk about the ratios of the side lengths in relation to θ. Since, sine (half-chord) is always the side opposite of the θ (central angle), hopefully students will come up with the idea that the sine θ can be described as $\frac{opposite}{hypotenuse}$. Then students may use the words $\frac{side\ next\ to\ the\ angle}{hypotenuse}$ or $\frac{beside}{hypotenuse}$ or $\frac{adjacent}{hypotenuse}$ to describe cosine θ, which is the same as sine of the angle that is the complementary angle of α. In Figure 1 $sine\ \theta = \frac{3}{\sqrt{34}}$ and $cosine\ \theta = \frac{5}{\sqrt{34}}$, and in Figure 2 $sine\ \alpha = \frac{5}{\sqrt{34}}$ and $cosine\ \alpha = \frac{3}{\sqrt{34}}$. Students should note that the sine of one angle is the cosine of the other and vice versa. Eventually students can move towards abbreviating sine and cosine.

Figure A:

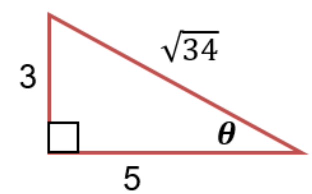
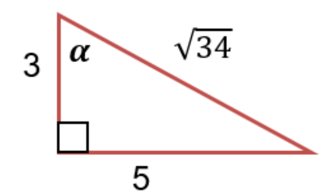


Figure B:



INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.SRT.6-8)

Understanding Trigonometric Notation

It is just a shorthand!

The sine-to-radius ratio of a right triangle whose central angle is θ is $\frac{length\ of\ opposite\ side\ from\ \theta}{length\ of\ hypotenuse}$.

~~The~~ sine-to-radius ratio ~~of a right triangle whose central angle is~~ θ is $\frac{\cancel{length\ of}\ opposite\ side\ from\ \theta}{\cancel{length\ of}\ hypotenuse}$.

sine-~~to-radius ratio~~ of θ is $\frac{opposite\ side\ from\ \theta}{hypotenuse}$

sine ~~of~~ $\theta = \frac{opposite\ \cancel{side\ from\ \theta}}{hypotenuse}$

$\sin\theta = \frac{opposite}{hypotenuse}$

In the past, trigonometric tables listed only sines of angles. To find the cosine of an angle, one would have to look up the sine of its complementary angle. Investigate sines and cosines of complementary angles, and guide students to discover that they are equal to one another. Trigonometry tables can still be used to emphasize the complementary nature of sine and cosine.

EXAMPLE

Use a trigonometric table to list the sine of each of the following angle measures:

- 30°
- 60°
- 20°
- 70°
- 10°
- 80°
- 25°
- 65°
- 15°
- 75°

Now list the cosine of each angle measure. What do you notice?

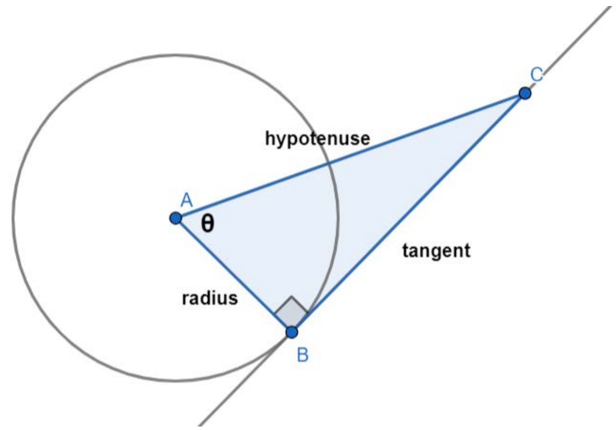
Tangent

Tangent means "touching line." Students should be able to relate the tangent of a circle to the tangent ratio in a right triangle. The concepts of trigonometric ratios as lines in a unit circle will be extended to the reciprocal ratios secant, cosecant, and cotangent in later courses.

EXAMPLE

Part 1

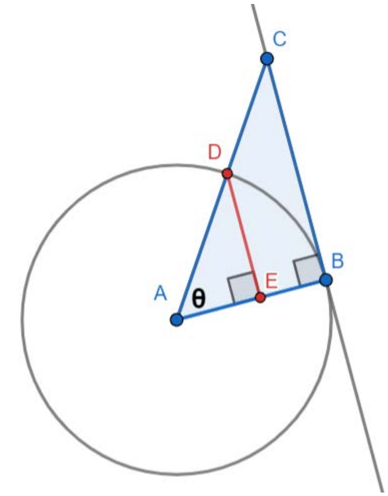
- **a.** What do you know about tangent lines?
- **b.** Draw a circle (or create one using dynamic geometric software) where the radius intersects the tangent line. Then use the radius and the tangent (in addition to a third line) to create a right triangle. Label the tangent line, radius, and the hypotenuse.
- **c.** There is also a tangent-to-radius ratio of a right triangle whose central angle is θ. Measure your triangle, and write the tangent-to-radius ratio.
- **d.** Now draw a triangle and circle that is geometrically similar to what you drew in part **b.** This time fix the radius at 1. Write a proportion comparing the two tangent-to-radius ratios.
- **e.** Why can the word *tangent* be both used to describe the tangent-to-radius ratio and the segment tangent to the radius of the circle when the radius equals 1?
- **f.** If the radius ≠ 1, what is needed to describe tangent θ?



Discussion: Students should recall the definition of a tangent line: a line which intersects a circle at exactly one point. They should also recall the theorem that if a line is tangent to a circle, then it is perpendicular to the radius drawn to the point of tangency. Students should recognize that the tangent-to-radius ratio equals the length of the tangent when the radius is 1 because $\frac{length\ of\ tangent\ segment}{1} = \frac{length\ of\ tangent\ segments}{length\ of\ radius}$ or $length\ of\ tangent = \frac{length\ of\ tangent\ segments}{length\ of\ radius}$. Therefore, all tangent ratios can be thought of as a description of the ratio of the length of tangent line segment to the length of the radius of a circle when the radius is 1. Make connections between this example and the sine-to-radius ratios and the complement of sine-to-radius ratio.

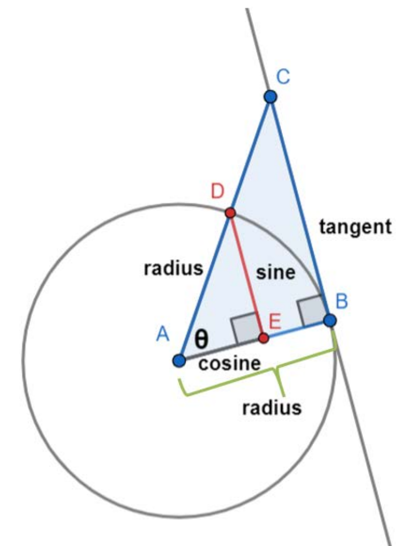
INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.SRT.6-8)

Part 2



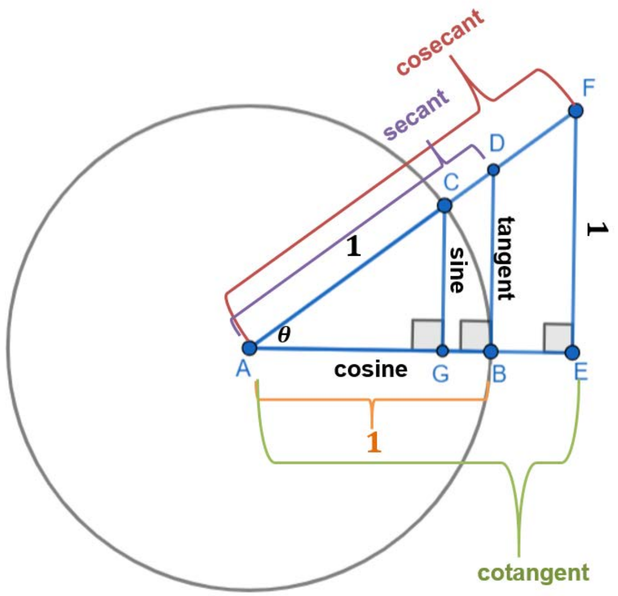
- **a.** In the diagram circle A has a radius of 1. Label the tangent and radius.
- **b.** What is the relationship between $\triangle ABC$ and $\triangle AED$? Explain.
- **c.** What implication does your answer in part **b.** have on the side length of the triangles?
- **d.** Use the tangent-to-radius ratio to write a proportion that is true for the two triangles.
- **e.** What other geometric term is $\overline{DE}$ known by?
- **f.** What other geometric term is $\overline{AE}$ known by?
- **g.** Fill in the blank using the terms for $\overline{DE}$ and $\overline{AE}$ in parts **e.** and **f.** $\frac{tangent}{radius} = $ __________
- **h.** Since the radius of the circle is 1, how can you define tangent?
- **i.** Instead of defining tangent as $\frac{sin\theta}{cos\theta}$, could you define it in terms of the position of the right triangle's legs from the central angle? Does it hold true for both $\triangle ABC$ and $\triangle AED$?
- **j.** How will your definition in part **i.** help you define tangent when the right triangle is not associated with a circle?
- **k.** Label the diagram using the words tangent, radius, sine, and cosine.

Discussion: Students should recognize that $\triangle ABC$ and $\triangle AED$ are similar because of AA-Triangle Similarity Criteria, so the side lengths are proportional. Students should write the proportion $\frac{tangent}{radius} = \frac{\overline{CB}}{\overline{AB}} = \frac{\overline{DE}}{\overline{AE}}$. Since $\overline{DE}$ is the half-chord sine, and $\overline{AE}$ is cosine, $\frac{tangent}{radius} = \frac{sine}{cosine} = \frac{sin\theta}{cos\theta}$, and since the radius is 1, $tangent = \frac{sin\theta}{cos\theta}$. The tangent-to-radius ratio can also be defined as $\frac{opposite}{adjacent}$, which can be useful when a right triangle is not connected to the circle. This triangle can be extended by adding lines for cosecant, secant, and cotangent for more advanced students (See diagram on the left of the next page).



INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.SRT.6-8)

On the left shows a diagram that represents sine, cosine, and tangent with the addition of secant, cosecant, and cotangent following from the definition of sine as the half chord. The diagram on the right is an alternative diagram.



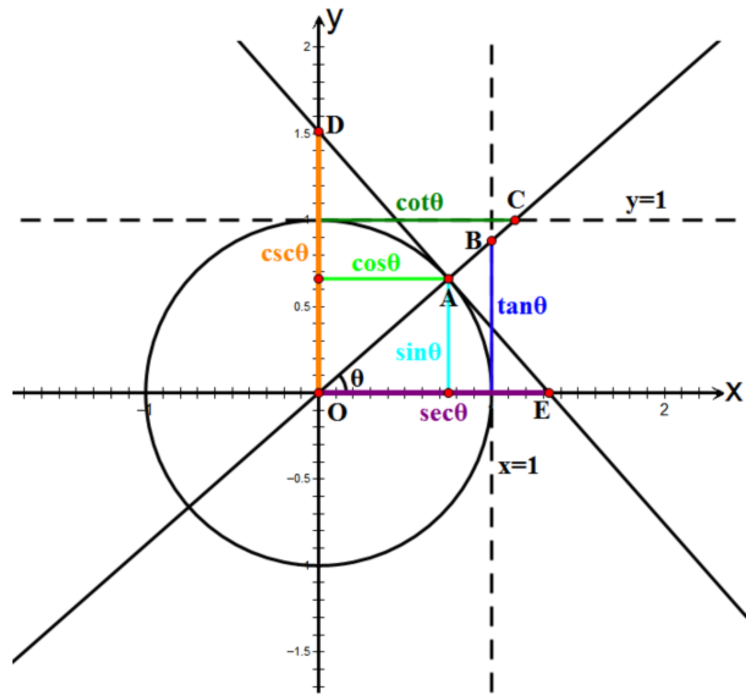


Image taken from [Trigonometric Functions](), Wikipedia.

TIP!

Historically, tangents were not part of trigonometry. Whereas sines and cosines focused on trigonometry and astronomy, tangents dealt with shadows. Triangle trigonometry began by determining the length of a shadow cast by a vertical stick, called a shadow stick or gnomon, given the angle of the sun because the line length was tangent, the function was known as tangent. An example of this is a sundial.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.SRT.6-8)

EXAMPLE

Draw or create at least two right triangles that meet the criteria for each of the following. Label their sides. Approximations are acceptable. You may use a calculator, and it may be helpful to use dynamic geometric software to create the triangles.

- **a.** sin 30°
- **b.** tan 45°
- **c.** sin 20°
- **d.** cos 40°
- **e.** sin 70°
- **f.** tan 60°

Discussion: The idea behind this example is to emphasize the ratio nature of sine, cosine, and tangent. Parts **a.** and **b.** are fairly easy as students can create any triangle whose side lengths have a ratio of $\frac{1}{2}$ for the opposite to hypotenuse (half-chord to radius) ratio in part **a.,** and whose side lengths have a ratio of 1 for the opposite to adjacent (sine to cosine) in part **b.** Parts **c.-f.** are more difficult. For part **c**. students can use their calculator to find the sine of 20 which is approximately 0.34. (It may be helpful to have students round to the nearest hundredth for this activity). Then they could draw a triangle with a leg of 17 and a hypotenuse of 50 or a leg of 0.34 and a hypotenuse of 1 or a leg of 8.5 and a hypotenuse of 25. *Note: This is also laying foundational understanding for advanced math courses.* Another approach could be to have the students build right triangles with a fixed acute angle using geometric software and have them manipulate the side lengths. If dynamic geometric software is used, make sure students see the equivalent ratios of the two sides. If the program allows it, students may want to measure their side lengths in thousandths or hundred thousandths to see the ratios.

GENERAL INFORMATION ABOUT TRIGONOMETRIC RATIOS

If one of the acute angles in a right triangle is known, then all right triangles with that acute angle are similar to one another. Because these triangles are similar to each other, their within-figure ratios are equal within this set of triangles. Students may discover that there are six possible ratios for comparing pairs of sides in these triangles. The emphasis in this course is sine, cosine, and tangent. Cosecant, secant, and cotangent will be explored in Algebra 2 but can be used as an extension during this course.

EXAMPLE

How many trigonometric ratios can there be in a right triangle? Explain.

Discussion: Students need to get to the idea that three sides can only be paired three different ways. If students differentiate the trigonometric ratios from their reciprocals, then they may find six different ratios. This can be tied into the fundamental counting principle of probability where $3 \cdot 2 = 6$.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.SRT.6-8)

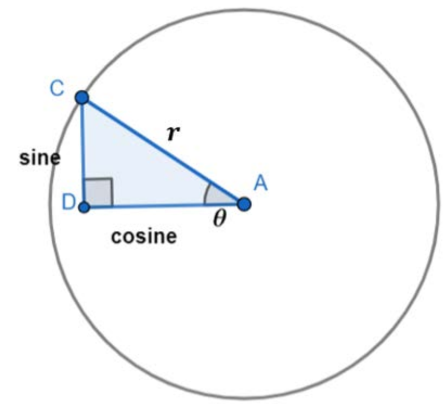
Have students explore using dynamic geometric software whether sine, cosine, and tangent have maximum values.

Observe that, as the size of the acute angle increases, sines and tangents increase while cosines decrease. Sines and cosines also have maximum values, whereas tangents do not.

EXAMPLE

Using geometric software, create a right triangle (with a hypotenuse of 1) inside a circle where the radius is the hypotenuse. Label the sine and cosine.

Figure 1



Part 1

Drag the intersection point of sine and the radius (point C in Figure 1) around the circle to change the size of θ.

- **a.** As θ increases, what happens to sine?
- **b.** As θ decreases, what happens to sine?
- **c.** Does sine have a maximum and/or minimum value? Explain?
- **d.** As θ increases, what happens to cosine?
- **e.** As θ decreases, what happens to cosine?
- **f.** Does cosine have a maximum and/or minimum value? Explain?
- **g.** How does sine behave in relation to cosine? Explain.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.SRT.6-8)

Part 2

Use geometric software, create a right triangle as shown. Drag the intersection point of the hypotenuse and tangent (point C in Figure 2) along the tangent line to change the size of θ.

- **a.** As θ increases, what happens to tangent?
- **b.** As θ decreases, what happens to tangent?
- **c.** Does tangent have a maximum and/or minimum value? Explain.

Figure 2

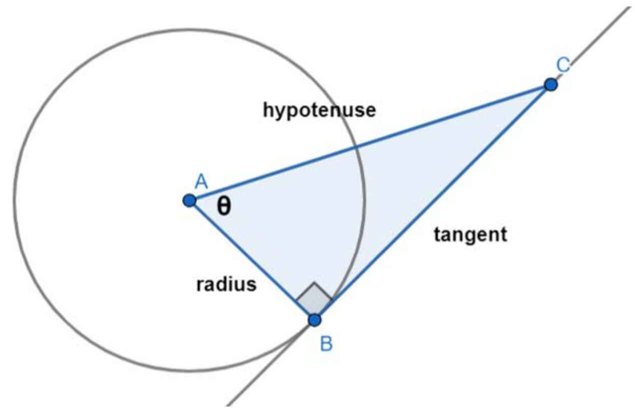
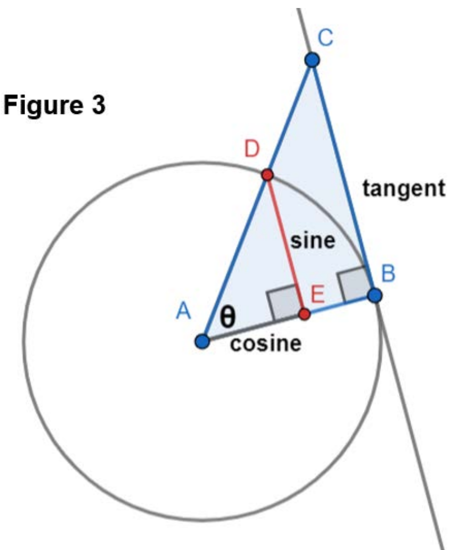


Figure 3



Part 3

Using geometric software, create two right triangles as shown. Drag the intersection point of the hypotenuse and tangent (point C in Figure 3) along the tangent line to change the size of θ.

- **a.** What happens to sine as tangent increases? Explain.
- **b.** What happens to sine as tangent decreases? Explain.
- **c.** What happens to cosine as the tangent increases? Explain.
- **d.** What happens to cosine as the tangent decreases? Explain.
- **e.** How do sine, cosine, and tangent relate to each other?

TIP! Some students do not realize that opposite and adjacent sides need to be identified with respect to a particular acute angle in a right triangle.

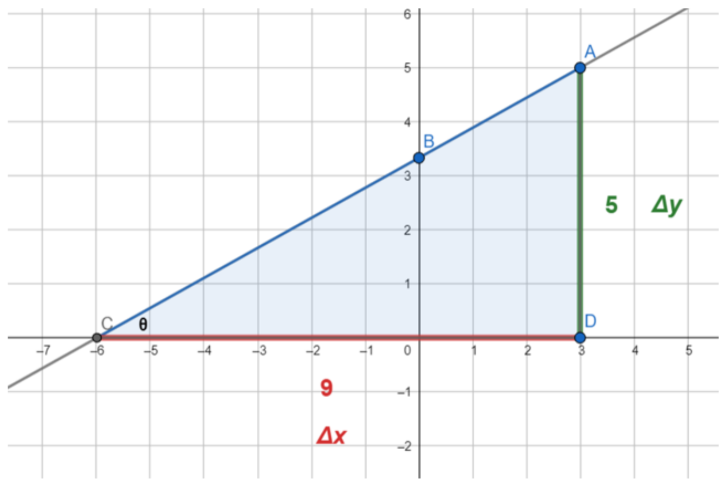
Some students incorrectly believe that the trigonometric ratios defined in this cluster apply to all triangles, but they are only defined for acute angles in right triangles.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.SRT.6-8)

Give opportunities for students to make connections between slope of a line and the tangent of the angle between the line and the horizontal axis.

EXAMPLE

- **a.** Draw a line on a coordinate plane.
- **b.** Find the slope of the line.
- **c.** Create a right triangle using your line as the hypotenuse and the horizontal axis and a vertical line as the two legs.
- **d.** Find the sine, cosine, and tangent of the acute angle created by the horizontal axis and the hypotenuse.
- **e.** What conclusions can you make? Justify your thinking.
- **f.** Compare your triangle to the triangles of your classmates. Does your conclusion generalize for any other examples? Explain.



$$\sin\theta = \frac{5}{\sqrt{106}} = \frac{5\sqrt{106}}{106}$$

$$\cos\theta = \frac{9}{\sqrt{106}} = \frac{9\sqrt{106}}{106}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{(5\sqrt{106})/106}{(9\sqrt{106})/106} = \frac{5}{9}$$

or

$$\tan\theta = \frac{o}{a} = \frac{5}{9}$$

$$m = \frac{\Delta y}{\Delta x} = \frac{5}{9}$$

Discussion: Students should make the connection that the tangent ratio $\left(\frac{sin\theta}{cos\theta}\right)$ is the same as the slope of the line, because the sine is the same as the vertical line of the slope triangle (Δy) and the cosine is the same as the horizontal line of the slope triangle (Δx).

SOLVING PROBLEMS USING RIGHT TRIANGLES

Special Right Triangles

In Geometry, students should be able to determine the trigonometric values of special right triangles; fluency is not the priority at this level. In Algebra 2, however, students will develop fluency of special right triangles such as 30°-60°-90° or 45°-45°-90°using the unit circle.

Have students apply the concept of similar triangles and the Pythagorean Theorem to find properties of special triangles such as 30°-60°-90° and 45°-45°-90°. It may be beneficial to expose more advanced students to rationalizing the denominator of simple radical expression (with a single term in the denominator).

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.SRT.6-8)

EXAMPLE

Part 1

a. Create an equilateral triangle inside a circle where the radius is a side of the triangle whose length is 1. Label the angle measures and side lengths.

b. Draw an altitude from one of the vertices to create two right triangles. Label the angle measures and side lengths of the right triangle that are apparent.

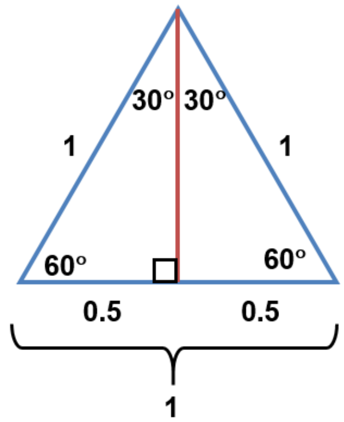
c. Find the length of the altitude. Explain what you did.

d. Use your knowledge of similar triangles to fill in the chart showing the relationships between sides of a 30°-60°-90° triangle. In the chart, ℓ refers to the length of the shorter leg, L refers to the length of the longer leg, and H refers to the length of the hypotenuse. Leave numbers in radical form.

e. Choose how you prefer to define the relationship of the sides of a 30°-60°-90° triangle (ℓ, L, or H). Draw a triangle to illustrate the relationship using your chosen preference. Then write the relationship in words using the words hypotenuse, shorter leg, and longer leg.

f. Draw and label the side lengths of the three similar 30°-60°-90° triangles using your preferred relationship.

g. Solve the following problems using your preferred relationship for 30°-60°-90° triangles.
- If the length of the hypotenuse is 10 find the lengths of the other two legs.
- If the length of the short leg is 4 find the length of the hypotenuse and the long leg.
- If the length of the long leg is 9 find the length of the short leg and the hypotenuse.

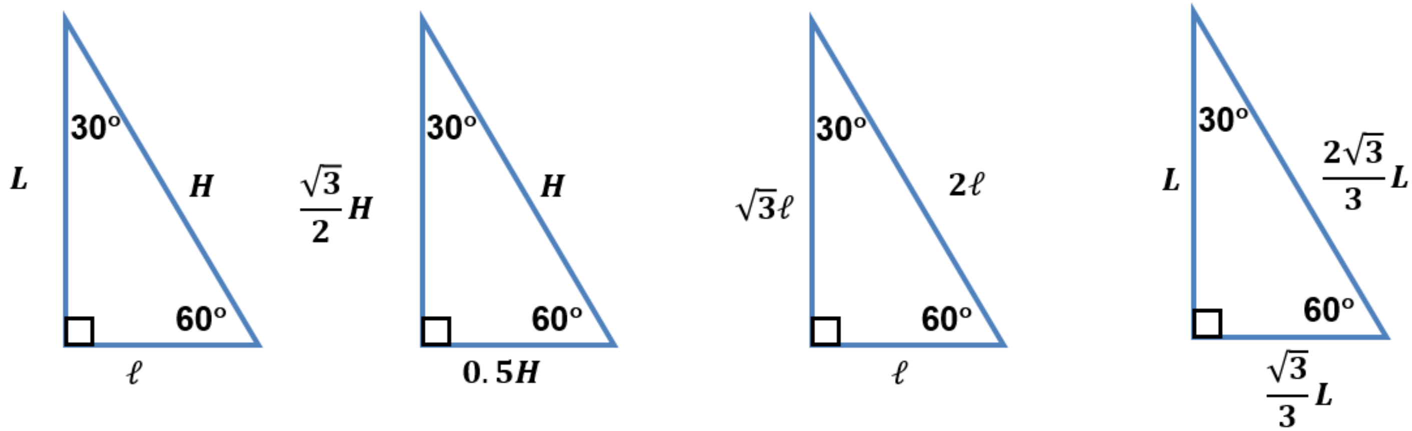


ℓ	L	H
		1
1		
	1	
		H
ℓ		
	L	

Discussion: In part **a.**, you may wish to remove the triangle from the circle and just have the students draw an equilateral triangle with side lengths of 1. If the triangle is presented in a circle, it reinforces the relationships between circles and triangles and builds informal foundations of the unit circle. Students may define the relationships in terms of ℓ, L, or H. There are different advantages to the students' choice. Typically, most Geometry textbooks define the relationship in terms of the short leg ℓ since the numbers are easier, especially if an equilateral triangle with a side of 2 is used to illustrate the relationship. However, the advantage of using H to define the relationship is that it will carry over to understanding of the unit circle in later courses. It is important for a teacher to describe the relationships using the words hypotenuse, shorter leg, and longer leg. For example, the length of the longer leg is $\sqrt{3}$ times that of the shorter leg and the length of the hypotenuse is twice the length of the shorter leg or the length of the shorter leg is half the length of the hypotenuse and the length of the longer leg is $\frac{\sqrt{3}}{2}$ times the length of the hypotenuse. (You may need to draw attention to the fact that $\frac{\sqrt{3}}{2}$ is less than one

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.SRT.6-8)

ℓ	L	H
0.5	$\frac{\sqrt{3}}{2}$	1
1	$\sqrt{3}$	2
$\frac{1}{\sqrt{3}}$ or $\frac{\sqrt{3}}{3}$	1	$\frac{2}{\sqrt{3}}$ or $\frac{2\sqrt{3}}{3}$
$0.5H$	$\frac{\sqrt{3}}{2}H$	H
ℓ	$\sqrt{3}\ell$	2ℓ
$\frac{1}{\sqrt{3}}L$ or $\frac{\sqrt{3}}{3}L$	L	$\frac{2}{\sqrt{3}}L$ or $\frac{2\sqrt{3}}{3}L$



Students who memorize the side length relationships of 30°-60°-90° triangles as $1:2:\sqrt{3}$ oftentimes incorrectly interpret $\sqrt{3}$ as the hypotenuse. It may be best to avoid this memory trick or draw attention to the fact that the value of $\sqrt{3}$ is less than 2, and therefore cannot be the hypotenuse which is always the longest side of the triangle.

Part 2

Using your preferred relationship for the side lengths of 30°-60°-90° triangles find the following. Leave values in radical form:

- **a.** $\sin 30°$
- **b.** $\cos 30°$
- **c.** $\tan 30°$
- **d.** $\sin 60°$
- **e.** $\cos 60°$
- **f.** $\tan 60°$

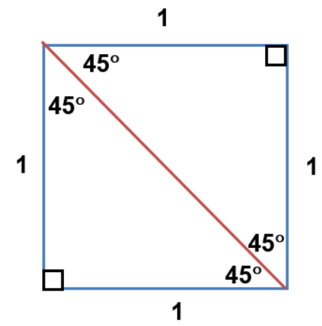
Discussion: Connect the 30°-60°-90° triangle relationships to the ratios needed for the unit circle in Algebra 2. This will not only lay some foundational understanding but also reinforce the finding of trigonometric ratios.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.SRT.6-8)

EXAMPLE

Part 1

a. Create a square with side lengths of 1. Draw a diagonal in the square creating two right triangles. Label the angle measures and the side of the triangle.
b. What kind of triangle is created? Explain why.
c. Find the missing side length in the triangle. Explain how you found it.
d. Use your knowledge of similar triangles to fill in the chart showing the relationships between sides of a 45°-45°-90° triangle. In the chart, ℓ refers to length of the legs, and H refers to the length of the hypotenuse. Leave numbers in radical form.
e. Choose how you prefer to define the relationship of the sides of a 45°-45°-90° (ℓ or H). Draw a triangle to illustrate the relationship using your chosen preference. Then write the relationship in words using the words hypotenuse and leg.
f. Draw and label the side length of three similar 45°-45°-90° triangles using your preferred relationship.
h. Solve the following problems using your preferred relationship for 45°-45°-90° triangles.
- If the length of the hypotenuse is 10 find the lengths of the other two legs.
- If the length of a leg is 4 find the length of the hypotenuse.



ℓ	H
	1
1	
	H
ℓ	

Discussion: Students may define the relationships in terms of ℓ or H. It is important for a teacher to describe the relationships using words such as hypotenuse and leg. (In this case since it is an isosceles triangle there is not a shorter or longer leg.) For example, the length of the hypotenuse is $\sqrt{2}$ times the length of the leg and the length of the leg is $\frac{\sqrt{2}}{2}$ times the length of the hypotenuse.

Part 2 (Extension)

Using your preferred relationship for the side lengths of the 45°-45°-90° triangles find the following. Leave values in radical form.
a. $\sin 45°$
b. $\cos 45°$
c. $\tan 45°$

Discussion: Connect the 45°-45°-90° triangle relationships to the ratios needed for the unit circle in Algebra 2. This will not only lay some foundational understanding but also reinforce the finding of trigonometric ratios.

Pythagorean Triples

Pythagorean triples (and their families) should be explored in connection with similar triangles to show that for example, the sine of an acute angle in a 3:4:5 triangle is equal to the sine of the corresponding angle in a 6:8:10 triangle.

Connecting the Pythagorean Theorem to Trigonometry

EXAMPLE

Find $\tan\theta$, given $\sin\theta = \frac{1}{3}$.

Discussion: Students could draw a right triangle labeling the opposite side of θ as 1and the hypotenuse as 3. Then they could use the Pythagorean Theorem to solve for the missing side length:

$$1^2 + b^2 = 3^2$$
$$1 + b^2 = 9$$
$$b^2 = 8$$
$$b = \sqrt{8} \approx 2.828$$

Thus $\tan\theta = \frac{1}{\sqrt{8}}$ or $\frac{\sqrt{8}}{8}$.

Application Problems

Right triangle trigonometry is one of the most applicable areas of mathematics. Give students the opportunity to solve many problems where they can apply trigonometric ratios in a real-world context. Use cooperative learning in small groups for discovery activities and outdoor measurement projects. Have students work on applied problems and projects, such as using clinometers and trigonometric ratios to measure the height of the school building or a flag pole. See the Instructional Resources/Tools sections for application problems involving right triangles.

When solving a right triangle, some students incorrectly use the right angle as a reference angle.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.SRT.6-8)

Instructional Tools/Resources

These tools and resources may be helpful for instruction. Before use, they should be evaluated at the local level to determine their appropriateness for instruction.

Manipulatives/Technology

- Cutouts of right triangles
- Rulers
- Protractors
- Compasses
- Scientific calculators
- Dynamic geometric software (Geometer's Sketchpad®, Desmos®, Cabri®, or GeoGebra®)
- Trig Trainer® instructional aids
- Clinometers (can be made by the students)
- Websites for the history of mathematics:
  - Trigonometric Functions
  - Trigonometric Course
  - Beyond SOH-CAH-TOA: An Example of How History Helps Us to Understand Trigonometry by Eureka Math.

Defining Trigonometric Ratios

- Defining Trigonometric Ratios by Illustrative Mathematics is a task where students use the notion of similarity to define the sine and cosine of an acute angle.
- Tangent of Acute Angles by Illustrative Mathematics is a task where students focus on studying values of tan x for special angles and conjecturing from these values how the function tan x varies.
- Trigonometry by A B Cron is a GeoGebra unit that helps students understand trigonometric functions.
- Sine and Cosine Explained Visually by Victor Powell is an interactive website that explains sine and cosine visually.
- Roadblocks to Success in Trigonometry is a worksheet that deals with students struggle of seeing opposites and connecting trigonometric ratios with similar figures.
- Trig Ratios Part 4 Sine by LearnWithJeff is a YouTube video that explores the meaning behind SOHCAHTOA and connects sine to a half-chord.
- How to Learn Trigonometry Intuitively from Better Explained is a website with an accompanying video on how to approach trigonometry intuitively.
- Slope and Tangent by Texas Instruments is an TInspire activity where students explore the relationship between the slope of a line and the tangent of the angle between the line and the horizontal axis.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.SRT.6-8)

The Complementary Relationship Between Sine and Cosine

- Sine and Cosine of Complementary Angles by Illustrative Mathematics is a task where students provide a geometric explanation for the relationship between sine and cosine.
- Trigonometric Function Values by Illustrative Mathematics is a task where students explore the relationship between sine and cosine for special benchmark angles.

Solving Problems Involving Right Triangles

- Neglecting the Curvature of the Earth by Illustrative Mathematics is a task where students apply tangents and circles of right triangles to a modeling situation.
- Ask the Pilot by Illustrative Mathematics is a task where students apply tangents and circles of right triangles to a modeling situation.
- Access Ramp by Achieve the Core is a CTE task where students design an access ramp using ADA requirements.
- Miniature Golf by Achieve the Core is a CTE task where students design a mini golf course.
- Range of Motion by Achieve the Core is a CTE task where students explore a man's range of motion after a bicycle accident.
- Launch Altitude Tracker by NASA has students use a simple altitude track to indirectly measure the altitude of rockets they construct.
- Problem 19: Beyond the Blue Horizon from NASA's Space Math III is an activity where students determine the height of a transmission antenna to insure proper reception.
- Solving Problems with Circles and Triangles from Mathematics Assessment Project is a lesson where students have to solve problem involving a triangle inscribed in a circle inscribed in a triangle.
- Calculating Volumes of Compound Objects by Mathematics Assessment Project has student use right triangles and their properties to solve volume problems.
- Solving Problems with Circles and Triangles by Mathematics Assessment Project is a lesson where students solve problems by determining the lengths of the sides in right triangles.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.SRT.6-8)

Curriculum and Lessons from Other Sources

- EngageNY, Geometry, Module 2, Topic E, Lesson 25: Incredibly Useful Ratios, Lesson 26: The Definition of Sine, Cosine, and Tangent, Lesson 27: Since and Cosine of Complementary Angles and Special Angles, Lesson 28: Solving Problems Using Sine and Cosine, Lesson 29: Applying Tangents, Lesson 30: Trigonometry and the Pythagorean Theorem are lessons that pertain to this cluster. *Note: Trigonometric Identities, Inverse Trigonometric ratios, and using trigonometric ratios and the Pythagorean to solve problems when an acute angle is not given has been moved to Algebra 2 in Ohio.*
- Mathematics Vision Project, Geometry, Module 4: Similarity and Right Triangle Trigonometry have tasks that pertain to this cluster.
- Georgia Standards of Excellence Curriculum Framework, Unit 3: Right Triangle Trigonometry has many tasks that pertain to this cluster.
- Illustrative Mathematics, Geometry, Unit 4: Right Triangle Trigonometry, Lesson 1: Angles and Steepness, Lesson 2: Half a Square, Lesson 3: Half a Triangle, Lesson 4: Ratios in Right Triangles, Lesson 5: Working with Ratios in Right Triangles, Lesson 6: Working with Trigonometric Ratios, and Lesson 7: Applying Ratios in Right Triangles, Lesson 8: Sine and Cosine in the Same Right Triangle are lessons that pertain to this cluster.

General Resources

- Arizona 7-12 Progression on Geometry is an informational resource for teachers. This cluster is addressed on page 17.
- Arizona High School Progression on Modeling is an informational resource for teachers. This cluster is addressed on page 17.
- High School Coherence Map by UnboundED is a map of the relationships among high school mathematics standards. Please note: This resource is aligned to the Common Core not Ohio's Learning Standards. Although the tool is useful, it needs to be checked with respect to Ohio's Learning Standards.
- van Hiele Model of Geometric Thinking is a pdf created by ODE that summarizes the van Hiele levels.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.SRT.6-8)

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STANDARDS	MODEL CURRICULUM (G.C.1-4)
Geometry **CIRCLES** **Understand and apply theorems about circles.** **G.C.1** Prove that all circles are similar using transformational arguments. **G.C.2** Identify and describe relationships among angles, radii, chords, tangents, and arcs and use them to solve problems. *Include the relationship between central, inscribed, and circumscribed angles and their intercepted arcs; inscribed angles on a diameter are right angles; the radius of a circle is perpendicular to the tangent where the radius intersects the circle.* **G.C.3** Construct the inscribed and circumscribed circles of a triangle; prove and apply the property that opposite angles are supplementary for a quadrilateral inscribed in a circle. (+) **G.C.4** Construct a tangent line from a point outside a given circle to the circle.	**Expectations for Learning** In middle school, students have worked with measurements of circles such as circumference and area. In this cluster, students extend their understanding of similarity to circles. Students solve problems using the relationships among the arcs and angles created by radii, chords, secants, and tangents. They will also construct inscribed and circumscribed circles of a triangle. The student understanding of this cluster begins at van Hiele Level 1 (Analysis) and moves to Level 2 (Informal Deduction/Abstraction). **ESSENTIAL UNDERSTANDINGS** • All circles are similar because one circle can be translated so that its center maps onto the center of the other and then dilated about the common center by the ratio of the radii. • The measure of an arc is equal to the measure of its corresponding central angle. • The measure of an inscribed angle is half the measure of its corresponding central angle. • Inscribed angles on a diameter of a circle are right angles (special case of inscribed angles). • A tangent is perpendicular to the radius at the point of tangency. • A secant is a line that intersects a circle at exactly two points. • A circumscribed angle is created by two tangents to the same circle from the same point outside the circle. • The center of the circumscribed circle is the point of concurrency of the perpendicular bisectors because it is equidistant from the vertices of the triangle. • The center of the inscribed circle is the point of concurrency of the angle bisectors because it is equidistant from the sides of the triangle. • While all triangles can be inscribed in a circle, a quadrilateral can be inscribed in a circle if and only if the opposite angles in the quadrilateral are supplementary. *Continued on next page*

STANDARDS	MODEL CURRICULUM (G.C.1-4)
	Expectations for Learning, continued **MATHEMATICAL THINKING** • Use accurate mathematical vocabulary. • Make connections between concepts, terms, and properties. • Recognize, apply, and justify mathematical concepts, terms, and their properties. • Solve mathematical and real-world problems accurately. • Determine reasonableness of results. • Consider mathematical units involved in a problem. • Make sound decisions about using tools. **INSTRUCTIONAL FOCUS** • Use transformational arguments to prove that all circles are similar. • Given a diagram, identify radii, chords, secants, and tangents, and the arcs and angles formed by them. • Solve mathematical and real-world problems involving angles and arcs formed by radii, chords, secants, and tangents. • Construct the angle bisectors of a triangle to locate the incenter, and then construct the inscribed circle. • Construct the perpendicular bisectors of a triangle to locate the circumcenter, and then construct the circumscribed circle. • Provide an informal argument for why the opposite angles of an inscribed quadrilateral are supplementary based on the arcs the angles intercept and their corresponding central angles. • Solve problems using the property that opposite angles are supplementary for a quadrilateral inscribed in a circle. • (+) Construct a tangent line from a point outside a given circle to the circle. *Continued on next page*

STANDARDS	MODEL CURRICULUM (G.C.1-4)
	Content Elaborations **OHIO'S HIGH SCHOOL CRITICAL AREAS OF FOCUS** • Geometry, Number 5, page 8 **CONNECTIONS ACROSS STANDARDS** • Experiment with transformations in the plane (G.CO.1). • Make geometric constructions (G.CO.12). • Understand similarity in terms of similarity transformations (G.SRT.2). • Find arc length and areas of sectors of circles (G.C.5).

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.C.1-4)

Instructional Strategies

Note: The model curriculum is not a collection of lessons nor a full curriculum; it does not suggest pace, sequence, or amount of time spent on topics. It provides information about a topic related to the standards including ideas for examples, strategies for teaching, common misconceptions, and possible connections between topics. In addition, the emphasis of the model curriculum is on instruction not on assessment; therefore, the examples in the Instructional Strategies section are written for instructional purposes not necessarily as assessment items. Examples may need to be rewritten to accommodate the needs of each individual classroom.

Standards for Mathematical Practice

This cluster focuses on but is not limited to the following practices:

MP.1 *Make sense of problems and persevere in problem solving.*

MP.3 *Construct viable arguments and critique the reasoning of others.*

MP.5 *Use appropriate tools strategically.*

It is important to reinforce the precise definition of a circle from G.CO.1 as it is the foundation for the rest of the learning in this cluster.

VAN HIELE CONNECTION

In Geometry students are expected to move from Level 1(Analysis) to Level 2 (Informal Deduction/Abstraction). In van Hiele Level 1 students may list all the properties of a shape, but not see the relationships between properties. Whereas van Hiele Level 2 can be characterized by the student doing some or all of the following:

- noticing relationships within and between properties and figures;
- forming meaningful definitions;
- justifying reasoning using informal arguments;
- drawing logical maps and diagrams; and/or
- not yet understanding formal deduction.

This cluster when combined with G.CO.9-11 will also move into Level 3 (Deduction) where students start to construct proofs and understand the necessity of proofs.

See the van Hiele pdf on ODE's website for more information about van Hiele levels.

SIMILARITY OF CIRCLES

Two figures are similar if there is a sequence of rigid motions and dilations that maps one figure onto the other. Another way to look at it is that a figure and its dilated image are similar. In previous clusters, students looked at similarity of polygons. Now they extend that understanding to circles.

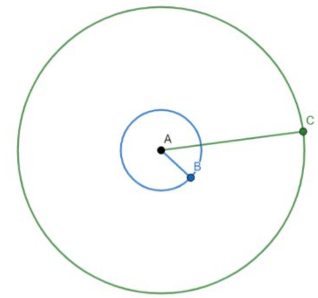
Given any two circles in a plane, guide students to discover the transformations that map one circle onto the other. It may be helpful to have students start with circles that have the same center using the center as the center of dilation. From there challenge students to prove that all circles are similar despite their location on the plane. Students should come to the conclusion that transformations must include a dilation (possibly with a scale factor of 1) but could also include a translation since a translation allows centers of any two circles to coincide. This experience can help students use transformation to understand all circles are similar.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.C.1-4)

EXAMPLE

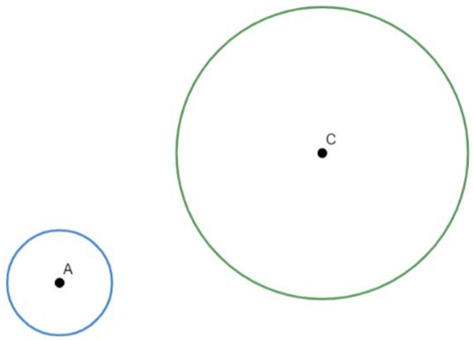
Part 1

Map the circle with radius AB onto the circle with radius AC. Explain your sequence of transformations.



Part 2

Map circle A onto circle C. Explain your sequence of transformations.



Part 3

Create any two circles in the plane. Name one circle R and the second circle J. Map circle R onto circle J. Explain your sequence of transformations.

Part 4

- **a.** Can any two circles be mapped onto each other? Explain.
- **b.** What transformation(s) must be included in order to map two noncongruent circles onto each other?
- **c.** Are all circles similar? Explain why or why not.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.C.1-4)

EXAMPLE

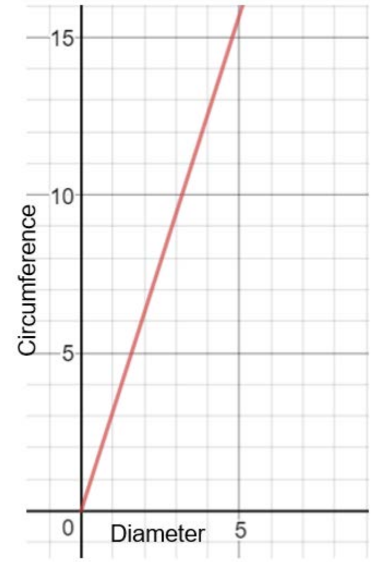
Marcus was cleaning out his old notebooks and found a graph he made from Pi Day in 7th grade. In the activity, he measured the circumference and diameter of various circular objects to find pi.

a. How is pi represented on the graph?

b. Write the equation of the line.

c. Using his graph as a guide and your knowledge of circles, list the circumference and diameter of possible circular objects he might have measured. Fill out your information in a table.

d. In his high school Geometry class yesterday, Marcus used rigid motions to prove that all circles are similar. How does what Marcus learned about pi in 7th grade relate to similarity of circles?



Discussion: Students should realize that π is represented by the slope or $\frac{circumference}{diameter}$. The equation of the line is $C = \pi d$. In 7th grade students learned that a line that goes through the origin represents a proportion and the slope of that line is the constant of proportionality. In this case π is the constant of proportionality which can also be called the scale factor. Because the scale factor is constant, the ratio of $\frac{circumference}{diameter}$ stays the same, so the relationship between the circumference and diameter of a circle is proportional, thus all circles are similar.

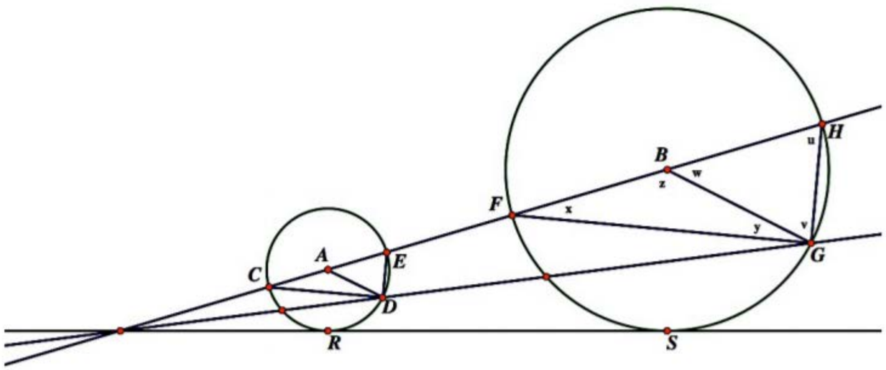


Image taken from Mathematics Vision Project, Geometry, Module 5 Circles: A Geometric Perspective.

EXAMPLE

Circle A is dilated by a scale factor of 2.5 about point O to produce circle B and $m\angle z = 131°$ and $\overline{AC} = 4.2$. Find the measure of the following:

a. $\angle x$

b. $\angle ADC$

c. $\angle w$

d. $\angle v$

e. $\angle AED$

f. $\overline{AD}$

g. $\overline{FG}$

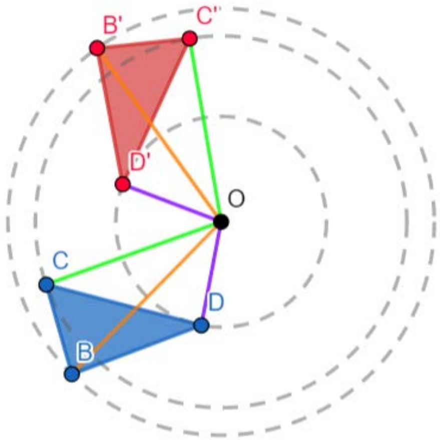
h. $\overline{AE}$

i. $\overline{ED}$

RELATIONSHIPS AMONG PARTS OF A CIRCLE

Students should be able to identify and describe relationships among angles, radii, chords, secants, tangents, and arcs to solve problems. For example, students should be able to find measures of inscribed angles, given the measure of the intercepted arc. For angles with a vertex outside of the circle, only the case of two tangents needs to be considered. A case formed by two secants or a secant and a tangent could be explored as an extension. Also, properties of congruent triangles and perpendicular lines to prove theorems about diameters, radii, chords, and tangent lines when appropriate.

The emphasis on G.C.2 is on using the relationships of the parts of a circle to solve problems. However, students should have some exposure to proving theorems such as Chord – Center, Arc – Chord Congruence, Secant Length, Angle – Chord, Angle – Secant, Tangent - Chord, Tangent - Secant, and Tangent Square, spend more time on the applicational aspect of the theorems;
Note how the study of circles can be used when dealing with rotations of figures. For example, students should see that points on a rotated figure move along circles centered around the same center. They should also notice that each vertex and its image are equidistant from the center of rotation since they are radii of the same rotated circle.

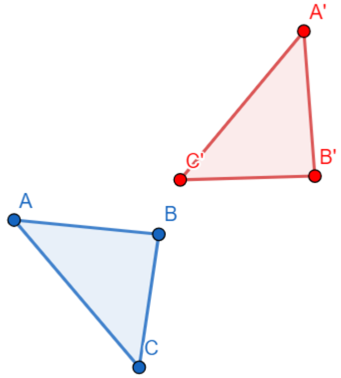


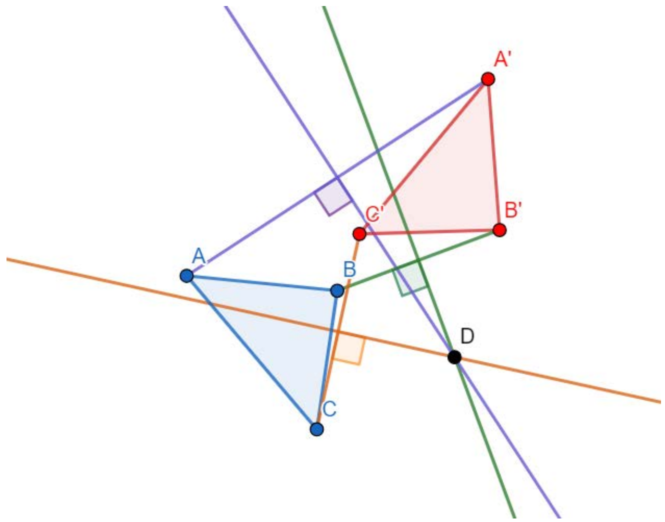
INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.C.1-4)

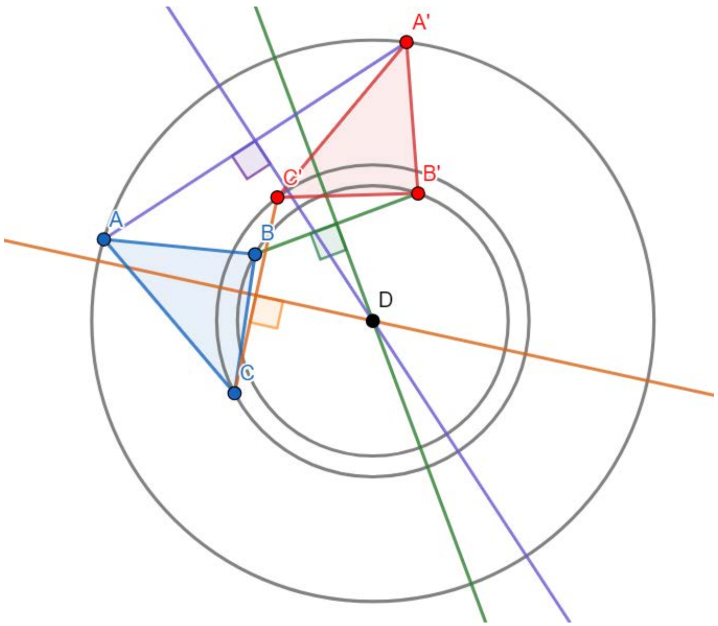
EXAMPLE

Finding the Center of Rotation

Find the center of rotation that maps $\triangle ABC$ onto $\triangle A'B'C'$.







