# Ohio's Model Curriculum | Mathematics

with Instructional Supports

## Math 1 Course

Ohio | Department of Education

# Mathematics Model Curriculum

## with Instructional Supports

# Math 1 Course

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# Introduction

## PURPOSE OF THE MODEL CURRICULUM

Just as the standards are required by Ohio Revised Code, so is the development of the model curriculum for those standards. Throughout the development of the standards (2016-17) and the model curriculum (2017-18), the Ohio Department of Education (ODE) has involved educators from around the state at all levels, Pre-K–16. The model curriculum reflects best practices and the expertise of Ohio educators, but it is not a complete a curriculum nor is it mandated for use. The purpose of Ohio's model curriculum is to provide clarity to the standards, a foundation for aligned assessments, and guidelines to assist educators in implementing the standards. The model curriculum is not a collection of lessons nor a full curriculum; it does not suggest pace, sequence, or amount of time spent on topics. It provides information about a topic related to the standards including ideas for examples, strategies for teaching, and possible connections between topics in addition to highlighting some misconceptions.

## COMPONENTS OF THE MODEL CURRICULUM

The model curriculum contains two sections: Expectations for Learning and Content Elaborations.

**Expectations for Learning:** This section begins with an introductory paragraph describing the cluster's position in the respective learning progression, including previous learning and future learning. Following are three subsections: Essential Understandings, Mathematical Thinking, and Instructional Focus.

- **Essential Understandings** are the important concepts students should develop. When students have internalized these conceptual understandings, application and transfer of learning results.
- **Mathematical Thinking** statements describe the mental processes and practices important to the cluster.
- **Instructional Focus** statements are key skills and procedures students should know and demonstrate.

Together these three subsections guide the choice of lessons and formative assessments and ultimately set the parameters for aligned state assessments.

**Content Elaborations:** This section provides further clarification of the standards, links the critical areas of focus, and connects related standards within a grade or course.

*Continued on next page*

## Introduction, continued

### COMPONENTS OF INSTRUCTIONAL SUPPORTS

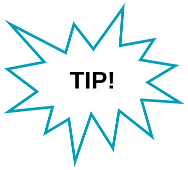
The Instructional Supports section contains the **Instructional Strategies** and **Instructional Tools/Resources** sections which are designed to be fluid and improving over time, through additional research and input from the field. The **Instructional Strategies** are descriptions of effective and promising strategies for engaging students in observation, exploration, and problem solving targeted to the concepts and skills in the cluster of standards. Descriptions of common misconceptions as well as strategies for avoiding or overcoming them and ideas for adapting instructions to meet the needs of all students are threaded throughout. The **Instruction Tools/Resources** are links to relevant research, tools, and technology. In our effort to make sure that our Instructional Supports reflect best practices, this section is under revision and will be published in 2018.

There are several icons that help identify various tips in the Instructional Strategies section:

 = a common misconception

 = a technology tip

 = a career connection



= a general tip which may include diverse leaner or English learner tips.

## Standards for Mathematical Practice—Math 1

The Standards for Mathematical Practice describe the skills that mathematics educators should seek to develop in their students. These practices rest on important "processes and proficiencies" with long-standing importance in mathematics education. The descriptions of the mathematical practices in this document provide **examples** related to the Math 1 course of how student "processes and proficiencies" will change and grow as students engage with and master new and more advanced mathematical ideas across the grade levels. For a more detailed description of the Standards of Mathematical Practice see page 4 of Ohio's Learning Standards for Math 1. These examples just highlight a few specific concepts where the practices may be applied. These examples in no way encompass all the applications of the mathematical practices involved in the course.

**MP.1 Make sense of problems and persevere in solving them.**
Students persevere when attempting to understand the differences between linear and exponential functions. They make diagrams of geometric problems to help make sense of the problems.

**MP.2 Reason abstractly and quantitatively.**
Quantitative reasoning entails habits of creating a coherent representation of the problem at hand; of considering the units involved; of attending to the meaning of quantities, not just how to compute them; and of knowing and flexibly using different properties of operations and objects.

**MP.3 Construct viable arguments and critique the reasoning of others.**
Students use formal and informal proofs to verify, prove, and justify geometric theorems with respect to congruence. These proofs can included paragraph proofs, flow charts, coordinate proofs, two-column proofs, diagrams without words, indirect proofs, or the use of dynamic software.

**MP.4 Model with mathematics.**
Students apply their mathematical understanding of linear and exponential functions to a variety of real-world problems, such as linear and exponential growth. Students also discover mathematics through experimentation and by examining patterns in data from real-world contexts.

**MP.5 Use appropriate tools strategically.**
Students develop a general understanding of the graph of an equation or function as a representation of that object, and they use tools such as graphing calculators or graphing software to create graphs in more complex examples, understanding how to interpret results. They construct diagrams to solve problems.
*Continued on next page*

## Standards for Mathematical Practice, continued

**MP.6 Attend to precision.**

Students use clear and precise definitions in discussion with others and in their own reasoning. They understand the meaning of the symbols they choose, including using the equal sign consistently and appropriately. They are careful about specifying units of measure and labeling to clarify the correspondence with quantities in a problem. They decide whether an equation represents a function by making sure that every input value corresponds to exactly one output value.