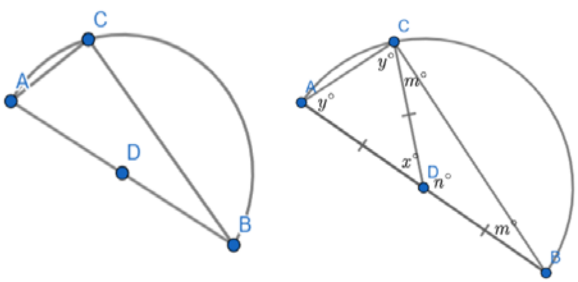
Discussion: To find the center of rotation students must find the perpendicular bisector of two or more segments that connect corresponding vertices. Students should justify why perpendicular bisectors are concurrent at the center of rotation. They should remember that each vertex and its corresponding vertex lie on the same circle. One way to justify the concurrency of perpendicular bisectors is to use the chords that connect the corresponding vertices and the idea that a perpendicular bisector includes all points equidistant from the endpoints. Since the endpoints of a chord are equidistant from the center of the circle, the perpendicular bisector must include the center of the circle.

GeoGebra can help show how to rotate a figure around a point.

- Create a figure and a point of rotations.
- Create a slider.
- Then use the transform rotation button.
- When placing the rotation substitute α in for the number of degrees. That connects the slider to the object and center of rotation.

Angle on the Diameter Inscribed on a Semi-Circle

Drawing another radius to the point on the circle creates two isosceles triangles each with congruent bases angles.



$x + n = 180$ supplementary angles
$x + 2y = 180$ sum of angles in a triangle
$n + 2m = 180$ sum of angles in a triangle
$x + 2y + n + 2m = 360$ addition
$2y + 2m = 180$ substitution
$y + m = 90$ division
Therefore, an inscribed angle intercepting the diameter must be a right angle.

Starting with the special case of an angle inscribed in a semicircle, use the fact that the angle sum of a triangle is 180° to show that this angle is a right angle. Dynamic geometric software could be a great tool for developing this concept. Students can grab a point on a circle and move it to see that the measure of the inscribed angle passing through the endpoints of a diameter is always 90°. It is important to note that having students construct the situations themselves can help solidify understanding of the relationships. Once students gain an intuitive sense of this concept using software, it may be useful to have students prove it using Thales' Theorem: If A, B, and C are distinct points on a circle where the line AB is a diameter, then the angle $\angle ACB$ is a right angle. Then extend the result to any inscribed angle.

CENTRAL AND INSCRIBED ANGLES

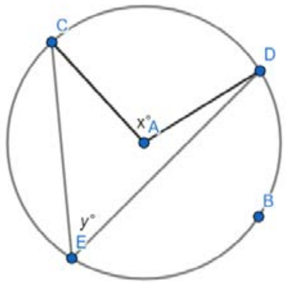
Have students explore the relationship between central angles and inscribed angles using dynamic geometric software. Once students observe that the measure of an inscribed angle is always half of the measure of the central angle that intercepts the same arc, have them prove why this relationship is true.

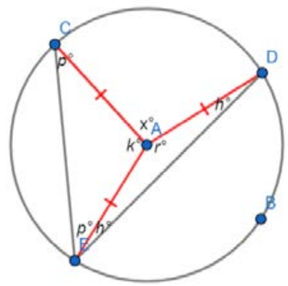
INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.C.1-4)

EXAMPLE

Proving the Relationship between Inscribed and Central Angles

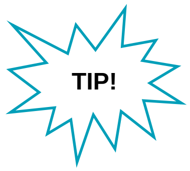
Prove that the measure of the inscribed angle $\angle CED$ is half of the measure of the central angle $\angle CAD$.





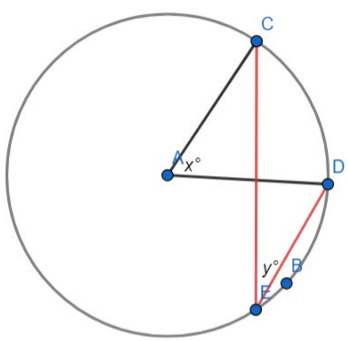
Discussion: One way to prove this theorem is to draw a radius of the circle through points A and E. This makes two isosceles triangles: $\triangle CAE$ and $\triangle DAE$. Since the base angles of isosceles triangles are congruent, the base angles can be labeled with the same variable. By the Angle Addition Postulate $m\angle CEA + m\angle DEA = m\angle CED$ or $p° + h° = y°$.

Some applications of central angles could include an analog clock, a Ferris wheel, a baseball park, etc.



Students should also realize that $x + k + r = 360°$ because the sum of the central angles of a circle equals 360°, $k + 2p = 180°$, and $r + 2h = 180°$ because the sum of the angles of a triangle equals 180°. Then students can use algebraic manipulation:

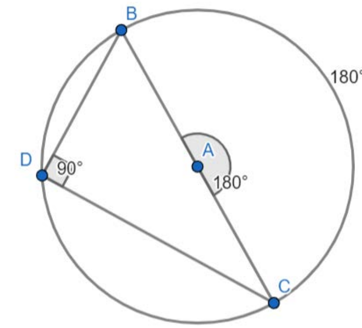
- $k + 2p = 180 \; and \; r + 2h = 180, so$
- $By\ the\ Substitution\ Property: k + 2p + r + 2h = 360$
- $By\ the\ Associative\ Property: k + r + 2p + 2h = 360$
- $Since: x + k + r = 360$
- $By\ the\ Substitution\ Property: x + k + r = k + r + 2p + 2h$
- $And: x = 2p + 2h$
- $By\ the\ Distributive\ Property\ x = 2(p + h)$
- $By\ the\ Substituion\ Property\ since\ p + h = y, then\ x = 2y$

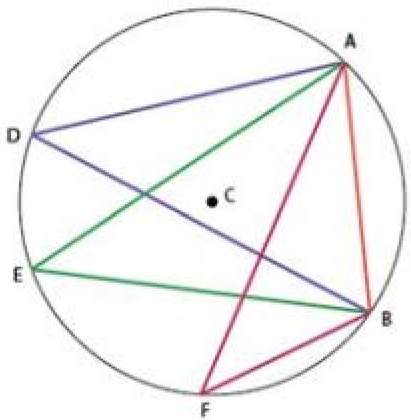


Next using the diagram at the right, students can prove that $x = 2y$ or $y = \frac{1}{2}x$ because $\angle CAD$ and $\angle DEC$ intercept the same arc.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.C.1-4)

Students sometimes confuse the relationships of the measures of the intercepted arc and the measures of the inscribed angles and central angles. Use the simple case of a diameter to help students remember which angle is equal to the arc measure, and which angle is half of the arc measure (inscribed angle).





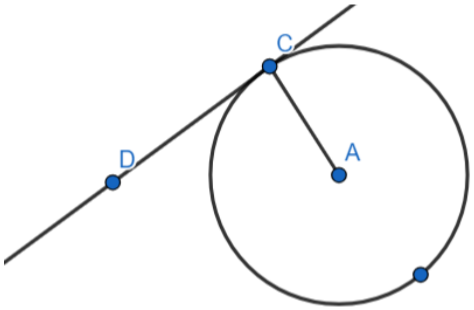
Inscribed Angles and Arc Length

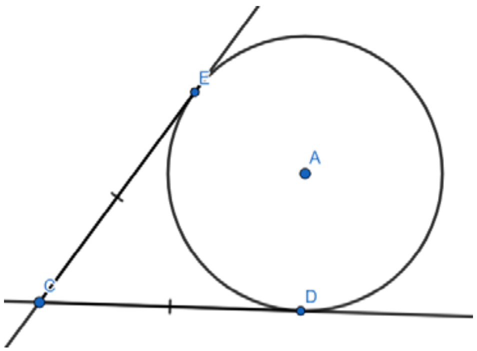
EXAMPLE

- Fold a paper plate (or coffee filter) to form a chord $\overline{AB}$ that does not intersect the center of the circle. Mark that chord with a colored pencil.
- Draw and label three inscribed angles where each ray intersects the endpoints of the chord. Use a different colored pencil to label each inscribed angle.
- Measure the inscribed angles using a protractor, or compare the angles using tracing paper.
- Ask students the following questions:
  - What did you notice about all the measures of the inscribed angles? Why is this true?
  - What do you know about the relationship between the measure of the central angle and the angular measure of the intercepted arc?
  - How do you think the measure of the inscribed angle relates to the angular measure of the intercepted arc?

Tangent Lines to a Circle

Have students explore the relationship between a tangent line and the radius at the point of tangency using dynamic geometric software. They should discover that a tangent line to a circle is perpendicular to the radius. After students discover this, challenge them to prove it. Students may use indirect proof (proof by contradiction) or they may use rigid motions by treating the line through the radius $\overline{AC}$ as the line of reflection. In addition, students can discover that from any external point, there are exactly two lines tangent to the circle. The distances from the exterior point to each point of tangency are equal. Students can be challenged to prove this using congruent triangles.



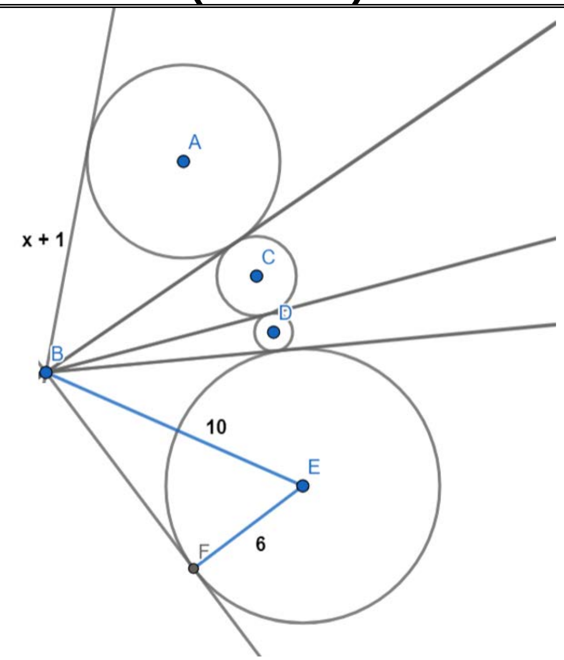


Students may incorrectly think they can tell by inspection whether a line intersects a circle at exactly one point. However, it may be beneficial to point out that a tangent line is the line perpendicular to a radius at the point where the radius intersects the circle (point of tangency).

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.C.1-4)

EXAMPLE

Find the value of *x*. Explain how you found your answer.



Discussion: Students should recognize that $\triangle BEF$ is a right triangle since a tangent line makes a perpendicular line with the radius at the point of tangency, so by the Pythagorean Theorem $\overline{BF}$ is 8 units. Since the distance from the exterior point to each point of tangency is the same, $\overline{BF}$ is congruent to the other segment that is tangent to circle E, which is also 8 units. The other line segment that is tangent to circle E through point B is also tangent to circle D. Therefore, the other line segment that is tangent to Circle D through point B is also congruent to $\overline{BF}$ by the Transitive Property. Since circle C and circle A share tangent lines through point B, the tangent of circle A through point B is also 8, so $x + 1 = 8$, and $x = 7$.

Circumscribed Angles

Explore that a circumscribed angle and the associated radii at the points of tangency create a kite with two right angles. Once students have a firm grasp of the fact that a tangent line creates a right angle to the radius at the point of tangency, they should be able to prove that the circumscribed angle and the central angle intercepting the same arc are supplementary.

CONSTRUCTIONS WITH CIRCLES

Standards G.C.3 and G.C4 connect with G.CO.12 and G.CO.13.

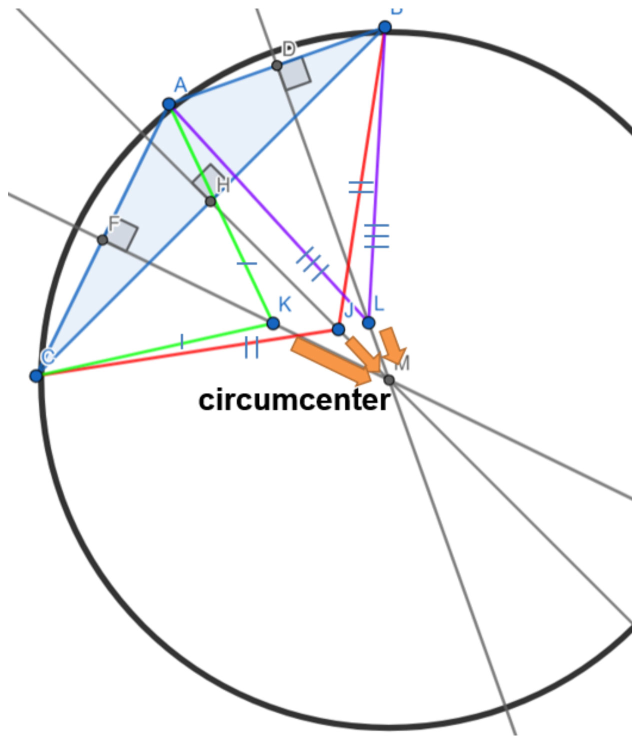
Construct the Inscribed and Circumscribed Circles of a Triangle

Use formal geometric constructions to construct perpendicular bisectors and angle bisectors of a given triangle. Their intersections are the centers of the circumscribed and inscribed circles. The center of the circumscribed circle is the circumcenter, and the center of the inscribed circle is incenter. Have students explain why the perpendicular bisectors create the circumcenter of the circumscribed circle and the angle bisectors create incenter of the inscribed circle.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.C.1-4)

Relate the perpendicular bisectors of the sides of a triangle to the theorem that states any point on the perpendicular bisector of a line segment is the same distance from the two endpoints. Students should then reason through this idea to connect it to the circumcenter of the circle.

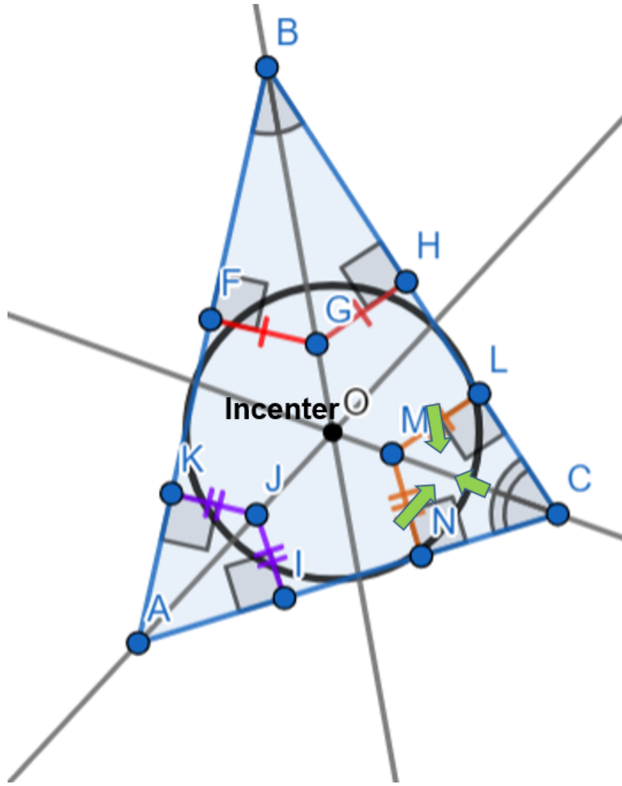
Students can use geometric software such as GeoGebra to explore the circumcenter and incenter of a circle by dragging points along perpendicular and angle bisectors. The diagram below shows a student's attempt to use GeoGebra to construct a circumscribed circle around a triangle by dragging points along the perpendicular bisectors to find the triangle's circumcenter.



- For $\triangle ABC$ the perpendicular bisectors are constructed.
- As points K, J, and L are dragged along their respective perpendicular bisectors, each point is equidistant from the endpoints of their respective segments, $\overline{AC}$, $\overline{BC}$, and $\overline{BA}$.
- When K, J, and L all reach point M (the point of concurrency), segments $\overline{CK}$, $\overline{AK}$, $\overline{CJ}$, $\overline{BJ}$, $\overline{AL}$ and $\overline{BL}$ (which are now $\overline{CM}$, $\overline{AM}$, and $\overline{BM}$) are congruent.
- Therefore, the segments become the radii and a circle can be drawn as point M is the center of the circle.
- Point M is also the circumcenter of $\triangle ABC$ as the circumcenter is the center of a circumscribed circle about the triangle.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.C.1-4)

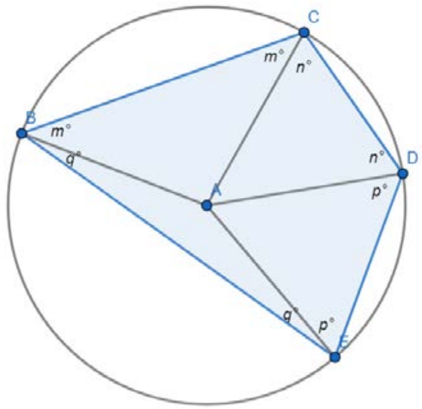
To understand the location of the incenter, students can relate the angle bisectors of a triangle to the theorem that states any point that lies on the bisector of an angle is equidistant from the sides of the angle. The diagram below shows a student's attempt to use GeoGebra to construct an inscribed circle inside a triangle by dragging points along the angle bisectors to find the triangle's incenter.

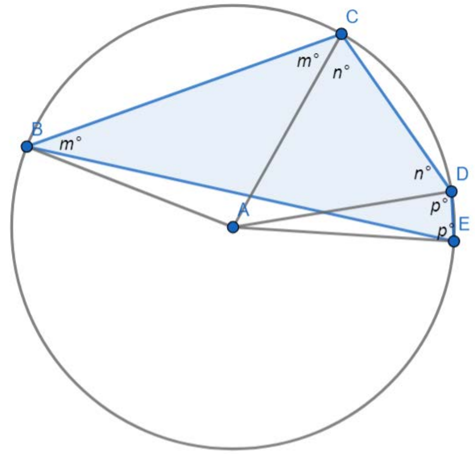


- For the incenter of $\triangle ABC$, the angle bisectors are constructed.
- As points $G, M,$ and J move along their respective angle bisectors, $\overline{BO}, \overline{OC},$ and $\overline{OA}$, $G, M,$ and J are all equidistant from the rays, $\overline{BF}, \overline{BH}, \overline{CL}, \overline{CN}, \overline{AI},$ and $\overline{AK}$ of their respective angles.
- When they all reach point O (the point of concurrency), all the segments, $\overline{FG}, \overline{HG}, \overline{LM}\ \overline{NM}, \overline{IJ},$ and $\overline{KJ}$ (which are now $\overline{FO}, \overline{HO}, \overline{LO}\ \overline{NO}, \overline{IO},$ and $\overline{KO}$) are congruent.
- Therefore, the segments become the radii of a circle, and a circle can be drawn with O as the center and $\overline{FO}, \overline{HO}, \overline{LO}\ \overline{NO}, \overline{IO},$ and $\overline{KO}$ are the radii.
- Point O is also the circumcenter of $\triangle ABC$ as the incenter is the center of a inscribed circle within the triangle.

Opposite Angles are Supplementary for Quadrilaterals Inscribed in a Circle
There are three possible methods of proving opposite angles of an inscribed quadrilateral are supplementary.

Method 1: Dissect an inscribed quadrilateral into four isosceles triangles formed by radii and the side of the quadrilateral. Since the measures of base angles of isosceles triangles are congruent, the base angles can be labeled with the same variable.





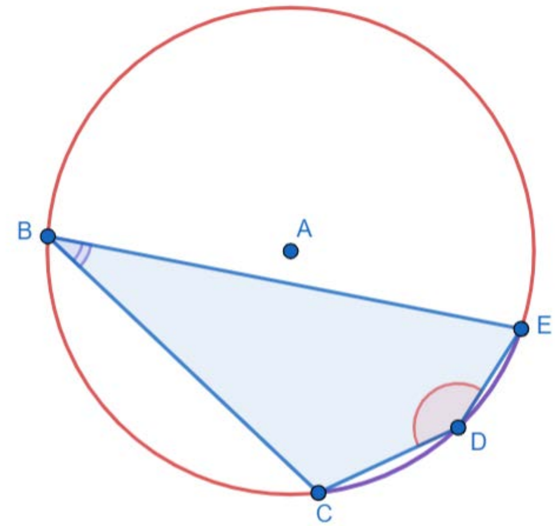
Discussion: Make sure to show different examples of quadrilaterals inscribed in a circle, so students see that the proof works despite the shape of the quadrilateral. Focus on the left diagram with the angles' measures indicated to prove that opposite angles of the inscribed quadrilateral are supplementary.

- $\angle CDE$ and $\angle CBE$ are opposite angles in the inscribed quadrilateral.
- $m\angle CDE = p + n$ and $m\angle CBE = m + q$ (Angle Addition Postulate)
- $2m + 2n + 2p + 2q = 360$ (The sum of all angles in the quadrilateral equals 360°.)
- $m + n + p + q = 180$ (Multiplication/Division Property of Equality)
- $(m + q) + (n + p) = 180$ (Associative Property)
- $m\angle CDE + m\angle CBE = 180$, (Substitution)
- Therefore, opposite angles are supplementary.

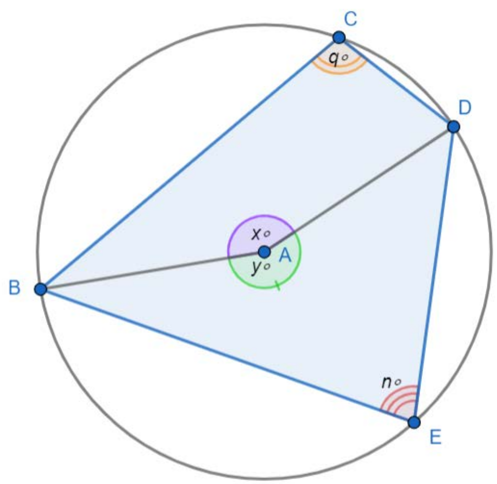
INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.C.1-4)

Method 2:

- Use a of pair opposite angles of a quadrilateral inscribed in a circle.
- A quadrilateral $BEDC$ is inscribed in a circle A. Angle CDE and angle EBC are opposite angles.
- $m\widehat{CDE} + m\widehat{EBC} = 360$ (The sum of all arc angular measures in a circle equal 360°.)
- $m\widehat{EBC} = 2m\angle CDE$ (The measure of the inscribed angle is $\frac{1}{2}$ the measure of the intercepted arc.)
- $m\widehat{CDE} = 2m\angle EBC$ (The measure of the inscribed angle is $\frac{1}{2}$ the measure of the intercepted arc.)
- $2m\angle CDE + 2m\angle EBC = 360$ (Substitution Property)
- $m\angle CDE + m\angle EBC = 180$ (Multiplication Property of Equality)
- Therefore, the sum of the measures of opposite angles in a quadrilateral is 180°, or the angles are supplementary.



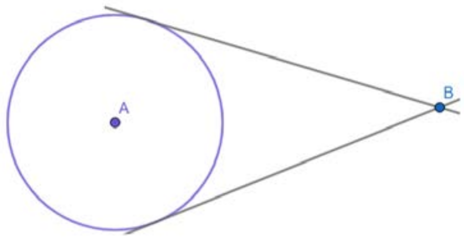
Method 3:
Deconstruct a quadrilateral inscribed in a circle into two quadrilaterals that have one angle as a central angle and the angle measures as indicated.



- $x + y = 360$ (2 angles that make up a circle equal 360°)
- $q = \frac{1}{2}y$ (The measure of an inscribed angle is $\frac{1}{2}$ the measure of its corresponding central angle.)
- $y = 2q$ (Multiplication Property of Equality)
- $n = \frac{1}{2}x$ (The measure of an inscribed angle is $\frac{1}{2}$ the measure of its corresponding central angle.)
- $x = 2n$ (Multiplication Property of Equality)
- $2q + 2n = 360$ (Substitituion Property)
- $q + n = 180$ (Multiplication/Division Property of Equality)
- Therefore, two opposite angles of a quadrilateral inscribed in a circle are supplementary.

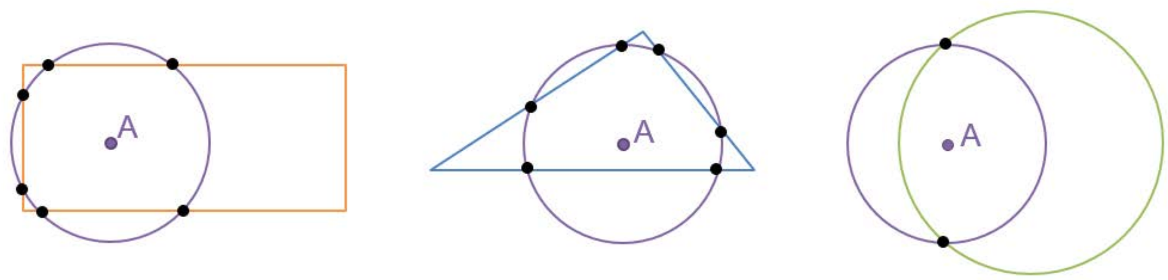
Construct a Tangent Line from a Point Outside the Given Circle to the Circle (+)
One way to help students reason through this construction is by using questioning techniques pushing the students to think through the logic of the construction.

EXAMPLE
Construct two distinct lines tangent to the circle A from a point B that is outside of the given circle.



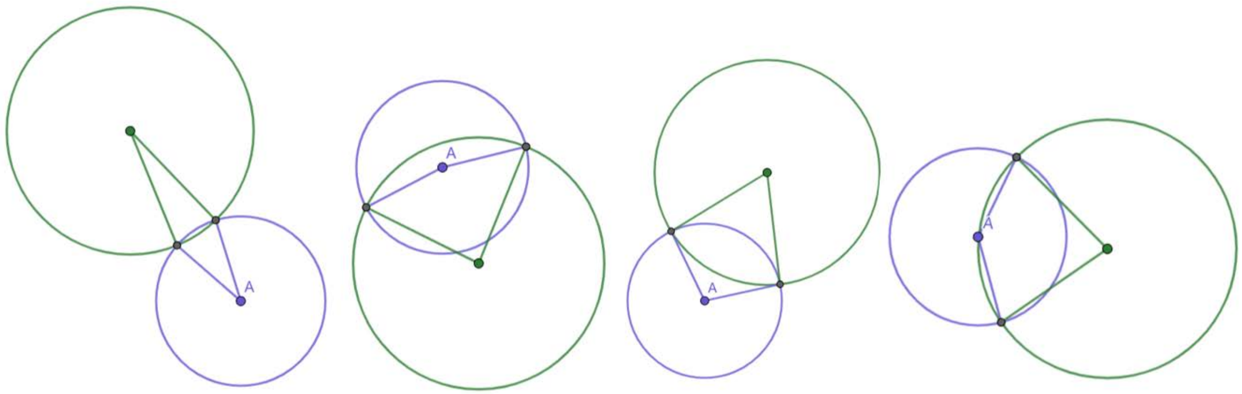
Discussion: [This activity can be demonstrated by using two transparencies, by geometric software or even by having students drag shapes on PowerPoint, Word, or on a Smartboard.] Start by having the students sketch the end product, so they keep the goal in mind. A sample conversation is recorded below.

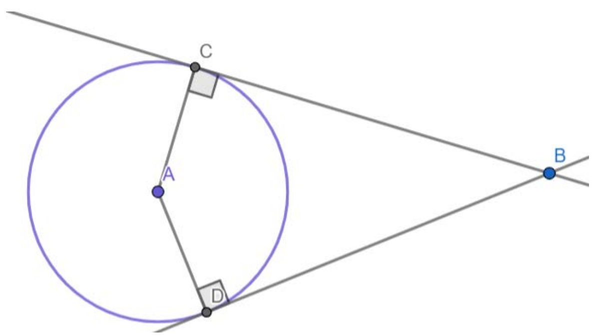
Teacher: This picture shows two tangent lines passing through a Point B that is outside the circle. Tell me what you know about tangent lines.
Student: A tangent intercepts a circle at only one point.
Teacher: What is that point called?
Student: The point of tangency.
Teacher: So, if you have two tangent lines, how many points of tangency should you have?
Student: Two.
Teacher: What are some shapes that we can construct that will intersect the circle at a maximum of two points? [Have students use geometric software to move some different shapes around to show that only circles can intersect another circle at a maximum of two points.]



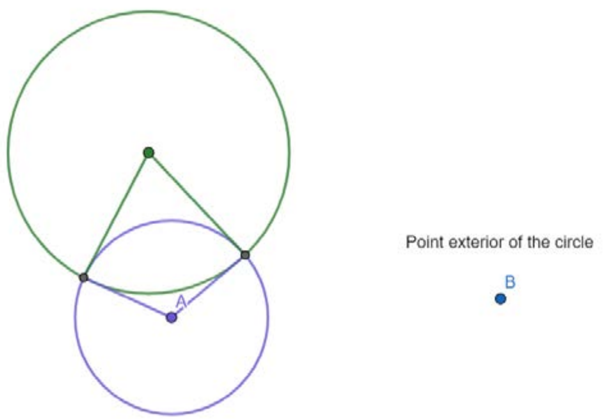
Student: Circles.
Teacher: As you move the circle around what do you notice about the intersection points?

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.C.1-4)





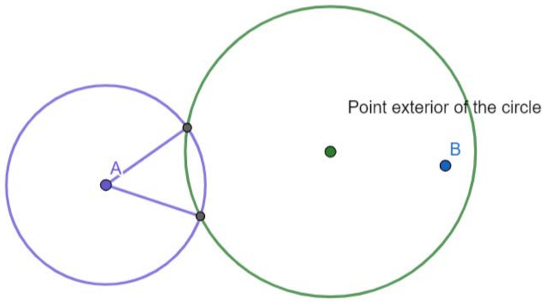
Student: They create radii using the center of both circles.
Teacher: Now recall the goal of our task to construct two tangent lines from a point outside the circle. How do the tangent lines and radii relate to one another?
Student: A tangent line should be perpendicular to the radius of a circle at the point of tangency.

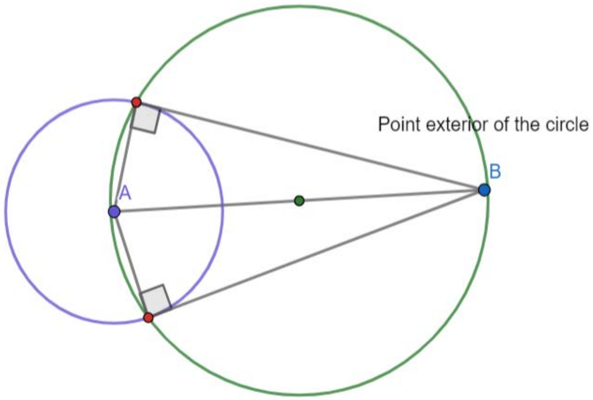


Teacher: So, you need your final construction to look like this (pictured on the right):
Teacher: So which circle's radius should be the focus?
Student: The original circle.
Teacher: How do you think you will get there with your intersecting circles? Will any intersection points of two intersecting circles create points of tangency?

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.C.1-4)

Student: No, they will have to line up.
Teacher: What do you mean? Will this work? Where do you have to place the circle and how big does the circle have to be in order to ensure that at the point of tangency the radii and the tangent lines will form right angles? Can you explain?





Student: What about if we create right angles in the new (green) circle and then line them up? We already learned the theorem that an inscribed triangle that is built on the diameter is a right angle, so we can create right triangles using that theorem. Therefore, the new circle (green) would have to pass through the center, A, of the original circle (purple) and the point B exterior to the original circle.

Teacher: So, if you didn't have geometric software, how would you construct two tangent lines using a compass and straight edge?

Student: You would have to create the diameter of a new circle connecting the exterior point (point B) and the center of the original circle. Then find the midpoint of the diameter, and use half the diameter as the radius to create the new intersecting (green) circle. Next you would have to find the intersection points of the two circles and connect those points to the given exterior point. Since the constructed triangles are right triangles, their legs are perpendicular. Then you will have correctly constructed a tangent line going the point exterior to the circle that is perpendicular to the circle at the point of tangency.

Constructing a tangent to a circle from a given point could also be addressed when inscribing triangles, quadrilaterals, and hexagons.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.C.1-4)

Instructional Tools/Resources

These tools and resources may be helpful for instruction. Before use, they should be evaluated at the local level to determine their appropriateness for instruction.

Manipulatives/Technology

- Tracing paper (patty paper)
- Transparencies
- Graph paper
- Ruler
- Protractor
- Compass
- Mira
- Computer dynamic geometric software (such as Geometer's Sketchpad®, Desmos, Cabri®, or GeoGebra®).
- Graphing calculators and other handheld technology such as TI-Nspire™.
- GeoGebra Geometry is the Geometry application of GeoGebra.

Similar Circles

- Similar Circles by Illustrative Mathematics is a task that has students prove that circles are similar using a coordinate plane. There is an attached GeoGebra file.

Arc Lengths and Central Angles

- It All Comes Full Circle by NASA is a lesson where students have to use properties to predict and monitor vehicle positions of the space shuttle and the International Space Station. They will calculate arc lengths and find the measures of central angles and use the properties of similarity.
- Folding Circles: Exploring Theorems through Paper Folding is an NCTM Illuminations lesson where students use paper circle or paper plates to explore theorems about circles. *NCTM now requires a membership to view their lessons.*

Tangents

- Tangent Lines and the Radius of a Circle is a task by Illustrative Mathematics that has students explain why the tangent line is perpendicular to the radius at the tangent line. It explains two possible approaches: reason directly from past results of plane geometry including the Pythagorean Theorem and a proof by contradiction or use the idea of rigid motions of the plane and the line of reflection.
- Neglecting the Curvature of the Earth is an Illustrative Mathematics task where students apply geometric concepts such as properties of tangents to modeling situations.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.C.1-4)

Angles Inscribed in Circles

- Right Triangles Inscribed in Circles and Right Triangles Inscribed in Circles II are Illustrative Mathematics tasks where students use properties of triangles to show that a triangle inscribed in a circle where one side is the diameter is a right triangle.

Constructing the Inscribed and Circumscribed Circles of a Triangle

- Locating Warehouse by Illustrative Mathematics is a task that has students inscribe a circle given a context.
- Placing a Fire Hydrant by Illustrative Mathematics is a task that has students apply properties of circumscribing a circle given a context.
- Inscribing and Circumscribing Right Triangles by Mathematics Assessment Project has students find the relationship between the radii of inscribed and circumscribed circles of right triangles.

Curriculum and Lessons from Other Sources

- EngageNY, Geometry, Module 5, Topic A, Lesson 1: Thales' Theorem, Lesson 2: Circles, Chords, Diameters, and Their Relationships, Lesson 3: Rectangles Inscribed in Circles, Lesson 4: Experiments with Inscribed Angles, Lesson 5: Inscribed Angle Theorem and Its Applications, Lesson 6: Unknown Angle Problems with Inscribed Angles in Circles are lessons that pertain to this cluster.
- EngageNY, Geometry, Module 5, Topic B, Lesson 7: The Measure of the Arc, Lesson 8: Arcs and Chords are lessons that pertain to this cluster.
- EngageNY, Geometry, Module 5, Topic C, Lesson 11: Properties of Tangents, Lesson 12: Tangent Segments, Lesson 13: The Inscribed Angle Alternate—a Tangent Angle, Lesson 14: Secant Lines; Secant Line that Meet Inside a Circle, Lesson 15: Secant Angle Theorem, Exterior Case, Lesson 16: Similar Triangle in Circle-Secant (or Circle-Secant-Tangent) Diagrams are lessons that pertain to this cluster.
- Mathematics Vision Project, Geometry, Module 5: A Geometric Perspective has many lessons that pertain to this cluster.
- Georgia Standards of Excellence Curriculum Frameworks, Geometry, Unit 4: Circles and Volume has many lessons that pertain to this cluster.
- Illustrative Mathematics, Geometry, Unit 7: Circles, Lesson 1: Lines, Angles, and Curves, Lesson 2: Inscribed Angles, Lesson 3: Tangent Lines, Lesson 4: Quadrilaterals in Circles, Lesson 5: Triangles in Circles, Lesson 7: Circles in Triangles are lessons that pertain to this cluster.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.C.1-4)

General Resources

- Arizona 7-12 Progression on Geometry is an informational document for teachers. This cluster is addressed on page 17.
- High School Coherence Map by UnboundED is a map of the relationships among high school mathematics standards. Please note: This resource is aligned to the Common Core not Ohio's Learning Standards. Although the tool is useful, it needs to be checked with respect to Ohio's Learning Standards.
- van Hiele Model of Geometric Thinking is a pdf created by ODE that summarizes the van Hiele levels.
- Prove that All Circles Are Similar 1 and Prove that All Circles Are Similar 2 are threads on Bill McCallum's Mathematical Musings blog that discuss similarity of circles.

References

- Common Core Standards Writing Team. (2016, March 24). *Progressions for the Common Core State Standards in Mathematics (draft). Grades 7–8, HS, Geometry*. Tucson, AZ: Institute for Mathematics and Education, University of Arizona.

STANDARDS	MODEL CURRICULUM (G.C.5)
Geometry **CIRCLES** **Find arc lengths and areas of sectors of circles.** **G.C.5** Find arc lengths and areas of sectors of circles. **a.** Apply similarity to relate the length of an arc intercepted by a central angle to the radius. Use the relationship to solve problems. **b.** Derive the formula for the area of a sector, and use it to solve problems.	**Expectations for Learning** In middle school, students are limited to working with measurements of circles such as circumference and area. This cluster spans Geometry/ Mathematics 2 and Algebra 2/ Mathematics 3. In Geometry/Mathematics 2, students are using part-to-whole proportional reasoning to find arc lengths and sector areas, in which the arc or central angle is measured in degrees. In Algebra 2/ Mathematics 3, students derive and use formulas relating degree and radian measure. The student understanding of this cluster aligns with van Hiele Level 2 (Informal Deduction/Abstraction). *Note: Since in Algebra 1 students focus on quadratics with leading coefficients of 1 with occasional uses of other simple coefficients, geometry standards should only apply to equations where the squared terms have a coefficient of 1 or occasionally other simple leading coefficients.* **ESSENTIAL UNDERSTANDINGS** • A central angle that turns through n one-degree angles is said to have an angle measure of n degrees. • The measure of an arc is equal to the measure of the corresponding central angle and is expressed in degrees, while the length of an arc is expressed in units of linear measure. • The arc length is a part of the circumference of a circle. • The ratio of the central angle to 360 degrees is equal to the ratio of the length of the arc to the circumference of the circle. • The sector area is a part of the area of a circle. • The ratio of the central angle to 360 degrees is equal to the ratio of the area of the sector to the area of the circle. • Because all circles are similar, if the radius of the circle is scaled by k, the corresponding arc length is multiplied by k and the sector area is multiplied by k^2. *Continued on next page*

STANDARDS	MODEL CURRICULUM (G.C.5)
	Expectations for Learning, continued **MATHEMATICAL THINKING** • Consider mathematical units involved in a problem. • Make connections between concepts and terms. • Generalize concepts based on patterns. • Use proportional reasoning (part to whole). • Draw a picture to make sense of a problem. • Solve real-world and mathematical problems accurately. • Plan a solution pathway. • Attend to the meaning of quantities. **INSTRUCTIONAL FOCUS** • Develop understanding of the formulas for arc length and area of a sector through derivation. • Solve problems using arc lengths and areas of sectors of circles. **Content Elaborations** **OHIO'S HIGH SCHOOL CRITICAL AREAS OF FOCUS** • Geometry, Number 5, page 8 **CONNECTIONS ACROSS STANDARDS** • Understand and apply theorems about circles (G.C.1-2). • Experiment with transformations in the plane (G.CO.1). • Explain volume formulas, and use them to solve problems (G.GMD.1). • Understand similarity in terms of similarity transformations (G.SRT.2). • Understand the relationships between lengths, areas, and volumes (G.GMD.5-6).

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.C.5)

Instructional Strategies

Note: The model curriculum is not a collection of lessons nor a full curriculum; it does not suggest pace, sequence, or amount of time spent on topics. It provides information about a topic related to the standards including ideas for examples, strategies for teaching, common misconceptions, and possible connections between topics. In addition, the emphasis of the model curriculum is on instruction not on assessment; therefore, the examples in the Instructional Strategies section are written for instructional purposes not necessarily as assessment items. Examples may need to be rewritten to accommodate the needs of each individual classroom

Standards for Mathematical Practice

This cluster focuses on but is not limited to the following practices:

***MP.1** Make sense of problems and persevere in solving them.*
***MP.2** Reason abstractly and quantitatively.*
***MP.4** Model with mathematics.*
***MP.6** Attend to precision.*
***MP.7** Look for and make use of structure.*
***MP.8** Look for an express regularity in repeated reasoning.*

ARC LENGTHS

Students need to understand that angle measure and angular arc measure are each an amount of rotation and are measured in the same units. The measure of the central angle of the entire circle as well as the angular arc measure of the entire circle are equal to 360°. One way to show this is to relate a central angle of 90° to the arc measure of a quarter of a circle.

Connect back to the 4^{th} grade standard 4.MD.5, where angles are defined as an amount of turning. The angular measure of an arc as a measure can be thought of in a similar way. For example, consider that doing a 180 is a half turn, 90 is a quarter turn, etc.

EXAMPLE

Part 1

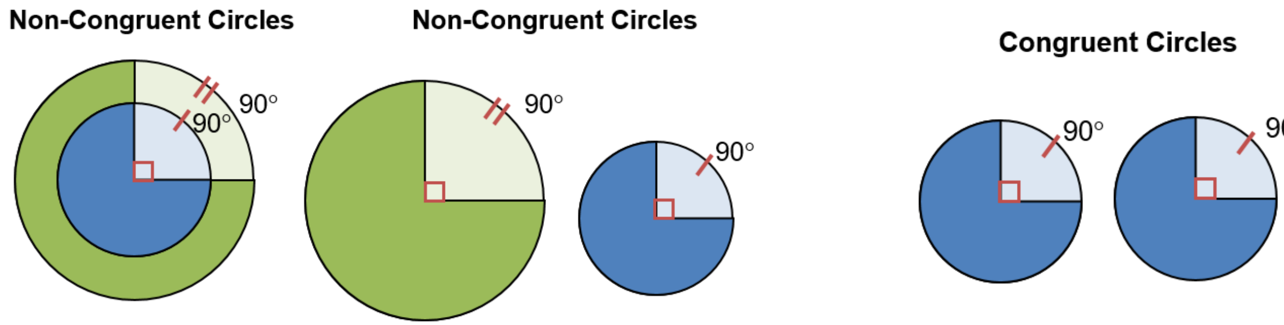
One method to model this is to have one student stand at the center of a circle and another student stand on the circumference. "Connect" the two students with a piece of string to represent the radius. As the student at the center turns through the central angle and does not move his position from the center, the student on the circle will rotate the same amount but will travel along the circle the distance equaling to the length of the arc.

Part 2

Now draw two concentric circles on the sidewalk and have three students hold the string. One student should continue to anchor the radius (string) at the center. The second student should stand on the outside concentric circle and hold the other end of the string. The third student should hold the string at the place where it intersects the inside concentric circle. Mark the starting point. As the student standing at the center turns through the central angle, both students will rotate the same amount, but will travel different lengths. Mark the end of the turn, and have students measure the arc length.

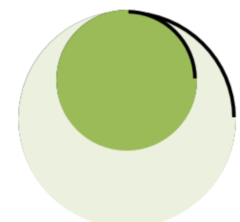
INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.C.5)

Discuss what it means for arcs to be congruent. Students need to understand that congruent central angles in non-congruent circles have non-congruent corresponding intercepted arcs. Therefore, congruent arcs will have equal angular measures and equal arc lengths and either be on the same or congruent circles.



Many students incorrectly think that arc measure (the amount of rotation about the center) and arc length (a distance along the arc) are the same. Show students two concentric circles that have the same angular measure but different arc lengths.

Some students may incorrectly think that the arc of a larger size circle will have the same size and shape (length and curvature) of a smaller circle. Have students map one circle onto the other (without using dilation), so they can see the overlap is not perfect because the curvatures are different.



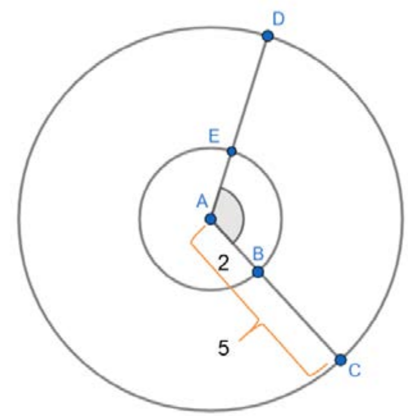
Discuss how arc length is related to circumference. Begin by calculating lengths of arcs that are simple fractional parts of a circle (e.g., $\frac{1}{6}$), and do this for circles of various radii so that students discover a proportional relationship. Students should discover that the ratio of the arc length to the circumference is proportional to the measure of the central angle to 360° or $\frac{arc\ length}{circumference\ (2\pi r)} = \frac{central\ angle\ \theta}{360°}$. The focus should be more on understanding the concepts of arc length rather than memorizing formulas.

In G.C.1 students discovered that all circles are similar. Therefore, the dilation of a circle by a scale factor of *k* results in a dilation of any fixed arc along the circle by the same scale factor, *k*, and the length of the dilated arc equals the product of the length of the arc and factor *k*. (This also connects to G.GMD.5 and 6 that discusses the proportional relationships between lengths and areas.)

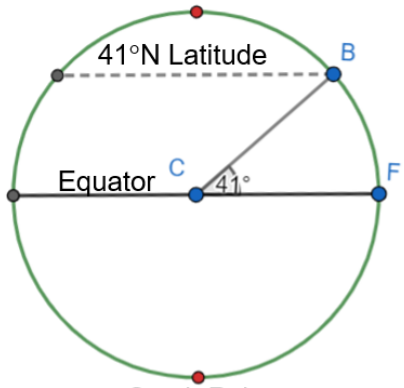
INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.C.5)

EXAMPLE

If the length of $\widehat{EB}$ is $\frac{4\pi}{3}$, how long is $\widehat{DC}$?



Discussion: Students should realize that since all circles are similar, the scale factor between two concentric circles shown is $\frac{AC}{AB}$ and the image of $\widehat{EB}$ is $\widehat{DC}$ whose length is $\left|\widehat{DC}\right| = \frac{AC}{AB}\left|\widehat{EB}\right|$ where $\left|\widehat{DC}\right|$ is the arc length of $\widehat{DC}$ and $\left|\widehat{EB}\right|$ is the arc length of $\widehat{EB}$. Therefore $\left|\widehat{DC}\right| = \frac{5}{2}\left(\frac{4\pi}{3}\right)$ or $\frac{10\pi}{3}$.



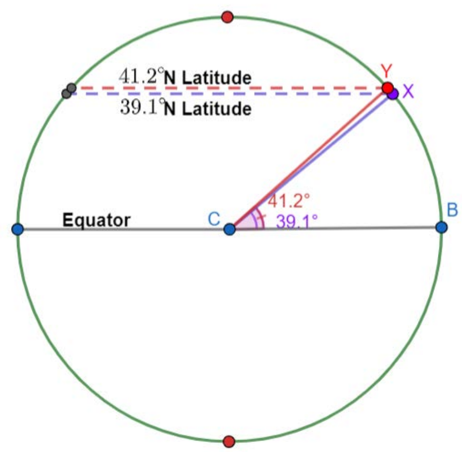
EXAMPLE

On the surface of Earth, the latitude of point B is equal to the measurement of the angle formed by point B, the center of Earth, C, and a second point F along the same line of longitude intersecting the equator. This relationship is depicted at the cross sectional at the left. The radius of the Earth is approximately 3,960 miles.

a. The latitude of Cincinnati, Ohio is 39.1°N. Approximately, how far away is Cincinnati from the equator?

b. Defiance, Ohio is due north of Cincinnati, Ohio. The latitude of Defiance, Ohio is 41.2°N. Approximately how far apart are the two cities?

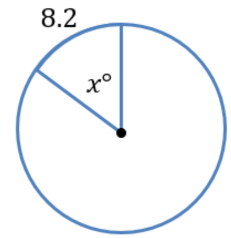
Discussion: In Part **a.,** to find the length of $\widehat{XB}$, students can use a proportion, $\frac{39.1°}{360°} = \frac{a}{2 \cdot 3960\pi}$, to find that the length of XB is 2,702.4 miles. In Part **b.,** to find the distance between the two cities, students need to find the arc length between the two altitudes. To find degree measure of the central angle, subtract the two given angles $41.2° - 39.1°$ to get $2.1°$. Then use a proportion to find the arc length: $\frac{2.1°}{360°} = \frac{a}{2 \cdot 3960\pi}$. After solving the proportion students should get approximately 145.1 miles between the two cities.



INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.C.5)

EXAMPLE

If the arc length intercepted by the central angle, *x*, is 8.2, what would be the length of the intercepted arc if you double *x*?



Discussion: Students should see the relationship between the central angle and the arc length. If the central angle doubles, the arc length will also double.

EXAMPLE

For a circle with each specified radius, use similarity to list the length of an arc intercepted by each of the following central angles.

	central angles							
radius	**30°**	**60°**	**90°**	**120°**	**150°**	**180°**	**270°**	**360°**
6	π							
12								
18								
24								
30								

a. Fill out the chart.
b. What patterns do you notice?
c. What conclusions can you make about arc length?

AREAS OF SECTORS OF CIRCLES

Discuss the relationship between the central angle and the area of a sector. Begin by calculating the areas of the sectors that are simple fractional parts of a circle (e.g. $\frac{1}{6}$), and do this for circles of various radii so that students discover a proportional relationship. Students should discover that the ratio of the area of the sector to the area of a circle is equal to the measure of the central angle to 360°, or $\frac{area\ of\ a\ sector}{area\ of\ a\ circle\ (\pi r^2)} = \frac{central\ angle\ \theta}{360°}$. The focus should be more on understanding the concepts of sector area rather than memorizing formulas.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.C.5)

Although, circle graphs are not mentioned in the standards, they are an application of sector area of a circle.

EXAMPLE

Take a survey of your classmates using categorical data with at least four categories. Create a circle graph illustrating your data. Then find the area of each sector. Make sure to show your calculations.

Discussion: After students finish the activity discuss with students why circle graphs are not always the best way to display data. Save the Pies for Dessert by Stephen Few of Perceptual Edge is an article that could be used for discussion.

TIP! Use coffee filters or paper plates to have students create sectors of circles.

EXAMPLE

Choose a local pizza company.

- **a.** How big is a slice of medium round pizza? How big is a slice of a large round pizza?
- **b.** Compare the cost per slice of a large round pizza versus a cost per slice of a medium round pizza.
- **c.** Which pizza is the better deal? Explain.

Discussion: This question is intentionally left open. Students can discuss the meaning of the word *big*. Does it mean longer, wider, more weight, more area, or heavier? Eventually settle on the idea of area of a sector. NPR has an article titled 74,476 Reasons You Should Always Get the Bigger Pizza and published on February 26, 2014 that could supplement this activity. Each pizza may have a different number of slices, styles of crusts, or thickness depending on crust style (factoring in volume). The suggestion is to explore how the best deal can be determined through examining the features of pizzas.

In G.C.1 students discovered the fact that all circles are similar, therefore the dilation of a circle by a scale factor of k results in area of a sector multiplied by a factor of k^2. (This also connects to G.GMD.5 and 6 that discusses the proportional relationships between lengths and areas.)

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.C.5)

EXAMPLE

For a circle with each specified radius, use similarity to list the area of a sector of a circle intercepted by each of the following central angles.

	central angles							
radius	**30º**	**60º**	**90º**	**120º**	**150º**	**180º**	**270º**	**360º**
6	3π							
12								
18								
24								
30								

a. Fill out the chart.
b. How does the area change when the central angle changes by a factor of *k*? Explain.
c. How does the area change when the radius changes by a factor or *k*? Explain.

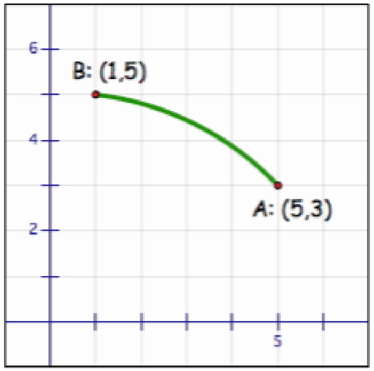
TIP! Sectors and segments are often used interchangeably in everyday conversation. Care should be taken to distinguish these two geometric concepts.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.C.5)

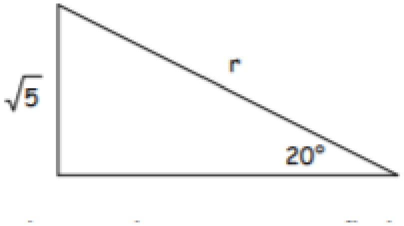
EXAMPLE

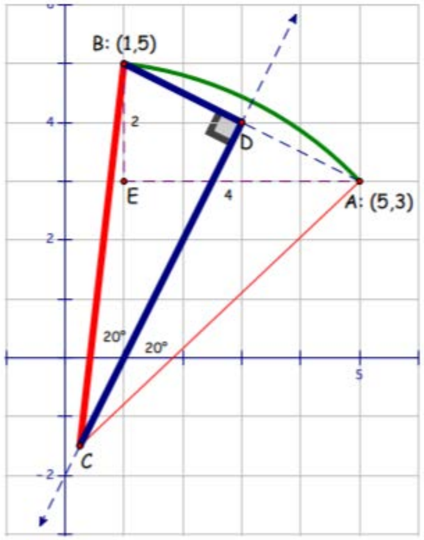
Arc to Area

The arc on the right has a measure of 40°, and its endpoints are at (1,5) and (5, 3). Find the area of the circle that contains the arc.



Discussion: This is an example of a problem of the week. There are several methods students can use to solve this problem. One method is to draw a chord connecting the two end points of the arc, and then draw CD that goes through the center of the circle and is a perpendicular bisector of a chord, which would cut the central angle of 40° in half. Then use the Pythagorean Theorem to find the length of $\overline{AB}$ (see purple dash lines) to get $2\sqrt{5}$ for the length of $\overline{AB}$. Since $\overline{BD}$ was formed by a perpendicular bisector, the length of $\overline{BD}$ is half the length of $\overline{AB}$ or $\sqrt{5}$. Triangle BDC is also a right triangle, so sine of half the central angle can be used to find the length of the radius $\overline{BC}$. Thus $\overline{BC} = \frac{\sqrt{5}}{\sin 20}$. So students can use $A = \pi r^2$ or $A = \pi\left(\frac{\sqrt{5}}{\sin 20}\right)^2$ which is approximately 134.28 square units. A more detailed explanation of this problem can be found on NCTM Problems of the Week Arc to Area.





INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.C.5)

Instructional Tools/Resources

These tools and resources may be helpful for instruction. Before use, they should be evaluated at the local level to determine their appropriateness for instruction.

Manipulatives/Technology

- Ruler
- Compass
- Protractor
- String
- Coffee Filters
- Paper Plates
- Computer dynamic geometric software (such as Geometer's Sketchpad®, Desmos, Cabri®, or GeoGebra®).
- Graphing calculators and other handheld technology such as TI-Nspire™.

Arc Length and Area of Sectors

- My Favorite Slice by CPalms is a lesson that uses pizzas as application to calculate the area of a sector.
- Measures of Arcs and Sectors by Rafranz D. is a lesson plan that connects arc length and circumference and a sector as a fractional part of a circle.
- Arc Lengths (Degrees) by Toh Wee Teck is a GeoGebra applet where students can explore arc length.
- Area of Sectors (Degrees) by Toh Wee Teck is a GeoGebra applet where students can explore area of sectors.
- Area Length and Sector Area by James Dunseith is a lesson where students apply similarity to find arc length and sector areas of circles with different radii.
- Save the Pies for Dessert by Stephen Few of Perceptual Edge is an article about why Pie Charts are not the best representative of graphical displays.
- Calculating Arcs and Areas of Sectors of Circle by Mathematics Assessment Project is a lesson where students compute perimeters, areas, and arc lengths of sectors and find the relationships between arc lengths and areas of sectors after scaling.

Curriculum and Lessons from Other Sources

- EngageNY, Geometry, Module 5, Topic B, Lesson 7: The Angle Measure of an Arc, Lesson 8: Arcs and Chords, Lesson 9: Arc Length and Areas of Sectors, and Lesson 10: Unknown Length and Area Problems are lessons that pertain to this cluster.
- Mathematics Vision Project, Geometry, Module 5: A Geometric Perspective has many lessons that pertain to this cluster.
- Georgia Standards of Excellence Curriculum Frameworks, Geometry, Unit 4: Circles and Volume has many lessons that pertain to this cluster.
- Illustrative Mathematics, Geometry, Unit 7, Lesson 8: Arcs and Sectors, Lesson 9: Part to Whole and Lesson 10: Angles, Arcs, and Radii are lessons that pertain to this cluster.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.C.5)

General Resources

- Arizona 7-12 Progression on Geometry is an informational resource for teachers.
- High School Coherence Map by UnboundED is a map of the relationships among high school mathematics standards. Please note: This resource is aligned to the Common Core not Ohio's Learning Standards. Although the tool is useful, it needs to be checked with respect to Ohio's Learning Standards.
- van Hiele Model of Geometric Thinking is a pdf created by ODE that summarizes the van Hiele levels.

References

- Cooper, L & Dennis, E. (August 2016). Estimating Earth's circumference with an app. *Mathematics Teaching in the Middle School, 22, (1*), 47-50.

STANDARDS	MODEL CURRICULUM (G.GPE.1)
Geometry **EXPRESSING GEOMETRIC PROPERTIES WITH EQUATIONS** **Translate between the geometric description and the equation for a conic section.** **G.GPE.1** Derive the equation of a circle of given center and radius using the Pythagorean Theorem; complete the square to find the center and radius of a circle given by an equation.	**Expectations for Learning** In middle school, students use the Pythagorean Theorem to find distances between points within the coordinate system. In the high school algebra, students complete the square to solve quadratic equations. In this cluster, students derive the equation of a circle using the Pythagorean Theorem. They also complete the square to find the center and radius of a circle. The student understanding of this cluster begins at van Hiele Level 2 (Informal Deduction/Abstractions) and moves to Level 3 (Deduction). **ESSENTIAL UNDERSTANDINGS** • The equation of a circle relates a fixed center, a fixed radius, and a set of variable points, which are the points on the circle. • Just as the distance formula is an application of the Pythagorean Theorem, so is the equation of a circle. **MATHEMATICAL THINKING** • Use precise mathematical language. • Discern and use a pattern or structure. • Make connections between concepts, terms, and properties within the grade level and with previous grade levels. • Justify relationships in geometric figures. • Use technology strategically to deepen understanding. • Solve routine and straightforward problems accurately. **INSTRUCTIONAL FOCUS** • Use the Pythagorean Theorem to derive the equation of a circle. • Given the equation of a circle that is not in standard form, find the center and radius of the circle by completing the square. *Continued on next page*

STANDARDS	MODEL CURRICULUM (G.GPE.1)
	Content Elaborations **OHIO'S HIGH SCHOOL CRITICAL AREAS OF FOCUS** • Geometry, Number 5, page 8 **CONNECTIONS ACROSS STANDARDS** • Prove that all circles are similar (G.C.1). • Prove theorems about triangles (G.SRT.4).

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.GPE.1)

Instructional Strategies

Note: The model curriculum is not a collection of lessons nor a full curriculum; it does not suggest pace, sequence, or amount of time spent on topics. It provides information about a topic related to the standards including ideas for examples, strategies for teaching, common misconceptions, and possible connections between topics. In addition, the emphasis of the model curriculum is on instruction not on assessment; therefore, the examples in the Instructional Strategies section are written for instructional purposes not necessarily as assessment items. Examples may need to be rewritten to accommodate the needs of each individual classroom.

Standards for Mathematical Practice
This cluster focuses on but is not limited to the following practices:
MP.2 *Reason abstractly and quantitatively.*
MP.4 *Model with mathematics.*
MP.5 *Use appropriate tools strategically.*
MP.6 *Attend to precision.*
MP.7 *Look for and make use of structure.*

VAN HIELE CONNECTION

In Math 1 students are expected to move from Level 2 (Informal Deduction/Abstraction) toward Level 3 (Deduction). in shapes and transformations.

Shapes

Van Hiele Level 2 can be characterized by the student doing some or all of the following with respect to shapes:

- noticing relationships within and between properties and figures;
- forming meaningful definitions;
- justifying reasoning using informal arguments;
- drawing logical maps and diagrams; and/or
- not yet understanding formal deduction.

Van Hiele Level 3 can be characterized by the student doing some or all of the following with respect to shapes:

- constructing proofs and understanding the necessity of proofs;
- understanding the role and relationships between of axioms, theorems, definitions, and postulates;
- differentiating between necessary and sufficient conditions; and/or
- making logical conclusions.

Transformations

Van Hiele Level 2 can be characterized by the student doing some or all of the following with respect to transformations:

- analyzing relationships within and between transformations;
- algebraically expressing transformations: reflections about the *x*- and *y*-axes, and rotations of 45°, 90°, and 180° about the origin; *Note: In high school, students should be given an opportunity to experience rotations around any point (not just the origin) and onto any angle, not just multiples of 90. Also, reflections are not limited to the x-and y-axes but could be about any line.*
- understanding and justifying how to use perpendicular bisectors of corresponding points to locate the turn center of a rotation;
- understanding how the composition of two reflections can be a translation or a rotation; and/or
- following a rationale if shown proof.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.GPE.1)

Van Hiele Level 3 can be characterized by the student doing some or all of the following with respect to transformations:

- Understanding and creating formal proofs using transformations.

RELATING THE DISTANCE FORMULA AND/OR THE PYTHAGOREAN THEOREM TO THE EQUATION OF A CIRCLE

The Instructional Strategies for G.GPE.4-7 address how to use the Pythagorean Theorem to derive the Distance Formula. It is up to a district to decide whether or not deriving the Distance Formula is best placed in G.GPE.7 or G.GPE.1. Regardless of where the Distance Formula is introduced, in this standard, students should use the Distance Formula to derive the equation of a circle. Review the definition of a circle as a set of points whose distance from a fixed point is constant.

EXAMPLE

Part 1.

a. If a circle has a radius of 3, how many circles can you make on a coordinate grid?
b. What information do you need to write the equation of a specific circle on the coordinate plane?

Part 2

a. If a circle has a center of $(4, -1)$, how many circles can you make on a coordinate grid?
b. What information do you need to write the equation of a specific circle on a coordinate plane?

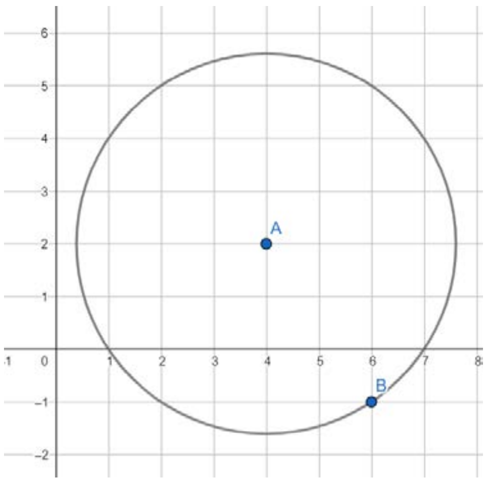
Discussion: Students should come to the conclusion that in order to define a circle both a center and radius needs to be stated.

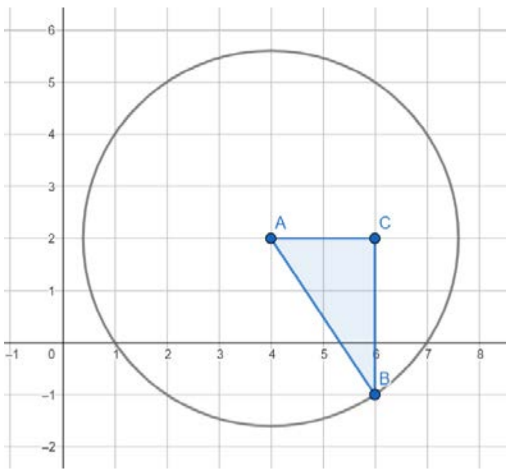
EXAMPLE

Part 1

a. How could you find the length of a radius of a circle in a coordinate plane?
b. Draw a circle on the coordinate plane and find the length of the radius of the circle.

Discussion: In our example on the right a student drew a circle with center $(4, 2)$ and a point on the circle $(6, -1)$. Students should connect the Pythagorean Theorem or the Distance Formula to the length of the radius. The length of $\overline{AB}$ is $\sqrt{(6-4)^2 + (-1-2)^2} = r$ to get $\sqrt{2^2 + (-3)^2} = \sqrt{13}$, so the radius is $\sqrt{13}$.

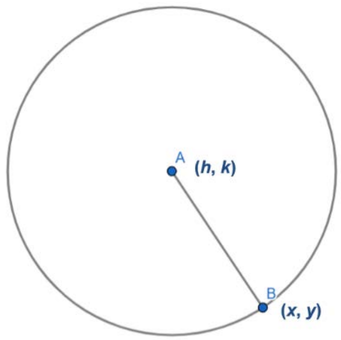




INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.GPE.1)

Part 2

Using the method you used in Part 1, create a formula for finding the radius of a circle with the center at (h, k) and with any arbitrary point on a circle is (x, y).



Discussion: Students should recognize that they can just substitute (*x*, *y*) and (*h*, *k*) for (x_1, y_1) and (x_2, y_2) into the Distance Formula, $D = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ respectively. So, the equation of the circle is $\sqrt{(x-h)^2 + (y-k)^2} = r$ or $(x-h)^2 + (y-k)^2 = r^2$.

Part 3

- **a.** Using the formula you derived in Part 2, write the equation of a circle with the center (4, 2) and a radius of $\sqrt{13}$.
- **b.** How is that similar to what you did in Part 1?
- **c.** Write an equivalent equation for the equation you wrote in part **a.**

Discussion: Draw attention the different forms of the equation of a circle such as center-radius (standard) form and general form.

TIP! The Distance Formula involves subscripted variables whose differences are squared and then added. The notation and multiplicity of steps can be a serious stumbling block for some students. Therefore, it is important that students conceptually understand the Distance Formula and connect it to the Pythagorean Theorem.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.GPE.1)

EXAMPLE

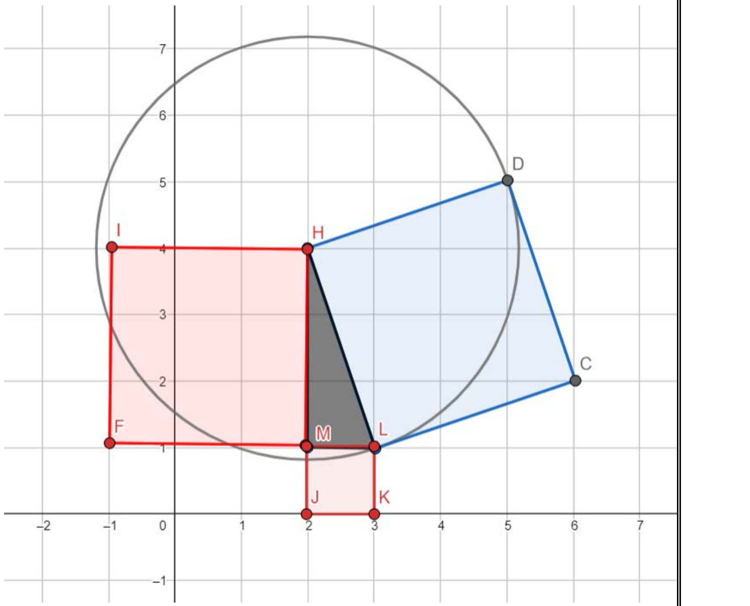
Use the diagram on the right to answer the questions:

a. What's the relationship among the red squares and the blue square?

b. The equation of the above circle is $(x-2)^2+(y-4)^2=10$.

- **i.** Which part of the diagram represents $(x-2)^2$?
- **ii.** Which part of the diagram represents $(y-4)^2$?
- **iii.** Which part of the diagram represents 10?
- **iv.** Which part of the diagram represents x?
- **v.** Which part of the diagram represents y?
- **vi.** Which part of the diagram represents $(2,4)$?
- **vii.** Which part of the diagram represents 2?
- **viii.** Which part of the diagram represents 4?
- **ix.** What relationship does point M have to point L?
- **x.** What relationship does point M have to point H?
- **xi.** Which part of the diagram represents $(x-2)$?
- **xii.** Which part of the diagram represents $(y-4)$?

c. Using geometric software create a circle and radius. Create a right triangle where the radius is the hypotenuse. Then create squares on each side of the triangles. Drag the endpoint around and see what happens.

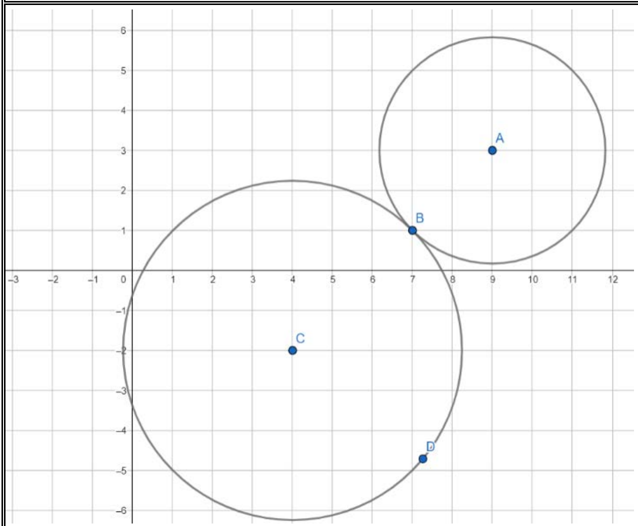


Discussion: Students should note that the sum of the areas of the red squares equals the area of the blue squares.

- **i.** $(x-2)^2$ is the area of the red square $MLKJ$.
- **ii.** $(y-4)^2$ is the area of the red square $HMFI$.
- **iii.** 13 is the area of the blue square $HDCL$.
- **iv.** x is the x -coordinate of point L (or technically any point on the circle).
- **v.** y is the y -coordinate of point L (or technically any point on the circle).
- **vi.** (2, 4) is the center of the circle.
- **vii.** 2 is the x-coordinate of the center of the circle and the distance point H is from the y-axis.
- **viii.** 4 is the y-coordinate of the center of the circle and distance point H is from the origin.
- **ix.** Point M has the same y-coordinate as L
- **x.** Point M has the same x-coordinate as H.
- **xi.** $(x-2)$ is the length of $\overline{LM}$
- **xii.** $(y-4)$ is the length of $\overline{HM}$.

Note: The bullet points above are not meant to be problems completed on a worksheet, but rather points of discussion that support the visual for the relationship between the Pythagorean Theorem and the equation of a circle.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.GPE.1)



EXAMPLE

What is the distance between the centers of the circles in the diagram?

Discussion: Draw attention to the fact that points $A, B,$ and C are collinear, and point B is between points A and C. Therefore, students could use the distance formula to find the distances between point A and point C. Some students may want to find the length of radius AB and radius BC and add them together. Discuss why either method works. Discuss how they would find the distance if the points are not collinear.

EXAMPLE

Write an equation of a circle where $(-8, 3)$ and $(-2, 1)$ are endpoints of the diameter of the circle.

Discussion: First, students can find the center by using the midpoint formula or some other method. Then, they can write the equation of the circle using the center and radius.

EXAMPLE

Find the radius of a circle centered at $(-2, 6)$ where one of the points on the circle is $(1, 3)$.

Discussion: Students can utilize the equation of the circle, the Pythagorean Theorem, or the Distance Formula to find the radius.

EXAMPLE

Write an equation of a circle that is tangent to $y = -1, y = -7$ and $x = 1$.

Discussion: There are two possible circles that satisfy the given conditions. Students may want to use a coordinate grid to help them visualize the process. One circle would be centered at $(4, -4)$ to the right of $x = 1$ and the other circle would be centered at $(-2, -4)$ to the left of $x = 1$. Both circles will have a radius of 3 that is half the distance from $y = -1$ and $y = -7$.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.GPE.1)

EXAMPLE

Part 1

What could be the coordinates of a point on a circle with a center at the origin and a radius of $\sqrt{26}$?

Part 2

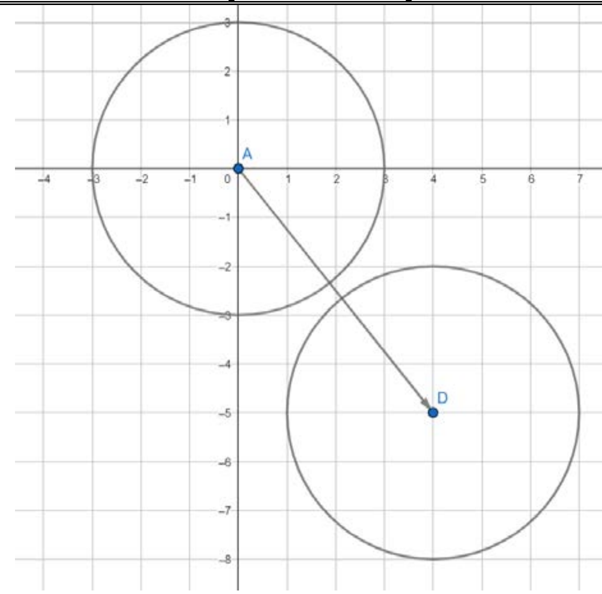
What could be coordinates of a point on a circle with a center of $(-4, 2)$ and a radius of $\sqrt{26}$?

Discussion: This is an example of an open-ended problem that could be solved a variety of ways with multiple solutions. One way that students can approach this is by writing an equation of the circle: $(x + 4)^2 + (y - 2)^2 = 26$. Students could visualize the Pythagorean Theorem and realize that 26 is the sum of two perfect squares. The perfect squares less than 26 are 1, 4, 9, 16, and 25. Only 25 and 1 equal 26. Therefore, $(x + 4)^2$or $(y - 2)^2$ has to equal 25 and the other squared binomial has to equal 1. If students chose to set $(x + 4)^2 = 25$ and $(y - 2)^2 = 1$, then $x = 1, -9$ and $y = 1, 3$. Therefore coordinates of a point on the circle could be $(1, 1), (1, 3), (-9, 1)$ or $(-9, 3)$. If students choose to set $(x + 4)^2 = 1$ and $(y - 2)^2 = 25$, then $x = -3, -5$ and $y = 7, -3$. Therefore the coordinates of the point on the circle could be $(-3, 7), (-3, -3), (-5, 7), (-5, -3)$. *Note: Students may choose to find non-integer coordinates.*

The formula for an equation of a circle can also be connected to translations of a circle using coordinate notation. A translation can represented by $(x, y) \rightarrow (x + 2,\ y - 3)$ to describe a figure that translates two units to the right in the horizontal direction and three units down in the vertical direction.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.GPE.1)

EXAMPLE



a. Draw a circle on the coordinate plane with a radius of your choosing and centered at the origin.
b. Write the equation of the circle you drew for part **a.**
c. Translate your circle 4 units to the right and 5 units down.
d. Describe your translation using coordinate notation.
e. Write the equation of a circle for the image you drew in part **c.**
f. Compare the coordinate notation of your translated image in part **d.** to the equation of the circle in part **e.** What do you notice?
g. Perform another translation of the original circle centered at the origin to anywhere in the coordinate plane.
h. Describe your translation in part **g.** using coordinate notation.
i. Write the equation of the translated circle from part **g.**
j. Compare the coordinate notation of your translated image in part **h.** to the equation of the circle in part **i.** What do you notice?
k. Explain how the equation of the circle is really just a way of describing a translated image of a circle centered at the origin. Look for connections between the equations of the two circles and the translations of the two circles?

Discussion:

a. A student may choose a radius of 3.
b. The equation of the circle for part **a.** would be $x^2 + y^2 = 9$.
d. Coordinate notation describing the translation in part **c.** would be $(x, y) \to (x + 4, y - 5)$.
e. The equation of the translated circle would be $(x - 4)^2 + (y + 5)^2 = 9$.
f. Students should recognize that there is a 4 and 5 present both in the coordinate notation of the transformation and the equation of the circle, but their corresponding signs are different.
g. By translating it a second time, they should see that the same pattern emerges.
h.-k. Since students choose their own translations for part **g.**, they could discuss with their classmates that this is a consistent finding. The big question then becomes "Why are the signs different?" For example, if the students used our example, they could make the observation that although the plane moves 4 units to the right (+4) and 5 units down (–5) to get the translated image back to the center, it would have to move 4 units to the left (–4) and 5 units up (+5).

Students oftentimes incorrectly think that the signs in front of *h* and *k* in the equation of the circle correspond to the signs of the coordinate of the center of the circle. Allow students to discover that they actually indicate how far the image is translated to a congruent circle centered at the origin. Therefore, the signs are opposite of the actual center of the circle.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.GPE.1)

EXAMPLE

The local hardware store sells pop-up sprinkler heads that have an adjustable 6 ft. to 15 ft. radius for $2.96 each. If your backyard measures 60 ft. by 100 ft., how many sprinklers should you place on your lawn to get maximum coverage and by spending the least amount of money? Where would you place the sprinklers and why?

- Illustrate the lawn in the first quadrant of the coordinate plane with one of the corners at the origin.
- Illustrate the placement of the sprinklers by drawing circles on your coordinate plane with the center and radius marked. Where would you place the sprinklers and why?
- Write an equation of the circle to represent each setting.
- How much will it cost?
- Compare your work to your classmates and discuss your findings. Justify your reasoning.

Discussion: This is a modeling problem, so there is more than one correct answer. Not only do students have to figure out where to place the sprinklers and decide on the radius of each sprinkler, they also have to determine if it is better to overlap sprinklers, so the lawn is completely watered or if it is better to leave small gaps and save money by not wasting water or buying extra sprinklers. Students may discuss if it is appropriate to water over the boundaries of their backyard which may hit a neighbor's lawn, a building, a garden, or a sidewalk. Teachers and students can also utilize Google maps to zoom in and out of farmland, so they can see this concept implemented at a larger scale.

FINDING THE CENTER AND RADIUS OF A CIRCLE BY COMPLETING THE SQUARE

Starting with any quadratic equation in two variables (x and y) in which the coefficients of the quadratic terms are equal, complete the squares in both x and y and obtain the equation of a circle in standard form. In Algebra 1, students used completing the square to solve quadratic equations and write quadratic equations in vertex form. They may need to review the algebraic method of completing the square and connect it to a geometric model. algebra tiles are helpful for students to visualize this problem. Connect with standards A.SSE.3, A.REI.4, and F.IF.8. See Algebra 1 Model Curriculum A.SSE.3-4 and A.REI.4 for more information on completing the square with algebra tiles.

TIP!

The method of completing the square is a multi-step process that takes time to assimilate. A geometric demonstration of completing the square can be helpful in promoting conceptual understanding.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.GPE.1)

EXAMPLE

a. How does the equation of the circle relate to the Pythagorean Theorem?

b. The areas of all three squares, the center, and the radius can be seen in center-radius form, $(x-h)^2+(y-k)^2=r^2$. If the equation, $x^2+y^2-2x+6y=3$, is in general form for the equation of the circle, can you see the expression of the area of the squares in this form? Can you see the coordinate of the center or the radius of the circle in this form?

c. Show the area of the squares with algebra tiles? Are the squares complete? If not, how can you complete them? Discuss as a class or within your group.

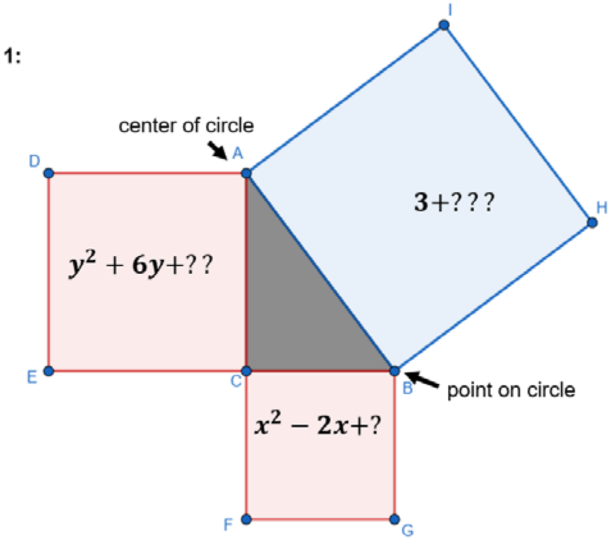
d. Find the length of each side of the triangle using the squares.

e. Write your equation in center-radius form.

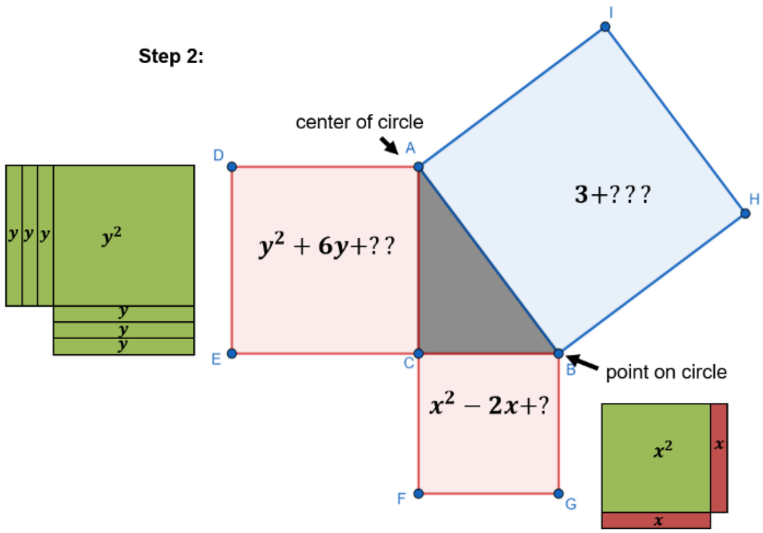
f. Identify the center and radius of the circle.

g. Draw the circle on the coordinate plane.

Step 1:



Step 2:



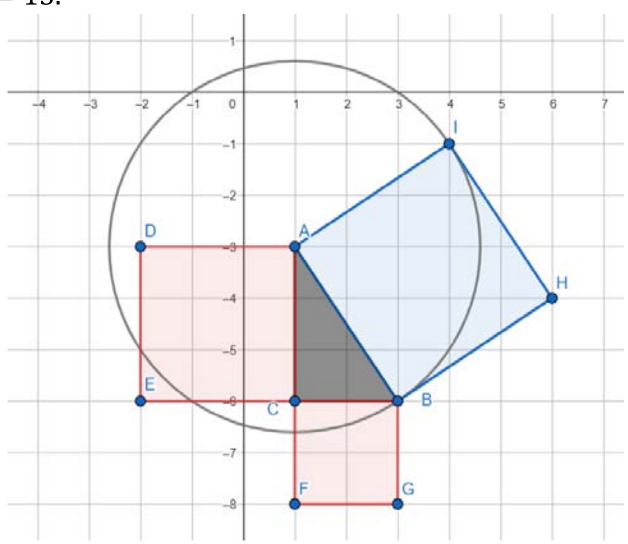
Discussion:

The purpose of this example is to use completing the square to convert an equation of a circle in general form to the center-radius form.

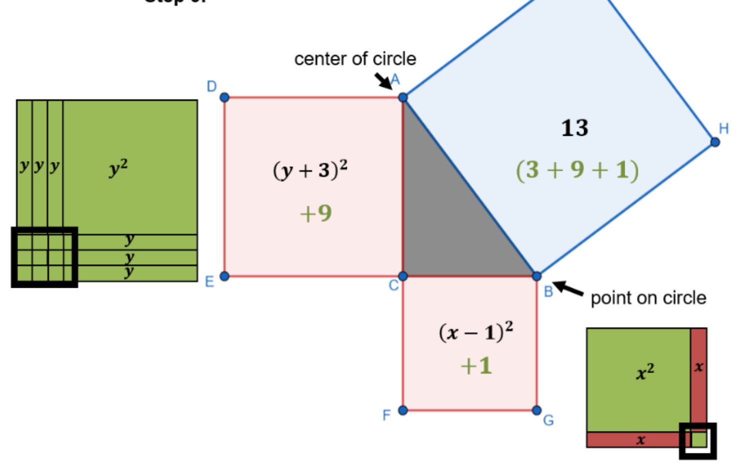
- In part **a.** students should be able to state that a right triangle can be formed where the hypotenuse is the radius. Thus, $(x-h)^2$ would represent the area of the square formed on one leg, and $(y-k)^2$ would represent the area of the square formed on the other leg, and r^2 would represent the area of the square formed on the hypotenuse. In group discussion, reinforce the fact that three squares need to be formed by each side of the triangle so that they satisfy the Pythagorean Theorem.
- In part **b.** students should realize that the general form does not lend itself to seeing the areas of the squares or the center or the radius.
- In part **c.** students should come up with the idea of completing the square. If they are struggling to get there, connect the concept to the visual proof of the Pythagorean Theorem where there is a square on each of the sides of the right triangle. Explain to students that the center of the circle and a point on the circle are endpoints of the hypotenuse. If students are having trouble visualizing it, create a circle with geometric software where the hypotenuse of a right triangle is the radius, and then rotate the triangle around the center of the circle.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.GPE.1)

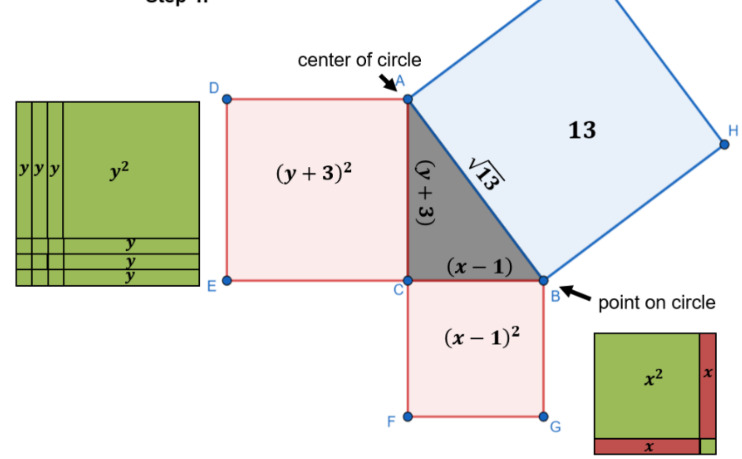
- In part **d.** ask students which terms would go in which square? The x-terms should be placed in the square below the horizontal line of the triangle, and the y-terms should be placed in the square next to the vertical line of the triangle, and the constant term should be placed in the square of the hypotenuse (See Step 1). Use algebra tiles to illustrate completing the square (See Step 2). Ask students how many unit squares are needed to complete each of the two smaller squares. Students should see that 9 more unit squares are needed for the red square that is represented by $y^2 + 6y + ??$ and 1 unit square is needed for red square that is represented by $x^2 - 2x + ?$. Since the sum of the areas of both red squares along the legs has to equal the area of the blue square along the hypotenuse, the extra area added to the red squares has to be added to the blue square (Step 3). Then students can see that the area of the blue square is $3 + 9 + 1 = 13$. They can also see that $x^2 - 2x + 1$ and $y^2 + 6y + 9$ are perfect squares, where side lengths are $(x - 1)$ and $(y + 3)$ respectively, so the area of the squares can be represented by $(x + 1)^2, (y + 3)^2$, , and 13 (Step 4).
- In part **e.**, students can write the equation of the circle in center-radius form as, $(x + 1)^2 + (y + 3)^2 = 13$.
- In part **f.**, student should identify the center as $(1, 3)$ and the radius as $\sqrt{13}$.



Step 3:



Step 4:



INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.GPE.1)

Students should also understand that for this course radii must be greater 0. In later courses students will explore circles with negative radii which are imaginary.

FINDING THE POINT(S) OF INTERSECTION OF A LINE AND CIRCLE

In A.REI.7 students found solutions to a system involving a linear equation and an equation of a parabola using graphing and substitution. Extend that concept to a system involving equations of lines and equations of circles both algebraically and by graphing. Students can also explore solving systems using a table on a graphing utility.

EXAMPLE

Find the intersection point(s) of graphs represented by the equations $y = x^2 + 4x + 1$ and $y = -x - 3$.

Discussion: The purpose of this example is to determine the ways a circle and a line can intersect. Build on this example so students can see that they can have 0, 1, or 2 common solutions.

EXAMPLE

- **a.** How many possible solutions are there of a system involving an equation of a line and a circle. Explain.
- **b.** How can you tell how many solutions a system involving an equation of a circle and a line would have?
- **c.** Find the intersection point of $(x+3)^2 + (y-2)^2 = 10$ and $y = 3x + 1$ algebraically. Then check your work using a graphing utility.

Discussion: The purpose of this activity is for students to determine the ways a circle and a line can intersect. They should be able to find the point(s) of intersection between a circle and a line.

Ask students how to graph a circle using a graphing utility. Have students rearrange the equations by connecting to A.REI.6. If students are using a graphing utility other than Desmos, lead them to discover that it will take two different equations $y_1 = \sqrt{r - (x-h)^2} - k$ and $y_2 = -\sqrt{r - (x-h)^2} - k$ to create a circle graph.

TIP!

Be precise with language of equations versus language of functions because not every quadratic equation represents a function, e.g., a circle is not a function.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.GPE.1)

Instructional Tools/Resources

These tools and resources may be helpful for instruction. Before use, they should be evaluated at the local level to determine their appropriateness for instruction.

Manipulatives/Technology

- Dynamic geometric software (Geometer's Sketchpad®, Cabri®, Desmos, or GeoGebra®)
- Graph paper

Equation of Circles

- Explaining the Equation of a Circle by Illustrative Mathematics is a task that has students derive the formula for an arbitrary circle in a plane.
- Slopes and Circles by Illustrative Mathematics is a task that has students figure out algebraically that point X is on the circle of diameter AB whenever $\angle AXB$ is a right angle.
- Earthquake is a Desmos activity where the goal is to find out whose houses are within five miles from the epicenter. This activity could be extended by having the student write the equation of a circle.
- Back to School: Deriving Circles with Desmos by Fishing4Tech is a blog that describes an activity using equation of circles and the context of earthquakes. Untitled Graph is a Desmos applet that accompanies this activity where students can write the equation of a circle for the first, second, and third waves of an earthquake.
- Intro: Equation of Circles by Lauren Olson is a Desmos activity that has students write equations of circles in standard and general form to match various descriptions and constraints.
- Sorting Equations of Circles 1 and Sorting Equation of Circles 2 are lessons from the Mathematics Assessment Project where students use the Pythagorean Theorem to derive the equation of a circle, translate between the geometric features of circles and their equations, and sketch a circle from its equation.

Applications using the Equation of a Circle

- All Wet by NCTM is a Problem of the Week where students explore a sprinkler's radius. It may have to be slightly adapted to stress the focus of this cluster. *NCTM now requires a membership to view their lessons.*
- How Can We Water All of That Grass by Robert Kaplisky is a modeling lesson involving sprinklers. It may have to be slightly adapted to stress the focus of this cluster.
- Landscape Irrigation: Circles upon Circles by Conceptual Learning Circles is a modeling lesson involving sprinklers. The extension has students write equations for the circles drawn.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.GPE.1)

Translation of Circles

- Translation of Circles is a GeoGebra activity that relates the translation of circles to the equation of a circle.

Algebra Tiles

- Algebra Tiles Applet by NCTM Illuminations is a link to a virtual algebra tiles applet. *NCTM now requires a membership to view their lessons.*
- Virtual Algebra Tiles is an applet from Michigan Virtual University that allows students to use algebra tiles. This applet allows for positive and negative representation of the tiles.
- CPM Tiles is an applet from the CPM Educational Program that allows students to use algebra tiles. The benefit of this applet is that students can change the dimensionality of x and y. However, it is limited by not allowing for a negative representation of the tiles.
- Algebra tile templates on the SMART Exchange has a variety of useful models that teachers can use if they have access to a SMART Board.

Curriculum and Lessons from Other Sources

- EngageNY, Geometry, Module 5, Topic D, Lesson 17: Writing an Equation for a Circle, Lesson 18: Recognizing Equations of Circles, Lesson 19: Equations for Tangent Lines to Circles are lessons that pertain to this cluster.
- Georgia Standards of Excellence, Geometry, Unit 5: Geometry and Algebraic Connections has many tasks that pertain to this cluster.
- Mathematics Vision Project, Geometry, Module 6: Connecting Algebra and Geometry has several tasks related to this cluster.
- Illustrative Mathematics, Geometry, Unit 6, Lesson 4: Distances and Circles, Lesson 5: Squares and Circles and Lesson 6: Completing the Square are lessons that pertain to this cluster.

General Resources

- Arizona 7-12 Progression on Geometry is an informational resource for teachers. This cluster is addressed on pages 17-18.
- High School Coherence Map by UnboundED is a map of the relationships among high school mathematics standards. Please note: This resource is aligned to the Common Core not Ohio's Learning Standards. Although the tool is useful, it needs to be checked with respect to Ohio's Learning Standards.
- van Hiele Model of Geometric Thinking is a pdf created by ODE that summarizes the van Hiele levels.

References

- Common Core Standards Writing Team. (2016, March 24). *Progressions for the Common Core State Standards in Mathematics (draft). Grades 7–8, HS, Geometry*. Tucson, AZ: Institute for Mathematics and Education, University of Arizona.

STANDARDS	MODEL CURRICULUM (G.GPE.4-7)
Geometry **EXPRESSING GEOMETRIC PROPERTIES WITH EQUATIONS** **Use coordinates to prove simple geometric theorems algebraically and to verify specific geometric statements.** **G.GPE.4** Use coordinates to prove simple geometric theorems algebraically and to verify geometric relationships algebraically, including properties of special triangles, quadrilaterals, and circles. *For example, determine if a figure defined by four given points in the coordinate plane is a rectangle; determine if a specific point lies on a given circle.* (G, M2) **G.GPE.5** Justify the slope criteria for parallel and perpendicular lines, and use them to solve geometric problems, e.g., find the equation of a line parallel or perpendicular to a given line that passes through a given point. **G.GPE.6** Find the point on a directed line segment between two given points that partitions the segment in a given ratio. **G.GPE.7** Use coordinates to compute perimeters of polygons and areas of triangles and rectangles, e.g., using the distance formula.★	**Expectations for Learning** In middle school, students find the distance between two points in a coordinate system; work with linear functions; solve linear equations; and apply the Pythagorean Theorem in the coordinate system. In addition, they use square root symbols to represent solutions to equations and they evaluate square roots of rational numbers. In this cluster, students use the coordinate system to justify slope criteria for parallel and perpendicular lines; partition line segments proportionally; and compute perimeters and areas of geometric figures. These strategies are used for proof of geometric relationships and properties. The student understanding of this cluster begins at van Hiele Level 2 (Informal Deduction/Abstractions) and moves to Level 3 (Deduction). **ESSENTIAL UNDERSTANDINGS** • Coordinate proof is a method that uses algebraic techniques to prove geometric theorems and properties. • Properties of geometric figures, especially special quadrilaterals, can be proven on a coordinate plane using lengths of segments, slopes of lines, and equations of lines. • Coordinate proof can be used to prove that figures are congruent or similar. • The slopes of parallel lines are equal, and the product of the slopes of perpendicular lines is –1, except for horizontal and vertical lines. • Partitioning a line segment into a given ratio is an application of similar triangles. *Continued on next page*

STANDARDS	MODEL CURRICULUM (G.GPE.4-7)
	Expectations for Learning, continued **MATHEMATICAL THINKING** • Use accurate mathematical vocabulary to represent geometric relationships. • Make connections between terms and formulas. • Recognize, apply, and justify mathematical concepts, terms, and their properties. • Compute using strategies or models. • Use formal reasoning with symbolic representation. • Determine reasonableness of results. • Solve multi-step problems accurately. • Discern and use a pattern or structure. **INSTRUCTIONAL FOCUS** • Know and use the distance formula. • Use coordinates to prove simple geometric theorems algebraically. • Verify geometric relationships algebraically. • Justify the slope criteria for parallel and perpendicular lines. • Write equations of parallel and perpendicular lines. • Solve geometric problems using the slopes of parallel and perpendicular lines. • Partition a line segment given a ratio. • Use coordinates to compute perimeters and areas. **Content Elaborations** **OHIO'S HIGH SCHOOL CRITICAL AREAS OF FOCUS** • Geometry, Number 4, page 7 • Geometry, Number 5, page 8 **CONNECTIONS ACROSS STANDARDS** • Know precise definitions (G.CO.1). • Prove geometric theorems (G.CO.9-10). • Understand and apply theorems about circles (G.C.1-4). • Prove theorems involving similarity (G.SRT.4-5). • Understand the relationships between lengths, areas, and volumes in similar figures (G.GMD.6). • Apply geometric concepts in modeling situations (G.MG.3).

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.GPE.4-7)

Instructional Strategies

Note: The model curriculum is not a collection of lessons nor a full curriculum; it does not suggest pace, sequence, or amount of time spent on topics. It provides information about a topic related to the standards including ideas for examples, strategies for teaching, common misconceptions, and possible connections between topics. In addition, the emphasis of the model curriculum is on instruction not on assessment; therefore, the examples in the Instructional Strategies section are written for instructional purposes not necessarily as assessment items. Examples may need to be rewritten to accommodate the needs of each individual classroom.

Standards for Mathematical Practice
This cluster focuses on but is not limited to the following practices:
MP.2 *Reason abstractly and quantitatively.*
MP.4 *Model with mathematics.*
MP.5 *Use appropriate tools strategically.*
MP.6 *Attend to precision.*
MP.7 *Look for and make use of structure.*

This cluster connects geometric to algebraic concepts by using the coordinate plane. The emphasis is on justification. The main skill that should be gleaned from these standards is for students to be able to explain reasons for their thinking and why/how something is true, much like in the ELA writing standards where *evidence* must be included for any claim. Students continue to use precise language and relevant vocabulary to justify steps in their work and construct viable arguments that defends their method of solution. See clusters G.CO.9-11 and G.SRT.4-5 for expectations surrounding proof.

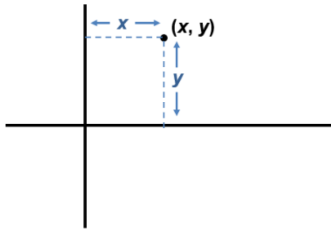
Formerly, students found the distance between two points using the Pythagorean Theorem. Now, using the Pythagorean Theorem, students should understand how to derive the Distance Formula. They should be able to explain how the Pythagorean Theorem, the slope formula, and the Distance Formula are connected.

TIP!

The Euclidean Distance Formula is the shortest possible path between two points in a plane. There are other distance formulas such as geodesic distance which is the shortest path along a curved surface such as the Earth. In this document, the Distance Formula will reference the Euclidean Distance Formula, but it might be helpful to have a discussion so that students are aware that other types of distance formulas exist.

TIP!

Show the dispalcements of points within a quadrant rather than on the *x*- and *y*- axes, so students can view the the points' distance *from* the axes in addition to its distances *along* the axes. Viewing an ordered pair as displacement of a point from the axes is more useful in proving theorems than viewing the ordered pair as horizontal and vertical distances along the axes.



INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.GPE.4-7)

VAN HIELE CONNECTION

In Geometry students are expected to move from Level 2 (Informal Deduction/Abstraction) toward Level 3 (Deduction). Van Hiele Level 2 can be characterized by the student doing some or all of the following:

- analyzing relationships within and between transformations;
- algebraically expressing transformations: reflections about the *x*- and *y*-axes, and rotations of 45°, 90°, and 180° about the origin; *Note: In high school, students should be given an opportunity to experience rotations around any point (not just the origin) and onto any angle, not just multiples of 90. Also, reflections are not limited to the x-and y-axes but could be about any line.*
- understanding and justifying how to use perpendicular bisectors of corresponding points to locate the turn center of a rotation;
- understanding how the composition of two reflections can be a translation or a rotation; and/or
- following a rationale if shown proof.

Van Hiele Level 3 can be characterized by the student doing some or all of the following:

- understanding and creating formal proofs using transformations.

See the van Hiele pdf on ODE's website for more information about van Hiele levels.

TIP!

It is important when connecting algebra concepts with geometry concepts that students do not lose sight of the underlying geometric meanings and reduce the geometric concepts to procedures.

GEOMETRIC THEOREMS WITH COORDINATES

TIP!

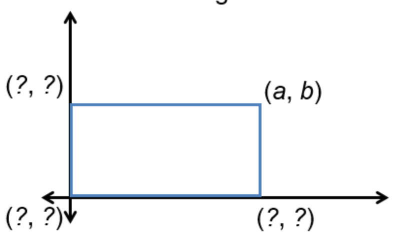
When approaching a coordinate proof with polygons, encourage students to choose a convenient placement of the axes on the coordinate place such as the origin or by placing a vertex on one of the axes or by using lines of symmetry. This will make calculation easier. It may be helpful to have students give the missing coordinates without introducing any new variables.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.GPE.4-7)

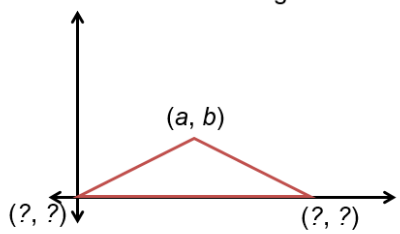
EXAMPLE

Find the missing coordinates. Only use the variables given in the diagram.

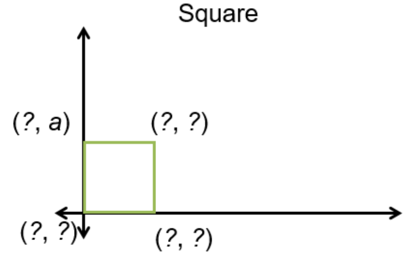
Rectangle



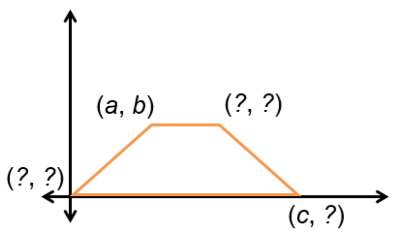
Isosceles Triangle



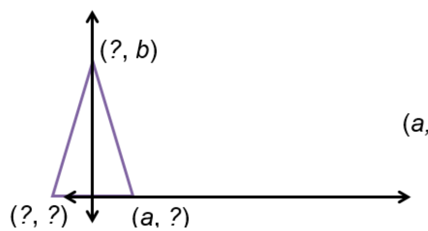
Square



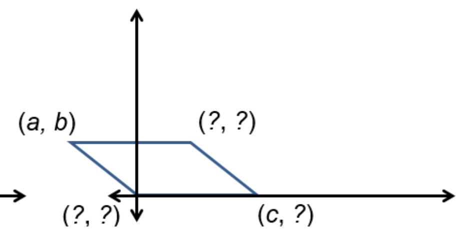
Isosceles Trapezoid



Isosceles Triangle



Parallelogram



Taken from Houghton Mifflin Geometry.

The part of geometry where students use algebraic proofs to prove geometric theorems is called analytic geometry. Students utilize the coordinate plane to prove geometric theorems algebraically. This includes the properties of special triangles among others. Students need to show all algebraic steps with justification that prove simple geometric theorems.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.GPE.4-7)

EXAMPLE

Part 1

- In your group, create a rectangle in the coordinate plane. Create three more rectangles. What are the coordinates of each vertex of your rectangle?
- In your group, create a quadrilateral that is not a rectangle in the coordinate plane. Create three more quadrilaterals that are not rectangles. What are the coordinates of each vertex of your quadrilateral?
- Discuss what you notice about the coordinate points that are vertices of rectangles compared to the coordinate points that are vertices of non-rectangles.
- Can you come up with any criteria to determine if four coordinate points make a rectangle?

Discussion: After students experiment in the coordinate plane with various groups of four points that make rectangles and those that do not, they may falsely come to the conclusion that two pairs of points share the same *x*-coordinate and that two pairs of vertices of a quadrilateral share the same *y*-value such as (1, 2), (1, 4), (6, 4), (6, 2). This sets them up for the thinking required in Part 2.

Part 2

- Do the coordinates (5, 6), (7, 4), (9, 10), and (11, 8) make a rectangle? Explain, justifying your answer using the coordinate plane.
- Do the criteria that you wrote to create rectangles in Part 1 hold true? If not, revise your criteria to determine if four points create a rectangle.
- Given four points in a coordinate plane, find a complete set of criteria to determine if these points create a rectangle.

Discussion: At this point some students may come up with the idea that the opposite sides have to have the same slope and that opposite sides are congruent. However, Part 3 pushes them to discover that a parallelogram that is not a rectangle also has opposite congruent sides with equal slopes.

Part 3

- Do (2, –1), (3, 2), (5, 0), and (4, –3) create a rectangle? Justify your answer with the coordinate plane.
- Does the criteria that you created for rectangles in Part 2 hold true? If not, revise your criteria to determine if four points create a rectangle.

Discussion: Part 3 pushes students to think that they must have some kind of criteria to create right angles in a rectangle. Therefore, not only do the opposite sides have to be parallel and thus have equal slopes, but the adjacent sides must be perpendicular, so they must have slopes that are opposite reciprocals.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.GPE.4-7)

Use slopes and the Distance Formula to solve problems about figures in the coordinate plane such as the following:

- Given three noncollinear points, decide if they are vertices of an isosceles, equilateral, or right triangle?
- Given four points, decide if they are vertices of a parallelogram, a rectangle, a rhombus, or a square?
- Given the equation of a circle and a point, does the point lie outside, inside, or on the circle?
- Given the equation of a circle and a point on it, find an equation of the line tangent to the circle at that point.

Students could explore the medians of a triangle intersecting at a point. This could be done using dynamic geometric software or patty paper. See the Instructional Tools/Resources for different applets and resources. However, students could also use coordinate geometry to prove that the medians of a triangle are concurrent. Students could also explore the orthocenter, the circumcenter, and the incenter of a triangle using geometric software and use coordinate geometry to prove their conjectures.

EXAMPLE

Part 1

Use geometric software to discover the relationship between the medians of a triangle.

Discussion: This exploration connects with G.CO.10. Students should discover that all three medians intersect at a single point inside the triangle, so they are concurrent. The concurrent point formed by the medians is called the centroid.

Part 2

Give students Doritos (or cardboard cutouts of triangles or both), and have them balance them on their finger. Have them mark the balancing point as best as they can. Ask them what they notice about the balancing point.

Discussion: Students should notice that the balancing point is the centroid of the triangle—the point of concurrency formed by the three medians.

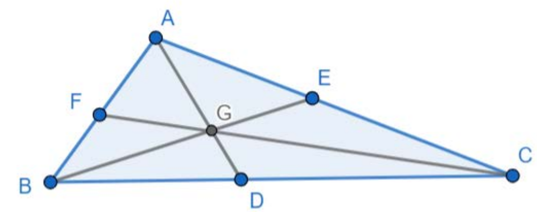
Example continued on next page

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.GPE.4-7)

Part 3

- Find the coordinates of the centroid of $\triangle TED$ with vertices $T(-3,3), E(5,\ 4)$ and $D(1,-1)$ in two different ways.
- How can you find the centroid of any triangle using vertices $A(x_1,y_1), B(x_2,y_2), C(x_3,y_3)$? Prove that your method is correct.

Discussion: The centroid of triangle TED is (1, 2). Although there are many ways to find the centroid (all of which are acceptable routes for students to pursue), the idea of balancing a Dorito should be fresh in their minds. After discussion, students may come to the idea that the balance point would be the mean of the *x*-coordinates, $\frac{x_1+x_2+x_3}{3}$, and the mean of the *y*-coordinates, $\frac{x_1+x_2+x_3}{3}$. Have them prove their ideas using geometric software. Another way to find the coordinates of the centroid is by creating a system of two equations describing the lines containing two of the medians. The centroid, or point of intersection of the median, is the solution of the system. An alternative way is to apply Ceva's Theorem which states that if D, E, and F are points on sides AB, BC, and CA, respectively, then lines AD, BE, and CF are concurrent if and only if $\frac{BD}{DC}\cdot\frac{CE}{EA}\cdot\frac{AF}{FB}=1$. (See the diagram at the right) Ceva's Theorem may be unfamiliar to many students, but proving it could be used as an extension.



Part 4

a. In $\triangle TED$ from Part 3, find the distance from each vertex to the centroid, and the distance of each midpoint to the centroid. What do you notice about the relationship between the distances?

b. Use geometric software to establish the relationship between the centroid and any median on the triangle

Discussion: Students should discover that the centroid divides each median into a ratio of 2:1, which means that the centroid will always be $\frac{2}{3}$ of the distance from the vertex o the midpoint of the opposite side. In this example, part **a.** and part **b.** could be switched depending what you want to emphasize. Having students find the distances first allows for reinforcement of the Distance Formula, and then use the geometric software to generalize and informally derive the theorem that the centroid divides each median into a ratio of 2:1. Using geometric software first, allows students to generalize and informally derive the theorem that the centroid divides each median into a ratio of 2:1, and then reinforce concepts of partitioning line segments.

Other simple coordinate proofs may include theorems such as the diagonals of a parallelogram bisect each other, the diagonals of a rectangle are congruent, the diagonals of a square are perpendicular, and a point on the perpendicular bisector of a segments it equidistant from the endpoints of the segment.

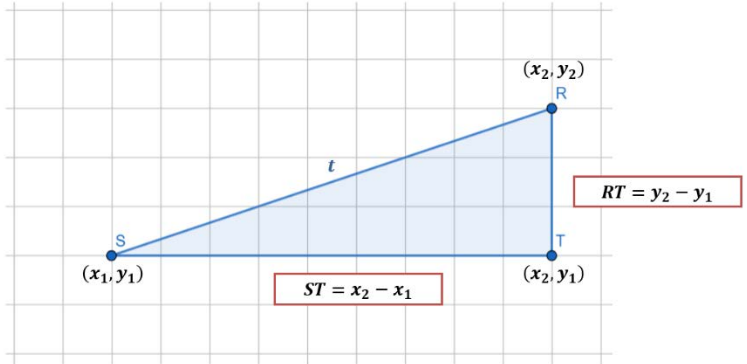
INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.GPE.4-7)

Students often incorrectly think that bisectors divide triangles in half or act as lines of symmetry. Misconceptions around bisectors can be challenging to correct. Students may have a difficult time drawing valid conclusions about angle bisectors, line segment bisectors, and perpendicular bisectors. When presented with a bisector ask students, "What is being bisected?" or "What type of bisector is this?". Then reinforce the definition for the appropriate type of bisector. Formally teaching about and distinguishing between bisectors before teaching students to prove may be more beneficial. Draw attention to the fact that a perpendicular bisector is a special type of line segment bisector.

THE DISTANCE FORMULA

In Grade 8, students used the Pythagorean Theorem to find the distance between any two points. Now students need to generalize those ideas to derive the Distance Formula.

Students should be able to see that if point S is located at $(x_1,\ y_1)$ and point R is located at $(x_2,\ y_2)$, then point T, that is a vertex of a right angle must be located at $(x_2,\ y_1)$ because $\overline{ST}$ is a horizontal line segment and $\overline{RT}$ is a vertical line segment. Therefore ST must be $x_2 - x_1$ and RT must be $y_2 - y_1$.



Since points S, T and R form a right triangle, then the Pythagorean Theorem can be applied as $(x_2 - x_1)^2 + (y_2 - y_1)^2 = t^2$ which can be rewritten as $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = t$ or the Distance Formula. Students in Geometry should be able to not only use the Distance Formula but explain how it is derived from and it is related to the Pythagorean Theorem.

Midpoint Formula

Given two points, use the Distance Formula to find the coordinates of the midpoint. Students should realize that given two points, they can derive the Midpoint Formula by finding the average of the coordinates. Generalize this for two arbitrary points to derive the Midpoint Formula; beginning on a number line may be helpful.

TIP!

For an extension, students can derive the Distance Formula and the Midpoint Formula for three-dimensions. They may also find the formula for the length of a diagonal of a box.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.GPE.4-7)

EXAMPLE

If the midpoint P of $\overline{RT}$ is (3, 1), and point R is (–2, 5), what are the coordinates of point T?

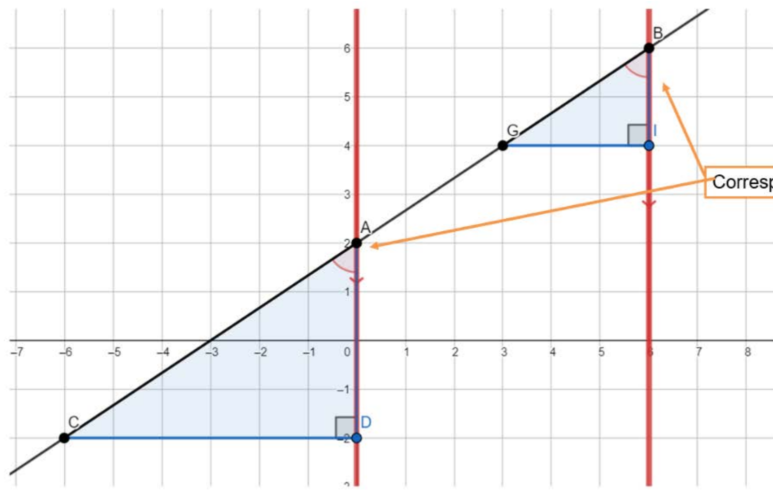
Discussion: Students should realize that because the horizontal distance between points R and P is 5 units, the same horizontal distance of 5 units to the right is from point P to point T. So, the x-coordinate of point T is $3 + 5 = 8$. The vertical distance between points R and P is 4 units. The same distance of 4 units down is from point P to point T. So, the y-coordinate of point T is $1 - 4 = -3$. Therefore, the coordinates of point T are $(8, -3)$.

SLOPE

Slope Formula

Use what students learned in Grade 8 about slope (8.EE.5-6) and connect it to the Slope Formula. Students can revisit the Slope Formula and connect the notion of slope triangles to similar triangle theorems.

Using a pair of slope triangles students can show that every nonvertical and nonhorizontal line has a nonzero constant slope that equals $\frac{y_2-y_1}{x_2-x_11}$. Students can choose any four distinct points, $A, B, C,$ and G, on the given line to create two non-congruent right triangles ADC and BIG. Since the triangles are formed by the given line and a pair of vertical and a pair of horizontal grid lines, there are two pairs of corresponding congruent angles formed by those lines and the transversal CB. This makes the two triangles similar by AA-Similarity Criteria. Since the triangles are similar, the sides of the triangle are proportional. Hence, $\frac{AD}{BI} = \frac{CD}{GI}$ or $\frac{AD}{CD} = \frac{BI}{GI}$. Because ratios $\frac{AD}{CD}$ and $\frac{BI}{GI}$ are equal, they can be denoted by the same constant, m, so that $\frac{AD}{CD} = m$, and $\frac{BI}{GI} = m$. So, in the right triangles m is a ratio between the length of a vertical leg to the length of a horizontal leg. Since the length of a vertical leg is the difference of the y -coordinates, Δy, of the corresponding endpoints and the length of the horizontal leg is the difference of x -coordinates, Δx of the corresponding endpoints, then $m = \frac{\Delta y}{\Delta x}$, where $\Delta y = y_2 - y_1$and $\Delta x = x_2 - x_1$ and (x_1, y_1) or (x_2, y_2) are the coefficients of any two fixed points on the line.



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INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.GPE.4-7)

Students may claim that a vertical line has infinite slope. This suggests that infinity is a number. Since applying the slope formula to a vertical line leads to division by zero, we say that the slope of a vertical line is undefined. Also, the slope of a horizontal line is 0. Students often say that the slope of vertical and/or horizontal lines is "no slope," which is incorrect.

Pay special attention to the slope of a line and its applications in analyzing properties of lines. Allow adequate time for students to review slopes and equations of lines. Use slopes and the Distance Formula to solve problems about figures in the coordinate plane such as the following:

- Given a line and a point not on it, find an equation of the line through the point that is parallel to the given line.
- Given a line and a point not on it, find an equation of the line through the point that is perpendicular to the given line.
- Given the equations of two non-parallel lines, find their point of intersection.

Parallel and Perpendicular Lines

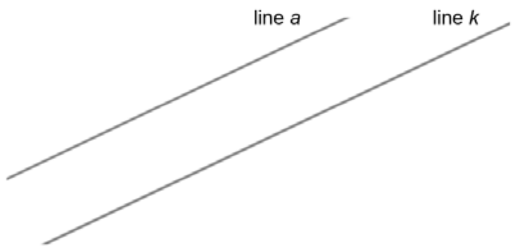
Discover the fact that pairs of parallel lines have equal slopes, and discover that pairs of perpendicular lines have slopes that are opposite reciprocals of each other and have a product of –1, except for a pair of vertical and horizontal lines.

Students need to provide a convincing argument for the slope criteria for parallel and perpendicular lines. This can be done via algebraic, geometric, transformational, or indirect proof methods.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.GPE.4-7)

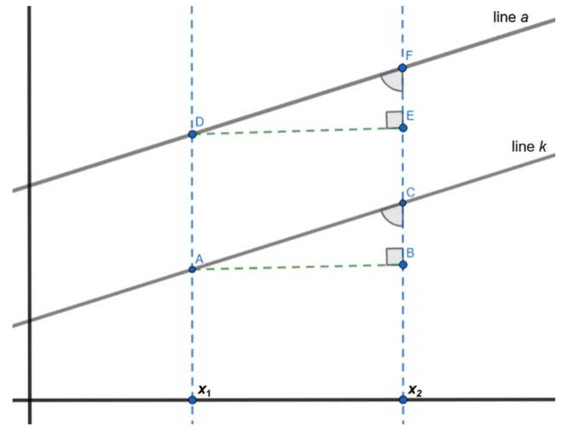
EXAMPLE

Given two parallel non-vertical lines a and k, prove that non-vertical parallel lines have the same slope.



Discussion:

1. Draw two distinct vertical lines through x_1 and x_2.
2.
   - On line k and line a, select two points with x_1 as their x-coordinate, label them A and D.
   - On line k and line a, select two points with 2 as their x-coordinate, label them C and F.
   - Draw a horizontal segment from D to the vertical line through x_2. Repeat for point A. This creates two right triangles as horizontal lines and vertical lines are perpendicular.
   - Label the third vertex of each right triangle as B and E.



3.
   - If $k \parallel a$ and $\overleftrightarrow{FB}$ is a transversal, then $\angle DFE \cong \angle ACB$ because they are corresponding angles.
   - $\angle DEF \cong \angle ABC$ because both are right angles.
   - Both horizontal legs, $\overline{DE}$ and $\overline{AB}$ are congruent because points are on the same vertical lines were chosen and therefore have the same distance, $x_2 - x_1$.
   - Therefore, $\triangle ABC \cong \triangle DEF$ by AAS.
   - $\overline{BC} \cong \overline{EF}$ by CPCTC.
4. Using the slope formula, the slope of line k is $m_1 = \frac{BC}{AB}$ and the slope of line a is $m_2 = \frac{EF}{DE}$.
5. Since $BC = EF$ and $AB = DE$, the slopes are equal ($m_1 = m_2$).

Use slopes and the Distance Formula to solve problems about figures in the coordinate plane.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.GPE.4-7)

EXAMPLE

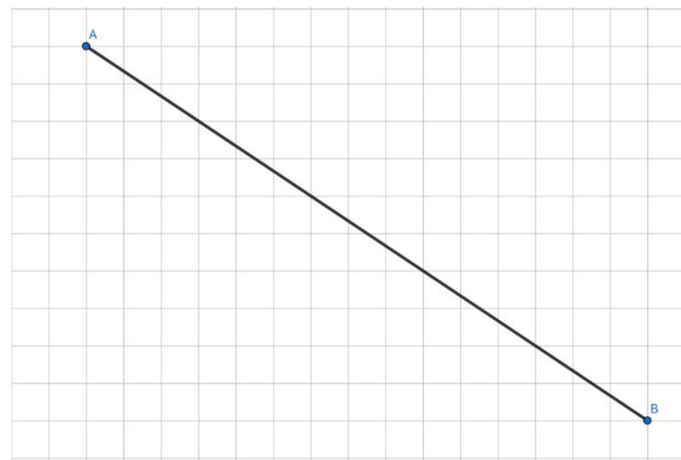
Analyzing Parallel and Perpendicular Lines

Discuss the differences and similarities between finding equations of lines that are parallel to a given line to those perpendicular to the same line.

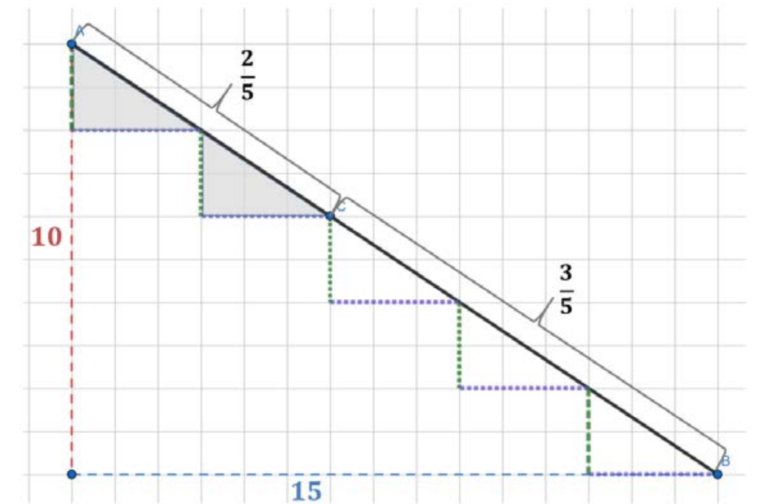
A POINT ON A DIRECTED LINE SEGMENT BETWEEN TWO GIVEN POINTS

Students will find the point that partitions a line segment into a given ratio.

EXAMPLE



- **a.** On the line segment $\overline{AB}$, Plot point C so that it is $\frac{2}{5}$ of the distance from point A to point B. Show that point C divides the segment into a ratio of 2:3.
- **b.** Find the ratio of $AC : AB$.
- **c.** Find the ratio of $BC : AB$.
- **d.** Find the ratio of $BC : AC$.
- **e.** Find the ratio of $AC : BC$.



Discussion: Note that the vertical distance (rise) from point A to point B is 10, and the horizontal distance (run) from point A to point B is 15. In order to partition the segment into 5 equivalent parts, these distances can be divided by 5 (since there are 5 sections or partitions) to show the smaller increment of down 2 and right 3. Since each increment is $\frac{1}{5}$ of the line, two increments would be $\frac{2}{5}$ of the line, so point C would be at the end of the second increment. That would leave 3 increments between points C and B, so $\overline{AB}$ would be divided into a ratio of 2:3. Some students may notice that since $\frac{2}{5}$ of 15 is 6 and $\frac{2}{5}$ of 10 is 4, using this information, they would just have to add 6 units to the horizontal direction and 4 units to the vertical direction.

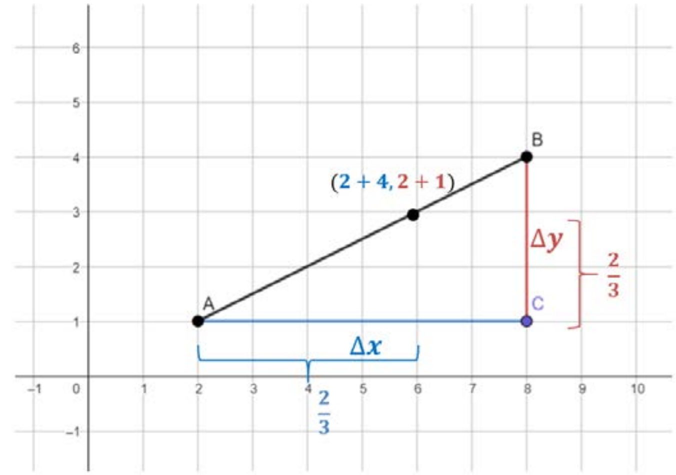
INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.GPE.4-7)

EXAMPLE

Part 1

Find the point D which is $\frac{2}{3}$ of the distance along of $\overline{AB}$ from $A(2,1)$ to $B(8,4)$.

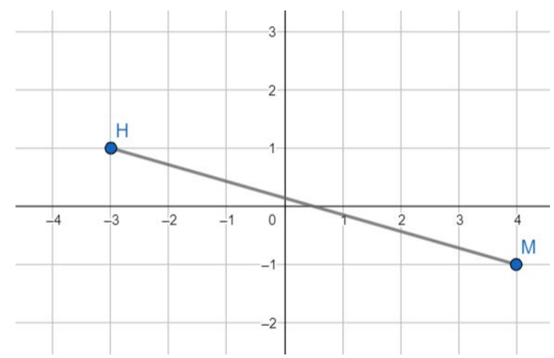
Discussion: One way to approach this problem is to think about the line segment in terms of the slope triangle and find $\frac{2}{3}$ of the horizontal distance (Δx), and $\frac{2}{3}$ of the distance along the vertical distance (Δy). The horizontal distance is $8-2=6$ and the vertical distance is $4-1=3$. Since $\frac{2}{3}$ of 6 is 4, then 4 units need to be added to $x_1=2$. Since $\frac{2}{3}$ of 3 is 2, then 2 units need to be added to $y_1=1$. Because $2+4=6$ and $1+2=3$, point D which is $\frac{2}{3}$ of the distance along of $\overline{AB}$ is $(6,3)$.



Part 2

Find the point A that is $\frac{2}{3}$ of the distance along of $\overline{HM}$ from $H(-3,1)$ to $M(4,-1)$.

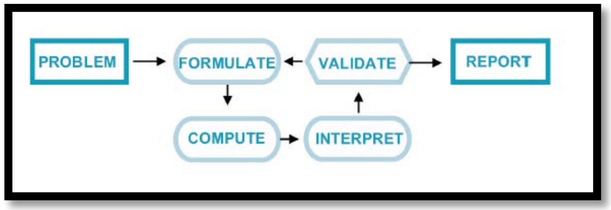
Discussion: Students are encouraged to make a diagram. This problem is a little more complex than the problem in Part 1, because the length of the line segment does not divide evenly and because the position of point A is down and to the right from point H along the line segment. Therefore, students have to rely more on generalizing the algebraic calculations than counting the slope from the graph. The horizontal distance of $\overline{HM}$ is $|4-(-3)|=7$ and the vertical distance is $-1-1=2$. Since point A is to the right from point H horizontally, $\frac{2}{3}$ of 7 is $\frac{14}{3}$, and $\frac{14}{3}$ needs to be added to H's x-coordinate -3 to get $\frac{5}{3}$. Likewise, since point A is lower than point H vertically, $\frac{2}{3}$ of 2 is $\frac{4}{3}$, and $\frac{4}{3}$ needs to be subtracted from H's y-coordinate 1 to get $-\frac{1}{3}$. Therefore, point A is $\left(\frac{5}{3},-\frac{1}{3}\right)$.



INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.GPE.4-7)

MODELING

G.GPE.7 is a modeling standard. See page 10 for more information about modeling.



COMPUTING PERIMETERS AND AREAS OF POLYGONS USING COORDINATES

Students should apply their knowledge of perimeter and area of polygons to the coordinate plane. This includes the application of the Distance Formula.

Use the Distance Formula to find the length of each side of a polygon whose vertices are known, and compute the perimeter of that figure.

EXAMPLE

An airplane flies from Cleveland Hopkins International Airport located at approximately 41.41°N, 81.85°W to Chicago O'Hare International Airport located at approximately 41.97°N and 87.90°W and then to Dallas/Fort Worth International Airport located at approximately 32.90°N and 97.04°W and then back to Cleveland Hopkins International Airport. Find the total distance the airplane flew.

Discussion: Point out to the students that in this situation, the use of Distance Formula can help to find the approximate distance that the airplane flew. The reason it is just an approximation is because the Distance Formula only applies to two-dimensional planes (not the flying kind); however, the earth is a three-dimensional sphere. Explain to students that in navigation the Haversine Formula is used which takes into account the distance along the circular arc between two points on a sphere using their longitudes and latitudes. Also explain that flight paths are rarely the shortest distance between two points since they have to accommodate different flight patterns for safety reasons. A simpler problem would be to utilize the map of a city (like Columbus or Cincinatti) on a coordinate plane to find the distance between locations on a map.

EXAMPLE

Given the vertices of a triangle or a parallelogram, find the equation of a line containing the altitude to a specified base and the point of intersection of the altitude and the base. Use the Distance Formula to find the length of that altitude and base, and then compute the area of the figure.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.GPE.4-7)

As students become proficient in using slopes and the Distance Formula to solve the kinds of problems suggested in this cluster, allow them to solve more complex problems with the aid of dynamic geometric software.

Instructional Tools/Resources

These tools and resources may be helpful for instruction. Before use, they should be evaluated at the local level to determine their appropriateness for instruction.

Manipulatives/Technology

- Graph paper
- Patty paper
- Scientific or graphing calculators
- Dynamic geometric software (Geometer's Sketchpad®, Cabri®, Desmos®, or GeoGebra®)

Geometric Theorems

- Medians Centroid Theorem (Proof without Words) by Tim Brzezinski is a GeoGebra applet that illustrates the theorem.

Distance Formula

- Introduction to Distance Formula by Lee-Anne Patterson is a Desmos activity that connects the Pythagorean Theorem to the Distance Formula.

Midpoint Formula

- A Midpoint Miracle by Illustrative Mathematics is a task where students have the opportunity to discover that joining the midpoints of a quadrilateral will form a parallelogram.

Parallel and Perpendicular Lines

- Equations of Polygon Sides by Greta is a Desmos activity where students apply their knowledge of equations of parallel and perpendicular lines to form quadrilaterals.
- Distance Between Parallel Lines by Patti is a Desmos activity to help students explore that parallel lines are equidistant at any point.
- Parallel and Perpendicular Lines by Bob Lochel is a Desmos activity where students explore parallel and perpendicular lines.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.GPE.4-7)

Parallel and Perpendicular Lines, continued

- Slope is a Mathematics Instructional Plan for Geometry created by Virginia's Department of Education.
- Equal Area Triangles on the Same Base II by Illustrative Mathematics is a task where students create different triangles with the same areas by applying properties of parallel lines.
- Parallel Lines in the Coordinate Plane by Illustrative Mathematics is a task where students prove the slope criterion for parallel lines.
- Triangles Inscribed in a Circle by Illustrative Mathematics is a task where students apply properties of perpendicular lines.
- Unit Squares and Triangles by Illustrative Mathematics is a task where students may apply properties of perpendicular lines as a solution strategy.
- When are Two Lines Perpendicular? by Illustrative Mathematics is a task where students explore the slope of perpendicular lines.
- Unit Square and Triangles by Illustrative Mathematics is a task where students apply coordinate geometry to find the area of a triangle inside a unit square.

Points of Concurrency

- Proofs that the Median of a Triangle are Concurrent by Michael McCallum from the University of Georgia is an explanation of the proof. It includes a link to a Geometer's Sketchpad demonstration.
- Triangles: Points of Concurrency by Tim Brzezinski is a GeoGebra webpage that has many applets about the points of concurrency in triangles.
- Points of Concurrency in Triangles by knwilson is a GeoGebra applet where students can see the orthocenter, circumcenter, incenter, and centroid.
- Points of Concurrency in a Triangle by Ed Bernal is a GeoGebra applet where students can drag the vertices of triangles and observe the point of concurrency.
- Finding Centroid by Julia Finneyfrock is a Desmos lesson where students find the centroid.

Partitioning Line Segments

- Finding Triangle Coordinates by Illustrative Mathematics is a task where students use similar triangles in order to study the coordinates of points which divide a line segment into a given ratio.
- Scaling a Triangle in the Coordinate Plane by Illustrative Mathematics is a task where students apply dilations to a triangle in the coordinate plane.

Polygons on the Coordinate Grid

- Squares on a Coordinate Grid by Illustrative Mathematics is a task where students use the Pythagorean Theorem to construct squares of different sizes on a coordinate grid.
- Triangle Perimeters by Illustrative Mathematics is a task where students apply the Pythagorean Theorem to calculate distances and areas.
- Equal Area Triangles on the Same Base I by Illustrative Mathematics is a task where students create different triangles with the same areas.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.GPE.4-7)

Curriculum and Lessons from Other Sources

- EngageNY, Geometry, Module 4, Topic A, Lesson 1: Searching a Region in the Plane, Lesson 2: Finding Systems of Inequalities That Describe Triangular and Rectangular Regions, Lesson 3: Lines that Pass Through Regions, Lesson 4: Designing a Search Robot to Find a Beacon are lessons that pertain to this cluster.
- EngageNY, Geometry, Module 4, Topic B, Lesson 5: Criterion for Perpendicularity, Lesson 6: Segments that Meet at Right Angles, Lesson 7: Equations for Lines Using Normal Segments, Lesson 8: Parallel and Perpendicular Lines are lessons that pertain to this cluster.
- EngageNY, Geometry, Module 4, Topic C, Lesson 9: Perimeter and Area of Triangles in the Cartesian Plane, Lesson 10: Perimeter and Area of Polygonal Regions in the Cartesian Plane, Lesson 11: Perimeters and Areas of Polygonal Regions Defined by Systems of Inequalities are lessons that pertain to this cluster.
- EngageNY, Geometry, Module 4, Topic D, Lesson 12: Dividing Segments Proportionately, Lesson 13: Analytic Proofs of Theorems Previously Proved by Synthetic Means, Lesson 14: Motion Along a Line—Search Robots Again (extension), Lesson 15: The Distance from a Point to a Line are lessons that pertain to this cluster.
- Mathematics Vision Project, Geometry, Module 6: Connecting Algebra and Geometry has a task that pertains to this cluster.
- Georgia Standards of Excellence Curriculum Frameworks, Geometry, Unit 5: Geometric and Algebraic Connections has many tasks that align to this cluster.
- Illustrative Mathematics, Geometry, Unit 6, Lesson 7: Distances and Parabolas, Lesson 10: Parallel Lines in a Plane, Lesson 11: Perpendicular Lines in the Plane, Lesson 12: It's All on the Line, Lesson 13: Intersection Points, Lesson 14: Coordinate Proof, Lesson 15: Weighted Averages, Lesson 16: Weighted Averages in a Triangle, Lesson 17: Lines in Triangles are lessons that pertain to this cluster.

General Resources

- Arizona 7-12 Progression on Geometry is an informational document for teachers. This cluster is addressed on pages 17-18.
- Arizona High School Progression on Modeling is an informational document for teachers.
- High School Coherence Map by UnboundED is a map of the relationships among high school mathematics standards. Please note: This resource is aligned to the Common Core not Ohio's Learning Standards. Although the tool is useful, it needs to be checked with respect to Ohio's Learning Standards.
- van Hiele Model of Geometric Thinking is a pdf created by ODE that summarizes the van Hiele levels.
- G-GPE.4 is a blog thread on Bill McCallum's Mathematical Musings blog is a discussion about the example listed in the standard G.GPE.4.
- G-GPE.6 is a blog thread on Bill McCallum's Mathematical Musings blog is a discussion about the rationale for including the specific skill listed in the standard.
- G.GPE.5 is a blog thread on Bill McCallum's Mathematical Musings blog is a discussion about the depth of understanding in this standard.
- Points of Concurrency is a blog thread on Bill McCallum's Mathematical Musings blog is a discussion about points of concurrency.
- Providing the Slope Criteria is a blog thread on Bill McCallum's Mathematical Musings blog is a discussion about circular reasoning in G.GPE.5 and G.CO.4.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.GPE.4-7)

References

- Common Core Standards Writing Team. (2016, March 24). *Progressions for the Common Core State Standards in Mathematics (draft). Grades 7–8, HS, Geometry*. Tucson, AZ: Institute for Mathematics and Education, University of Arizona.
- Common Core Standards Writing Team. (2013, July 4). P*rogressions for the Common Core State Standards in Mathematics (draft). High School, Modeling.* Tucson, AZ: Institute for Mathematics and Education, University of Arizona.
- Jurgensen, R., Brown, R., & Jurgensen, J. (1985). *Geometry: Teacher's Edition.* Houghton Mifflin Geometry: Boston, MA.
- Matsuura, R. & Sword, S. (February 2015). Illuminating coordinate geometry with algebraic symmetry. *Mathematics Teacher*, *108, (6),* 470-473.

STANDARDS	MODEL CURRICULUM (G.GMD.1, 3)
Geometry **GEOMETRIC MEASUREMENT AND DIMENSION** **Explain volume formulas, and use them to solve problems.** **G.GMD.1** Give an informal argument for the formulas for the circumference of a circle, area of a circle, and volume of a cylinder, pyramid, and cone. *Use dissection arguments, Cavalieri's principle, and informal limit arguments.* **G.GMD.3** Use volume formulas for cylinders, pyramids, cones, and spheres to solve problems.★	**Expectations for Learning** In middle school, students use established circumference, area, and volume formulas for two- and three-dimensional figures. Instead of using area and volume formulas rotely, students in this cluster give informal justifications for these formulas and use them to solve problems. The student understanding of this cluster begins at van Hiele Level 1 (Analysis) and moves to Level 2 (Informal Deduction/Abstraction). **ESSENTIAL UNDERSTANDINGS** • A three-dimensional solid can be viewed as a stack of layers. • If all of the layers of a three-dimensional solid have the same area, then the volume is the area of the base times the height. • The volume remains unchanged when layers parallel to the base in a three-dimensional solid are shifted. • A cone's volume is $\frac{1}{3}$ of the volume of a cylinder if their base areas are equal and their heights are congruent. • A pyramid's volume is $\frac{1}{3}$ of the volume of a prism if their base areas are equal and their heights are congruent. • Volume, like area, is additive, so to find the volume of a composite figure, cut the figure into pieces of known volume and add or subtract as appropriate. • The cross sections of a cylinder are circles of equal area. • The cross sections of a prism are congruent to the base, so therefore the areas are equal. **MATHEMATICAL THINKING** • Draw a picture or create a model to make sense of a problem. • Make and modify a model to represent mathematical thinking. • Attend to meaning of quantities. • Consider mathematical units involved in a problem. • Solve real-world and mathematical problems accurately. • Determine reasonableness of results. • Use informal reasoning. *Continued on next page*

STANDARDS	MODEL CURRICULUM (G.GMD.1, 3)
	Expectations for Learning, continued **INSTRUCTIONAL FOCUS** • Explain and justify the formulas for the circumference of a circle, area of a circle, and volume of a cylinder, pyramid, and cone. • Apply volume formulas in real-world and mathematical problems. ★ • (+) Informally apply Cavalieri's principle for the formulas for the volume of a sphere and other solid figures. **Content Elaborations** **OHIO'S HIGH SCHOOL CRITICAL AREAS OF FOCUS** • Geometry, Number 3, page 6 **CONNECTIONS ACROSS STANDARDS** • Experiment with transformations in the plane (G.CO.1). • Understand and apply theorems about circles (G.C.2, 5). • Understand the relationships between lengths, areas, and volumes (G.GMD.5-6). • Apply geometric concepts in modeling situations (G.MG.1-3).

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.GMD.1, 3)

Instructional Strategies

Note: The model curriculum is not a collection of lessons nor a full curriculum; it does not suggest pace, sequence, or amount of time spent on topics. It provides information about a topic related to the standards including ideas for examples, strategies for teaching, common misconceptions, and possible connections between topics. In addition, the emphasis of the model curriculum is on instruction not on assessment; therefore, the examples in the Instructional Strategies section are written for instructional purposes not necessarily as assessment items. Examples may need to be rewritten to accommodate the needs of each individual classroom.

Standards for Mathematical Practice
This cluster focuses on but is not limited to the following practices:
MP.3 *Construct viable arguments and critique the reasoning of others.*
MP.4 *Model with mathematics.*
MP.5 *Use appropriate tools strategically.*
MP.7 *Look for and make use of structure.*
MP.8 *Look for and express regularity in repeated reasoning.*

In Grade 8, students were required to know and use the formulas for volumes of cylinders, cones, and spheres. In this cluster those formulas are derived by a combination of concrete demonstrations and formal reasoning. Students who eventually go on to study calculus will formally derive formulas for the volume of a pyramid, cone, sphere and other solids using definite integrals.

Connect G.GMD.4 with this cluster as it is closely related.

It might be helpful to tie in concepts of area related probability concepts such as unions, intersections, subsets, and complements in S.CP.1. For example, the area of the union of two regions is the sum of the areas minus the area of the intersection. Also, the area of the difference of two regions, where one is contained in the other or subset, is the difference of the areas.

VAN HIELE CONNECTION

In Geometry students are expected to move from Level 1(Analysis) to Level 2 (Informal Deduction/Abstraction). Van Hiele Level 2 can be characterized by the student doing some or all of the following:

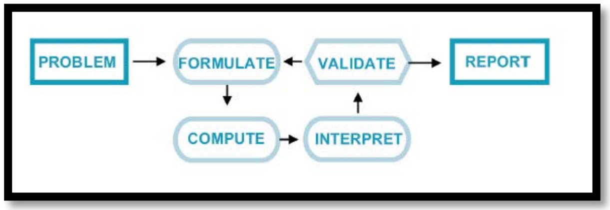
- fully understanding the procedures and formula for determining measurement of simple, familiar shapes;
- generalizing unit-measure enumeration to include fractional units;
- understanding and making unit conversions; and/or
- generalizing measurement to non-squares for area and non-cubes for volume.

See the van Hiele pdf on ODE's website for more information about van Hiele levels.

MODELING

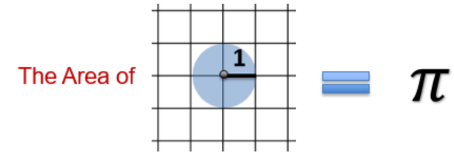
G.GMD.3 is a modeling standard. See page 10 for more information about modeling.

Formulas are mathematical models of relationships among quantities. Some formulas describe laws of nature such as gravity, others are measurements of one quantity in terms of another. This ties in with A.CED.4 that students learned in Algebra 1.



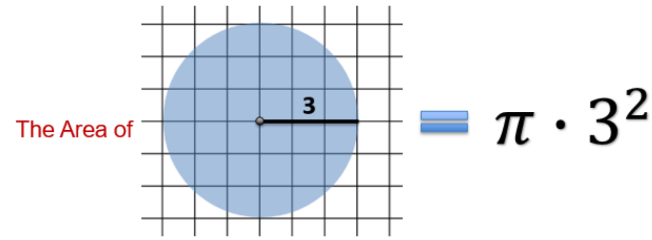
INFORMAL ARGUMENTS FOR FORMULAS

Students should draw a picture, write a paragraph, demonstrate, or describe orally the rationale for the development of area, circumference, and volume formulas for circles, cylinders, pyramids, cones, and spheres. Give the opportunities for students to explore multiple ways to derive area and volume formulas.



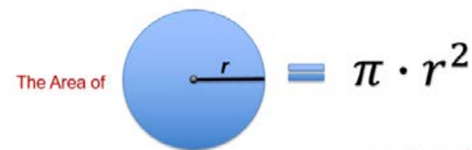
Circles

Building on the concepts of scale factor (G.GMD.6) and on the fact that all circles are similar (G.C.1), students should know that when a figure is scaled by a factor of *x*, the area changes by a factor of x^2. Define the constant π as the area of the region inside a circle when the radius is 1 unit.

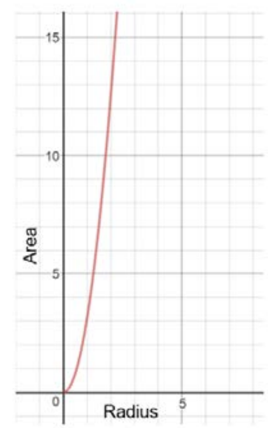


EXAMPLE

- π can be defined as the area of the region inside a circle when the radius is 1 unit. Draw an example of π using this definition.
- Enlarge a circle by a scale factor of your choosing and draw a picture representing your circle. How does the radius of the circle change? How does the area of the circle change?
- Enlarge a circle by a scale factor of r and draw a picture representing your new circle. How does the radius of the circle change? How does the area of the circle change?
- How do your pictures represent the formula for area of a circle?
- Why is it beneficial to define π in this way?



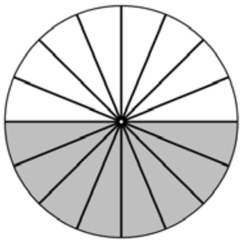
Discussion: Enlarging a circle by a scale factor of r means that the radius changes by a scale factor of r, and the area changes by a scale factor of r^2; hence the formula for the area of a circle is $A = \pi r^2$. Draw students' attention to the fact that although there is a relationship between the area and radius, it is not directly proportional; the graph of $A = \pi r^2$ is not a straight line, but rather it is squarely proportional because the graph is not a straight line.

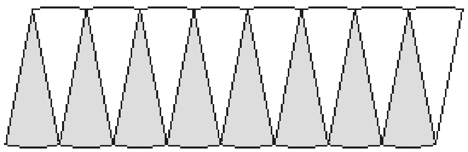


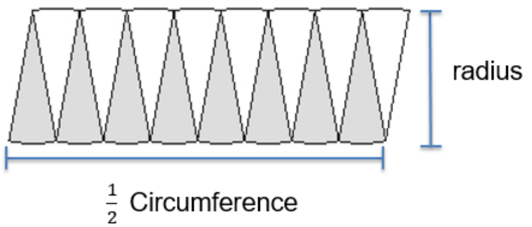
INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.GMD.1, 3)

TIP! Refer to the area of the circle as the area of the region inside the circle, since the circle itself is just the curved outline of the figure.

Another visual for understanding the area of a circle can be modeled by cutting up a circular disc (for example a paper plate) into pieces along diameters and reshaping the pieces into a parallelogram. Cut a cardboard circular disk into 8 congruent sectors, and rearrange the pieces to form a shape that looks like a parallelogram with two scalloped edges. Repeat the process with 16 sectors and note how the edges of the parallelogram look "straighter." Discuss what would happen in the case where the number of sectors becomes infinitely large.







Ask students to identify what represents the circumference and the radius in their new parallelogram-looking shape. Identifying the radius gives cause for conversations. Students should come to the conclusion that the parallelogram's height is the radius and its length is $\frac{1}{2}$ of the circumference. Therefore, another formula for area is $A = \frac{1}{2}Cr$.

Since students already found in the previous example that the area of circle is $A = \pi r^2$ or $A = \pi rr$, then by substitution $\frac{1}{2}Cr = \pi rr$. After dividing each side by r, the formula simplifies to $\frac{1}{2}C = \pi r$. They can then solve for C by multiplying each side by 2 to get $C = 2\pi r$. Since $2r = d,\ C = 2\pi r$ is equivalent to $C = d\pi$.

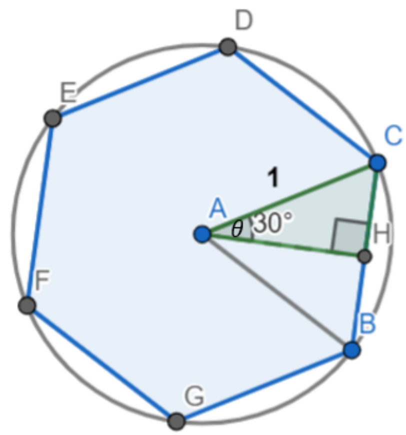
INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.GMD.1, 3)

Students can use informal limit arguments to find the area of a circle such as finding the area of polygons inscribed in circles as the number of sides increases.

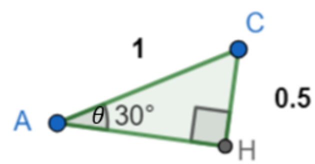
EXAMPLE

Part 1

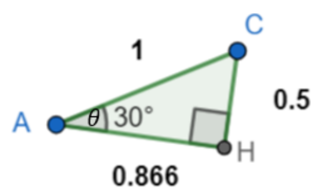
- Find the area of a hexagon inscribed in a circle with a radius of 1.
- How does the area of the inscribed hexagon compare to the area of the circle? Explain.



$$\theta = \frac{360}{6} \div 2 = 30°$$



$$\text{half–chord} = \sin 30° = \frac{x}{1} = 0.5$$



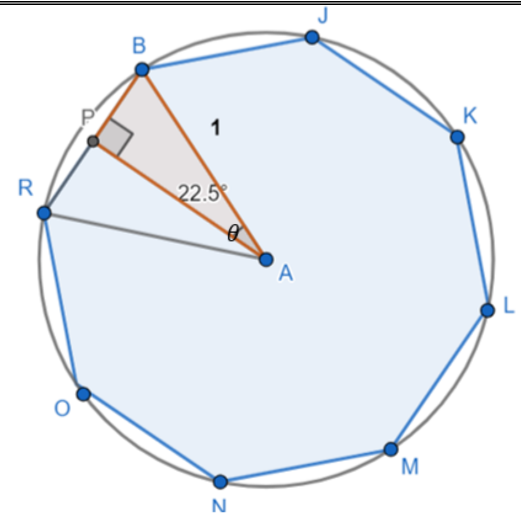
$$\text{altitude} = \cos 30° = \frac{x}{1} \approx 0.866$$

$$A_{hexagon} \approx 12\left(\frac{1}{2}(0.5)(0.866)\right) = 2.598$$

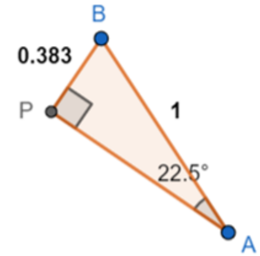
INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.GMD.1, 3)

Part 2

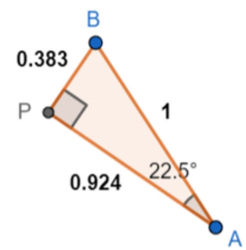
- Find the area of an octagon inscribed in a circle with a radius of 1.
- How does the area of the inscribed hexagon in Part 1 compare to the area of the octagon? Explain.
- Which area is closer to the area of the circle: the hexagon in part 1 or the octagon in part 2? Explain.
- What type of regular polygons would have an area that would be the closest to the area of the circle? Explain.



$$\theta = \frac{360}{8} \div 2 = 22.5°$$



$$\text{half–chord} = \sin 22.5° = \frac{x}{1} \approx 0.383$$



$$\text{altitude} = \cos 22.5° = \frac{x}{1} \approx 0.924$$

$$A_{octagon} \approx 16\left(\frac{1}{2}(0.383)(0.924)\right) = 2.831$$

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.GMD.1, 3)

Part 3

- Use an Excel spreadsheet to show the area of different regular polygons inscribed in a circle with a radius of 1 as the number of sides increases.
- How can you use this information to determine the area of a circle? Explain.

	A	B	C	D	E	F
1	Number of Sides	Angle Measure	Sine Side	Altitude or Cosine	Area	
2	6	30	0.5	0.866025	2.598076	
3	7	25.71429	0.433884	0.900969	2.73641	
4	8	22.5	0.382683	0.92388	2.828427	
5	9	20	0.34202	0.939693	2.892544	
6	10	18	0.309017	0.951057	2.938926	
7	11	16.36364	0.281733	0.959493	2.973524	
8	12	15	0.258819	0.965926	3	
9	13	13.84615	0.239316	0.970942	3.020701	
10	14	12.85714	0.222521	0.974928	3.037186	
11	15	12	0.207912	0.978148	3.050525	
12	16	11.25	0.19509	0.980785	3.061467	
13	17	10.58824	0.18375	0.982973	3.070554	
14	18	10	0.173648	0.984808	3.078181	

Discussion: The goal of this activity is for students to see that as they increase the number of sides of a regular polygon inscribed in a circle, the area of the polygon approaches that of a circle. The reason a unit circle is used is that in a unit circle, the area equals π (which is how π was defined in an earlier example). It also gives students a chance to apply what they have learned in trigonometry using a unit circle. One way to fill out the Excel Spreadsheet is as follows:

- Type the number "6" in cell A2. Type the formula "=A2+1" into cell A3, and drag down the formula for Column A.
- To get the angle measure of central angle, in B2 type the formula "=360/(2*A2)" or another equivalent formula. Then drag the formula through the rest of the column.
- To get sine which is the half-chord in a circle with the radius of 1, type "=SIN(RADIANS(B2))" in cell C2. Then drag the formula through the rest of the column.
- Students can either use the Pythagorean Theorem to find the altitude (or apothem) or realize that the cosine is the altitude in a right triangle with a radius of 1. If students recognize this, they can type =COS(RADIANS(B2)) into cell D2 and then drag the formula through the rest of the column.
- To find the area of the triangle, students can type "=C2*D2*A2" into cell E2, and then drag the formula through the rest of the columns. *Note: Some students may want to have more columns showing more steps, so the formula may not be quite as condensed. Individual student methods should be encouraged.*

In middle school students should have discovered that the circumference is a little more than three times the diameter of the circle. Now they should build on that to explore the formula for circumference.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.GMD.1, 3)

EXAMPLE

Part 1

- Find the perimeter of a hexagon inscribed in a circle with a radius of 1.
- How does the perimeter of the inscribed hexagon compare to the circumference of the circle? Explain.

Part 2

- Find the perimeter of an octagon inscribed in a circle with a radius of 1.
- How does the perimeter of the inscribed hexagon in Part 1 compare to the perimeter of the octagon? Explain.
- Which perimeter is closer to the circumference of the circle: the hexagon in Part 1 or the octagon in Part 2? Explain.
- What type of regular polygons would have a perimeter that would be the closest to the circumference of the circle? Explain.

Part 3

- Using the spreadsheet from the previous example show what happens to the perimeter of different regular polygons inscribed in a circle with a radius of 1 as the number of sides increases.
- How can you use this information to determine the circumference of a circle? Explain.

Discussion: Students should realize that in a circle with a radius of 1, the circumference is 2π. Therefore, as the number of sides in a regular polygon inscribed in a circle increases, the perimeter of the regular polygon approaches the circumference or 2π which is approximately 6.283185307. One way to fill out the Excel Spreadsheet is as follows:

- Since each side is a chord, the perimeter of a regular polygon is twice the half-chord (sine) times the number sides. In cell F2, type "=2*C2*A2" or another equivalent formula. *Note: Some students may want to have more columns showing more steps shown, so the formula may not be quite as condensed. Individual student methods should be encouraged.*

	A	B	C	D	E	F	G
1	Number of Sides	Angle Measure	Sine Side	Altitude or Cosine	Area	Perimeter	
2	6	30	0.5	0.866025	2.598076	6	
3	7	25.71429	0.433884	0.900969	2.73641	6.074372348	
4	8	22.5	0.382683	0.92388	2.828427	6.122934918	
5	9	20	0.34202	0.939693	2.892544	6.15636258	
6	10	18	0.309017	0.951057	2.938926	6.180339887	
7	11	16.36364	0.281733	0.959493	2.973524	6.198116251	
8	12	15	0.258819	0.965926	3	6.211657082	
9	13	13.84615	0.239316	0.970942	3.020701	6.222207271	
10	14	12.85714	0.222521	0.974928	3.037186	6.230586151	
11	15	12	0.207912	0.978148	3.050525	6.237350725	
12	16	11.25	0.19509	0.980785	3.061467	6.242890305	
13	17	10.58824	0.18375	0.982973	3.070554	6.247483606	
14	18	10	0.173648	0.984808	3.078181	6.251334396	
15	19	9.473684	0.164595	0.986361	3.084645	6.254594431	
16	20	9	0.156434	0.987688	3.09017	6.257378602	
17	21	8.571429	0.149042	0.988831	3.094929	6.259775179	
18	22	8.181818	0.142315	0.989821	3.099058	6.261852884	
19	23	7.826087	0.136167	0.990686	3.102663	6.263665858	
20	24	7.5	0.130526	0.991445	3.105829	6.265257227	
21	25	7.2	0.125333	0.992115	3.108624	6.266661678	
22	26	6.923077	0.120537	0.992709	3.111104	6.267907373	
23	27	6.666667	0.116093	0.993238	3.113314	6.269017363	
24	28	6.428571	0.111964	0.993712	3.115293	6.270010662	

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.GMD.1, 3)

EXAMPLE

Wind a piece of string or rope upon itself to form a circular disk (like a circular rug) or use an onion with concentric rings. Cut the figure along its radius. Stack the pieces to form a triangular shape with base *C* ($2\pi r$) and altitude *r*. Again, discuss what would happen if the string or onion ring became thinner and thinner so that the number of pieces in the stack became infinitely large. Then calculate the area of the triangle to derive the formula $A = \pi r^2$.

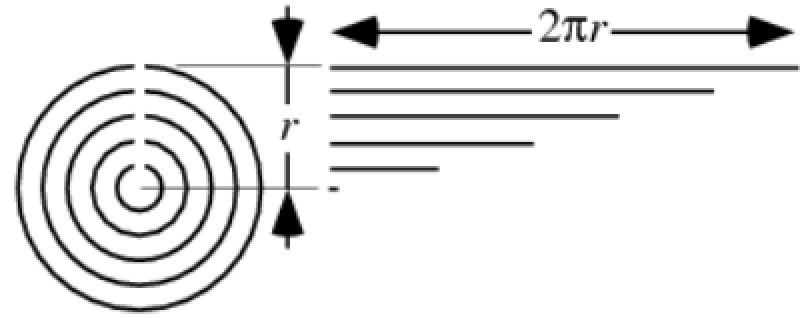


Image from Calculus Proof for the Area of a Circle from Math Stack Exchange.

Pi

Have students investigate the history of pi. There are several major questions to be answered:

- When and why was the symbol π chosen to represent this number?
- How did the "formula" for the area of a circle evolve?
- How is it possible to compute more than a billion digits of the number π?
- What is meant by saying that π is a transcendental number?

Students need to understand that π is an irrational number that can be placed on a number line. Discuss the difference between usefulness and precision. Students need to understand that 3.14 or $\frac{22}{7}$ are approximations of π, and although the approximations are useful, they are not the most precise. Throughout history different fractional values other than $\frac{22}{7}$ were used for π. Many times the fraction chosen to represent π was the most useful, not necessarily the most precise. That is why many people use 3.14 (or $\frac{277}{150}$) instead of $\frac{22}{7}$ even though $\frac{22}{7}$ is technically more precise. Even the π button on the calculator is just an approximation of π as the value is infinitely long. The level of precision needed depends on the application that is being used. For example, a construction worker may need more precision in building a large structure such as a skyscraper versus a small structure such as a small patio. A physicist might need more precision than a carpenter, since there needs at least 30 digits of π to approximate the size of the universe. Context must be taken into account when determining the level of precision that is needed.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.GMD.1, 3)

Many students want to incorrectly think of infinity as a number. Avoid this by talking about a quantity that becomes larger and larger with no upper bound.

Students may incorrectly believe that π is an exact number such as 3.14 or $\frac{22}{7}$ rather than understanding that 3.14 and $\frac{22}{7}$ are just approximations of π. Students can do activities with measuring circles or by placing π on number lines to break this misconception.

Cylinders

There are varying definitions of cylinders. Some mathematicians emphasize the distinction between solids and surfaces which can be distinguished in three-dimensional figures. A surface is the boundary of a three-dimensional figure. The region enclosed by the surface together with the surface itself is a solid. The differences between a brick and an empty box illustrate this concept. These mathematicians define prisms and cylinders (or rather circular cylinders) as subsets of general cylinders (cylindrical surface). A cylindrical solid can also be viewed as the set of points between a region (base) and its translated image in space, including the region and its image. Using this definition a cylinder and a prism are special cases of a cylindrical solid. On the other hand, circular cylinder and prism are also defined as the surfaces of a cylindric solid whose base is a circle and polygon respectively. A cylindrical solid may be formed using any shape as a base not just circles and polygons. *Note: This definition of a cylinder and prism mentions surfaces; however the common use of the term refer to solids. It is like defining polygons and other closed figures such as circles by their boundaries but not their interior points.* See Cylinder by Wolfram MathWorld and Cylinder by Wikipedia.

Others define a cylinder as any solid whose cross sections are perpendicular to some axis running through the solid are all the same. Again, in this viewpoint prisms and circular cylinders (what many commonly call cylinders) are subsets of cylinders.

Another viewpoint is to define a cylinder and a prism as separate solids defined by the shape of their bases: A cylinder is a solid with two congruent circular bases and a prism is a solid with two congruent polygon bases. In this instance a circular cylinder can be thought of as a prism where the base has an infinite number of sides. It is up to each district/teacher to decide how to define three-dimensional figures for their students.

Regardless, there is a connection between prisms and cylinders, so it is appropriate to teach prisms and figures with two congruent irregular parallel bases even though these types of figures are not explicitly called out in the high school standards. These types of irregular figures connect to the modeling standards. See standard G.MG.1.

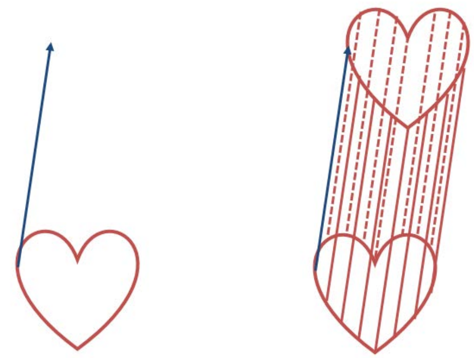
As one figure can by defined as a subset of the other, therefore it may make more sense to generalize the formulas as $V = Bh$, where B is the area of the Base. This formula will be true for all cylindrical solids instead of the figure specific formulas: $V_{Cylinder} = \pi r^2 h$ and $V_{prism} = lwh$. Also, the volume can be viewed as the product of the area of the preimage region and the distance between the planes of the bases formed by the preimage and image. It may be helpful to connect with G.CO.14 and have students create a hierarchy of three-dimensional figures to show the nuances between the definitions.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.GMD.1, 3)

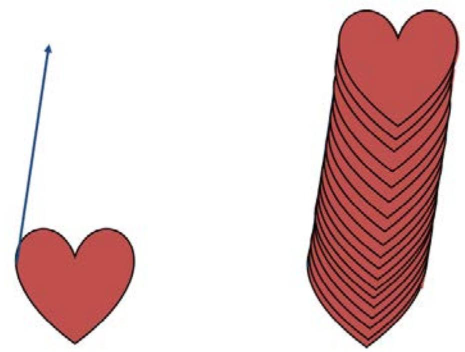
Cylinders can be explored in terms of translations.

EXAMPLE

- Create a surface by translating the figure along a directed line segment (vector).



- Create a solid by translating a region along a directed line segment (vector).



- Compare and contrast the two resulting figures in part **a.** and part **b.**
- How could you find the volume of the two figures?
- How does the volume of the two figures compare?

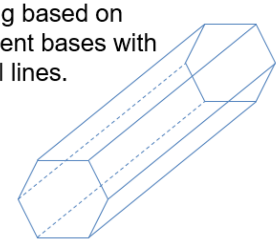
Discussion: The benefit of this type of problem is that it builds upon transformational geometry that has been emphasized in this course. It also has the ability to lay a foundation for students who will be pursuing advanced mathematics courses. Students should compare cases and come to the realization that the first figure is just a surface. It is "empty" like a jar, can, or empty box. The second figure is a solid made up of disks like a stack of tissues or a stack of coins or a mold of Jell-O. Part **b.** can be connected to G.GMD.4 with respect to the cross section that is parallel to the base. The volume of the figures can be found using $V = Bh$ which is the same formula that can be used for cylinders and prisms, so to calculate the volumes, the area of the base (preimage) and the height would be needed. Regardless of whether the figure is a solid or a surface (empty solid), the volume is the same. This problem can be extended to a discussion of surface areas. Surface area could be illustrated by using a piece of notebook paper to show the lateral area in combination with the two bases. *Note: Although not needed for this problem, students may want to find the area of the heart in order to calculate the volume. In order to find the area of the heart, they would need to use dissection arguments.*

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.GMD.1, 3)

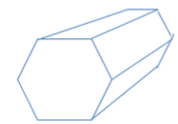
TIP!

It may be helpful to have students draw examples of prisms and cylinders based on definitions and then based on perspective. Have them explain any differences. For example, the corresponding edges on the bases of the perspective drawing would not be parallel. Instead they would be drawn as if they intersected at the horizon. The figure would change depending on the viewer's position.

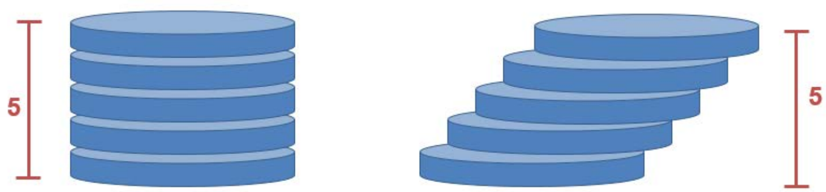
Drawing based on congruent bases with parallel lines.



Perspective drawing with non-parallel lines



Understanding the volume of cylinders builds upon the concepts of G.GMD.4: Identify the shapes of two-dimensional cross-sections of three-dimensional objects. As students identify the shapes of the cross-sections that are parallel to the base, they can make the connection that figures are just composed of layers where the height could represent the number of layers. In prisms and cylinders the layers are figures congruent to the base and in pyramid figures the layers are figures that are similar to the base.



Students should also explore Cavalieri's principle. Introduce Cavalieri's principle using a concrete model, such as a deck of cards. Use Cavalieri's principle with cross sections of cylinders, pyramids, and cones to justify their volume formulas. It can also be shown by stacking coins to make a right cylinder, and then shifting the coins to make a oblique cylinder illustrating that the volume remains constant.

Although the standard only calls for the volume of cylinders, it may be beneficial to also have students find the surface area of cylinders as finding the surface area of cylinders is missing from the standards.

Invite an architect or an engineer to visit the class to demonstrate some uses of cylinders, pyramids, cones or spheres in their work.

Pyramids and Cones (Cases of Conic Solids)

Just as some mathematicians consider prisms and circular cylinders special cases of generalized cylinders, some mathematicians consider circular cones and pyramids special cases of general cones. One definition of a conic solid is the set of points between the vertex and all the points of its base (which can be any shape) together with the vertex and the base. Using this definition, a cone is the surface of a conic solid whose base is a circle, and a pyramid is the surface of a conic solid whose base is a polygon.

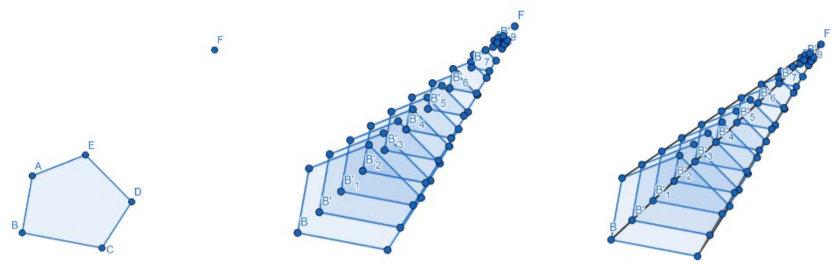
A conic solid can also be defined in terms of dilations. A conic solid is the set of points between a region and its dilated image in space, including the region and its image. In this course the scale factor of the dilated image is between 0 and 1.

Another definition of a conic solid is the set of all line segments from the apex to the base. Some mathematicians make a distinction between a cone and a pyramid by defining the figures in terms of their base. Again, in this instance a cone can be thought of a pyramid whose base has infinitely many sides. It is up to a district to determine how to define figures; however, using more inclusive definitions of generalized cones allows students to generalize the formula $V_{cone} = \frac{1}{3}Bh$, where B is the area of the Base, to a variety of figures. For more information regarding defining cones, see Cone by Wikipedia and Generalized Cone by Wolfram Mathworld.

Just as a cylinder can be related to cross sections congruent to the base that is created by translations, a cone can be related to the cross sections similar to the base, that is created by dilations with a scale factor *k* (where $1 > k > 0$) about a fixed center of dilation.

EXAMPLE

- Using dynamic geometric software create a shape and a center of dilation outside the figure.
- Dilate the original figure around the dilation point with a scale factor between 0 and 1.
- Choose a different scale factor between 0 and 1 and dilate the original figure again.
- Repeat the process in part **c.** an additional 12 times.
- Connect any vertices in your figure. What do you notice?
- Justify any observations you had in part **e.**

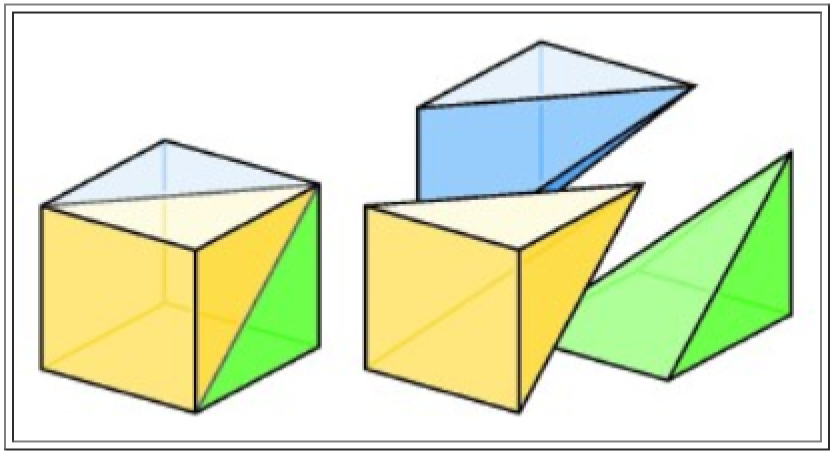


Discussion: Students should come to the conclusion that a conic solid or pyramid was formed (depending on the terminology used in the classroom). Each cross section is a dilation of the original 2D shape by a scale that is less than 1 that pushes the plane towards the center of dilation. This is because each of the cross sections are similar figures to the original figure (which becomes the base) and dilations less than 1 pull the plane towards the center of dilation. Discuss why a dilation maps a plane to a parallel plane. An interesting extension for students who will be pursuing advanced math courses would be to ask students "What would the cone would look like if the scale factor was not limited to positive numbers?"

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.GMD.1, 3)

For pyramids and cones, the presence of the factor $\frac{1}{3}$ in the formulas for volumes need some explanation. Using a set of plastic shapes, pour liquid or sand from one shape into another to informally demonstrate the relationship between the capacity (volume) of a cone and the capacity (volume) of a cylinder with the same base and height and a pyramid and prism with the same base and height. Another way to help students understand presence of the factor $\frac{1}{3}$ for pyramids is by using Geoblocks®. The set includes three pyramids with equal bases and altitudes that will stack to form a cube. An algebraic approach involves the formula for the sum of squares ($1^2 + 2^2 + \ldots + n^2$).

Although pouring water from pyramids into cubes (and cones into cylinders) is a good informal argument to illustrate that the volume of a pyramid is $\frac{1}{3}$ the volume of the prism with the same base and height, a more formal argument is disecting a cube into congruent pyramids. After the presence of the coefficient $\frac{1}{3}$ has been justified for the formula of the volume of the pyramid ($A = \frac{1}{3}Bh$), it can be argued that it must also apply to the formula of the volume of the cone by considering a cone to be a pyramid that has a base with infinitely many sides. There are applets and lesson plans with pyramid net templates in the Instructional Resources/Tools section.



Three congruent pyramids meet along a diagonal of a cube.

Image taken from Volume Patterns for Pyramids by Brown University
http://www.math.brown.edu/~banchoff/Beyond3d/chapter2/section02.html

TIP! The inclusion of the coefficient $\frac{1}{3}$ in the formulas for the volume of a pyramid or cone and $\frac{4}{3}$ in the formula for the volume of a sphere remains a mystery for many students. In high school, students should attain a conceptual understanding of where these coefficients come from. Concrete demonstrations, such as pouring water from one shape into another should be followed by more formal reasoning.

Using arguments based on informal limits can also help students understand the formula for a pyramid or cone.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.GMD.1, 3)

EXAMPLE

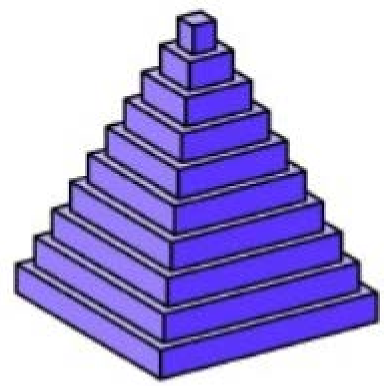
Note: This example is conducive to using a spreadsheet.

Part 1

- Draw or create a cube that is 10 cm x 10 cm x 10 cm.
- Find the volume of your cube.

Part 2

- Draw or create a pyramid with base that is 10 cm by 10 cm units and a height of 10 cm.
- Then draw or create a pyramid with base that is 10 cm by 10 cm consisting of 10 layers where each layer is 1 cm in height.
- What do you know about each of the layers?
- Find the volume of your figure.
- How does the drawing of your figure in part **b.** compare to your pyramid in part **a.**? Is there any way to get a more accurate pyramid?
- Draw or create a pyramid with base that is 10 cm by 10 cm consisting of 20 layers where each layer is 0.5 cm in height.
- Does the volume of your pyramid in part **f.** match the value of your pyramid in part **a.**? How could you get it closer?
- As a class decide how many layers you want in your structure to get it as close as you can to your figure in part **a**. You may want to use a spreadsheet to determine your calculations.
- What happens if you keep making thinner and thinner layers?
- How does your volume compare to the original cube in Part 1?
- What do you think the formula for a pyramid should be?



Discussion: Students should realize that each successive layer is a square with shorter sides. Since all squares are similar, they can calculate the volume by adding the layers together: $100 + 81 + 64 + 49 + 36 + 25 + 16 + 9 + 4 + 1 = 385\ cm^3$. Students should realize that although the volume of the figure in part **b.** is close to that of their pyramid in part **a.**, it is not exact because there are "gaps" between the two figures. Cavalieri's principle can also be applied to cones and pyramids. Through discussion, student should be able to come up with the idea that the more layers there are, the more closely the volume of the pyramid will align to the pyramid in part **a.** A pyramid with 20 square layers would have the volume of $(10^2 + 9.5^2 + 9^2 + 8.5^2 + 8^2 + 7.5^2 + 7^2 + 6.5^2 + 6^2 + 5.5^2 + 5^2 + 4.5^2 + 4^2 + 3.5^2 + 3^2 + 2.5^2 + 2^2 + 1.5^2 + 1^2 + 0.5^2 +)0.5 = 358.75\ cm^3$. As a class decide how precise you want to be. For example, suppose your students want 100 layers that are 0.1 cm high. In a spreadsheet label column A in cell A1 with the title "Base Length" and type "10" into cell A2. In cell A3 type "=A2 – 0.1." then drag the formula for 100 or so rows. Then label column B in cell B1 with the title "Area of the Base". Type "=A2^2" in cell B2 and drag down the formula. Label cell A103 as "Sum of Area Square", and write the formula "=Sum(B2:B102)" in B103. Then multiply that by the height of the layers which is 0.1 or use the formula "=B103*0.1" in another cell which would result in 338.35 cm^3. If students are more ambitious, you could repeat the same procedure in

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.GMD.1, 3)

the spreadsheet using 1,000 layers to get 333.8335 cm^3. Students should realize that as the number of layers increases and the difference between the side lengths of each of the subsequent layers get smaller and smaller the volume approaches $333.\overline{3}$ cm^3, which is $\frac{1}{3}$ the volume of the original cube in part 1.

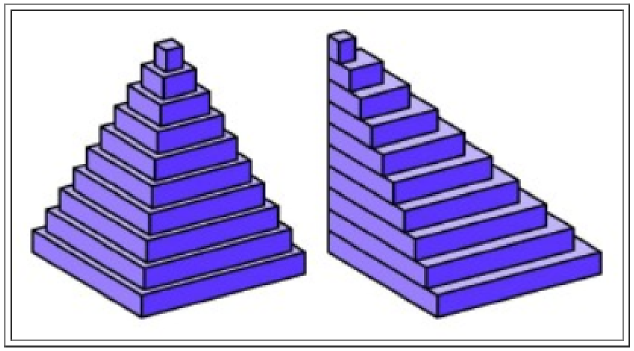
Part 3.

- Draw or create a cylinder with a radius of 10 cm and a height of 10 cm.
- Find the volume of your cylinder.

Part 4

- Draw or create a cone with a radius of 10 cm and a height of 10 cm.
- Draw or create a cone with layers like you did for the pyramid in Part 2. Find its volume.
- Use your spreadsheet to find the volume of your cone.
- How does the volume of the cone compare to the volume of the cylinder in Part 3?

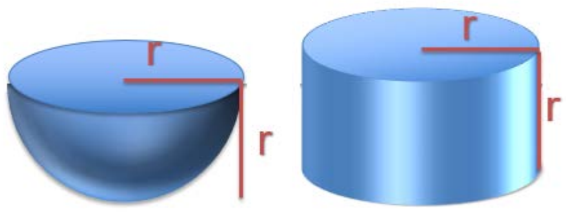
Note: Part 4 is similar to the process in part 2 with a circular base instead of a square base. The spreadsheet formulas need to be adjusted accordingly.



The same thin slabs approximate the volume of both a centered square-based pyramid and a pyramid with the same base and its top over one corner.

Image taken from Volume Patterns for Pyramids by Brown University

Students can generalize the formula for a pyramid to a cone using dissection arguments, transformations of layers or informal limit arguments.



Spheres

Consider using the following argument for the volume of a sphere. It may help to have students visualize a hemisphere "inside" a cylinder with the same height and "base." The radius of the circular base of the cylinder is also the radius of the sphere and the hemisphere. The area of the "base" of the cylinder and the area of the section created by the division of the sphere into a hemisphere is πr^2. The height of the cylinder is also r, so the volume of the cylinder is πr^3. Students can see that the volume of the hemisphere is less than the volume of the cylinder and more than half the volume of the cylinder. Illustrating this with concrete materials such as rice or water will help students see the relative difference in the volumes. At this point, students can reasonably accept that the volume of the hemisphere of radius r is $\frac{2}{3}\pi r^3$ and therefore volume of a sphere with radius r is twice that or $\frac{4}{3}\pi r^3$. There are several websites with explanations for students who wish to pursue the reasons in more detail, including the YouTube video Visualizing the Volume of a Sphere Formula: Deriving the Algebraic Formula with Animations by Kyle Pearce.

USING FORMULAS TO SOLVE PROBLEMS

This section involves geometric modeling. The formulas for volumes of cylinders, pyramids, cones and spheres can be applied to a wide variety of problems such as finding the capacity of a pipeline; comparing the amount of food in cans of various shapes; comparing capacities of cylindrical, conical and spherical storage tanks; using pyramids and cones in architecture, etc. Use a combination of concrete models and formal reasoning to develop conceptual understanding of the volume formulas. See the Instructional Resources/Tools section for examples of problems.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.GMD.1, 3)

Instructional Tools/Resources

These tools and resources may be helpful for instruction. Before use, they should be evaluated at the local level to determine their appropriateness for instruction

Manipulatives/Technology

- Rope or string
- Concrete models of circles cut into sectors and cylinders, pyramids, cones and spheres cut into slices.
- Rope or string
- Geoblocks® or comparable models of solid shapes
- Volume relationship set of plastic shapes
- Web site on Archimedes and the volume of a sphere,
- Dynamic geometric software (Geometer's Sketchpad®, Desmos®, Cabri®, or GeoGebra®)
- Video: *The Story of Pi* from Project MATHEMATICS!

Circumference of a Circle

- Circumference of a Circle by Illustrative Mathematics is a task where students find the circumference of a circle highlighting two different approaches: similar triangles and similarity of circles.
- Circumference = ? (Animation) by Tim Brzezinski is a GeoGebra applet illustrates the meaning of pi in terms of circumference divided by diameter.

Area of a Circle

- Area of a Circle by Illustrative Mathematics is a task where students find the area of a circle using Archimedes argument.
- Circle Area (by Peeling), Circle Area (by Peeling!), Area of a Circle (by Peeling), Area of a Circle-without Words (Animation 38) by Tim Brzezinski are GeoGebra applets that illustrate the formula for the are a circle by peeling.
- Area of a Circle by minoomath is a GeoGebra applet that shows how to find the area of a circle using a dissection argument.

Pi

- Improving Approximation for Pi with GeoGebra by tchoi99 is a GeoGebra Applet that allows students to test different values for polygon sides and compare the approximations of pi between Archimedes' and Snell-Huygens' methods.

Cavalieri's Principle

- Oblique and Right Pyramid-Cavalieri by Ted Coe is a GeoGebra applet that illustrates Cavalieri's principle.
- Cavalieri's Principle by Andreas Lindner is a GeoGebra applet that illustrates Cavalieri's principle.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.GMD.1, 3)

Cylinders and Prisms

- Volume Formulas for Cylinders and Prisms by Illustrative Mathematics is a task where students establish formulas for cylinders and prisms based on dissections.
- Volume of a Special Pyramid by Illustrative Mathematics is a task where students calculate the volume of a specific pyramid with a square base imbedded in a cube.
- Unwrapping a Cylinder by Tim Brzezinski is a GeoGebra applet that illustrates the surface area of cylinders.
- Unwrapping a Cylinder: Revamped! by Tim Brzezinski is a GeoGebra applet that illustrates the surface area of cylinders with an augmented reality addition.
- Right Triangular Prism! and Adjustable Triangular Prism by Tim Brzezinski is a GeoGebra applet that allows students to manipulate prisms in order to help them make sense of 2-D drawings of 3-D figures.
- Three Figures That Form a Cube by Wolfram Demonstrations Project is an applet that shows how a pyramid is $\frac{1}{3}$ of a cube.
- You Pour, I Choose, by Dan Meyer is a 3-act task that explores the volume of two different cylinders.
- Water Tank by Dan Meyer is a 3-act task that explores filling up a water tank.

Pyramids and Cones

- Trisecting the Cube into 3 Pyramids, Volume of Pyramids by Anthony OR 柯志明 is a GeoGebra applet that shows dissecting a cube into 3 pyramids.
- Square Pyramid: Underlying Anatomy by Tim Brzezinski is a GeoGebra applet that illustrates the difference between the height and the lateral height.
- Net of a Cone and Curved Surface Area of Cones (Combined Version) by Anthony OR 柯志明 is a GeoGebra applet that illustrates the surface area of a cone.
- Dissecting the Cube by Philip Busse and Beth McNabb is a lesson plan where students cut-out nets of square pyramids, make the pyramids, and then use them to make cubes thereby allowing them to compare the volumes of a cube and a pyramid.
- TI Online User Group October 2014 has 5 TI-Nspire geometry files. One of which is Pyramids and Cones which explores the volume formulas in connection with a cube.
- Arguments for Volume Formulas for Pyramids and Cones by Anthony Carruthers is a Better Lesson plan where students give informal arguments for volume.
- Prisms and Pyramids: How Many Pyramids Does It Take to Fill the Prism? by Kyle Pearce from Tap Into Teen Minds is a 3-act task where students explore the relationship between a pyramid and cube.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.GMD.1, 3)

Spheres

- Volume of Spheres and Volume of Spheres with Proofs by Anthony OR 柯志明 is a GeoGebra applet that shows the formula for the volume of a sphere.
- A Sphere in a Cylinder by Ted Coe is a GeoGebra applet that shows how to make sense of the formula for the volume of a sphere.
- TI Online User Group October 2014 has 5 TI-Nspire geometry files. One of which is Volume of a Sphere which uses Cavalier's Principal to find the volume of a sphere.
- Cones and Spheres: How Many Cones Does It Take to Fill the Sphere? by Kyle Pearce from Tap Into Teen Minds is a 3-act task where students explore the relationship between a cone and sphere.

Using Formulas to Solve Problems

- Greenhouse Management by Achieve the Core is a CTE task where students need to help their manager produce a crop of Easter Lilies. In one of the questions, students have to find the volume of a cylindrical pot.
- World's Largest Coffee Cup by Dan Meyer is a 3-act task where students need to apply their knowledge of the volume formula for a cylinder. Tap Into Teen Minds: Hot Coffee has accompanying resources for the task.
- Big Nickel by Andrew Stadel from Tap Into Teen Minds is a 3-act task where students need to find the volume of a giant nickel.
- Mix, Then Spray by Kyle Pearce from Tap Into Teen Minds is a 3-act task where students need to find the volume of spray bottle by using composite figures.
- Mustard Mayhem: How Many Bottles of Mustard Was Used? by Kyle Pearce from Tap Into Teen Minds is a 3-act task where students need to use the volume in a bottle of mustard to compare it to the amount of mustard in condiment cups.
- Water Tank: How Long Will It Take to Fill Up the Water Tank? by Dan Meyer is a 3-act task where students need to find how long it will take to fill a water tank.
- You Pour, I Choose: Which Glass Contains More Soda? by Dan Meyer is a 3-act task where students need to figure out which glass has more pop.
- Centerpiece by Illustrative Mathematics is a task where students have to apply the formula for the volume of a cylinder to model a real-life scenario.
- Doctor's Appointment by Illustrative Mathematics is a task where students have to analyze a real-life scenario involving a cone.
- Volume Estimation by Illustrative Mathematics is a modeling task where students have to apply volume formulas to a figure that is neither quite a cylinder nor a cone.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.GMD.1, 3)

Using Formulas to Solve Problems, continued

- The Great Egyptian Pyramids by Illustrative Mathematics is a task where students have to find the volume of a pyramid or using the volume to find a pyramid's base and height.
- Cubed Cans by NCTM Illuminations is a lesson where students try to find out how many cans fit in a prism. *Note: NCTM now requires a membership to view their lessons.*
- Sand Castles from Mathematics Vision Project and its accompanying teacher notes in partnership with the Utah State Office of Education has students explore the area and volume of pyramids.

Curriculum and Lessons from Other Sources

- EngageNY, Geometry, Module 3, Topic A, Lesson 1: What is Area?, Lesson 2: Properties of Area, Lesson 4: Proving the Area of the Disk are lessons that pertain to this cluster.
- EngageNY, Geometry, Module 3, Topic B, Lesson 5: Three-Dimensional Space, Lesson 6: General Prisms and Cylinders and Their Cross-Sections, , Lesson 8: Definition and Properties of Volume, , Lesson 10: The Volume of Prisms and Cylinders and Cavalieri's Principle, Lesson 11: The Volume Formula of a Pyramid and Cone, Lesson 13: How Do 3D Printers Work? are lessons that pertain to this cluster.
- Georgia Standards of Excellence Curriculum Frameworks, Geometry, Unit 4: Circles and Volumes has many tasks that pertain to this cluster.
- Mathematics Vision Project, Geometry, Module 7: Modeling with Geometry has several tasks that pertain to this cluster.
- Illustrative Mathematics, Geometry, Unit 4, Lesson 11: Approximating Pi is a lesson that pertains to this cluster.
- Illustrative Mathematics, Geometry, Unit 5: Solid Geometry, Lesson 9: Cylinder Volumes, Lesson 10: Cross Sections and Volume, Lesson 11: Prisms Practice, Lesson 12: Prisms and Pyramids, Lesson 13: Building a Volume Formula for a Pyramid, Lesson 14: Working with Pyramids, Lesson 15: Putting all the Solids Together, Lesson 16: Surface Area and Volume, Lesson 18: Volume and Graphing are lessons that pertain to this cluster.
- Illustrative Mathematics, Geometry, Unit 7, Lesson 10: Angles, Arcs and Radii is a lesson that pertains to this cluster.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.GMD.1, 3)

General Resources

- Arizona 7-12 Progression on Geometry is an instructional research for teachers. This cluster is addressed on page 19.
- Arizona High School Progression on Modeling is an informational resource for teachers. This cluster is addressed on page 17.
- High School Coherence Map by UnboundED is a map of the relationships among high school mathematics standards. Please note: This resource is aligned to the Common Core not Ohio's Learning Standards. Although the tool is useful, it needs to be checked with respect to Ohio's Learning Standards.
- van Hiele Model of Geometric Thinking is a pdf created by ODE that summarized the van Hiele levels.
- "Know the Formula" by Bill McCallum (one of the authors of the Common Core) is a thread from his blog Mathematical Musings. It gives some clarification surrounding the intent of knowing the formulas and some of the other issues surrounding the Geometry standards. Note that his blog refers to the Common Core, and not Ohio's Learning standards. During the standards revision, Ohio may have addressed some of these concerns with different outcomes than he suggests.

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STANDARDS	MODEL CURRICULUM (G.GMD.4)
Geometry **GEOMETRIC MEASUREMENT AND DIMENSION** **Visualize relationships between two-dimensional and three-dimensional objects.** **G.GMD.4** Identify the shapes of two-dimensional cross-sections of three-dimensional objects, and identify three-dimensional objects generated by rotations of two-dimensional objects.	**Expectations for Learning** In middle school, students identify cross-sections as a result of slicing right rectangular prisms and pyramids. In this cluster, which supports the previous cluster, students extend the identification of cross-sections to include other three-dimensional solids. Students will also identify three-dimensional objects created when a two-dimensional object is rotated about a line. The student understanding of this cluster begins at van Hiele Level 1 (Analysis) and moves to Level 2 (Informal Deduction/Abstraction). **ESSENTIAL UNDERSTANDINGS** • Two-dimensional figures can be used to understand three-dimensional solids. • A three-dimensional figure can be created by rotating a two-dimensional figure about a line. **MATHEMATICAL THINKING** • Draw a picture or create a model to make sense of a problem. • Use technology strategically to deepen understanding. • Make connections between concepts, terms, and properties. **INSTRUCTIONAL FOCUS** • Identify two-dimensional cross-sections of three-dimensional objects. • Identify three-dimensional objects formed by rotations of two-dimensional objects. *Continued on next page*

STANDARDS	MODEL CURRICULUM (G.GMD.4)
	Content Elaborations **OHIO'S HIGH SCHOOL CRITICAL AREAS OF FOCUS** • Geometry, Number 3, page 6 **CONNECTIONS ACROSS STANDARDS** • Experiment with transformations in the plane (G.CO.1, 5). • Understand congruence in terms of rigid motions (G.CO.6). • Explain volume formulas, and use them to solve problems (G.GMD.1, (+) 2, 3). • Understand the relationships between lengths, areas, and volumes (G.GMD.5-6). • Apply geometric concepts in modeling situations (G.MG.1-3). • Classify and analyze geometric figures (G.CO.14).

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.GMD.4)

Instructional Strategies

Note: The model curriculum is not a collection of lessons nor a full curriculum; it does not suggest pace, sequence, or amount of time spent on topics. It provides information about a topic related to the standards including ideas for examples, strategies for teaching, common misconceptions, and possible connections between topics. In addition, the emphasis of the model curriculum is on instruction not on assessment; therefore, the examples in the Instructional Strategies section are written for instructional purposes not necessarily as assessment items. Examples may need to be rewritten to accommodate the needs of each individual classroom.

Standards for Mathematical Practice
This cluster focuses on but is not limited to the following practices:
MP.1 *Make sense of problems and persevere in solving them.*
MP.4 *Model with mathematics.*
MP.5 *Use appropriate tools strategically.*

This cluster should be connected to standards G.GMD.1, 3. Slices of rectangular prisms and pyramids were explored in Grade 7. In high school, the concept is extended to a wider class of solids. Students who eventually take calculus will learn how to compute volumes of solids by methods involving cross-sections.

VAN HIELE CONNECTION

In Geometry students are expected to move from Level 1(Analysis) to Level 2 (Informal Deduction/Abstraction). In van Hiele Level 1 can be characterized by the student doing some or all of the following:

- showing a greater degree of attention to properties of shapes and solids;
- building 3D figures from 2D images and 2D drawings from 3D figures;
- viewing a figure from front, back, left, and right positions of solids;
- visualizing cross-sections when slicing solids;
- comparing solids based on properties;
- using observation as a basis for explanations; and/or
- understanding that a movement is made, then after observing the result, the next movement is selected, etc.

Whereas van Hiele Level 2 can be characterized by the student doing some or all of the following:

- using mathematical analysis of solids before performing any movements;
- giving informal justifications based on isolated properties; and/or
- analyzing beginning and final positions and making a plan to transform figures using a sequence of movements.

(Note: The significant difference between Level 1 and Level 2 is using logical reasoning. Level 1 activities can be turned into Level 2 by logical reasoning).

See the van Hiele pdf on ODE's website for more information about van Hiele levels.

TWO-DIMENSIONAL CROSS SECTIONS OF THREE-DIMENSIONAL OBJECTS

Two-dimensional cross sections of three-dimensional objects have many useful applications. Biologists uses cross sections to study cells, and geologists use cross sections to illustrate layers of the earth. Using cross sections can help students understand the concept of volume and helps build spatial reasoning.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.GMD.4)

Slice various solids to illustrate their cross sections. This could be done using clay, play-dough, hard-boiled eggs, fruit such as oranges, or even a Jell-O mold. Rubber bands may also be stretched around a solid to show a cross section. There are many useful applets listed in the Instructional Tool/Strategies section. It may be helpful to start with the cross sections in G.GMD.1,3 and continue in G.GMD.4. Ask students to find possible cross-sections of a cube. A square cross-section is the most obvious. Finding cross-sections in the shape of triangles, parallelograms, rectangles and hexagons may be more challenging.

Post-It notes or tag boards can be used to illustrate cross sections as well.

TIP! Students may believe the only cross section for a three-dimensional figure is parallel or perpendicular to the base, when in fact there are many other ones as well. *Note: some sources such as EngageNY make a distinction between a slice and a cross section. Ohio does not make such a distinction.*

TIP! Students may have difficulty distinguishing shadows/projections from slices/cross sections.

TIP! Use 3-D Printers to illustrate cross sections to reinforce concepts of volume.

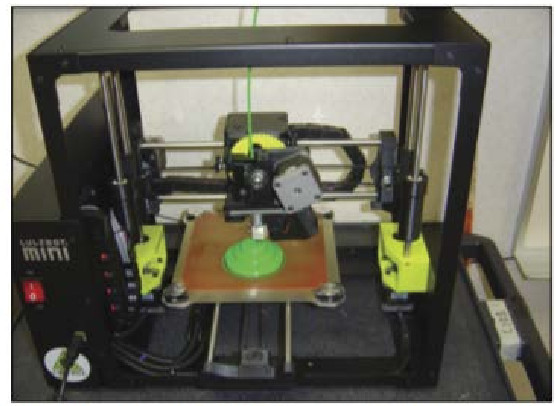


Fig. 3 We used a LulzBot Mini™ 3D printer to print models for volumes of revolution and cross section.

Images taken from Popelka, S. & Langlois, J. (March 2018). Getting out of Flatland.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.GMD.4)

Have students explore the sequence of cross-sections parallel to the base of 3D-objects. The thickness of the cross-sections may vary but as cross-sections become thinner, the lateral faces of the three-dimensional objects get smoother and the figure looks more and more like the original object.

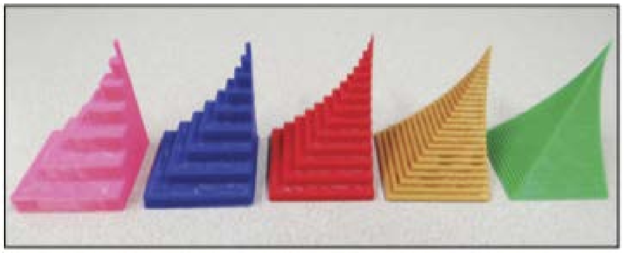


Fig. 5 These models for volume by cross sections show the progression from 7 squares on the left to a "smooth" surface on the right.

TIP!

The book *Flatland: A Romance of Many Dimensions*, by Edwin Abbot can be used to illustrate the idea of cross sections and dimensions.

THREE-DIMENSIONAL OBJECTS GENERATED BY ROTATIONS OF TWO-DIMENSIONAL OBJECTS

Manufacturers can create objects such as axels, funnels, pills, bottles, and pistons by rotating an object around an axis of revolution. Graphic designers can also use these methods to generate images. The simplest of these types of solids is a cylinder where a rectangle is rotated about an axis coinciding to one side of a rectangle.

Students may have difficulty grasping that a rotation of a two-dimensional object creates a circular base for the three-dimensional object, therefore it is vital that students have concrete experiences rotating two-dimensional objects. Use a two-dimensional shape, and spin it to show its three-dimensional representation. An example of this is taping a shape to a pencil and using a drill to spin it. Another method is to cut a half-inch slit in the end of a drinking straw, and insert a cardboard cutout shape. Rotate the straw and observe the three-dimensional solid of revolution generated by the two-dimensional cutout. Party decorations like a "honeycomb paper bell" that starts as a two-dimensional shape and fans to be a three-dimensional shape are good visual representations of this effect.

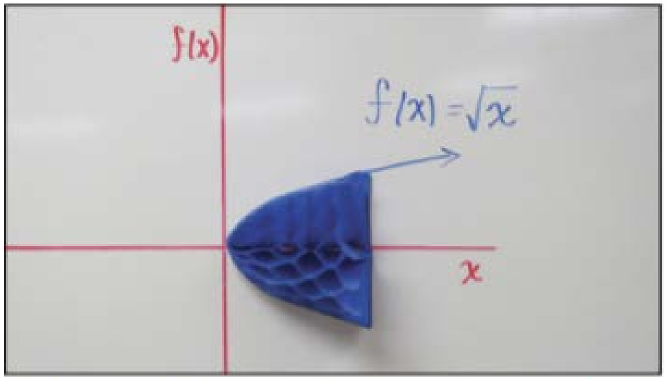


Fig. 1 Honeycomb cut-outs can be used to model a surface of revolution.

Images taken from Popelka, S. & Langlois, J. (March 2018). Getting out of Flatland.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.GMD.4)

TIP!

Encourage students to make drawings of solids with highlighted cross sections or drawings of solids generated by rotation two-dimensional shapes.

MODELING CONNECTION

People use cross sections in real-life to model different situations which enables them to answer questions or solve problems. One example is telling the age of a tree by the rings in its stump (or, in mathematics terms, cross-sections of the tree). Another example is when the Thai soccer team was stuck in a cave in Thailand (See diagram at the right). They used cross sections to show the size of the different areas of the cave. Cross sections are also used in science and social studies to illustrate the world around us. Use illustrations of cross sections from other content areas such as eyes, volcanoes, and cells to illustrate their usefulness.

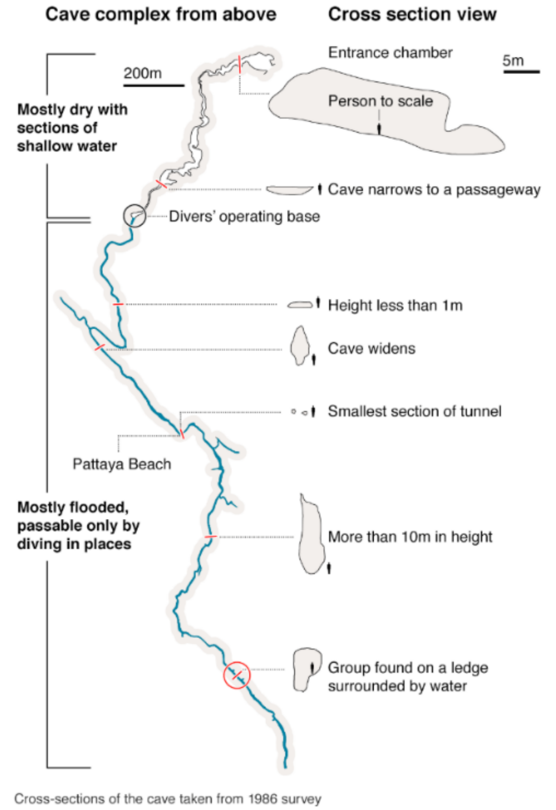


Image taken from BBC News, [Cave Rescue: Four More Boys Rescued by Thai Navy Divers] on July 9, 2018.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.GMD.4)

Instructional Tools/Resources

These tools and resources may be helpful for instruction. Before use, they should be evaluated at the local level to determine their appropriateness for instruction.

Manipulatives/Technology

- Concrete models of solids such as cubes, pyramids, cylinders, and spheres. Include some models that can be sliced, such as those made from Styrofoam.
- Rubber bands
- Cardboard cutouts of 2-D figures (e.g. rectangles, triangles, circles)
- Drinking straws
- Geometric software
- *Flatland: A Romance of Many Dimensions*, by Edwin Abbott
- Web sites, that illustrate geometric models. Some examples are The Geometry Junkyard and Wolfram Mathworld

Two-dimensional Cross Sections of Three-dimensional Objects

- Sections of Rectangular Prisms (Cuboids) by Anthony OR 柯志明 is a GeoGebra applet that shows the cross sections of a rectangular prism.
- Sections of Triangular Prisms by Anthony OR 柯志明 is a GeoGebra applet that shows the cross sections of a triangular prism.
- Sections of Cylinders by Anthony OR 柯志明 is a GeoGebra applet that shows the cross sections of a cylinder.
- Sections of Rectangular Pyramids by Anthony OR 柯志明 is a GeoGebra applet that shows the cross sections of a rectangular pyramid.
- Sections of Triangular Pyramids by Anthony OR 柯志明 is a GeoGebra applet that shows the cross sections of a triangular pyramid.
- Sections of Cones by Anthony OR 柯志明 is a GeoGebra applet that shows the cross sections of a cones.
- Sections of Spheres by Anthony OR 柯志明 is a GeoGebra applet that shows the cross sections of a spheres.
- Exploring Sections of Cubes by Anthony OR 柯志明 is a GeoGebra applet that shows the cross sections of a cubes.
- Cube Dissection Problem by Anthony OR 柯志明 is a GeoGebra applet that has students figure out how many pieces would be obtained by dissecting a cube with planes through all the diagonals of the six faces.
- The Flatland Game by Julian Fleron and Volker Ecke in the NSF funded book, *Discovering the Art of Geometry*, is game where students determine the identity of a solid object from a series of parallel cross sections taken at regular intervals.
- Slicing a Cube by NCTM Illuminations is a task that has students answer questions about a cross section of a cube. *Note: NCTM now requires a membership to view their lessons.*
- Ants Marching by NCTM Illuminations is a task that has students answer questions about a cross section of a cube. *Note: NCTM now requires a membership to view their lessons.*
- Cross Section Flyer by Shodor is an applet that shows cross sections and can be manipulated for various cross sections of the same three-dimensional figure.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.GMD.4)

Two-dimensional Cross Sections of Three-dimensional Objects, continued

- Can You Cut It? Slicing Three-Dimensional Figures is a CPalms lesson where students sketch, model, and describe cross-sections formed by a plan passing through a three-dimensional figure.
- Sections of a Cube by AssocTeachers math is a YouTube video that shows cutting through a cube in a number of different ways and examining the cross section of each.
- Math Shorts Episode 8—Slicing Three Dimensional Figures by PlanetNutshell is a YouTube video about slicing three-dimensional figures.
- Tennis Balls in a Can by Illustrative Mathematics is a task where students explore cross sections in the context of tennis balls in a can.
- Problem 6: Plasticine Geometry from the University of Waterloo is a worksheet that has students predict shapes of cross sections and then find an actual cross section using fishing line and plasticine.

Three-dimensional Objects Generated by Rotations of Two-Dimensional Objects

- Rotate Triangle by Sobarrera is a GeoGebra applet that shows a cone formed by rotating a triangle.
- Rotate Rectangle by Sobarrera is a GeoGebra applet that shows a cylinder formed by rotating a rectangle.
- Rotate Circle by Sobarrera is a GeoGebra applet that shows a sphere formed by rotating a cylinder.
- Rotate Triangle by Sobarrera is a GeoGebra applet that shows the figure formed by rotating a trapezoid.
- Investigation 6.4.4 Creating Solids of Rotation Using GeoGebra is a lesson where students rotate figures using GeoGebra.

Curriculum and Lessons from Other Sources

- EngageNY, Geometry, Module 3, Topic B, Lesson 5: Three-Dimensional Space, Lesson 6: General Prisms and Cylinders and Their Cross-Sections, Lesson 7: General Pyramids and Cones and Their Cross-Sections and Lesson 13: How Do 3D Printers Work? are lessons that pertain to this cluster. are lessons that pertain to this cluster.
- Georgia Standards of Excellence Curriculum Frameworks, Geometry, Unit 4: Circles and Volumes has many tasks that pertain to this cluster.
- Mathematics Vision Project, Geometry, Module 7: Modeling with Geometry has several tasks that pertain to this cluster.
- Illustrative Mathematics, Geometry, Unit 5: Solid Geometry, Lesson 1: Solids of Rotation, Lesson 2: Slicing Solids, Lesson 3: Creating Cross Sections by Dilating are lessons that pertain to this cluster.

General Resources

- Arizona 7-12 Progression on Geometry is an informational resource for teachers.
- High School Coherence Map by UnboundED is a map of the relationships among high school mathematics standards. Please note: This resource is aligned to the Common Core not Ohio's Learning Standards. Although the tool is useful, it needs to be checked with respect to Ohio's Learning Standards.
- van Hiele Model of Geometric Thinking is a pdf created by ODE that summarizes the van Hiele levels.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.GMD.4)

References

- Cochran, J., Cochran, X., Laney, K., & Dean, M. (May 2016). Expanding geometry understanding with 3D printing. *Mathematics Teaching in the Middle School*, *21, (9),* 534-542.
- Popelka, S. & Langlois, J. (March 2018). Getting out of Flatland. *Mathematics Teacher*, *111, (5),* 352-400.

STANDARDS	MODEL CURRICULUM (G.GMD.5-6)
Geometry **GEOMETRIC MEASUREMENT AND DIMENSION** **Understand the relationships between lengths, areas, and volumes.** **G.GMD.5** Understand how and when changes to the measures of a figure (lengths or angles) result in similar and non-similar figures. **G.GMD.6** When figures are similar, understand and apply the fact that when a figure is scaled by a factor of k, the effect on lengths, areas, and volumes is that they are multiplied by k, k^2, and k^3, respectively.	**Expectations for Learning** In middle school, students solve problems involving two-dimensional similar figures and calculate the volumes of three-dimensional figures. In this cluster, students extend their knowledge of similarity to explore and understand how changes to length or angle measure in one figure will result in similar or non-similar figures. Students will also understand the effect that a scale factor has on the length, area, and volume of similar figures and use this relationship to solve problems. The student understanding of this cluster begins at van Hiele Level 1 (Analysis) and moves to Level 2 (Informal Deduction/Abstraction). **ESSENTIAL UNDERSTANDINGS** • Changes to the lengths and/or angle measures of a figure result in similar and non-similar figures. • When changes to a figure result in similar figures with a scale factor of k, the lengths of the resulting figures are a multiple of k. • When changes to a figure result in similar figures with a scale factor of k, the areas of the resulting figures are a multiple of k^2. • When changes to a figure result in similar figures with a scale factor of k, the volume of the resulting figures are a multiple of k^3. **MATHEMATICAL THINKING** • Use precise mathematical language. • Draw a picture or create a model to make sense of a problem. • Determine reasonableness of results. • Solve multi-step problems accurately. • Plan a solution pathway. • Solve mathematical and real-world problems accurately. • Consider mathematical units involved in a problem. • Attend to the meaning of quantities. • Recognize, apply, and justify mathematical concepts, terms, and their properties. • Generalize concepts based on patterns. *Continued on next page*

STANDARDS	MODEL CURRICULUM (G.GMD.5-6)
	Expectations for Learning, continued **INSTRUCTIONAL FOCUS** • Classify objects as similar or non-similar when the lengths or angles of figures are changed. • Explain the types of changes to a figure that result in similar and non-similar figures. • Use geometry and algebra to explain how length, area, and volume are affected when scaling is applied. • Solve problems involving length, area, and volume of figures under scaling. **Content Elaborations** **OHIO'S HIGH SCHOOL CRITICAL AREAS OF FOCUS** • Geometry, Number 6, page 9 **CONNECTIONS ACROSS STANDARDS** • Explain volume formulas, and use them to solve problems (G.GMD.1, (+) 2, 3). • Understand similarity in terms of similarity transformations (G.SRT.1-2). • Apply geometric concepts in modeling situations (G.MG.2-3).

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.GMD.5-6)

Instructional Strategies

Note: The model curriculum is not a collection of lessons nor a full curriculum; it does not suggest pace, sequence, or amount of time spent on topics. It provides information about a topic related to the standards including ideas for examples, strategies for teaching, common misconceptions, and possible connections between topics. In addition, the emphasis of the model curriculum is on instruction not on assessment; therefore, the examples in the Instructional Strategies section are written for instructional purposes not necessarily as assessment items. Examples may need to be rewritten to accommodate the needs of each individual classroom.

Standards for Mathematical Practice

This cluster focuses on but is not limited to the following practices:

***MP.2** Reason abstractly and quantitatively.*

***MP.4** Model with mathematics.*

***MP.5** Use appropriate tools strategically.*

***MP.7** Look for and make use of structure.*

VAN HIELE CONNECTION

In Geometry students are expected to move from Level 1(Analysis) to Level 2 (Informal Deduction/Abstraction) and even touches on Level 3 (Deduction). Van Hiele Level 2 can be characterized by the student doing some or all of the following:

- fully understanding the procedures and formula for determining measurement of simple, familiar shapes;
- generalizing unit-measure enumeration to include fractional units;
- understanding and making unit conversions; and/or
- generalizing measurement to non-squares for area and non-cubes for volume.

Van Hiele Level 3 can be characterized by the student doing some or all of the following:

- comparing shapes by property-preserving transformations/decompositions;
- using geometric properties and variables to understand and solve problems involving formulas for non-familiar shapes; and/or
- comparing length, area, and volume by using geometric properties or transformations.

See the van Hiele pdf on ODE's website for more information about van Hiele levels.

SCALING

Students need to understand that scaling produces similar figures, but for figures to remain similar, the scale needs to be applied to each dimension. Give students time to build things by scaling. Rep-tiles, pattern-blocks, Legos, and cubes may help. A rep-tile is a shape that can be dissected into copies of the same shape that is used in tessellations. (More resources on rep-tiles can be found in the Instructional Tools/Resources section.) The goal is to help students achieve an intuition about scaling so that they think "I doubled the side lengths, so I get four figures that are the same as the original," etc. Students should explore non-similar solids as well. For example, multiply the length and width of a prism but not the height results in non-similar solids.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.GMD.5-6)

Many students do not see a difference between the unit of measures in the three different dimensions, and so they struggle with the correct use of unit of measure for the dimension they should be working in. For example, they may confuse square inches with cubic inches. Teachers could use multiple visual examples to help physically demonstrate lengths are one-dimensional, areas are two-dimensional, and volumes are three-dimensional and tie this into appropriate unit for the measurement.

EXAMPLE

Part 1

a. Create similar triangles using rep-tiles.
b. Create similar parallelograms using rep-tiles.

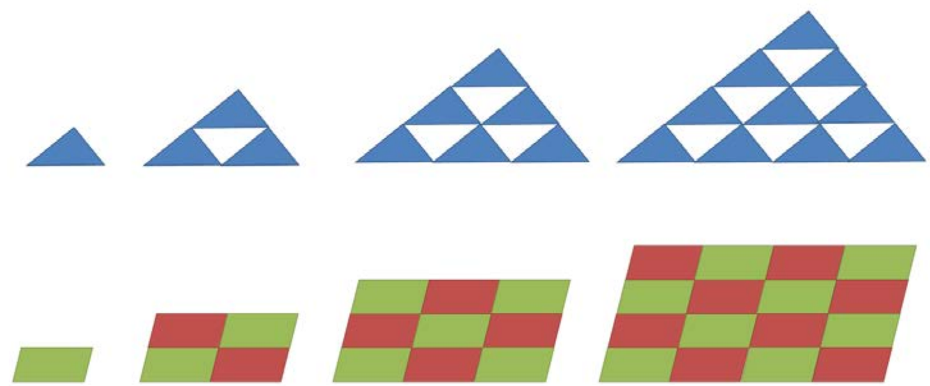
Discussion: The purpose of this activity is that students can understand that to properly scale a figure, the figure must scale by the same factor in all dimensions. Once students are able to visualize this, they will be ready for questions about perimeter, area, and volume.

Part 2:

Create a figure made out of tiles and change the side lengths of the figure as stated below:

a. How many tiles did you get when you double the side lengths of the figure? Explain.
b. How many tiles did you get when you triple the side lengths of the figure? Explain.
c. How many tiles did you get when you quadruple the side lengths of the figure? Explain.
d. How many tiles would you get if you changed the side length by a factor of k? Explain.
e. How does the perimeter in two similar figures relate?
f. How does the area in two similar figures relate?

Discussion: If you double the side lengths of a triangle (as shown in the first diagram below), then four of the original triangles can fit in the new figure. If you triple the side lengths of a triangle, then 9 triangles can fit in, etc. This may be clearer for parallelograms (as shown in the second diagram), but it is harder to see with triangles because the triangles require rotation. These diagrams are built using figures called "rep-tiles." In addition to the geometric (visual), you can also use an algebraic representation. Think about the area formulas for parallelograms and triangles. If you dilate the parallelogram, you multiply the base by k and the height by k, then the perimeter is multiplied by k and the area is multiplied by k^2. Students should come to the conclusion that the number of tiles changes by a factor of k^2.



INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.GMD.5-6)

Part 3

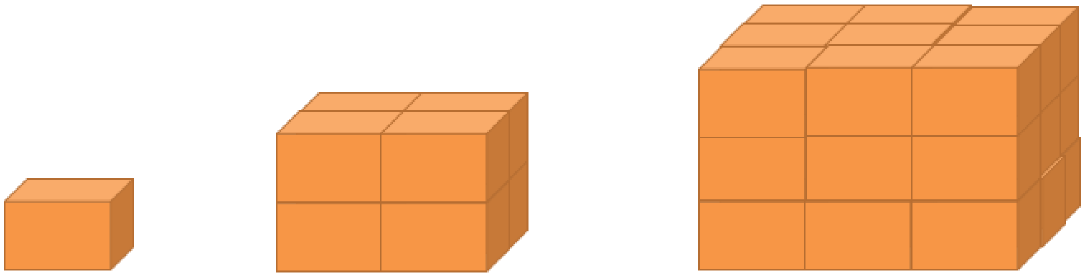
- **a.** Create similar figures to a 2 by 2 Lego brick.
- **b.** Create similar figures to a 2 by 3 Lego brick.
- **c.** Create similar figures to a 2 by 4 Lego brick.

Discussion: The purpose of this activity is that students can understand that to properly scale a three-dimensional figure, the figure must scale by the same factor in all three dimensions. Many students will incorrectly only want scale in two directions.

Part 4

- **a.** How many Legos do you get when you double the side lengths of the figure? Explain.
- **b.** How many Legos do you get when you triple the side lengths of the figure? Explain.
- **c.** How many Legos do you get when you quadruple the side lengths of the figure? Explain.
- **d.** How many Legos would you get if you changed the side length by a factor of k? Explain.
- **e.** How does the volume in two similar figures relate? Explain.
- **f.** How does the surface area of the two figures relate? Explain.

Discussion: If you double the side lengths of a prism (as shown in the first diagram below), then eight Legos fit in the new figure. If you triple the side lengths of a prism, then 27 Legos can fit in, etc. The idea in Parts 1 and 2 can be extended to 3D figures by multiplying the length, width, and height of a prism by the same scale factor. Explain what happens to the surface area (multiplied by k^2) and volume (multiplied by k^3). Students should come to the conclusion that the number of Legos changes by a factor of k^3 and can be connected to volume.



INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.GMD.5-6)

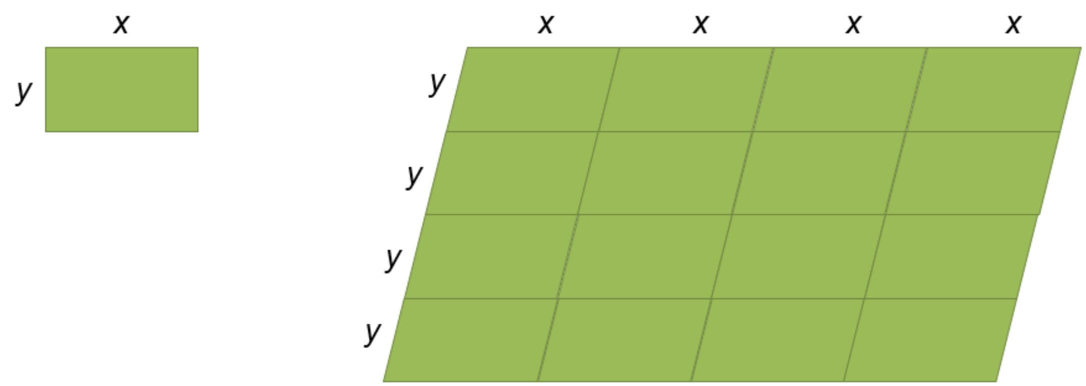
The misconception of units extends into the use of proportions. Students assume that if the ratio of the lengths is *a*:*b* then the ratio of the areas or volumes is also *a*:*b*, which is incorrect. They may also incorrectly think that objects that are dilated with a scale factor *k*, then $\frac{a}{b} = \frac{a^2}{b^2} = \frac{a^3}{b^3}$. That is why it is important for students to do hands-on activities to show that if an object grows multiplicatively in all directions or it grows the same amount in the other direction or dimension. For example, if a rectangle with side lengths a and b doubles by a scale factor of *k*, the proportions describing the side length would be $\frac{a}{b} = \frac{2a}{2b}$, but the area would be changed by a factor of k^2. Teachers should stress that proportions only work if both ratios are in the same dimension. For example, $\frac{3cm}{4cm} = \frac{6cm}{8cm}$ or $\frac{1cm^2}{4cm^2} = \frac{12cm^2}{48cm^2}$, but $\frac{4cm}{8cm} \neq \frac{12cm^2}{24cm^2}$.

COMPARING SIMILAR AND NON-SIMILAR FIGURES

Students should explore what changes would create similar figures compared to non-similar figures. This connects to dilations in G.SRT.1-3. Since in similar figures corresponding angles are congruent and sides are proportional, changes that do not keep angles congruent and sides proportional result in non-similar figures. An example of this would be an additive change versus a multiplicative change. It is also important for students to discover that in order to keep similarity, the multiplicative change must be applied to all dimensions. Students should discover that this concept applies to three-dimensional figures as well as two-dimensional figures. It is important for students to know that although congruent corresponding angles prove triangle similarity, this does not hold true for other polygons. For example, adding the same amount to the sides of a rectangle results in non-similar rectangles, yet the corresponding angles are still congruent. See Model Curriculum 7.G.1-3 for scaffolding ideas about similar figures and see Model Curriculum 6.RP.1-3 and 7.RP.1-3 for more information about multiplicative versus additive relationships.

EXAMPLE

Marissa said the two figures on the right are similar because their sides are proportional. Is she correct? Explain.



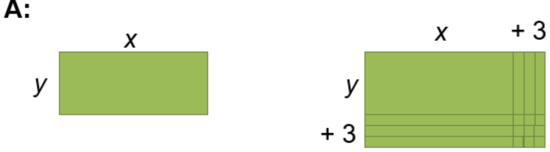
Discussion: Although there are four copies of the shape in each direction students make think the figures are similar. They may also say they are not similar because "I can see they aren't." Push students to more precisely state that the corresponding angles are not congruent, so the figures are not congruent.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.GMD.5-6)

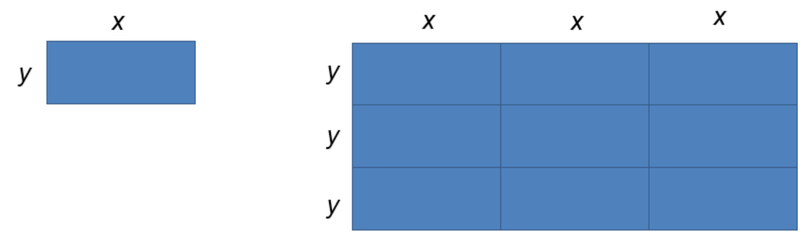
EXAMPLE

Which of the following situations shows a set of similar figures? Explain why each set either represents or does not represent similar figures.

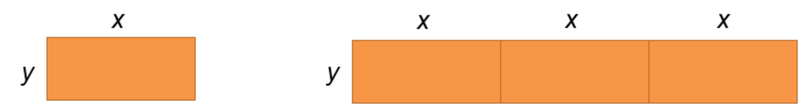
Set A:



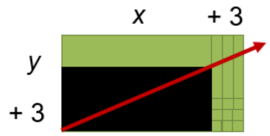
Set B:



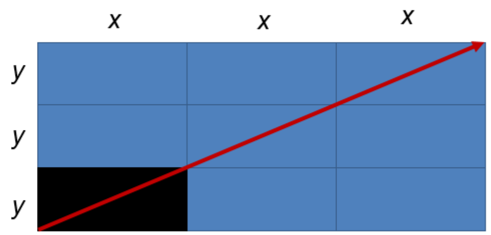
Set C:



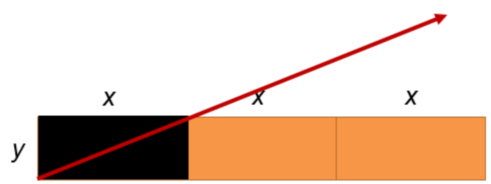
Set A:



Set B:



Set C:



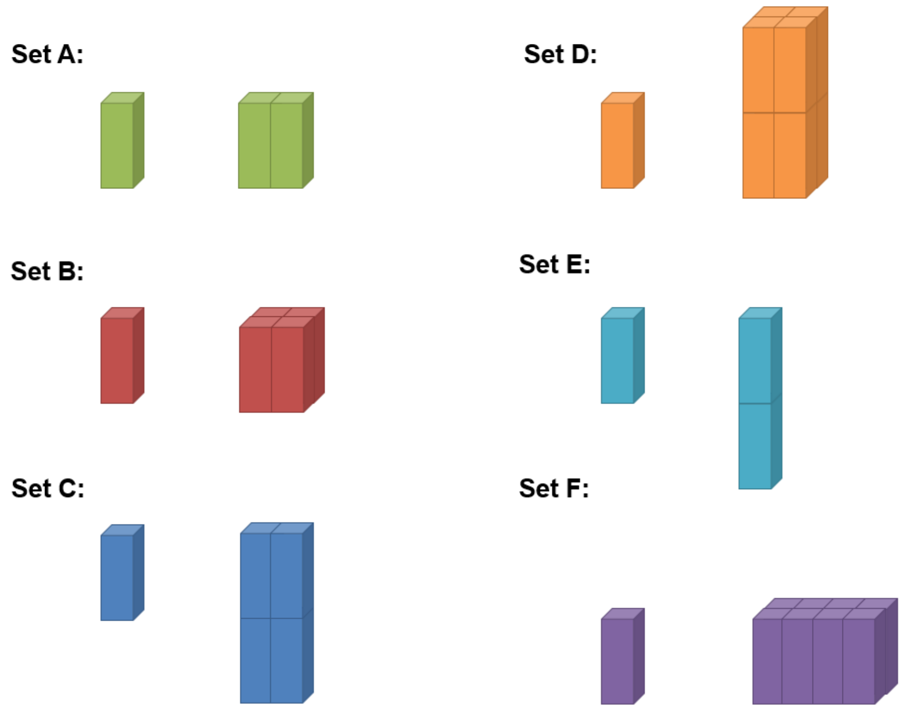
Discussion:

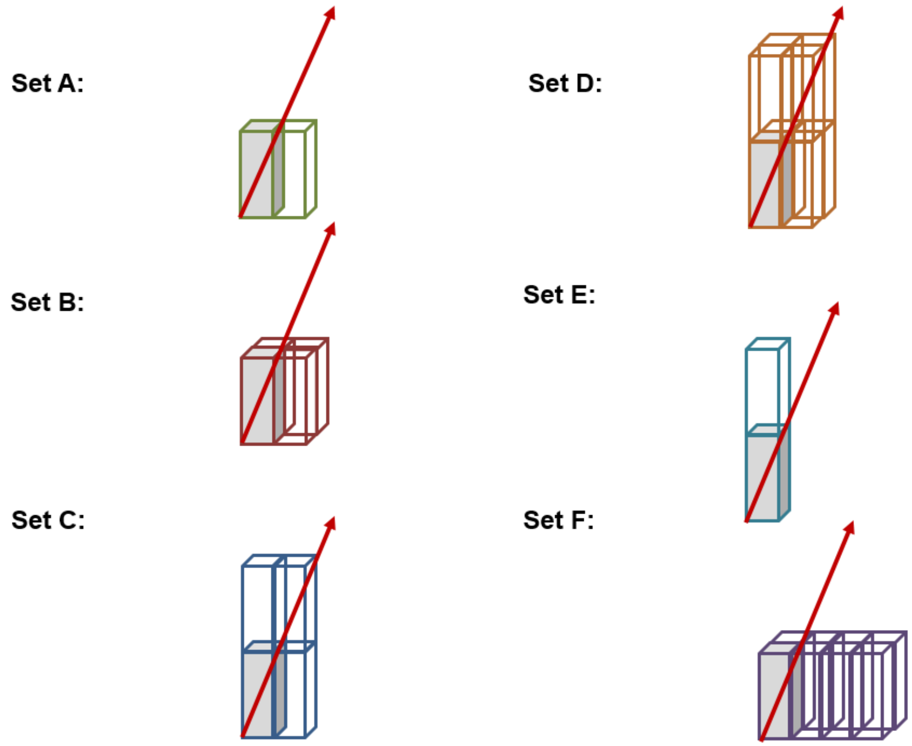
- Students should come to the realization that Set A does not show similar figures because it shows an additive relationship, and therefore the sides are not proportional. Set B shows a set of similar figures because both sides are proportional and angle measures are congruent. Set C does not show a set of similar figures because it the figure only changes in one-direction.
- Draw a connection between similar figures and dilations using the nesting method. Nested figures share a common vertex and vertex angle. Students can see that when nesting the figures atop of one another, only Set B has a line of dilation that goes through both corresponding vertices.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.GMD.5-6)

EXAMPLE

Which of the following situations (on the left) show a set of similar figures? Explain why each set either represents or does not represent similar figures.



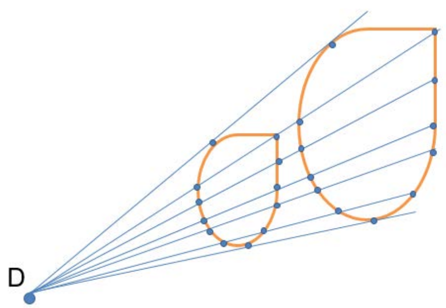


Discussion:

- Students may give a variety of explanations about which sets show similar figures. However, Set D is the only set that represents similar figures because the dimensions increase by a factor of 2 in each direction and the angles remain congruent.
- Students can also use the nesting method to help them in this situation as well. A line that connects the left-front vertex and the back-right vertex of the original prism should also go through the left-front vertex and the back-right vertex of the resulting prism. This is shown in the diagram by Set D. Have students explain why this is so by relating it to concepts of dilations.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.GMD.5-6)

This idea of using rays generating from the same point of dilation that pass through corresponding points also generalized for irregular figures.



Change the orientation of similar figures and have students explain why they are still similar. This prevents them from creating a misconception that similar figures must have the same orientation.

Fundamental Theorem of Similarity

There are several versions of the Fundamental Theorem of Similarity. It is up to each district to choose how they want to name the Fundamental Theorem of Similarity. The important thing is that students understand the concepts underlying both versions. Here are two examples:

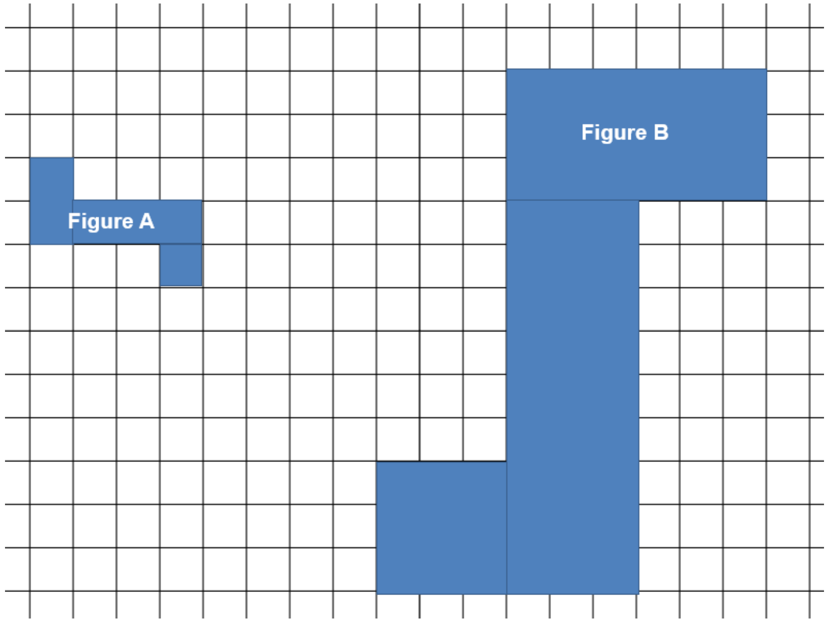
The University of Chicago Mathematics Project: Geometry textbook states:
If $G \sim G'$ and k is the ratio of similitude, then

- **a.** $Perimeter\ (G') = k \cdot Perimeter\ (G)$ or $\frac{Perimeter\ (G')}{Perimeter\ (G)} = k$;
- **b.** $Area\ (G') = k^2 \cdot Area\ (G)$ or $\frac{Area\ (G')}{Area\ (G)} = k^2$; and
- **c.** $Volume\ (G') = k^3 \cdot Volume\ (G)$ or $\frac{Volume\ (G')}{Volume\ (G)} = k^3$

H. Wu states (whose thoughts EngageNY is based upon): "Given a dilation with center O and a scale factor r, then for any two points P and Q in the plane so that $O, P,$ and Q are not collinear, the lines PQ and $P'Q'$ are parallel, where $P' = Dilation(P)$ and $Q' = Dilation(Q)$, and furthermore, $|P'Q'| = r|PQ|$."

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.GMD.5-6)

TIP! The object's unit of measure can help students remember the scale factors associated with each measurement. Since area is measured in square units, k is squared. Since volume is measured in cubic units, k is cubed.



EXAMPLE

Figure A and B are similar. Figure B has been created by rotating and dilating Figure A by a scale factor of 3.

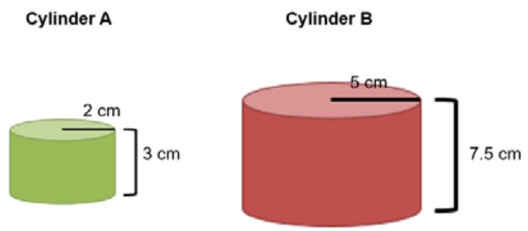
a. If the area of Figure A is 6 cm^2, what is the area of Figure B? How can you find out without counting the squares of Figure B?

b. Check to see if you are correct.

Discussion: Students should make the connection that area changes by a scale factor of k^2, so the new area would be $6 \cdot 3^2$ or 54 cm^2.

EXAMPLE

Cylinder B is the result of similarity transformation of Cylinder A.



a. What is the ratio of the radii for the two figures?

b. What is the ratio of the surface area for the two figures?

c. What is the ratio of the volume for the two figures?

Discussion: The ratio of the radii for the two cylinders is $\frac{5}{2}$. Although students could calculate the surface areas and volumes of the cylinders to find the ratios, some students may realize that the area changes by a factor of k^2 under a similarity transformation and volume changes by a factor of k^3, so the ratio of the surface area is $\frac{25}{4}$ and the ratio of the volume is $\frac{125}{8}$.

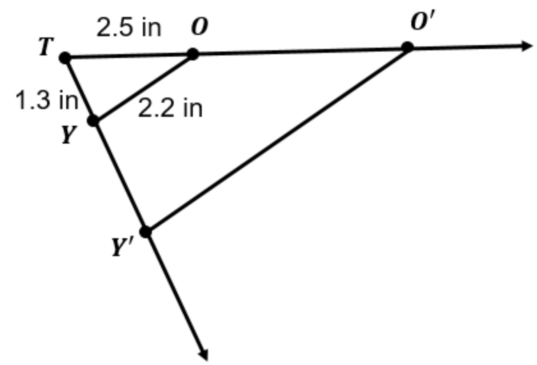
INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.GMD.5-6)

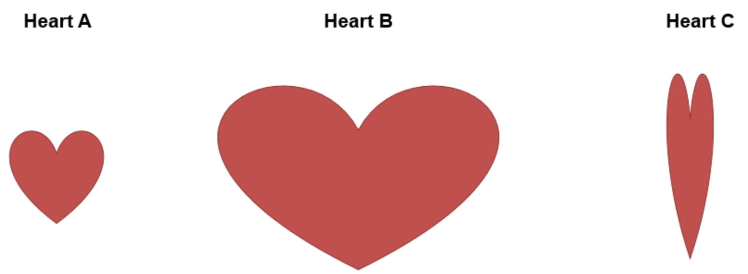
EXAMPLE

Note: Figure not drawn to scale

a. A dilation was performed on $\overline{OY}$ from the center of dilation T with a scale factor of 2.9. What is the perimeter of $\triangle TO'Y'$?

b. What is the length of $\overline{YY'}$?





EXAMPLE

a. If Heart A has an area of 16 in^2, and Heart B's height was doubled, and its width was tripled, what would the area of Heart B be?

b. If Heart A has an area of 16 in^2, and Heart C's height was doubled, and its width was halved, what would the area of Heart C be?

c. Are any of the hearts similar? Explain.

EXAMPLE

What would Lebron James look like if he was average height? Scale 6'8" LeBron down to 5'10."

Discussion: Lebron would be reduced by a scale factor of $\frac{7}{8}$ (6'8'' is 80 in and 5'10'' is 70 in). Students can explore what happens to his waist-size (original waist size multiplied by $\frac{7}{8}$ because waist size is a length), amount of fabric in jersey (original amount of fabric is multiplied by $\frac{7}{8}$ squared because amount of fabric is an area), etc.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.GMD.5-6)

EXAMPLE

a. Marcus wants to carpet a room that is 25 yd^2. Draw a scale drawing of the room in inches.
b. Fill in the blank: 1 yd^2 = ___ft^2 = ___in^2
c. Julia has a pool that has a capacity of 375 m^3. Draw a scale model in mm.
d. Fill in the blank: 1 m^3 = ___ cm^3 = ___ mm^3

EXAMPLE

Create a scale model of something in real-life. Include the original dimensions, your scale, and the new dimensions.

Instructional Tools/Resources

These tools and resources may be helpful for instruction. Before use, they should be evaluated at the local level to determine their appropriateness for instruction.

Manipulatives/Technology

- Tracing paper (patty paper)
- Transparencies
- Graph paper
- Ruler
- Protractor
- Compass
- Legos
- Pattern blocks or other shapes that can be tiled
- Computer dynamic geometric software (such as Geometer's Sketchpad®, Desmos, Cabri®, or GeoGebra®).
- Graphing calculators and other handheld technology such as TI-Nspire™.

Rep-Tiles

- Rep-Tiles by Mela Hardin is an activity where students discover and explore algebraic expressions through a special kind of tiling of the plane.
- Rep-Tiles by NCTM Illuminations is a lesson plan using rep-tiles. *Note: NCTM now requires a membership to view their lessons.*
- Rep-Tiles, Or How Mathematicians Start to Puzzle and Open Up Questions by Maxwell's Demon Vain Attempts to Construct Order is a blog that shows examples of rep-tiles.
- Rep-Tile by Wolfram MathWorld gives examples of rep-tiles.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.GMD.5-6)

Fundamental Theorem of Similarity

- Explore Fundamental Theorem of Similarity by HJ is a GeoGebra applet where students can explore FTS.
- Fundamental Theorem of Directly Similar Figures by Steve Phelps is a GeoGebra applet where students can explore FTS.

Scaling

- Geometry: Nested Similar Triangles by Texas Instruments is a lesson where students explore nested triangles to determine similarity.
- Scaling Away by NCTM Illuminations is a lesson where students compute the surface area and volume of a scale model. *Note: NCTM now requires a membership to view their lessons.*
- Scaling the City: Ground Truthing the Size of SimCity Objects by NCTM Illuminations is a lesson where students compare computer dimensions of objects to dimensions in real-life. *Note: NCTM now requires a membership to view their lessons.*
- Demonstrate the Effects of Scaling on Volume by Terry Lee Lindenmuth is a GeoGebra applet that allows students to scale a prism.
- Volume and Surface Areas of Similar 3D Figures by Anthony OR 柯志明 is a GeoGebra applet that allows students to scale a prism.
- 6.3 Investigating Connections between Measurements and Scale Factors of Similar Figures by NCTM is a lesson where students scale shapes on their applet and can see the corresponding graph of the scale factors. *Note: NCTM now requires a membership to view their lessons.*
- Blue Squares and Beyond by NCTM is a lesson where students construct and interpret figures using scale factors.

Curriculum and Lessons from Other Sources

- EngageNY, Grade 8, Module 3, Topic A, Lesson 4: Fundamental Theorem of Similarity and Lesson 5: First Consequences of FTS are lessons that pertain to this cluster.
- EngageNY, Geometry, Module 2, Topic A, Lesson 5: Scale Factors is a lesson that pertains to this cluster.
- EngageNY, Geometry, Module 3, Topic A, Lesson 3: The Scaling Principle for Areas is a lesson that pertains to this cluster.
- EngageNY, Geometry, Module 3, Topic B, Lesson 9: Scaling Principle for Volumes is a lesson that pertains to this cluster.
- Illustrative Mathematics, Geometry, Unit 5: Solid Geometry, Lesson 4: Scaling and Area, Lesson 5: Scaling and Unscaling, Lesson 6: Scaling Solids, Lesson 7: The Root of the Problem, and Lesson 8: Speaking of Scaling are lessons that pertain to this cluster.

General Resources

- Arizona 7-12 Progression on Geometry is an informational resource for teachers. This cluster is addressed on page 19.
- High School Coherence Map by UnboundED is a map of the relationships among high school mathematics standards. Please note: This resource is aligned to the Common Core not Ohio's Learning Standards. Although the tool is useful, it needs to be checked with respect to Ohio's Learning Standards.
- van Hiele Model of Geometric Thinking is a pdf created by ODE that summarizes the van Hiele levels.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.GMD.5-6)

References

- Coxford, A., Usiskin, Z., & Hirschorn, D. (1991). *The University of Chicago Mathematics Project: Geometry, Teacher's Edition.* Scott Foresman and Company, Glenview: IL.
- Ronau, R., Meyer, D. Crites, T., & Doughtery, B. (2015*). Putting essential understanding of geometry into practice*. The National Council of Teachers of Mathematics: Reston, VA.
- Snapp, B. & Findell, B. (2016). *Parallels in Geometry*. Creative Commons. Retrieved from https://people.math.osu.edu/findell.2/ParallelsInGeometry2016.pdf
- Wu, H. (2013). Teaching geometry in grade 8 and high school according to common core standards. Retrieved from https://math.berkeley.edu/~wu/CCSS-Geometry_1.pdf

STANDARDS	MODEL CURRICULUM (G.MG.1-3)
Geometry **MODELING WITH GEOMETRY** **Apply geometric concepts in modeling situations.** **G.MG.1** Use geometric shapes, their measures, and their properties to describe objects, e.g., modeling a tree trunk or a human torso as a cylinder.★ **G.MG.2** Apply concepts of density based on area and volume in modeling situations, e.g., persons per square mile, BTUs per cubic foot.★ **G.MG.3** Apply geometric methods to solve design problems, e.g., designing an object or structure to satisfy physical constraints or minimize cost; working with typographic grid systems based on ratios.★	**Expectations for Learning** In middle school, students work with nets, area, and volume; use appropriate tools to represent situations; and solve real-life and mathematical problems. In this cluster, students make sense of the world around them by using geometric models and their properties to solve more sophisticated problems. The student understanding of this cluster begins at van Hiele Level 1 (Analysis) and moves to Level 2 (Informal Deduction/Abstraction). **ESSENTIAL UNDERSTANDINGS** • Composite figures can be analyzed by approximating them with traditional two- and three-dimensional figures. • Many real-life scenarios are related to length, area, and volume. **MATHEMATICAL THINKING** • Use accurate mathematical vocabulary to represent geometric relationships. • Make connections between terms and properties. • Recognize, apply, and justify mathematical concepts, terms, and their properties. • Use formal reasoning with symbolic representation. • Determine reasonableness of results. • Use proportional reasoning. • Plan a solution pathway. • Connect mathematical relationships to real-world encounters. • Draw a picture or create a model to represent a problem. **INSTRUCTIONAL FOCUS** • Use geometric shapes, their measures, and their properties to describe objects. • Identify useful quantities for modeling situations. • Apply concepts of density based on area and volume. • Solve design problems geometrically. • *Continued on next page*

STANDARDS	MODEL CURRICULUM (G.MG.1-3)
	Content Elaborations **OHIO'S HIGH SCHOOL CRITICAL AREAS OF FOCUS** • Geometry, Number 2, page 5 • Geometry, Number 3, page 6 • Geometry, Number 5, page 8 **CONNECTIONS ACROSS STANDARDS** • Solving problems involving right triangles (G.SRT.8). • Use volume formulas to solve problems (G.GMD.3-4, 6). • Use coordinates to compute perimeters and areas (G.GPE.7).

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.MG.1-3)

Instructional Strategies

Note: The model curriculum is not a collection of lessons nor a full curriculum; it does not suggest pace, sequence, or amount of time spent on topics. It provides information about a topic related to the standards including ideas for examples, strategies for teaching, common misconceptions, and possible connections between topics. In addition, the emphasis of the model curriculum is on instruction not on assessment; therefore, the examples in the Instructional Strategies section are written for instructional purposes not necessarily as assessment items. Examples may need to be rewritten to accommodate the needs of each individual classroom.

Standards for Mathematical Practice

This cluster focuses on but is not limited to the following practices:

***MP.1** Make sense of problems and persevere in solving them.*

***MP.2** Reason abstractly and quantitatively.*

***MP.4** Model with mathematics.*

***MP.5** Use appropriate tools strategically.*

***MP.7** Look for and make use of structure.*

VAN HIELE CONNECTION

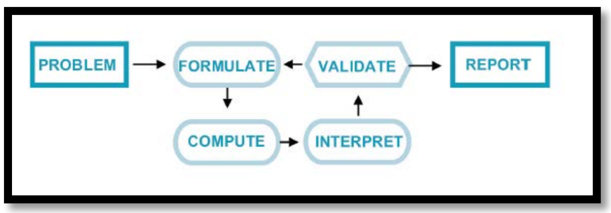
In Geometry students are expected to move from Level 1(Analysis) to Level 2 (Informal Deduction/Abstraction) and towards Level 3(Deduction).

Level 2 can be characterized by the student doing some or all of the following:

- fully understanding the procedures and formula for determining measurement of simple, familiar shapes;
- generalizing unit-measure enumeration to include fractional units;
- understanding and making unit conversions; and/or
- generalizing measurement to non-squares for area and non-cubes for volume.

Level 3 can be characterized by the student doing some or all of the following:

- comparing shapes by property-preserving transformations/decompositions;
- using geometric properties and variables to understand and solve problems involving formulas for non-familiar shapes; and/or
- comparing length, area, and volume by using geometric properties or transformations.



MODELING

This cluster is dependent on the modeling standards. See page 10 for more information about modeling.

In the standards, modeling means using mathematics or statistics to describe (i.e., model) a real-world situation to deduce additional information about the situation by mathematical or statistical computation and analysis. Modeling in high school uses problems that are not precisely formulated and may not necessarily have a "correct" answer. When making models, students need to figure out not only what to include in the model, but also what to exclude. Students also should be able to analyze and communicate the limitations of the model that they choose.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.MG.1-3)

TIP! The Consortium for Mathematics and Its Applications (COMAP) offers a High School Mathematical contest in Modeling which offers students the opportunity to compete in a team setting using mathematics to present solutions to real-world modeling problems. See their website for more information.

Genuine mathematical modeling typically involves more than one conceptual category. For example, modeling a herd of wild animals may involve geometry, measurement, proportional reasoning, estimation, probability and statistics, functions, and algebra. It would be somewhat misleading to try to teach a unit with the title of "modeling with geometry." Instead, these standards can be woven into several other content clusters.

The challenge for teaching modeling is finding problems that are interesting and relevant to high school students and, at the same time, solvable with the mathematical tools at the students' disposal.

TIP! Modeling activities are a good way to show connections among various branches of STEM fields. The sciences (e.g., phyiscs) use mathematics to model real-world phenomena.

DESCRIBING OBJECTS WITH GEOMETRIC MODELS

Provide students with opportunities to understand when real-life scenarios are related to length, area, and volume. For example, heating is related to volume, crop coverage is related to area, and traffic accidents are related to miles driven using probability concepts. Once they build this understanding, then they can use mathematics such as geometric shapes, their measures, and their properties to describe objects or situations. See the Instructional Resources/Tools section for examples.

Technology such as graphing calculators and spreadsheets can help students work with large amounts of data to make models of the real world.

Connecting Geometric Modeling with Probability.

Geometric modeling concepts can be integrated into probability situations. One example is the game of darts. The Instructional Resources/Tools sections has several links connecting probability, darts, and modeling at various levels.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.MG.1-3)

EXAMPLE

Part 1

- Diameter of playing circle: $13\frac{1}{4}$ inches
- Width of inner and outer rings: $\frac{1}{4}$ inches
- Diameter including the inner ring: $6\frac{1}{8}$inches
- Diameter of outer (double) bullseye: $1\frac{1}{4}$ inches
- Diameter of inner (triple) bullseye: $\frac{1}{2}$ inch

If Randy has no skill at darts and all his throws are random but hit the dartboard, find the theoretical probability of hitting the following targets:

a. inner bullseye
b. outer bullseye
c. area between the inner ring and outer bullseye
d. area between inner ring and outer ring
e. inner ring in any section
f. outer ring in any section
g. area of a section between the outer bullseye and inner ring
h. area of a section between the inner ring and outer ring

Discussion: This activity can be extended by having students create their own dartboard.

Information from this example taken from https://www.quia.com/files/quia/users/kimminr/GameDevelopment/Dartboard-Geometry.pdf_3

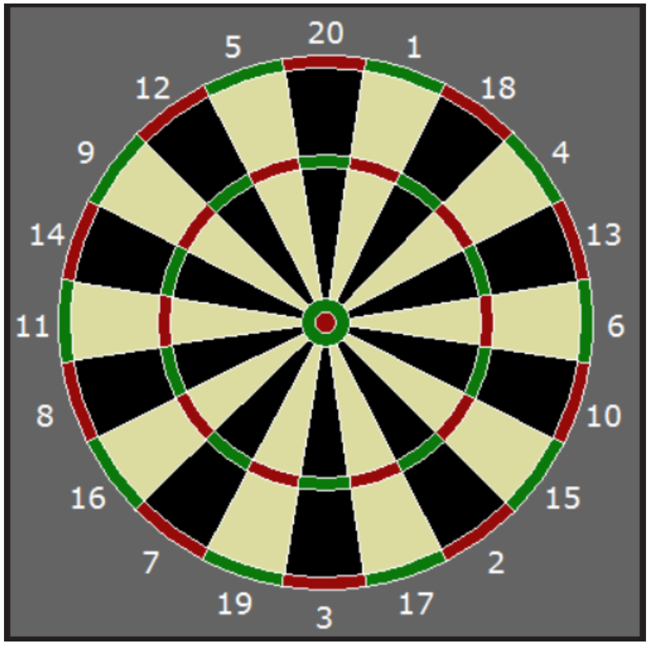


Image taken from dartboards. http://datagenetics.com/blog/january12012/index.html

Part 2

Darts can be scored with point values ranging from 1 to 20 depending on the sector with the corresponding number. If a dart lands in the outer ring, it is double the sector value, and if it lands in the inner ring it is triple the sector value. The inner bullseye is 50 points and the outer bullseye is 25 points.

a. Find the probability of the following assuming that the throwing is random but lands on the board:
  - three inner bullseyes in a row
  - the triple 20 three times in a row
  - an inner bullseye, an inner ring of 17, and the black sector of 10

b. Where should someone aim when they play darts? Explain.

Discussion: There is no simple answer to part **b.**, for it depends on the skill of the dart player. This will lead to an interesting classroom discussion. Is it better to aim towards the center or towards the outer circle and miss the board entirely? A Geek Plays Darts, a blog by DataGenetics has some interesting ideas for extension.

CONCEPTS OF DENSITY

Students should understand and be able to apply concepts of density based on area and volume in modeling situations.

EXAMPLE

Counting the Crowd

The Vatican said an historic 3.7 million people were at the Sunday event, an eye-popping number that would have made it the second largest papal Mass ever. . . . The problem was, the count released by Vatican and Brazilian officials was a guesstimate that statisticians say grossly inflated the crowd figures. The research director of Datafolha, one of Brazil's top polling and statistic firms, said that based on the size of the crowd area and reasonable density estimates, he would put Sunday's turnout at between 1.2 million and 1.5 million people. . . .

McPhail [an emeritus professor of sociology at the University of Illinois who has studied crowd counts for four decades] first chuckled when he heard the Vatican's crowd estimate in an area of Copacabana beach and adjoining streets that encompassed about 497,000 square meters (594,400 square yards).

By the Vatican's count, the crowd density throughout the entire area would have been 7.4 people per square meter, which wouldn't allow for movement of any kind, let alone the jumping, arm waving, singing and dancing seen at the papal events. . . .

Source: "Numbers Don't Add Up for Papal Mass Crowd Count," Associated Press, Aug. 3, 2013, http://newsinfo.inquirer.net/457893/numbers-dont-add-up-for-papal-mass-crowd-count

1. The article states that 3.7 million people may have attended the papal Mass in an area of 497,000 square meters. By this count, what was the crowd density? Does this number match that stated in the clip?
2. Assume that the crowd area was a rectangular region with the beach along one side. The full article states that the beach is 4 km long. How wide was the crowd area?
3. The firm Datafohla and other researchers say that a good rule is 2 to 3 people per m^2 at a packed event. Given the size of the crowd area, find an interval to estimate the number of people at the event.
4. The research director for Datafohla estimated that there were between 1.2 million and 1.5 million people at the event. According to these estimates, what would be a reasonable range of the density of the crowd?
5. An example from the Programme for International Student Assessment (PISA) reads:
   At a rock concert, a rectangular field 100 m × 50 m was reserved for fans to stand. The concert was sold out. Approximately how many fans were in attendance? Given your answers to the previous questions, determine which of the answers is the most reasonable.
   - **a.** 2,000
   - **b.** 5,000
   - **c.** 20,000
   - **d.** 50,000
   - **e.** 100,000
6. If the general rule is 2 to 3 people per m^2 in a packed event, what is a better estimate of the number of people at the concert in question 5?
7. Determine a good rule for determining the density of people at a packed event.

Example taken from Ebert, D., Morriss, J, Broude, E., & Lesser, R. (October 2015). Counting the crowd; taxing the garden. *Mathematics Teacher,* 109, (3), 168-171.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.MG.1-3)

EXAMPLE

How many people would be in the world if the world had the same population density as—

- **a.** Cleveland
- **b.** Columbus
- **c.** Lima
- **d.** Chillicothe
- **e.** Rio Grande
- **f.** Ohio
- **g.** your city

Discussion: Students will have to look up and/or calculate population density of these cities to answer the question.

TIP! Use masking tape to mask out square meters on the floor and have students stand in the squares to give students a concrete idea of population density.

APPLY GEOMETRIC METHODS TO SOLVE PROBLEMS

Students need to apply geometric methods to solve design problems. This is a good opportunity to make connections to careers and even possibly utilize the career tech standards. Applications could include the following:

- Amount/cost of flooring needed for a house or room
- Amount/cost of paint needed for a room or rooms
- Amount/cost of concrete needed to pour concrete for a patio
- Amount/cost of fencing needed to fence in a yard
- Amount/cost of heating or cooling a large central in BTUs/cubic feet.

Initially modeling problems may need to be scaffolded, but true modeling occurs once the scaffolding is removed. As students gain more exposure to modeling problems the scaffolding should be reduced. The GAIMME Report, Appendix D has some ideas on how to assess modeling in the classroom.

Three-act tasks can also be used in the classroom to promote modeling. See the Instruction Tools/Resources section for ideas.

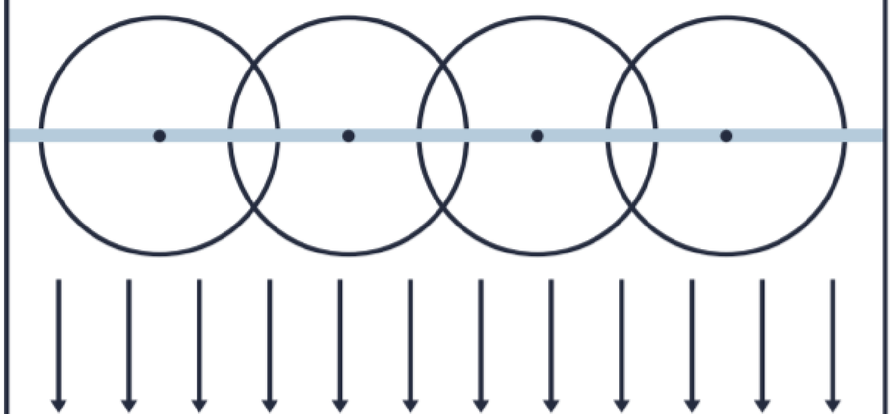
Students may incorrectly believe the process of solving may have only one pathway or that there is always one solution to a problem. Give students messy problems to confront this misconception.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.MG.1-3)

Students may incorrectly believe that answers must always be whole numbers, yet in real-life contexts decimal or fractional solutions occur all the time. It is important to give students problems that result in non-whole number answers.

EXAMPLE

A linear irrigation system consists of a long water pipe set on wheels that keep it above the level of the plants. Nozzles are placed along the pipe, and each nozzle sprays water in a circular region. The entire system moves slowly down the field at a constant speed, watering the plants beneath as it moves. You have 300 feet of pipe and 6 nozzles available. The nozzles deliver a relatively uniform spray to a circular region 50 feet in radius. How far apart should the nozzles be placed to produce the most uniform distribution of water on a rectangular field 300 feet wide?

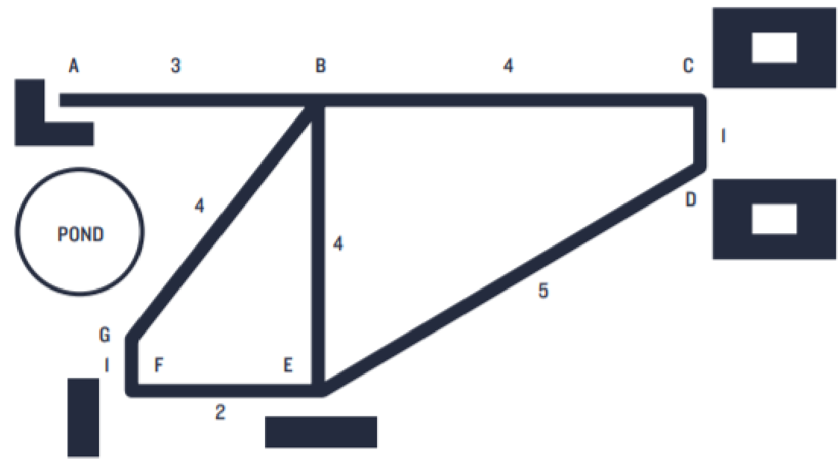


PROBLEM 3.4: THE IRRIGATION PROBLEM

Example taken from the GAIMME report.

EXAMPLE

A map of a portion of a college campus is shown below. The map shows the walking paths and dormitories in this section of campus and the approximate distances (in 100 feet) between locations (the distance between the dorm at D and the dorm at E is 500 feet). Your roommate has convinced you to open a hot dog cart during lunch hours on weekends at one of the intersections along the walkways. You would like the location to be as convenient as possible for the students. Where on campus should you set up your stand?



PROBLEM 3.7: THE HOT DOG CART [10]

Example taken from the GAIMME report.

Discussion: These problems are discussed on pages 74-75 and page 78-79 of the GAIMME report.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.MG.1-3)

Instructional Tools/Resources

These tools and resources may be helpful for instruction. Before use, they should be evaluated at the local level to determine their appropriateness for instruction.

Manipulatives/Technology

- Graphing calculators
- Dynamic geometric software (Geometer's Sketchpad®, Desmos®, Cabri®, or GeoGebra®)
- Rulers
- Protractors
- Compasses

Three-Act Tasks

- Captain's Wheel by When Math Happens is a 3-act task where students answer questions about spinning a wheel.
- Lava Field by When Math Happens is a 3-act task where students answer questions about how long it will take lava to cover a field.
- Closest to the Pin by When Math Happens is a 3-act task where students answer questions about which golf ball is closest to the hole.
- Equidistant Arena by When Math Happens is a 3-act task where students answer questions about where to place an arena.
- Will the Rims Fit? by When Math Happens is a 3-act task where students answer questions about fitting rims onto a car.
- Will the Court Fit? by When Math Happens is a 3-act task where students answer questions about whether a basketball court would be able to fit on a stage.
- Pancakes by When Math Happens is a 3-act task where students answer questions about how many pancakes can be made.
- How Many Houses? by When Math Happens is a 3-act task where students answer questions about how many houses are in a development.
- Air Mattress by When Math Happens is a 3-act task where students answer questions about how long it will take to fill an air mattress.
- Meatballs by Dan Meyer is a 3-act task where students answer questions about whether adding meatballs to a pot of spaghetti will make it overflow. Here is an explanation on how to teach the task.
- Car Caravan by Dan Meyer is a 3-act task where students answer questions about how many toy cars are in a circle. Here is some commentary on the tasks.
- World's Largest Hot Coffee by Dan Meyer is a 3-act task where students answer questions about how many gallons of coffee are in the largest coffee cup.
- Apple Mothership by Dan Meyer is a 3-act task where students answer questions about how many square feet each employee gets.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.MG.1-3)

Describing Objects with Geometric Models

- Toilet Roll by Illustrative Mathematics is a task where students use modeling to deduce algebraic relationships between variables stemming from geometric constraints.
- The Lighthouse Problem by Illustrative Mathematics is a task where students model phenomena on the surface of the Earth.
- Tennis Balls In a Can by Illustrative Mathematics is a task where students explore tennis balls in a can.
- Hexagonal Pattern of Beehives by Illustrative Mathematics is a task where students explore the design of beehives.
- Tilt of Earth's Axis and the Four Seasons by Illustrative Mathematics is a task where students relate their weather experiences with a simple geometric model to explain why the seasons occur.
- Solar Eclipse by Illustrative Mathematics is a task where students apply their knowledge of similar triangles to a solar eclipse.
- Modeling Motion: Rolling Cups by Mathematics Assessment Project where students produce a model to illustrate rolling a cup.
- Simpson Sunblocker by YouCubed has students explore geometric proportionality in the context of Mr. Burns placing a circular disk to block the sun over Springfield.

Concepts of Density

- How Many Leaves on a Tree? and How Many Leaves on a Tree? (Version 2) by Illustrative Mathematics is a task where students have to make a reasonable estimate for something that is too large to count.
- How Many Cells Are in the Human Body? by Illustrative Mathematics is a task where students have to apply concepts of mass, volume, and density in a real-world context.
- How Thick Is a Soda Can? Variation 1 and How Thick is a Soda Can? Variation 2 by Illustrative Mathematics is a task where students apply concepts of density to find the thickness of a soda can.
- Archimedes and the King's Crown by Illustrative Mathematics is a task where students combine the ideas of ratio and proportion with the context of density of matter.
- Indiana Jones and the Golden Statue by Illustrative Mathematics is a task where students are introduced to the subtle use of density and units related to density.
- A Ton of Snow by Illustrative Mathematics is a task where students examine a mathematical statement about the mass of snow.
- Density Word Problems by Khan Academy are word problems related to concepts of density.
- Where is Everybody? by Jessica Woolard is an NCTM Illuminations lesson where students explore the population densities in Canada and the United States. *NCTM now requires a membership to view their lessons.*
- Why is California So Important?, How Could that Happen?, and A Swath of Red by Kimberly Morrow-Leong is an NCTM Illuminations unit about the electoral college and population density. *NCTM now requires a membership to view their lessons.*
- Density and Specific Gravity—Practice Problems from The Math You Need, When You Need It is a website with math problems that make a distinction between density and weight.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.MG.1-3)

Apply Geometric Methods to Solve Problems

- Access Ramps by Achieve the Core is a CTE task where students design an access ramp.
- Fences by Achieve the Core is a CTE task where students add a fence to a pool.
- Framing a House by Achieve the Core is a CTE task where students frame a house.
- Storage Sheds by Achieve the Core is a CTE task where students build storage sheds.
- Stairway by Achieve the Core is a CTE task where students design a stairway for a custom home.
- Miniature Golf by Achieve the Core is a CTE task where students design three new holes of a golf course.
- Grain Storage by Achieve the Core is a CTE task where students have to design a new storage facility for grain.
- Ice Cream Cone by Illustrative Mathematics is a task where students develop a formula for surface area and estimate the maximum number of wrappers that could be cut from a rectangular piece of paper.
- Curriculum Burst 102: A Dart Probability by James Tanton, MAA Mathematician in Residence, is a lesson about the probability of a dart hitting a dart board.
- Dart Board Geometry by Quia is lesson on geometric modeling and probability using a dartboard.
- Another Dartboard by NZMaths connect geometric modeling and probability to the area under a parabola and above the *x*-axis.
- A Geek Plays Darts by DataGenetics is a blog that connects geometric modeling and probability and extends to more advanced topics.
- Darlene's Dart Board by NCTM is a problem of the week pertaining to a dart board. *NCTM now requires a membership to view their lessons.*
- Optimal Shooting Angle by Datagenetics is a blog that discusses the optimal shooting angle of a soccer ball.

Curriculum and Lessons from Other Sources

- EngageNY, Geometry, Module 2, Topic A, Lesson 1: Scale Drawings is a lesson that pertains to this cluster.
- EngageNY, Geometry, Module 2, Topic C, Lesson 19: Families of Parallel Lines and the Circumference of the Earth and Lesson 20: How Far Away Is the Moon? are lessons that pertain to this cluster.
- EngageNY, Geometry, Module 3, Topic B, Lesson 13: How Do 3D Printers Work? is a lesson that pertains to this cluster.
- Georgia Standards of Excellend and Curriculum Frameworks, Geometry, Unit 5: Geometric and Algebraic Connections has many tasks that pertain to this cluster.
- Mathematics Vision Project, Geometry, Module 7: Modeling with Geometry has several tasks that pertain to this cluster.
- Illustrative Mathematics, Geometry, Unit 3, Lesson 2: Scale of the Solar System is a lesson that pertains to this cluster.
- Illustrative Mathematics, Geometry, Unit 5: Solid Geometery, Lesson 17: Volume and Density and Lesson 18: Volume and Graphing are lessons that pertain to this cluster.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.MG.1-3)

General Resources

- Arizona 7-12 Progression on Geometry is an informational resource for teachers. This cluster is addressed on page 17.
- Arizona High School Progression on Modeling is an informational resource for teachers. This cluster is addressed on page 17.
- High School Coherence Map by UnboundED is a map of the relationships among high school mathematics standards. Please note: This resource is aligned to the Common Core not Ohio's Learning Standards. Although the tool is useful, it needs to be checked with respect to Ohio's Learning Standards.
- van Hiele Model of Geometric Thinking is a pdf created by ODE that summarizes the van Hiele levels.

References

- Common Core Standards Writing Team. (2019). *Progressions for the Common Core State Standards for Mathematics (draft February 8, 2019).* Tucson, AZ: Institute for Mathematics and Education, University of Arizona.
- Ebert, D., Morriss, J, Broude, E., & Lesser, R. (October 2015). Counting the crowd; taxing the garden. *Mathematics Teacher,* 109, (3),
- *GAIMME: Guidelines for Assessment and Instruction in Mathematical Modeling Education,* Sol Garfunkel and Michelle Montgomery, editors, COMAP and SIAM, Philadelphia, 2016. View the entire report, available freely online, at https://siam.org/Publications/Reports/Detail/Guidelines-for-Assessment-and-Instruction-in-Mathematical-Modeling-Education
- *Sourcebook of Applications of School Mathematics,* compiled by a Joint Committee of the Mathematical Association of America and the National Council of Teachers of Mathematics (1980).
- *Mathematics: Modeling our World*, Course 1 and Course 2, by the Consortium for Mathematics and its Applications (COMAP).
- *Geometry & its Applications* (GeoMAP). An exciting National Science Foundation project to introduce new discoveries and real-world applications of geometry to high school students. Produced by COMAP.
- Williams, N. & Bruels, C. (February 2011). Target geometry and probability using a dartboard. *Mathematics Teaching in Middle School*, *16, (6),* 375-378.

STANDARDS	MODEL CURRICULUM (S.CP.1-5)
Statistics and Probability **CONDITIONAL PROBABILITY AND RULES OF PROBABILITY** **Understand independence and conditional probability, and use them to interpret data.** **S.CP.1** Describe events as subsets of a sample space (the set of outcomes) using characteristics (or categories) of the outcomes, or as unions, intersections, or complements of other events ("or," "and," "not").★ **S.CP.2** Understand that two events A and B are independent if and only if the probability of A and B occurring together is the product of their probabilities, and use this characterization to determine if they are independent.★ **S.CP.3** Understand the conditional probability of A given B as $^{P(A \text{ and } B)}/_{P(B)}$, and interpret independence of A and B as saying that the conditional probability of A given B is the same as the probability of A, and the conditional probability of B given A is the same as the probability of B.★ *Continued on next page*	**Expectations for Learning** In middle school, students develop basic probability skills including probability as relative frequencies; probabilities of compound events; the development a uniform/non-uniform probability model; and the use of tree diagrams. Also, students are introduced to two-way frequency tables in middle school. However, students' only prior exposure to the concept of independence was in S.ID.5 (Algebra 1/Math 1). This cluster focuses on the concept of independence between two categorical variables. It also focuses on the understanding of independence rather than symbolic notation and formulas. Fluency with independence is expected by the end of Geometry/Math 2. **ESSENTIAL UNDERSTANDINGS** • Approximations for the true probability of an event can be found by looking at the long-run relative frequency. • The sample space of a probability experiment can be modeled with a Venn diagram. • The union of an event and its complement represent the entire sample space. • The intersection of an event and its complement represent the empty set. • Conditional probability is the probability of event A occurring given that event B has occurred. It is denoted by $A\|B$ and is read "A given B." • Two events occurring in succession are said to be independent if the outcome of one event has no effect on the outcome of the other, e.g., a coin tossed twice. Otherwise, the events are dependent, e.g., two cards are drawn in succession from a standard deck of cards. • The intersection of two sets A and B is the set of elements that are common to both set A and set B. It is denoted by $A \cap B$ and is read "A intersection B" as well as "A and B." • The union of two sets A and B is the set of elements, which are in A or in B or in both. It is denoted by $A \cup B$ and is read "A union B" as well as "A or B." • If A and B are events that have no outcomes in common $(A \cap B \neq 0)$, they are said to be mutually exclusive. Mutually exclusive events cannot occur together. *Continued on next page*

<table>
<tr><th>STANDARDS</th><th>MODEL CURRICULUM (S.CP.1-5)</th></tr>
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S.CP.4 Construct and interpret two-way frequency tables of data when two categories are associated with each object being classified. Use the two-way table as a sample space to decide if events are independent and to approximate conditional probabilities. *For example, collect data from a random sample of students in your school on their favorite subject among math, science, and English. Estimate the probability that a randomly selected student from your school will favor science given that the student is in tenth grade. Do the same for other subjects and compare the results.★*

S.CP.5 Recognize and explain the concepts of conditional probability and independence in everyday language and everyday situations. *For example, compare the chance of having lung cancer if you are a smoker with the chance of being a smoker if you have lung cancer.★*

</td><td>

Expectations for Learning, continued

MATHEMATICAL THINKING

- Use appropriate vocabulary.
- Attend to precision.

INSTRUCTIONAL FOCUS

- Recognize and explain for two successive events, whether the outcome of the first event affects the outcome of the second event.
- Recognize and justify conceptually whether two events are independent.
- Make connections between conditional probability and independence. Recognize sample space subsets in everyday contexts.
- Identify an event and its complement.
- Identify which components of the sample space represent the union and intersection of two events.
- Explain what a conditional probability means within a context.
- Distinguish between a conditional probability (A given B) and the probability of an intersection (A and B).
- Use a two-way frequency table to determine the following:
    - conditional probabilities;
    - probabilities of the sample space subsets;
    - event independence by comparing joint probabilities (P(A and B)) and the product of the separate probabilities (P(A) × P(B)); and
    - event independence by comparing the conditional probability (P(A given B)) and the probability P(A).

Content Elaborations

OHIO'S HIGH SCHOOL CRITICAL AREAS OF FOCUS

- Geometry, Number 1, page 3

CONNECTIONS ACROSS STANDARDS

- This will lead into the cluster (S.CP.6-9) which includes the calculations of conditional probabilities, and the use of probability formulas and set notation with probability.

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INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (S.CP.1-5)

Instructional Strategies

Note: The model curriculum is not a collection of lessons nor a full curriculum; it does not suggest pace, sequence, or amount of time spent on topics. It provides information about a topic related to the standards including ideas for examples, strategies for teaching, common misconceptions, and possible connections between topics. In addition, the emphasis of the model curriculum is on instruction not on assessment; therefore, the examples in the Instructional Strategies section are written for instructional purposes not necessarily as assessment items. Examples may need to be rewritten to accommodate the needs of each individual classroom.

Standards for Mathematical Practice
This cluster focuses on but is not limited to the following practices:
MP.4 *Model with mathematics.*
MP.6 *Attend to precision.*
MP.7 *Look for and make use of structure.*

GAISE MODEL

The GAISE model is a framework for teaching students to be statistically literate in a world that is driven by data. The GAISE model has four steps.

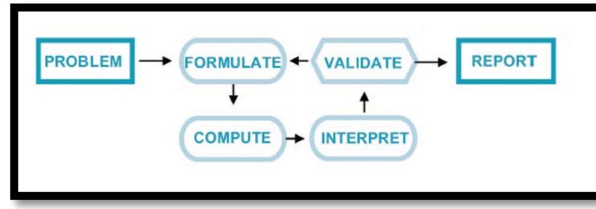
Step 1: Formulate Questions
Step 2: Collect Data
Step 3: Analyze Data
Step 4: Interpret Data

This cluster provides an opportunity for continued use of the GAISE model which is foundational for statistics taught in courses from middle school and beyond Algebra 2. Students in Geometry should be formulating their own questions of interest including starting to form questions that make a generalization about a population. They should be collecting data and begin discussing random selection and random assignment. This includes describing potential sources of error in data collection and analyzing if a sample is representative of the population.

EXAMPLE

Students can conduct a survey designed to explore possible independence or association between two events. Keep in mind that the data collected should ideally be qualitative as probability analysis is usually more appropriate for qualitative data than other methods of analysis. Quantitative data is better suited for other high school courses as it relates to other clusters (S.ID and S.IC). After students have collected their data from the surveys, they can organize it into a two-way frequency table. Then they can consider the resultant probabilities to evaluate whether their events are independent or associated. The project may also culminate in a presentation and/or paper summarizing their process and work.

MODELING



The Standard for Mathematical Practice (SMP.4), *modeling* is important for working with all forms of probability. In fact, all the standards in Statistics and Probability conceptual category are modeling standards. Instruction should stress the usefulness and applicability to real-world scenarios and using data driven probabilities in context. Students should use the probability tools addressed in this cluster to model real-life information. See page 10 for more information about modeling.

Students should be able to interpret and explain concepts of probability, including those expressed in percentages, in real-life situations.

TIP!

Many games that use playing cards or number cubes can be used to explore sample space. Students in middle school may have explored why 7 is a lucky number based on viewing the sample space of rolling two number cubes. At this level, it may be fun to use familiar games such as Yahtzee, Farkle, or even Monopoly to explore more advanced concepts of probability.

CHARACTERISTICS OF OUTCOMES

The "A and B" and "A or B" language is easy for students to misinterpret. With additional contextual language they can make sense of it, but will take a lot of practice to master the vocabulary of "or," "and," "not" with the mathematical notation of union (∪) and intersection (∩). Districts may choose which notation to use for a complement. Commonly used notations for the complement of event A include A^C, A', or $\bar{A}$.

In Algebra students learned about compound inequalities using the words "and" and "or." Make the connection between the solutions of compound inequalities and unions and/or intersections of sets. Begin by developing the concept that an "and" statement is an overlap (intersection) and that an "or" statement is a "union" by using familiar categorical contexts, emphasizing the words "union" and "intersection." After students become familiar with the contexts of "and" and "or" move to real-world examples modeled by compound inequalities that are solved and then graphed on a number line.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (S.CP.1-5)

EXAMPLE

Part 1

Have two students list their favorite fruits.

Beth:

Jennifer:

a. What are the favorite fruits of Beth *and* Jennifer?
b. What are the favorite fruits of Beth *or* Jennifer?

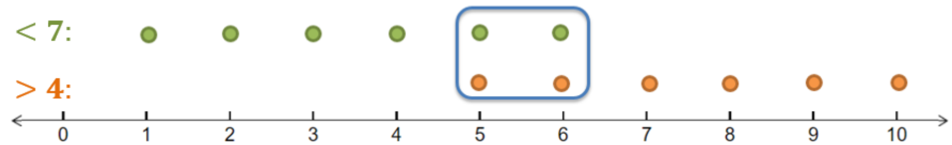
Discussion: Draw attention to the fact that the word "and" refers to the overlap (intersection) of the two sets of fruit. Therefore, the solution to part **a.** is bananas and oranges since both girls list those fruits as one of their favorites. The solution in part **b.** is each fruit listed either by Beth *or* Jennifer which would be strawberries, oranges, grapes, apples, bananas, cherries, pears, lemons, which would be a union of both sets of fruit. Notice that repeated fruit that is common to both sets is only listed once, not twice.

Part 2

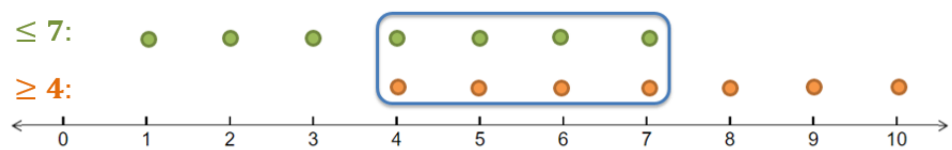
a. Use a number line to show the whole numbers between 1-10 that are bigger than 4 *and* smaller than 7?
b. Use a number line to show the whole numbers between 1-10 that are bigger than or equal to 4 *and* smaller than or equal to 7?
c. Use a number line to show the whole numbers between 1-10 that are bigger than 4 *or* smaller than 7?

Discussion: Students should build off the idea behind "and" and "or" in the previous fruit context. Draw attention to the fact that "and" refers to the overlap of the solutions between the two sets and "or" refers to each number listed in either set. So, the solution to part **a.** is 5 and 6. The solution to part **b.** is 4, 5, 6, and 7. The solution to part **c.** is 1, 2, 3, 4, 5, 6, 7, 8, 9, and 10. Notice that the numbers 5 and 6 are listed only listed once, not twice.

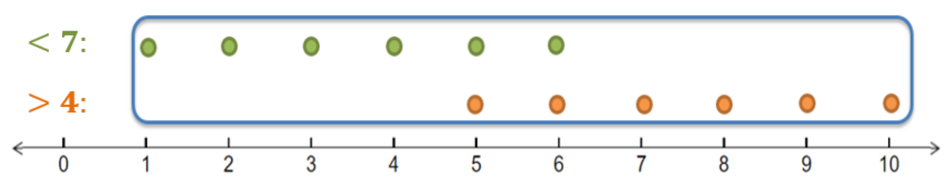
Bigger than 4 ***and*** smaller than 7



Bigger than or equal to 4 ***and*** smaller than or equal to 7



Bigger than 4 ***or*** smaller than 7



INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (S.CP.1-5)

Part 3

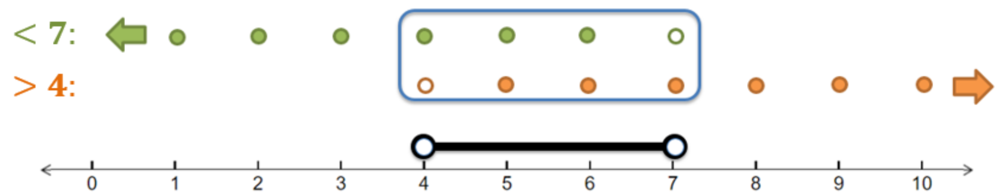
a. Use a number line to show all numbers satisfying the conditions $x > 4\ and\ x < 7$.

b. Use a number line to show all numbers satisfying the conditions $x \geq 4\ and\ x \leq 7$.

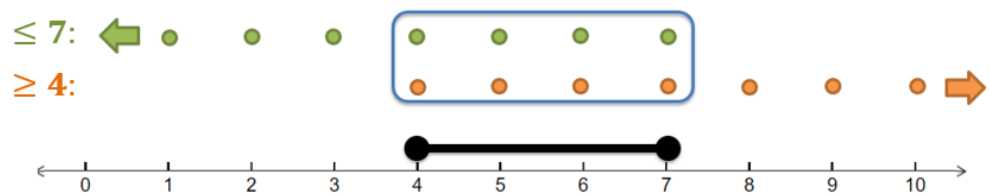
c. Use a number line to show all numbers satisfying the conditions $x > 4\ or\ x < 7$.

Discussion: Connect the inequalities to the previous two examples. Draw attention to the fact that "and" still mean an overlap or intersection of the two statements. The solution to part **a.** is $4 < x < 7$, and the solution to part **b.** is $4 \leq x \leq 7$. The solution to part **c.** is all real numbers since the graph covers the entire number line. *Note: The solutions can be written in other formats, e.g., set builder notation, interval notation, or inequalities.*

Greater than 4 ***and*** less than 7



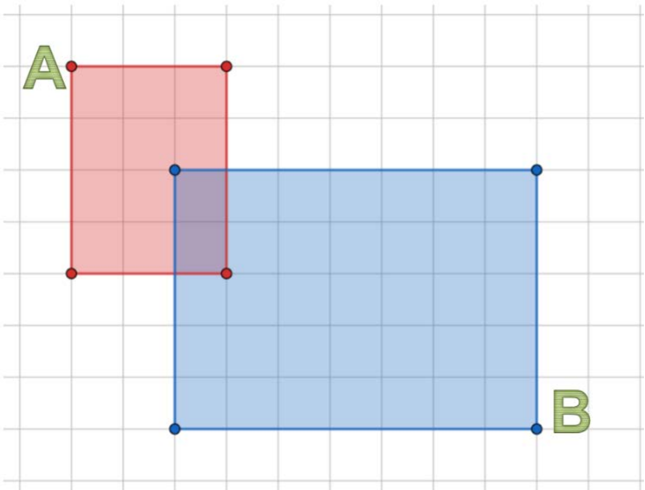
Greater than or equal 4 ***and*** less than or equal to 7



INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (S.CP.1-5)

To help students understand probability and Venn diagrams, it may be helpful to connect the idea of intersections and unions to area models. This will set them up for the Addition Rule in standard S.CP.6.

EXAMPLE



Part 1

a. What is the area of region A?
b. What is the area of region B?
c. What is the area of region A *and* B?
d. If A and B are regions in the plane, then $A \cap B$ denotes the intersection of the two regions, that is, all points that lie in both A and B. What is $A \cap B$?

Discussion: Students have been doing composite shapes since at least middle school and may have developed inaccurate language. However, now is the time to push for more precision in language. Some students will say that the area is the area of both rectangles minus the overlapping regions. Others may say it is only the overlapping region. This is a good discussion to have. Bring them back to the previous example regarding the fruit. The region of A and B is the overlapping piece because it includes both. Therefore, the area of A and B or the intersection of A and B (or $A \cap B$) is 2 units.

Part 2

a. What is the area of region A *or* region B?
b. Do you count the overlapping squares? Explain.
c. If A and B are regions in the plane, then $A \cup B$ denotes the union of the two regions (i.e., all points that lie in A or in B), including points that lie in both A and B. What is $A \cup B$?

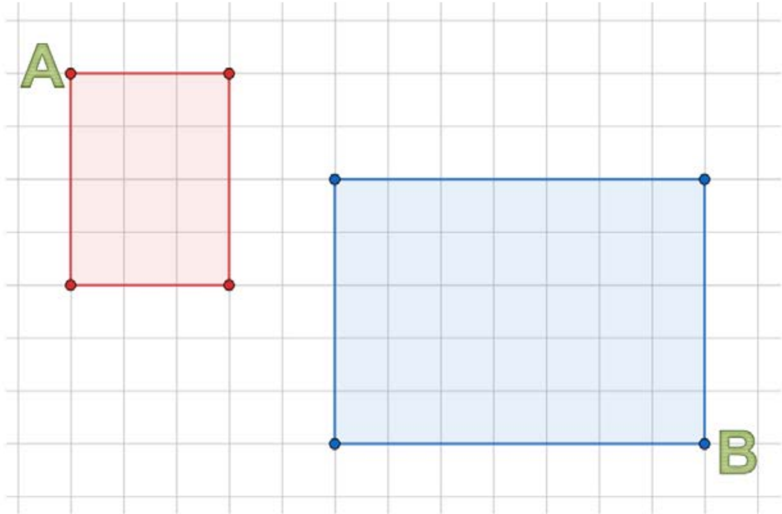
Discussion: Again, bring students back to the concept of the fruit. Draw attention to the fact that when students were finding the area of composite shapes there were actually finding the area of the union of the two shapes. Students should realize that just as they never counted the overlapping section twice in composite shapes, neither should they count twice the overlapping region in union. Note that often times the area of region A *or* B is also called the union of A *or* B and notated $A \cup B$. The area of $A \cup B$ is 45 square units.

TIP! Even though in the past students may have used the word "and" to refer to both shapes, they really should have been using the word "or," since from a mathematical standpoint the use of "and" is incorrect.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (S.CP.1-5)

EXAMPLE

a. What is the area of region A?
b. What is the area of region B?
c. What is the area of region A *and* B?
d. What is the area of region A *or* B?



Discussion: In this example the events are disjoint, so their intersection in the empty set and their union is the sum of the areas of the two rectangles.

TIP! To help students represent the probability symbols, show them that the sign for union, ∪, looks like a capital U. The sign for intersection ∩ looks like an A for "And."

The Standard for Mathematical Practice (SMP.7), *making use of structure* is important for working with all forms of probability. Specifically, students may use the structure of Venn diagrams to build two-way frequency tables and vice-versa.

Venn diagrams provide a way to display a sample space visually.

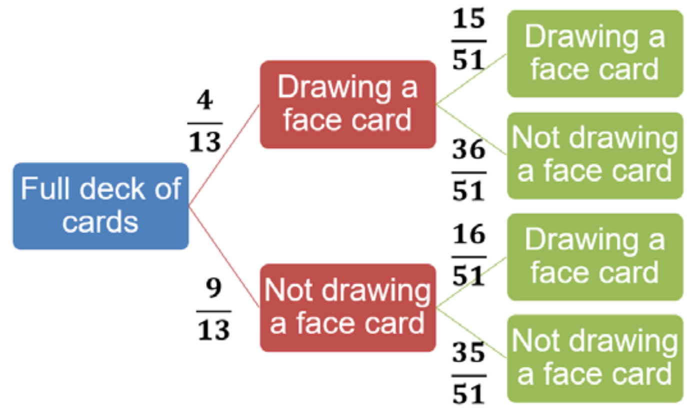
A good activity for working with sample space is through shading different sections of a Venn diagram to represent the various subsets of the sample space. Another way is to identify the subset's sample space represented by a shaded diagram. The Math Vision Project link in the *resources* section below provides excellent activities of this type.

Two-way frequency tables can also be used to identify all probabilities addressed in this cluster including marginal, union, intersection, conditional, and complementary probabilities. They allow students to use table data to evaluate independence of the events represented. It is also recommended that students practice transitioning back and forth between Venn diagrams and two-way frequency tables as well as other data displays of the teachers' choosing.

Although students are familiar with tree diagrams from middle school, the standards in this cluster do not mention the use of tree diagrams which is the traditional way to treat conditional probabilities. Instead, probabilities of conditional events are to be found using a two-way table wherever possible. However, tree diagrams may be a helpful tool for some problems, but students may have difficulty realizing that the second set of branches are conditional probabilities.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (S.CP.1-5)

Students may incorrectly believe that multiplying across branches of a tree diagram has nothing to do with conditional probability, when in fact a tree diagram is set up to illustrate conditional probabilities.



The probability of drawing two face cards successively without replacement is $\frac{4}{13} \cdot \frac{15}{51} = \frac{20}{221}$.

TWO-WAY TABLES

There are two-types of frequency tables: frequency tables and relative frequency tables. In Grade 8 students worked mostly with frequency tables. In Algebra 1, they were exposed to and expected to work with relative frequency tables.

Frequency Table

	More than 60 min	60 min or less	TOTAL
Phone	61	48	109
No Phone	11	40	51
TOTAL	72	88	160

Relative Frequency Table

	More than 60 min	60 min or less	TOTAL
Phone	$^{61}/_{160} \approx$ 0.38 = 38%	$^{48}/_{160} \approx$ 0.30 = 30%	$^{109}/_{160} \approx$ 0.68 = 68%
No Phone	$^{11}/_{160} \approx$ 0.07 = 7%	$^{40}/_{160} \approx$ 0.25 = 25%	$^{51}/_{160} \approx$ 0.32 = 32%
TOTAL	$^{72}/_{160} \approx$ 0.45 = 45%	$^{88}/_{160} \approx$ 0.55 = 55%	$^{160}/_{160}$ = 1.00 = 100%

TIP!

Calculating relative frequencies compared to frequencies can be difficult to understand. See https://opinionator.blogs.nytimes.com/2010/04/25/chances-are/. It may be helpful for students to transform a relative frequency table into a frequency table with the total as 1,000. Although, not exact, the numbers will be close enough for this level of mathematics.

Marginal Frequency
Row totals and column totals constitute the marginal frequencies. These are located in the margins of the table and can also be calculated by adding across columns or rows.

	More than 60 min	60 min or less	TOTAL
Phone	61	48	109
No Phone	11	40	51
TOTAL	72	88	160

Add

Add

	More than 60 min	60 min or less	TOTAL
Phone	61	48	109
No Phone	11	40	51
TOTAL	72	88	160

Joint Frequency
Joint frequency is where the two variables "join" such as keeping the phone in the bedroom and taking longer than 60 minutes to fall asleep. These can be found in the body of the table. Joint frequencies can be connected to the term "intersection."

Using a two-way table begins with calculation of marginal probabilities. Conditional probabilities and determination of independent events follow. A Bayes' Problem presented in a tree-diagram context (where a question asks for a particular prior event having happened in the first set of branches when the given information is about what specifically happened in the second set of branches) becomes straightforward when using a two-way table, while also avoiding a lot of confusing tree-diagram notation.

RELATIONSHIPS BETWEEN SETS AND PROBABILITY

Intersection
EXAMPLE
A survey was given asking about the social network people prefer. The results are summarized in the table.

	Snapchat	Facebook	Total
Ages 18-24	75	15	90
Ages 25+	12	92	104
Total	87	107	194

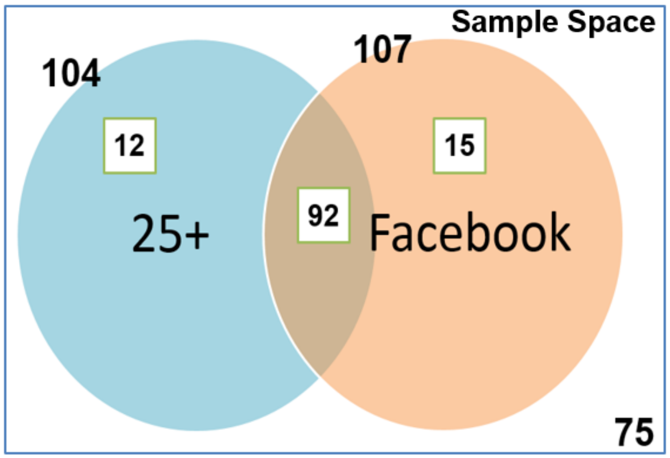
Example continued on next page

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (S.CP.1-5)

Once the data are organized into a table, the table can be used to find the probability that someone prefers Facebook AND is 25+ which can be written P(FB ∩ 25) which would be $\frac{92}{194}$ or approximately 47.42%.

	Snapchat	Facebook	Total
Ages 18-24	75	15	90
Ages 25+	12	92	104
Total	87	107	194

Note: Using two-way frequency tables should serve as a more formalized extension of the informal usage in cluster S.ID.5 from Algebra 1.



The data can also be represented by a Venn diagram, where the intersection of 92 is the shaded region. The total number of people in the survey could be found by adding up the frequencies in each region: the blue region representing those who were 25+ and did not prefer Facebook (12), the peach region representing those who were not 25+ but did prefer Facebook (15), the intersection of the regions representing the participants who were 25+ and prefer Facebook (92), and those not included in the Venn diagram (75) to get a total of 194.

Since intersections involve the word "and" students often incorrectly think the probability of two events happening together is greater than the probability of each individual event. However, it is actually smaller since it is a more specific (restrictive) event.

Union

The Venn diagram can also be used to find the probability if someone prefers Facebook OR is 25+ which can be written P(FB ∪ 25), by adding up all the numbers in the parts of the circles and dividing by the total number of people surveyed: $\frac{(12+92+15)}{194}$. (See previous example for complete data.)

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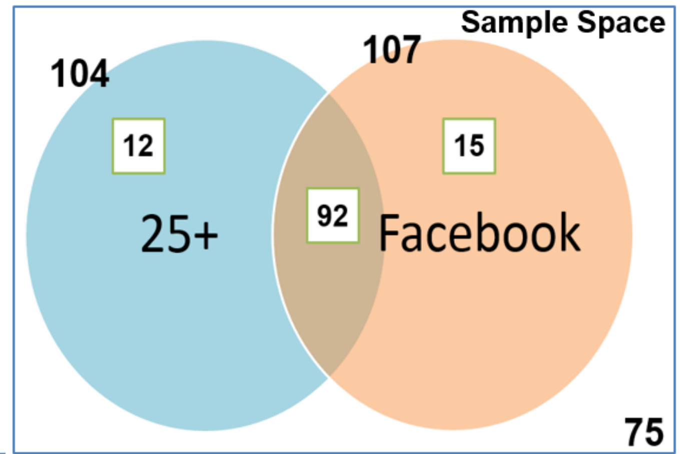
INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (S.CP.1-5)

Students should be able to make the connection between the Venn diagram and the table.



	Snapchat	Facebook	Total
Ages 18-24	75	15	90
Ages 25+	12	92	104
Total	87	107	194

For more information on how to use the table to find the probability in a union, see cluster S.CP.6-9.

Since unions involve the word "or," students often incorrectly think that they have to choose between two events or that they will lose an event, so they incorrectly think the probability is smaller than either of the events individually. However, it is minimally at least equal to the probability of the larger probability of the two events.

EXAMPLE

If the probability of liking soccer is 43% and the probability of being a freshman is 26%.

- Is the probability of liking soccer *or* being a freshman? Choose the most accurate statement.
  - **a.** 43%
  - **b.** 26%
  - **c.** At least 43%
  - **d.** At least 26%
  - **e.** At most 43%
  - **f.** At most 26%
- Is the probability of liking soccer *and* being a freshman? Choose the most accurate statement.
  - **a.** 43%
  - **b.** 26%
  - **c.** At least 43%
  - **d.** At least 26%
  - **e.** At most 43%
  - **f.** At most 26%

Discussion: The probability of liking soccer *or* being a freshman is at least 43%, whereas The probability of liking soccer *and* being a freshman is at most 26%.

Complement

Some problems can be solved much faster using the complement of the event. The complement of the event is the set of all outcomes that are not the event. The $P(event) + P(complement) = 1$. This formula can be rearranged to $P(event) = 1 - P(complement)$ or $P(complement) = 1 - P(event)$ depending on the situation. Commonly used notations for the complement of event A include A^C, A', or $\bar{A}$.

EXAMPLE

- **a.** What is the probability of rolling a 4 when rolling a number cube?
- **b.** What is the probability of not rolling a 4 when rolling a number cube?
- **c.** Would rolling a 4 be better, worse, or the same chance if you had two rolls? Explain.
- **d.** What is the probability of rolling at least one 4 when rolling a number cube twice?
- **e.** You wanted to roll a 4. You rolled the number cube, but rolled a 5. What is the chance that you will roll a 4 if you roll again?

Discussion:

- Part **a.** and part **b.** are $P(4) = \frac{1}{6}$ and $P(not\ 4) = \frac{5}{6}$.
- Part **c**. lends itself to discussion that should set the stage for more in-depth analysis in Part **d.**
- Part **d.** gets tricky as it would be easier to solve the problem using complement: $P(not\ 4, not\ 4) = \frac{5}{6} \cdot \frac{5}{6} = \frac{25}{36} \approx 69.4\%$, so the probability of rolling a 4 when rolling the number cube twice is $P(rolling\ at\ least\ one\ 4) = 1 - \frac{25}{36} = \frac{11}{36} \approx 30.6\%$, so you are more likely to roll a 4 if you have two chances to roll.
- Since probability only deals with what is not yet known and the outcomes of rolling a number cube is independent, the first roll does not affect the second roll. Therefore, the first failure in part **e**. is irrelevant, and $P(4) = \frac{1}{6}$.

Disjoint/Mutually Exclusive

When two events are *mutually exclusive,* also known as *disjoint*, the intersection of the events is the empty set. For example, possible student grades in a course are mutually exclusive because a student cannot have both an 'A' and a 'B' in a course at the same time.

Students may incorrectly think that independent and mutually exclusive events are the same thing. In reality, *independence* is a probability concept, which means that the occurrence of one event does not affect the occurrence of another. In contrast, *mutually exclusive* is an event concept, which means that the two events cannot occur at the same time.

CONDITIONAL PROBABILITY

In the Standard for Mathematical Practice *precision* (SMP.6), it is important for working with conditional probability. Attention to the definition of an event along with the writing and use of probability function notation are important requisites for communication of that precision. For example, given event $A(female)$ and event $B(survivor)$, what does $P(A|B)$ mean?

	More than 60 min	60 min or less	TOTAL
Phone	61	48	109
No Phone	11	40	51
TOTAL	72	88	160

Conditional probabilities are determined by focusing on a specific row or column of the table. For example, if a person sleeps with their phone, what is the probability it will take him or her more than 60 minutes to fall asleep? ($^{61}/_{109} \approx 56\%$.).

Notice the example above is different than "What is the probability that a person that takes more than 60 minutes to fall asleep sleeps with their phone?" ($^{61}/_{72} \approx 85\%$). *Note: The use of a vertical line for the conditional "given" is not intuitive for students and they often confuse the events $B|A$ and $A|B$.*

TIP! For conditional probabilities, students should recognize that in the formula $P(A|B) = \frac{P(A \cap B)}{P(B)}$ the numerator represents the value belonging to both events and the denominator is the total of the "given" event. For example, if event A is more than 60 minutes to fall asleep and event B is sleeping with a phone, then $P(More\ than\ 60\ min|Phone) = \frac{P(A \cap B)}{P(B)} = \frac{61}{109}$. Emphasize the distinction between this probability as compared to the probability of the intersection of two events—sleeping with their phone and taking more than 60 minutes to fall asleep—which would be $\frac{61}{160}$; students often confuse the two.

Exploring situations where the outcomes of tests can be false-positive is a good real-world example of conditional probabilities. See the Instructional Tools/Resources for ideas. A good example could be discussing the usefulness of medical testing despite the possibility of false positives using probability to defend a stance.

Students often find identifying a conditional difficult when the problem is expressed in words in which the word "given" is omitted. For example, find the probability that a female is a survivor. It may be helpful to have students rewrite conditional probabilities using different verbal expressions to show flexibility in language. **TIP!**

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (S.CP.1-5)

Conditional probability is addressed in S.CP.3, S.CP.5, and S.CP.6.

- Standard S.CP.3 defines and calculates conditional probability using mathematical symbolism. The probability, A *given* B, are stated with respect to the original (whole) sample space.
- Standard S.CP.5 conceptualizes and explains in words the conditional probability of S.CP.3.
- Standard S.CP.6 differs from S.CP.3 in that it forces the student to consider that event B is the sample space (which is reduced compared to the whole sample space) and that students need to use the part of B that belongs to A.

EXAMPLE

Roll two fair number cubes (one number cube is red and the other is green). Let event A represent the sum of the rolled numbers on the number cube equals 8 or more, and event B is both number cubes rolled numbers are prime numbers.

a. What are the possible outcomes of event A?
b. What are the possible outcomes of event B?
c. What is the probability of event A?
d. What is the probability of event B?
e. What is $P(A \textit{ given } B)$?

Discussion:

- The possible outcomes of event A are $\{(2,6)(3,5),(3,6),(4,4),(4,5)(4,6),(5,3),(5,4),(5,5)(5,6),(6,2),(6,3),(6,4),(6,5),(6,6)\}$.
- The possible outcomes of event B are $\{(2,2),(2,3),(3,2),(2,5),(5,2),(3,3),(3,5),(5,3),(5,5)\}$.
- The probability of event A is $\frac{15}{36}$ and the probability of event B is $\frac{9}{36}$.
- S.CP.3 emphasizes that the $P(A \textit{ given } B)$ is $P(A \textit{ and } B)$ divided by $P(B)$. Since $(A \textit{ and } B) = [(3,5),(5,3),(5,5)]$, then $P(A \textit{ and } B)$ is $\frac{3}{36}$ because there are 36 parings for rolling two number cubes. Therefore, $P(A \textit{ given } B)$ can be calculated by $\frac{3}{36} \div \frac{9}{36} = \frac{3}{9}$ or $\frac{1}{3}$. Notice the probabilities are with respect to the original sample space of 36 outcomes. S.CP.6 emphasizes the reduced sample space of event $B = [(2,2),(2,3),(3,2),(2,5),(5,2),(3,3),(3,5),(5,3),(5,5)]$. Each outcome has the probability $\frac{1}{9}$. Then $P(A \textit{ given } B)$ is the number of outcomes of A that belong to the 9 outcomes of B, namely 3 of them, $(3,5),(5,3),(5,5)$ that is $\frac{3}{9}$ or $\frac{1}{3}$.

INDEPENDENCE

Students in Grade 8 learn how to find association between two quantitative variables using the Quadrant Count Ratio (QCR). Then in Algebra 1, they used a comparable method called the Agreement-Disagreement-Ratio (ADR) which can be employed for categorical data in a 2 by 2 table for two Yes-No variables. It is calculated by taking the sum of the agreements minus the sum of the disagreements divided by the total or

ADR $= \frac{(a+d)-(b+c)}{T}$.

Agreement

Disagreement

Question 2

Question 1

	Yes	No	Row Total
Yes	a	b	$r_1 = a + b$
No	c	d	$r_2 = c + d$
Column Total	$c_1 = a + c$	$c_2 = b + d$	$T = a + b + c + d$

Tests for Independence

- $P(A \cap B) = P(A) \cdot P(B)$
- $P(A|B) = P(A)$
- $P(B|A) = P(B)$
- $P(B|A) = P(B|NotA)$

Now, students are building on the concept of association to understand independence. Compared to association, independent events have a very precise definition written in terms of probabilities of occurrence in the framework of having defined a sample space. Association can be thought of as dependence. *If two events are independent, then their variables are not associated. If two variables are dependent on each other, then there is some association (although it may be weak).* There are several ways to test for independence.

Using Conditional Probabilities

However, it is far more intuitive to introduce the independence of two events in terms of conditional probability (stated in Standard S.CP.3), especially where calculations can be performed in two-way tables: $P(A|B) = P(A)$ or $P(B|A) = P(B)$ In other words, if knowing that B has occurred does not affect A occurring, then the events are independent and vice versa. Referring to the table in the example on page 296, comparing $P(L)$ to $P(L|D)$ or comparing $P(D)$ to $P(D|L)$ will evaluate whether selecting diet is independent from selecting lemonade.

Step 1: $P(A|B) = \frac{P(A \cap B)}{P(B)}$

Step 2: $P(A|B) \cdot P(B) = P(A \cap B)$

Step 3: If events A and B are independent, then $P(A|B) = P(A)$.

Step 4: Therefore, $P(A) \cdot P(B) = P(A \cap B)$.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (S.CP.1-5)

EXAMPLE

Is the probability that a person who sleeps with their phone (Event B) and who takes more than 60 minutes to fall asleep (Event A) independent?

	More than 60 min	60 min or less	TOTAL
Phone	61	48	109
No Phone	11	40	51
TOTAL	72	88	160

Discussion:

Method 1: One way to show that events A and B are independent is to compare $P(A|B)$ with $P(A)$. The probability of a randomly selected person who takes more than 60 minutes to fall asleep given the person sleeps with his or her phone is $^{61}/_{109} \approx 56\%$, and the probability that a person sleeps more than 60 minutes is $^{72}/_{160} \approx 45\%$. Since $56\% \neq 45\%$ the events are not independent (or in this case dependent). Students can also compare $P(B|A)$ with $P(B)$.

Method 2: Another way to show that events A and B are independent is to compare $P(A|B)$ with $P(A|Not\ B)$. The probability of a randomly selected person who takes more than 60 minutes to fall asleep given the person sleeps with his or her phone is $^{61}/_{109} \approx 56\%$. The probability of a randomly selected person who takes more than 60 minutes to fall asleep given the person does NOT sleep with his or her phone is $^{11}/_{51} \approx 22\%$. Since $56\% \neq 22\%$ the events are not independent (or in this case dependent). Students can also compare $P(B|A)$ with $P(B|NotA)$.

There are many good problems that can appeal to students' sensitivities of fairness and justice in society. Students can formulate their questions that concern how certain characteristics of their own identity groups are viewed by society and understand how conditional probability is often misunderstood by society as whole.

Using Product of the Probabilities

The independence of two events is defined in Standard S.CP.2 using the notion of intersection: $P(A \cap B) = P(A) \cdot P(B)$. Two events, A *and* B, are independent if the fact that A occurs does not affect the probability of B occurring. Students can derive this formula by rearranging the rule for conditional probability which is more intuitive.

TIP!

Students may want to know why they multiply when finding the probability of two events. It may be clearer to think about probability as the fraction of the time that something will happen. If event A happens $\frac{1}{2}$ of the time, and event B happens $\frac{1}{3}$ of the time, and events A and B are independent, then event B will happen $\frac{1}{3}$ of the times that event A happens. To find $\frac{1}{3}$ of $\frac{1}{2}$, multiply the probabilities. The probability that events A and B both happen is $\frac{1}{6}$. Note also that adding two probabilities will give a larger number than either of them; but the probability that two events both happen cannot be greater than either of the individual events, so it would make no sense to add probabilities in this situation. From Dr. Peterson, http://mathforum.org/library/drmath/view/74065.html

Using the Knowledge of the Complement

Students can also test for independence comparing $P(A|B) = P(A|not\ B)$. In other words, if one event does not change whether we know that the other event has occurred or not occurred then $P(A|B) = P(A|not\ B)$.

EXAMPLE

One hundred people are surveyed regarding their favorite soft drink from the choices {Diet Cola, Diet Lemonade, Regular Cola, Regular Lemonade}. Results are shown in the table.

	Cola (C)	Lemonade (L)	Total
Diet (D)	20	13	33
Regular (R)	24	43	67
Total	44	56	100

- Test for independence of the probability of preferring to drink Cola compared to preferring to drink Diet:
  - o using the product of their probabilities;
  - o using conditional probabilities; and
  - o comparing the probability of an event to the probability of its complement.
- If the events are not independent, figure out if there is a strong or weak association based on past methods such as the Agreement-Disagreement Ratio (ADR).

Discussion:

a. The probability that people prefer Cola is $\frac{44}{100}$ or 44% or 0.44. The probability that people prefer Diet is $\frac{33}{100}$ or 33% or 0.33. Therefore if preferring Cola is independent of preferring Diet, 44% of the 33% of the Diet drinkers should have preferred Diet Cola or (0.33)(0.44) should equal 0.20, (see upper left cell in the table) but instead it equals 0.1452. Since 0.1452 does not equal 0.20, the events are not independent.

b. If the Event C is independent from the Event D then the probability of preferring Cola should be equivalent to the probability of preferring Cola given that the drink is Diet. Since P(Cola) = 44% and the P(Cola given Diet) is $\frac{20}{33}$ or 60.6% and 44% ≠ 60.6% the events are not independent.

c. If an event is independent, the probability of drinking Cola given that the drink is diet should be equivalent to the probability of drinking Cola given that the drink is Regular. If the probabilities of the two events are equal, this would prove that they are not associated. Since P(Cola given Diet) is $\frac{20}{33}$ or about 60.6% and the P(Cola given Regular) is $\frac{24}{67}$ or about 35.8%, the events are dependent.

d. After students explore the three methods of independence, they should realize that the events are not independent, so they have an association. They could use the ADR or Pearson's Coefficient for quantitative variables to test for strength of the association. In this case there is an association, but it is a weak association. Discuss when it would be appropriate to test for independence versus when it would be more appropriate to figure out the strength of the association.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (S.CP.1-5)

Note: On one hand S.CP.4 looks at the same questions as would be asked in 8.SP.4 and S.ID.5 but from a probability point of view. The difference between relative frequency and probability is very subtle as we define probability as long-run relative frequency. So, whereas S.ID.5 and 8.SP.4 are answering questions within the given data set using relative frequencies, S.CP.4 is randomly sampling a subject from the data set and asking for the chance that the chosen subject satisfies some classification of interest. The probability is assigned by calculating the respective relative frequency. On the other hand, S.CP.4 muddies the waters by asking about whether or not two events defined in the two-way table are independent. S.ID.5 and 8.SP.4 talk about association, a more vague concept than independence. Independent events have a very precise definition written in terms of probabilities of occurrence in the framework of having defined a sample space. The definition of independent events using the conditional probability point of view plays a prominent part here. Therefore, S.ID.5 is basically a repeat of 8.SP.4. Note that S.CP.4 asks similar questions to those of S.ID.5 and 8.SP.4; however in S.CP.4 the questions are now written in terms of a) random sampling of subjects, b) considering the table as a sample space with values of the variables considered as simple events of the sample space, and c) asking if the events are independent events (a more formal representation of questions regarding association asked in S.ID.5 and 8.SP.4).

Instructional Tools/Resources

These tools and resources may be helpful for instruction. Before use, they should be evaluated at the local level to determine their appropriateness for instruction.

Manipulatives/Technology

- Surveys
- Census Data
- StatTrek is a good website to clarify the vocabulary in this cluster.
- "Chances Are" by Steven Strogatz is an interesting article about probability.
- Understanding Uncertainties: Visualizing Probabilities by Mike Pearson and Ian Short from +Plus Magazine in an article that illustrates probabilities using pictures.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (S.CP.1-5)

Events and Sample Space

- Return to Fred's Fun Factory (with 50 cents) by Illustrative Mathematics is a task where students address standards regarding sample space, independence, probability distributions, and permutations/combinations. Some key components of this task challenge common misconceptions surrounding probability.
- Describing Events by Illustrative Mathematics is a task where students review the definition of sample space and events.
- "Go" in the news—Man Versus Machine by YummyMath is a lesson that has students figure out how many plays are possible in the game *Go* and apply the concept of the Fundamental Counting Principle.
- Too Early in the Day for So Many Choices by YummyMath is a lesson that has students figure out how many possible combinations of hot chocolate Dunkin' Donuts has.
- Doritos Roulette: Hot or Not? How Many Chips are Hot? by Tap Into Teen Minds is a 3-act task that explores how many hot Doritos are in a bag.
- Darius Washington—Free Throws for the Win: Will Darius Washington Score Enough Free Throws to Force Overtime or a Win? by Tap Into Teen Minds is a 3-act task that explores probability.
- Chance Experiments by Desmos is an introduction to probability using a spinner game.
- Probability: Union and Intersection by Jennifer Vadnais is a Desmos activity where students explore unions and intersections.

Independence

- Cards and Independence by Illustrative Mathematics is a task where students explore the concept of independence of events.
- The Titanic 2 by Illustrative Mathematics is a task where students explore the concepts of independence. This is the 2nd task in a series of three.
- Rain and Lightning by Illustrative Mathematics is a task where students explore the concept of independence of events and conditional probability.
- Lucky Envelopes by Illustrative Mathematics is a task where students explore the concept of independence of events.
- Finding Probabilities of Compound Events is a task where students explore the concept of independence of events.
- Breakfast Before School by Illustrative Mathematics is a task where students recognize and explain independence in everyday situations.
- The Egg Roulette Game by Amanda Walker from Statistics Education Web (STEW) is a probability lesson that follows the GAISE model analyzing Jimmy Fallon playing the Egg Roulette with celebrities. The first part of this lesson has to do with the cluster. The second part connects to statistical concepts. There is a note on how to scaffold down the activity as well.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (S.CP.1-5)

Conditional Probability

- The Titanic 1 by Illustrative Mathematics is a task where students explore the concepts of probability as a fraction of outcomes and using two-way data tables with emphasis on understanding conditional probability. This is the first task in a series of three.
- False Positives by Illustrative Mathematics is a task where students explore a common fallacy where two conditional probabilities are confused.
- Representing Conditional Probabilities 1 and Representing Conditional Probabilities 2 by Mathematics Assessment Project is task where students demonstrate their understanding of conditional probabilities, represent events as a subset of a sample space and communicate their reasoning.
- Representing Probabilities: Medical Testing by Mathematics Assessment Projects is a lesson where students understand and calculate conditional probability based on the real-life situation of medical testing.
- False Positives by Achieve the Core is a CTE Task that integrates the concept of probability into medical testing surrounding false positives.
- The False-Positive Paradox as a Class Activity/Discussion Point by Like Teaching—Assume *m* is positive is hands-on activity that can introduce the idea of false positive.
- Conditional Probability and Crime by Cris Wellington and Anne Quinn published in NCTM's Mathematics Teacher, Volume 109, (2) in September 2015. *NCTM now requires a membership to view their lessons.*

Two-Way Tables

- The Titanic 3 by Illustrative Mathematics is a task where students make conclusions based on data using a two-way table. This is the 3rd task in a series of three.
- How Do You Get to School? by Illustrative Mathematics is a task where students use a two-way table to calculate a probability and a conditional probability.
- Two-Way Tables and Probabilities by Illustrative Mathematics is a task where students use a two-way table to calculate a probability and a conditional probability.
- A Sweet Task by Elizabeth Fiedler, Maryann Huey, Brandon Jenkins, and Sharon Flinspach from Statistical Education Website (STEW) is a lesson based on the GAISE model where students gain an understanding of how to create a two-way frequency table and calculate probabilities based on colors of candy.
- Probability and 2-Way Tables-TPS4e Chapter 5 by Bob Lochel is a Desmos activity on two-way tables and probability.
- Performance Assessment Task: Winning Spinners by Inside Mathematics is a task where students need to demonstrate an understanding of the concept of constructing and interpreting two-way tables as a sample space.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (S.CP.1-5)

Curriculum and Lessons from Other Sources

- Georgia Standards of Excellence Curriculum Frameworks, Geometry, Unit 6: Applications of Probability has many tasks that align to this cluster.
- UC San Diego's Computer Science and Engineering page has a variety of probability problems that can be adapted for the course by using two-way tables, Venn diagrams, and tree diagrams.
- EngageNY, Algebra 2, Module 4, Topic A, Lesson 1: Chance Experiments, Sample Spaces, and Events, Lesson 2: Calculating Probabilities of Events Using Two-Way Tables, Lesson 3: Calculating Conditional Probabilities and Evaluating Independence Using Two-Way Tables, Lesson 4: Calculating Conditional Probabilities and Evaluating Independence Using Two-Way Tables, Lesson 5: Events and Venn Diagrams are many tasks that align to this cluster.
- Illustrative Mathematics, Geometry, Unit 8: Conditional Probability, Lesson 1: Up to Chance, Lesson 2: Playing with Probability, Lesson 3: Sample Spaces, Lesson 4: Tables of Relative Frequencies, Lesson 5: Combing Events, Lesson 7: Related Events, Lesson 8: Conditional Probabilities, Lesson 9: Using Tables for Conditional Probabilities, Lesson 10: Using Probabilities to Determine if Events are Independent, Lesson 11:Probabilities in Games are lessons that pertain to this cluster.
- The Mathematics Vision Project, Secondary Math Two, Module 9: Probability has many tasks that align to this cluster.
- The University of Florida has an Open Learning Textbook on Biostatistics that has good explanations about probability.
- Probability through Data: Interpreting Results from Frequency Tables by P. Hopfensperger, H. Kranendonk, R. Scheaffer is a module from Dale Seymour Publications.
- Probability Models by P. Hopfensperger, H. Kranendonk, R. Scheaffer is a pdf of a module by Dale Seymour Publications.

General Resources

- Arizona's High School Progression on Statistics and Probability is an informational document for teachers. This cluster is addressed on pages 13-15.
- Arizona's High School Progression on Modeling is an informational document for teachers. Statistics and Probability is discussed on page 10.
- High School Coherence Map by UnboundED is a map of the relationships among high school mathematics standards. Please note: This resource is aligned to the Common Core not Ohio's Learning Standards. Although the tool is useful, it needs to be checked with respect to Ohio's Learning Standards.
- Statistics Teacher is an online journal published by the American Statistical Association (ASA) and NCTM. It includes articles, lesson plans, and professional development opportunities.
- Significance is a magazine that demonstrate the practical use of statistics and shows how statistics benefits society.
- Chance is a magazine about statistics for anyone who is interested in using data to advance science, education, and society.
- Levels of Conceptual Understanding in Statistics (LOCUS) is an NSF funded project that has assessment questions around statistical understanding.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (S.CP.1-5)

References

- Common Core Standards Writing Team. (2016, March 24). *Progressions for the Common Core State Standards in Mathematics (draft). Grades 6-8, Statistics and Probability*. Tucson, AZ: Institute for Mathematics and Education, University of Arizona.
- Common Core Standards Writing Team. (2013, July 4). *Progressions for the Common Core State Standards in Mathematics (draft). High School, Modeling*. Tucson, AZ: Institute for Mathematics and Education, University of Arizona.
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- Schloemer, C. (April 2016). Mutually exclusive, complementary, or *Mathematics Teacher, 109, (8)*, 640.
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- Shaughnessy, J., Chance, B, & Kranendonk, H. (2009). *Focus in High School Mathematics: Reasoning and Sense Making in Statistics and Probability*, Reston, VA: National Council of Teachers of Mathematics.

STANDARDS	MODEL CURRICULUM (S.CP.6-9)
Statistics and Probability **CONDITIONAL PROBABILITY AND RULES OF PROBABILITY** **Use the rules of probability to compute probabilities of compound events in a uniform probability model.** **S.CP.6** Find the conditional probability of A given B as the fraction of B's outcomes that also belong to A, and interpret the answer in terms of the model.★ **S.CP.7** Apply the Addition Rule, P(A or B) = P(A) + P(B) − P(A and B), and interpret the answer in terms of the model.★ (+) **S.CP.8** Apply the general Multiplication Rule in a uniform probability model[G], P(A and B) = P(A)·P(B|A) = P(B)·P(A|B), and interpret the answer in terms of the model.★ (G, M2) (+) **S.CP.9** Use permutations and combinations to compute probabilities of compound events and solve problems.★ (G, M2)	**Expectations for Learning** In middle school, students develop basic probability skills including probability as relative frequencies; probabilities of compound events; development of a uniform/non-uniform probability model; and the use of tree diagrams. Also, they are introduced to two-way frequency tables in middle school. Now in Geometry/Math 2 this cluster formalizes the concepts of conditional probability and independence in S.CP.1-5. The focus of this cluster is developing the Addition Rule and the (+) Multiplication Rule in everyday contexts. Although permutations and combinations are part of Geometry/Math 2 for students who pursue advanced mathematics, these concepts would also be appropriate in a fourth-year course. Exploration of the Fundamental Counting Principle and factorials (!) may also be addressed in a fourth-year course. **ESSENTIAL UNDERSTANDINGS** • Compound probabilities model real-world scenarios and must be interpreted within a context. • The conditional probability of A *given* B is the fraction of B's outcomes that also belong to A. This can be expressed by $P(A\|B) = P(A \cap B)/P(B)$. • The addition rule is $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$ and can also be expressed as $P(A \cup B) = P(A) + P(B) - P(A \cap B)$. • (+) The Multiplication Rule is $P(A \text{ and } B) = P(A) * P(B\|A) = P(B) * P(A\|B)$. • (+) Permutations and combinations are strategies for counting the outcomes of a sample space. **MATHEMATICAL THINKING** • Use precise mathematical language. • Look for and make use of structure. • Compute accurately and efficiently. *Continued on next page*

STANDARDS	MODEL CURRICULUM (S.CP.6-9)
	Expectations for Learning, continued **INSTRUCTIONAL FOCUS** • Recognize and justify mathematically whether two events are independent. • Generalize probability rules using patterns. • Compute probabilities accurately and efficiently. • (+) Recognize and apply counting methods to compute probabilities. • Compute the conditional probability of A given B. • Interpret and explain conditional probability within a context. • Apply the Addition Rule. • Interpret and explain the Addition Rule within a context. • (+) Apply the Multiplication Rule. • (+) Interpret and explain the Multiplication Rule within a context. • (+) Know and explain the difference between a permutation and a combination. • (+) Calculate probabilities using permutations and combinations. • (+) Interpret and explain probabilities using permutations and combinations within a context. **Content Elaborations** **OHIO'S HIGH SCHOOL CRITICAL AREAS OF FOCUS** • Geometry, Number 1, page 3 **CONNECTIONS ACROSS STANDARDS** • Understand independence and conditional probability (S.CP.1-5).

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (S.CP.6-9)

Instructional Strategies

Note: The model curriculum is not a collection of lessons nor a full curriculum; it does not suggest pace, sequence, or amount of time spent on topics. It provides information about a topic related to the standards including ideas for examples, strategies for teaching, common misconceptions, and possible connections between topics. In addition, the emphasis of the model curriculum is on instruction not on assessment; therefore, the examples in the Instructional Strategies section are written for instructional purposes not necessarily as assessment items. Examples may need to be rewritten to accommodate the needs of each individual classroom.

Standards for Mathematical Practice

This cluster focuses on but is not limited to the following practices:

***MP.2** Reason abstractly and quantitatively.*

***MP.4** Model with mathematics.*

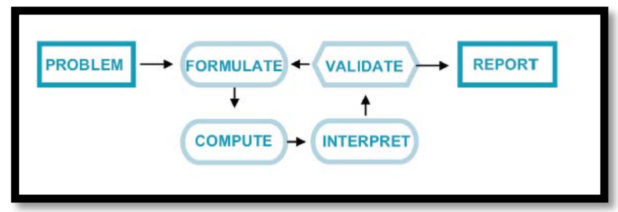
***MP.6** Attend to precision.*

The Standard for Mathematical Practice (SMP.6), *attending to precision* is a vital component in this cluster. Precision is important from reading and understanding the probability question to then selecting the proper rules of probability to use in answering the question.

The Standard for Mathematical Practice (SMP.2), *quantitative and abstract reasoning* is applicable to this cluster because of the need to represent the data symbolically and manipulating those symbols to make sense of the Addition Rule (S.CP.7) and the Multiplication Rule (+S.CP.8).

MODELING

The Standard for Mathematical Practice (SMP.4), *modeling* is critical to all probability standards because emphasis must be placed on the real-world context and applicability of what students are learning. In this cluster, students explore the rules that model probabilities. See page 10 for more information about modeling.



CONDITIONAL PROBABILITY

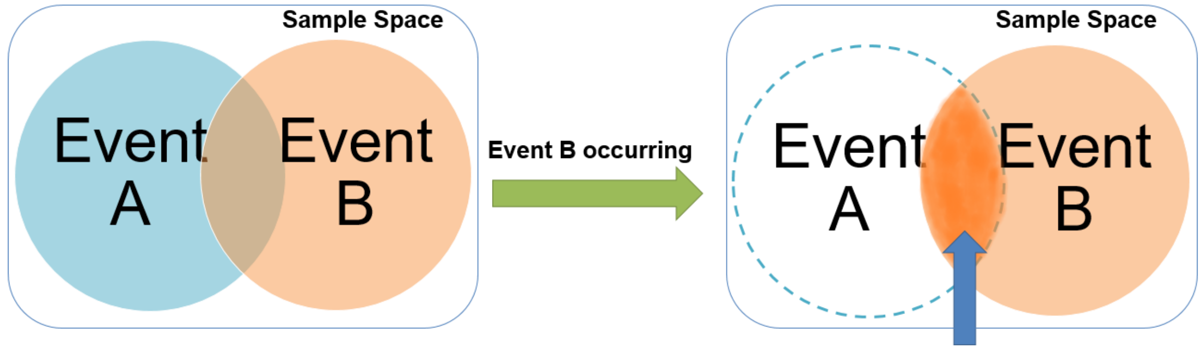
Standard S.CP.6 ties into S.CP.3 and S.CP.5 in the previous cluster and should be taught together. Whereas S.CP.3 emphasizes computation from the original sample space, S.CP.6 emphasizes computation from the reduced sample space. The standard S.CP.6 emphasizes conditional probability of A given B as the fraction of B's outcomes that belong to A. It can be illustrated using a Venn diagram or table. See cluster S.CP.1-5 for more information differentiating between the standards regarding conditional probability.

Students may incorrectly believe that the probability of A and B is always the product of the two events individually, not realizing that one of the probabilities may be conditional. Emphasize to students that they need to read the problem carefully and differentiate whether the events are independent or dependent. From there they can figure out the probability. Using Venn diagrams and two-way tables may help students gain conceptual understanding.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (S.CP.6-9)

In situations involving unions and intersections, the sample space includes both Event A and Event B. However, in situations involving conditional probability the sample space changes once Event B occurs; the only part of Event A that is left to be considered is the part that overlaps with Event B.

Viewing conditional probability from this perspective is more intuitive than the formula $P(A|B) = \frac{P(A \text{ and } B)}{P(B)}$. Students can use their understanding of sample space in this cluster to derive the rule in S.CP.2: If and only if $P(A \text{ and } B) = P(A) \cdot P(B)$, then the events are independent. Notice that standard S.CP.6 calls for the interpretation of the answer in terms of the model, so when students are asked to find the conditional probability, they should be given a contextual situation.



The sample space changes to only include the sample space of Event B. The only part of Event A's outcomes that are left, is those represented by the overlap with Event B.

	Event B	Event Not B	Total
Event A	Event A and Event B	Event A and Not B	Outcomes of Event A
Event Not A	Event B and Event Not A	Event Not A and Event Not B	Outcomes of Event Not A
Total	Outcomes of Event B	Outcomes of Event Not B	Total Sample Space

Event B occurring

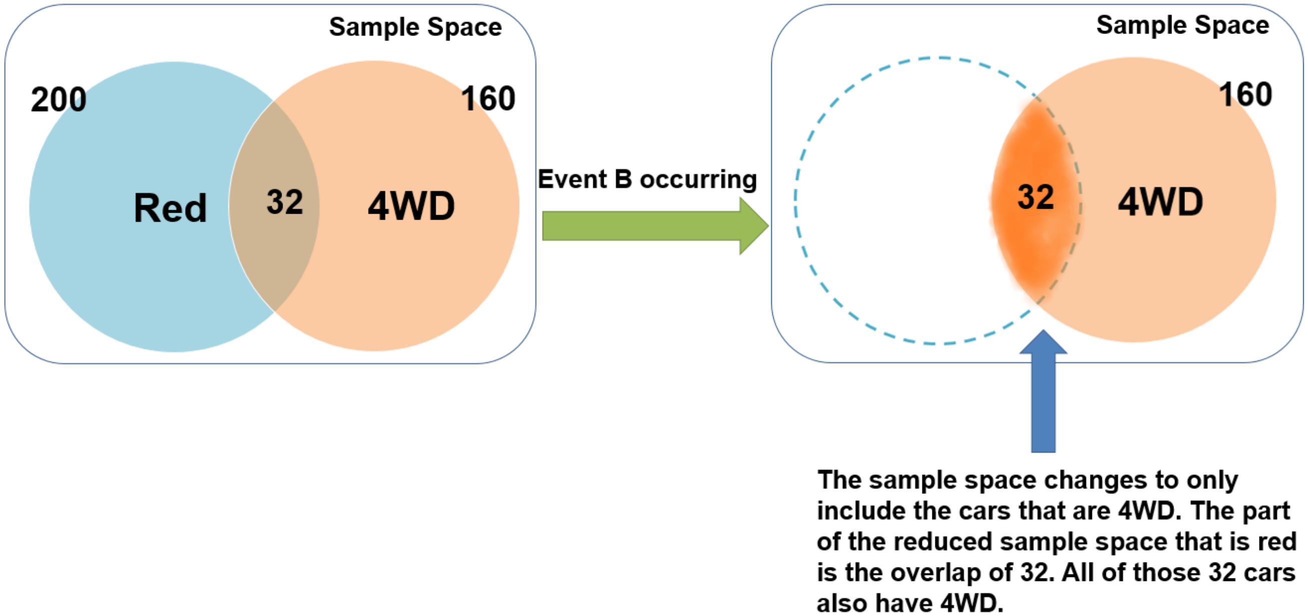
	Event B		
Event A	Event A and Event B		
Event Not A	Event B and Event Not A		
Total	Reduced Sample Space determined by Event B		

Sample Space changes to only include Event B. The only part of Event A that is left is the space that overlaps Event B.
Note: Event B can be in a row or column.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (S.CP.6-9)

EXAMPLE

Twenty percent of the cars on the used car lot were red. Sixteen percent of the cars have four-wheel drive. 3.2% of cars are red and have four-wheel drive. If the probability of car being red and a car having four-wheel drive are independent, and Chloe randomly chooses a car with 4WD what is the probability that it is red?



Discussion: The problem is stated in relative frequencies. Many students and adults struggle with calculating probabilities using relative frequencies. Although, students may choose to do this problem with relative frequencies (which would be slightly more precise). For this course they may wish to convert the relative frequencies to frequencies. When converting relative frequencies to frequencies the grand total could be in any multiple of 10 such as 100, 1,000, 10,000 etc., but for greater precision the grand total needs to be at least 1,000 for this course. Since the events are independent, the first step is to determine the probability of choosing a car that is red and is four-wheel drive. To find that students can multiply 0.16 and 0.20 using relative frequencies or $\frac{160}{1{,}000}$ and $\frac{200}{1{,}000}$ if converting to frequencies. They should get either 0.032 or 32 depending on their method. Using a Venn diagram or a two-way table, students can then see that the sample space changes to 0.16 or 160 (depending on whether relative frequencies or frequencies are used). From there they can calculate the conditional probability of $\frac{0.032}{0.16}$ or $\frac{32}{160} = 0.20 \text{ } or \text{ } 20\%$.

	Red	Not Red	Total
4WD	32	128	160
Not 4WD	168	672	840
Total	200	800	1,000

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (S.CP.6-9)

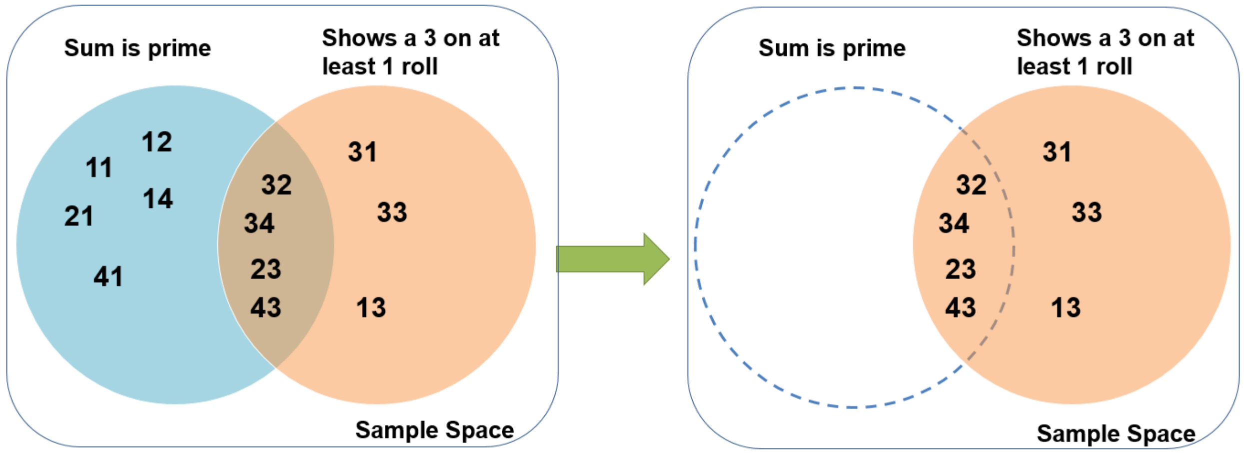
Identifying whether a probability is conditional when the word "given" is omitted can be difficult for students. For example, the wording "of students that have 'A's who are freshmen" is a way of stating "students who have an 'A' given they are freshmen."

Students can play games with number cubes to help reinforce conditional probability and explain sample space.

EXAMPLE

a. If a balanced tetrahedron with faces 1, 2, 3, 4 is rolled twice, what is the probability that the sum of the rolled numbers is prime (A) of those that show a 3 on at least one roll (B)?

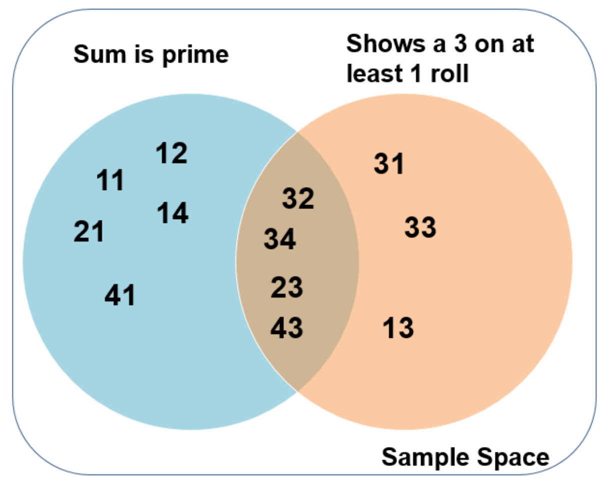
b. What is the probability that the sum of two rolls of a fair tetrahedron is prime (A) or at least one of the rolls is a 3 (B)?



Discussion:

a. Deciding whether the situation translates into $P(A\ and\ B)$, $P(A\ or\ B)$, or $P(A|B)$ may be problematic for students. The question can be rephrased as finding the probability that the sum is prime (A) given at least one roll shows 3 (B). One way to calculate the probability is to count the elements of B by listing them if possible. In this example, there are 7 paired outcomes {(3,1), (3,2), (3,3), (3,4), (1,3), (2,3), (4,3)} in Event B. Of those 7 there are 4 whose sum is prime {(3,2), (3,4), (2,3), (4,3)} which is the intersection of events $A\ and\ B$. Hence in the long run, 4 out of 7 times of rolling a fair tetrahedron twice, the sum of the two rolls will be a prime number under the condition that at least one of its rolls shows the digit 3. Showing the outcomes in a Venn Diagram may be helpful.

b. The wording in part **b.** is more obvious because of the presence of "or" in the sentence. It translates into $P(A\ or\ B)$ which is denoted as $P(A \cup B)$ in set notation. The sample space of a tetrahedron is 16, since each tetrahedra has 4 sides and $4 \cdot 4 = 16$. Again, it is often useful to appeal to a Venn Diagram in which A consists of the pairs as $A = \{(1,1),(1,2),(1,4),(2,1),(2,3),(3,2),(3,4),(4,1),(4,3)\}$ and $B = \{(1,3),(2,3),(3,1),(3,2),(3,3),(3,4),(4,3)\}$. Finding $P(A\ or\ B)$, by adding probability $P(A)$ and $P(B)$ is incorrect because the set of outcomes for $P(A)$ and $P(B)$ would include a duplicate of the two events, namely $23, 32, 34$ and 43. So $P(A\ or\ B)$ is $\frac{9}{16} + \frac{7}{16} - \frac{4}{16} = \frac{12}{16}$ or 75%, so 75% of the time, the result of rolling a fair tetrahedron twice will result in the sum being prime, or at least one of the rolls showing a 3, or perhaps both will occur.



ADDITION RULE

The Addition Rule is a formal way to find the union of two events: $P(A \cup B) = P(A) + P(B) - P(A \cap B)$. Students can derive both the Addition Rule and the Multiplication Rule as an extension. Permutations and combinations also provide opportunities for enrichment.

Though the use of strictly numeric problems can be used in this cluster as a platform for practicing with the rules, it is strongly recommended that students also be exposed to a healthy amount of contextual problems.

EXAMPLE

One hundred people are surveyed regarding their favorite soft drink from the choices {Diet Cola, Diet Lemonade, Regular Cola, Regular Lemonade}. Results are shown in the table. What is the probability that a person likes Cola or that they like Diet drinks?

	Cola (C)	Lemonade (L)	Total
Diet (D)	20	13	33
Regular (R)	24	43	67
Total	44	56	100

Discussion: These data were also used in S.CP.1-5. This example is asking for the union of $(A \cup B)$. Students can approach this task in a couple of ways. One way is to use the joint frequencies in the body of the table $\frac{20+13+24}{100}$ to get $\frac{57}{100}$. Another way is to use marginal frequencies, $\frac{33+44}{100}$, and then subtract the overlap of the joint frequency $\frac{20}{100}$ to get $\frac{57}{100}$. Although, the first way may be more intuitive when using the table, the second way leads to the deriving of the Addition Rule: $P(A\ or\ B) = P(A) + P(B) - P(A\ and\ B)$. The Addition Rule may be more efficient when the probabilities are not in tabular form.

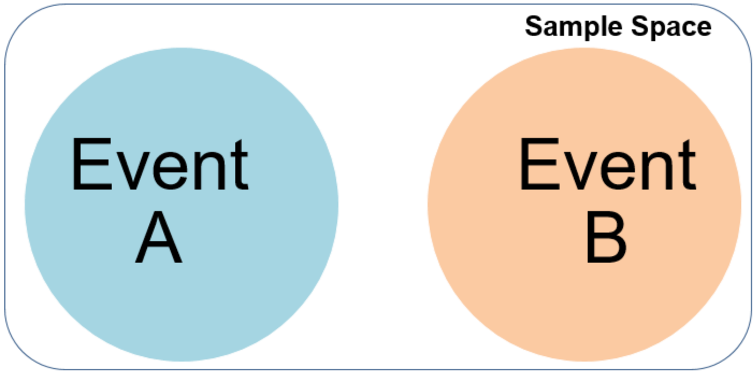
	Cola (C)	Lemonade (L)	Total
Diet (D)	20	13	33
Regular (R)	24	43	67
Total	44	56	100

	Cola (C)	Lemonade (L)	Total
Diet (D)	20	13	33
Regular (R)	24	43	67
Total	44	56	100

Students may incorrectly believe that the probability of A or B is always the sum of the two events individually not taking into account the overlap. Use Venn diagrams and two-way tables to confront this misconception.

Disjoint (Mutually Exclusive) Events

Disjoint events are events that cannot happen at the same time. For example, a person cannot be in New York City and in Hawaii at the same time. They are mutually exclusive, so their intersection is 0. Since there is no overlap, when a situation calls for the union of two events, one simply adds the probabilities of the two events together: $P(A \cup B) = P(A) + P(B)$, since $P(A\ and\ B) = 0$. If one of the two events must happen, then the events are complementary. For example, when rolling a number cube, one can either roll a "6" or "not a 6." In this case, rolling a "6" is an event and rolling "not 6" is its complement. The sum of their probabilities equals one.



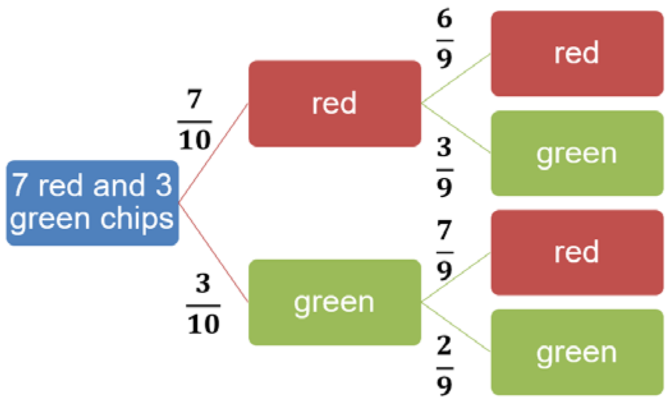
Sometimes students incorrectly think that disjoint events are independent, but they are not independent unless one event is impossible. Disjoint events never occur at the same time; their intersection is impossible. The occurrence of one event prohibits the occurrence of another. For example, being a teenager and being a senior citizen cannot happen at the same time because no one can be both a teenager and senior citizen. In contrast, independent events are unrelated. For example, being a teenager and being born in March are independent events.

MULTIPLICATION RULE (+)

It should be noted that the Multiplication Rule in Standard S.CP.8 is designated as a plus (+) standard. Whereas the Specific Multiplication Rule in S.CP.2 $P(A\ and\ B) = P(A) \cdot P(B)$ only works for independent events, the General Multiplication Rule $P(A\ and\ B) = P(A) \cdot P(B|A)$ in S.CP.8 works for either independent or dependent events. In reality, $P(A\ and\ B) = P(A) \cdot P(B)$ is just a subcase of $P(A\ and\ B) = P(A) \cdot P(B|A)$ when A and B are independent, since when A and B are independent $P(A|B) = P(A)$ and $P(B|A) = P(B)$. Note: Even though S.CP.8 is a (+) standard, it is connected to the Addition Rule in S.CP.7 and the *Specific Multiplication Rule* in S.CP.2.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (S.CP.6-9)

The Multiplication Rule $P(A \text{ and } B) = P(A) \cdot P(B|A)$ is useful when the sample space changes. It is best introduced in a two-stage setting in which A denotes the outcome of the first stage, and B, the second. For example, suppose a jar contains 7 red and 3 green chips. If one draws two chips without replacement from the jar, the probability of getting a red followed by a green is $P(red\ on\ first, green\ on\ second) = P(red\ on\ first) \cdot P(green\ on\ second\ given\ a\ red\ on\ first) = \frac{7}{10} \cdot \frac{3}{9} = \frac{21}{90}$ or $\frac{7}{30}$. Demonstrated on a tree diagram indicates that the conditional probabilities are on the second set of branches. It may be helpful to students to point out that sum of all the probabilities of $P(A \text{ and } B)$ for each branch equal 1.



The probability of drawing a red on the first draw and a green on the second draw without replacement is $\frac{7}{10} \cdot \frac{3}{9} = \frac{7}{30}$.

- $1st\ branch: \frac{7}{10} \cdot \frac{6}{9} = \frac{42}{90}$
- $2nd\ branch: \frac{7}{10} \cdot \frac{3}{9} = \frac{21}{90}$
- $3rd\ branch: \frac{3}{10} \cdot \frac{7}{9} = \frac{21}{90}$
- $4th\ branch: \frac{3}{10} \cdot \frac{2}{9} = \frac{6}{90}$
- $Sum\ of\ branches: \frac{42}{90} + \frac{21}{90} + \frac{21}{90} + \frac{6}{90} = 1$

EXAMPLE

In a standard playing deck, what is the probability that you will lay down two kings in row.

Discussion: The probability of getting the first king is $\frac{4}{52}$; however, once you lay down a king, there are only 3 kings left out of 51 cards, so the probability of the second king is $\frac{3}{51}$. Therefore the $P(king\ and\ king) = \frac{4}{52} \cdot \frac{3}{51}$ which equals $\frac{12}{2652}$ or $\frac{1}{221}$.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (S.CP.6-9)

EXAMPLE

One hundred people are surveyed regarding their favorite soft drink from the choices {Diet Cola, Diet Lemonade, Regular Cola, Regular Lemonade}. Results are shown in the table.

	Cola (C)	Lemonade (L)	Total
Diet (D)	20	13	33
Regular (R)	24	43	67
Total	44	56	100

a. Is the probability that a person likes regular Cola and Diet drinks independent?

b. What is the probability that if a person likes Cola and then he or she likes Diet drinks?

c. Does the rule $P(C \text{ and } D) = P(C) \cdot P(D)$ work in this circumstance? Explain.

d. Explain why the General Multiplication Rule $P(C \text{ and } D) = P(C) \cdot P(D|C)$ works to find $P(C \text{ and } D)$.

e. Does the other version of the General Multiplication Rule $P(C \text{ and } D) = P(D) \cdot P(C|D)$ also work to find $P(C \text{ and } D)$? Explain.

Discussion:

- Students can choose to use any of the four rules for independence to determine that the events are dependent.
- The probability of $P(D|C) = \frac{20}{44}$ or approximately 45%, whereas $P(D) = \frac{33}{100} = 33\%$. Since $33\% \neq 45\%$, the events are not independent.
- In this example, Looking at the upper left cell of the table students can easily see that the intersection of Cola and Diet Drinkers is $\frac{20}{100}$ or 20%. No, the Multiplication Rule $P(C \text{ and } D) = P(C) \cdot P(D)$, does not hold true because $\frac{44}{100} \cdot \frac{33}{100} = \frac{363}{2500}$ does not equal $\frac{20}{100}$. Therefore students should realize that the events are dependent.
- Using the General Multiplication Rule students can see that $P(C \text{ and } D) = \frac{44}{100} \cdot \frac{20}{44} = \frac{20}{100}$ which is the same value that the upper left cell in the table indicates as the intersection. One reason that students may state is that the rule works because the numerator of one of the fractions will equal the denominator of the other. Push them to explain if it will always work. They may generalize this observation by showing that $P(C) \cdot P(D|C) = P(D)$. For example, $P(C) \cdot P(D|C) = \frac{\text{Space of Event } C}{\text{Total Space}} \cdot \frac{\text{Space of Event } D}{\text{Space of Event } C} = \frac{\text{Space of Event } D}{\text{Total Space}}$.
- The same holds true for the other version of the formula $P(C \text{ and } D) = \frac{33}{100} \cdot \frac{20}{33} = \frac{20}{100}$.

PERMUTATIONS AND COMBINATIONS (+)

When addressing (+)S.CP.9, be aware that students have likely had no prior exposure to many of the prerequisites. For example, if listing of outcomes is not possible, the counting techniques such as Fundamental Counting Principle, permutations, or combinations may be required. Students should understand the difference between permutations (a number of different ordered arrangements of a fixed set of elements) and combinations (a number of unordered fixed sets of elements taken from the given set), and then, use permutations and combinations to find probabilities of compound events.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (S.CP.6-9)

Instructional Tools/Resources

These tools and resources may be helpful for instruction. Before use, they should be evaluated at the local level to determine their appropriateness for instruction.

Manipulatives/Technology

- Surveys
- Census Data
- StatTrek is a good website to clarify the vocabulary in this cluster.
- "Chance Are" by Steven Strogatz is an interesting article about probability.
- Understanding Uncertainties: Visualizing Probabilities by Mike Pearson and Ian Short from +Plus Magazine in an article that illustrates probabilities using pictures.
- Roll Dice Online is an applet for rolling dice.
- Dice and Spinners is an applet for dice and spinners. The number of sides on the dice can be changed.
- Virtual Dice by Curriculumbits.com is an applet for rolling dice.

Conditional Probability

- The Egg Roulette Game by Statistics Education Web (STEW) is a lesson that follows the GAISE model. It uses a probability game and computer simulation to explore the law of large numbers, conditional events, sampling distributions, and the central limits theorems using Jimmy Fallon's bit from the Late Night Show.
- Stick or Switch? by NCTM Illuminations is a lesson that explores the probability of compound events using methods such as tree diagrams and area models. *NCTM now requires a membership to view their lessons.*
- Three Shots: Should You Ever Foul at the Buzzer? By Mathalicious is a lesson where students calculate the conditional probability of a win or loss for the defensive team, given that they foul or do not foul.
- Conditional Probability and Probability of Simultaneous Events by Shodor is an activity where students explore conditional probability.
- The Dog Ate My Homework! by NRICH Math models the interpretation of statistics for testing involving false positives.
- Who is Cheating? by NRICH Math models the interpretation of statistics for testing. It can be used to establish the difference between $P(A\ given\ B)$ and $P(B\ given\ A)$.
- Conditional Probability is Important for All Students! by NRICH Math is an article that explains the importance of conditional probability.
- The Titanic 1, The Titanic 2, and The Titanic 3 by Illustrative Mathematics is a series of tasks developing conditional probability.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (S.CP.6-9)

Conditional Probability, continued

- How Do You Get to School? by Illustrative Mathematics uses a two-way table to calculate conditional probability.
- POM: Friends You Can Count On by Inside Mathematics is a series of open-ended tasks. Tasks D and E align to this grade level and cluster. Task D has students calculate the probabilities for simple and compound events. Task E has to do with false positives.
- POM: Got Your Number by Inside Mathematics is a series of open-ended tasks. Tasks D and E align to this grade level and cluster. Tasks D and E have students think about probabilities to determine a detailed strategy for playing the game.
- POM: Party Time by Inside Mathematics is a series of open-ended tasks. Task D aligns to this grade level and cluster. Tasks D has students reason about a fair game.

Addition Rule

- Odd or Even? The Addition and Complement Principles of Probability is a lesson that follows the GAISE model and analyzing the game *Odd or Even*?
- Coffee and Mom's Diner by Illustrative Mathematics is a task where students use the addition rule to compute a probability.
- Rain and Lightning by Illustrative Mathematics is a task where students explore the Addition Rule as part of the task.
- The Addition Rule by Illustrative Mathematics is a task where students to develop the addition rule for calculating the probability of the union of two events.

Multiplication Rule

- Games for Teaching Probability #3: Conditional Probability and the Multiplication Rule is a blog by Board Game Geek that explains modifying the game *No Thanks* to teach the Multiplication Rule. The game No Thanks is explained on this thread.
- P5: Playing Craps by Bowling Green State University is a lesson that integrated the Multiplication Rule in the context of craps.
- Multiplication Rule Probability: Definition, Examples is an article by StatisticsHowTo that describes the difference between the general and specific multiplication rule.
- Independence and Dependence by NRICH Math is an article that discusses, independence, dependence, and sampling with and without replacement including the Multiplication Rule.
- POM: Diminishing Returns by Inside Mathematics is a series of open-ended tasks. Task E challenges students to work with repeating decimals and probabilities to calculate the probability of being born male in an urban North American location.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (S.CP.6-9)

Permutations and Combinations

- POM: Cubism by Inside Mathematics is a series of open-ended tasks. Task E has student analyze permutations of a Rubik's Cube.
- POM: Rod Trains by Inside Mathematics is a series of open-ended tasks. Tasks D and E explore combinations of trains.
- Random Walk III and Random Walk IV by Illustrative Mathematics is part of a progression of tasks, starting with Random Walk and Random Walk II which stressed the function aspect of this situation, transitioning to the probability and statistics side.
- Alex, Mel, and Chelsea Play a Game by Illustrative Mathematics is a task where students combine the concept of independent events with computational tools for counting combinations, requiring fluent understanding of probability in a series of independent events.
- Return to Fred's Fun Factory (with 50 cents) by Illustrative Mathematics is a task where students address concepts regarding sample space, probability distributions, and permutations/combinations.

Curriculum and Lessons from Other Sources

- Georgia Standards of Excellence Curriculum Frameworks, Geometry, Unit 6: Applications of Probability has many tasks that align to this cluster.
- UC San Diego's Computer Science and Engineering page has a variety of probability problems that can be adapted for the course by using two-way tables, Venn diagrams, and tree diagrams.
- EngageNY, Algebra 2, Module 4, Topic A, Lesson 2: Calculating Probabilities of Events Using Two-Way Tables, Lesson 3: Calculating Conditional Probabilities and Evaluating Independence Using Two-Way Tables, Lesson 4: Calculating Conditional Probabilities and Evaluating Independence Using Two-Way Tables, Lesson 5: Events and Venn Diagrams, Lesson 6: Probability Rules, and Lesson 7: Probability Rules are many tasks that align to this cluster.
- EngageNY, Precalculus and Advanced Topics, Module 5, Topic A, Lesson 1: The General Multiplication Rule, Lesson 2: Counting Rules—The Fundamental Counting Principle and Permutations, Lesson 3: Counting Rules—Combinations, Lesson 4: Using Permutations and Combinations to Compute Probabilities are tasks that align to this cluster.
- The Mathematics Vision Project, Secondary Math Two, Module 9: Probability has many tasks that align to this cluster.
- Illustrative Mathematics, Geometry, Unit 8, Lesson 6: The Addition Rule,
- The University of Florida has an Open Learning Textbook on Biostatistics that has good explanations about probability.
- Probability through Data: Interpreting Results from Frequency Tables by P. Hopfensperger, H. Kranendonk, R. Scheaffer is a module from Dale Seymour Publications.
- Probability Models by P. Hopfensperger, H. Kranendonk, R. Scheaffer is a pdf of a module by Dale Seymour Publications.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (S.CP.6-9)

General Resources

- The Mathematics of Games of Pure Chance and Games of Pure Strategy by Sam Smith is a pdf explaining the probability of game theory.
- Arizona's High School Progression on Statistics and Probability is an informational document for teachers. This cluster is addressed on pages 15-17.
- Arizona's High School Progression on Modeling is an informational document for teachers. Statistics and Probability is discussed on page 10.
- High School Coherence Map by UnboundED is a map of the relationships among high school mathematics standards. Please note: This resource is aligned to the Common Core not Ohio's Learning Standards. Although the tool is useful, it needs to be checked with respect to Ohio's Learning Standards.
- Statistics Teacher is an online journal published by the American Statistical Association (ASA) and NCTM. It includes articles, lesson plans, and professional development opportunities.
- Significance is a magazine that demonstrate the practical use of statistics and shows how statistics benefits society.
- Chance is a magazine about statistics for anyone who is interested in using data to advance science, education, and society.
- Levels of Conceptual Understanding in Statistics (LOCUS) is an NSF funded project that has assessment questions around statistical understanding.

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Acknowledgments

Meghan Arnold
Teacher,
Miami East Local Schools, SW

Alex Blohm
Teacher,
South-Western City Schools, C

Kathleen Cooey
Curriculum Specialist/Coordinator,
West Geauga Local Schools, NE

Brett Doudican
Teacher, Dayton Early College, SW

Diane Farmer
Teacher,
Cleveland Catholic Dioceses, NE

Shari Fergusen
Teacher,
Tri-County North Local Schools, SW

Brad Findell (AC)
Higher Education,
Ohio State University, C

Ali Fleming (WG)
Teacher, Bexley City Schools, C

Christina Hamman
Curriculum Specialist/Coordinator, Medina City Schools, NE

Aryn Hinkle
Teacher,
Perrysburg Exempted Village, NW

Margaret (Peggy) Kasten (AC)
Retired Educator, C

Endora Kight Neal (WG)
Curriculum Specialist/Coordinator, Cleveland Metropolitan School District, NE

John Kramer
Teacher,
Flex High Charter School, C

Robert Krauss
Teacher, Manchester Local School District, SW

Emily Meister
Teacher,
Grandview Heights Schools, C

Stephen Miller
Curriculum Specialist/Coordinator, Summit County ESC, NE

Stephanie Minor
Teacher,
Worthington School District, C

Tara Nicely
Teacher, Chillicothe City Schools, C

Marla Norman
Teacher, Lebanon City Schools, SW

Steve Phelps
Teacher, Madeira City Schools, SW

Stephanie Putzier
Curriculum Specialist/Coordinator, Mahoning County ESC, NE

Tess Rivero (WG)
Teacher, Bellbrook-Sugarcreek Local Schools, SW

Parameswaran Sivagurunathan
Teacher, Cleveland Metropolitan School District, NE

Patricia Talarczyk
Teacher,
Mentor Exempted Village, NE

Heather VerStreate
Teacher, Mason City Schools, SW

Jennifer Walls (WG)
Teacher, Akron Public Schools, NE

Verdie Williams
Teacher, Pickaway Ross Joint Vocational School District, C

*(WG) refers to a member of the Working Group and (AC) refers to a member of the Advisory Committee in the Standards Revision Process.