**MP.7 Look for and make use of structure.**

Students recognize the significance of attributes of geometric figures such as the importance of an existing line or seeing the need for drawing an auxiliary line for solving problems. They also can step back for an overview and shift perspective. They can see complex objects, such as some algebraic expressions, as single objects or as being composed of several objects.

**MP.8 Look for and express regularity in repeated reasoning.**

Students see that the key feature of a line in the plane is an equal difference in outputs over equal intervals of inputs, and that the result of evaluating the expression $\frac{y_2 - y_1}{x_2 - x_1}$ for points on the line is always equal to a certain number $m$. Therefore, if $(x, y)$ is a generic point on this line, the equation $m = \frac{y - y_1}{x - x_1}$ will give a general equation of that line.

## Modeling

Modeling links classroom mathematics and statistics to everyday life, work, and decision-making. Modeling is the process of choosing and using appropriate mathematics and statistics to analyze empirical situations, to understand them better, and to improve decisions. Quantities and their relationships in physical, economic, public policy, social, and everyday situations can be modeled using mathematical and statistical methods. When making mathematical models, technology is valuable for varying assumptions, exploring consequences, and comparing predictions with data.

A model can be very simple, such as writing total cost as a product of unit price and number bought, or using a geometric shape to describe a physical object like a coin. Even such simple models involve making choices. It is up to us whether to model a coin as a three-dimensional cylinder, or whether a two-dimensional disk works well enough for our purposes. Other situations—modeling a delivery route, a production schedule, or a comparison of loan amortizations—need more elaborate models that use other tools from the mathematical sciences. Real-world situations are not organized and labeled for analysis; formulating tractable models, representing such models, and analyzing them is appropriately a creative process. Like every such process, this depends on acquired expertise as well as creativity.

Some examples of such situations might include the following:

- Estimating how much water and food is needed for emergency relief in a devastated city of 3 million people, and how it might be distributed.
- Planning a table tennis tournament for 7 players at a club with 4 tables, where each player plays against each other player.
- Designing the layout of the stalls in a school fair so as to raise as much money as possible.
- Analyzing the stopping distance for a car.
- Modeling a savings account balance, bacterial colony growth, or investment growth.
- Engaging in critical path analysis, e.g., applied to turnaround of an aircraft at an airport.
- Analyzing risk in situations such as extreme sports, pandemics, and terrorism.
- Relating population statistics to individual predictions.

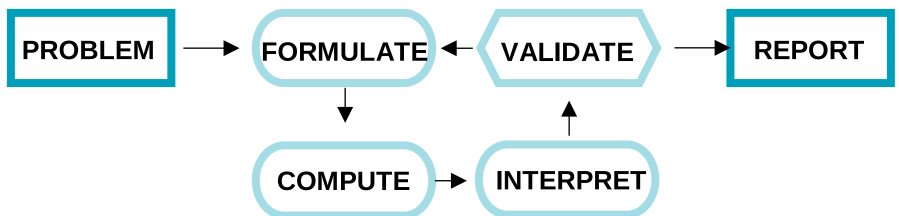
In situations like these, the models devised depend on a number of factors: How precise an answer do we want or need? What aspects of the situation do we most need to understand, control, or optimize? What resources of time and tools do we have? The range of models that we can create and analyze is also constrained by the limitations of our mathematical, statistical, and technical skills, and our ability to recognize significant variables and relationships among them. Diagrams of various kinds, spreadsheets and other technology, and algebra are powerful tools for understanding and solving problems drawn from different types of real-world situations.

*Continued on next page*

## Modeling, continued

One of the insights provided by mathematical modeling is that essentially the same mathematical or statistical structure can sometimes model seemingly different situations. Models can also shed light on the mathematical structures themselves, for example, as when a model of bacterial growth makes more vivid the explosive growth of the exponential function.

The basic modeling cycle is summarized in the diagram. It involves (1) identifying variables in the situation and selecting those that represent essential features, (2) formulating a model by creating and selecting geometric, graphical, tabular, algebraic, or statistical representations that describe relationships between the variables, (3) analyzing and performing operations on these relationships to draw conclusions, (4) interpreting the results of the mathematics in terms of the original situation, (5) validating the conclusions by comparing them with the situation, and then either improving the model or, if it is acceptable, (6) reporting on the conclusions and the reasoning behind them. Choices, assumptions, and approximations are present throughout this cycle.




In descriptive modeling, a model simply describes the phenomena or summarizes them in a compact form. Graphs of observations are a familiar descriptive model—for example, graphs of global temperature and atmospheric $CO_2$ over time.
Analytic modeling seeks to explain data on the basis of deeper theoretical ideas, albeit with parameters that are empirically based; for example, exponential growth of bacterial colonies (until cut-off mechanisms such as pollution or starvation intervene) follows from a constant reproduction rate. Functions are an important tool for analyzing such problems. Graphing utilities, spreadsheets, computer algebra systems, and dynamic geometry software are powerful tools that can be used to model purely mathematical phenomena, e.g., the behavior of polynomials as well as physical phenomena.

**Modeling Standards**
Modeling is best interpreted not as a collection of isolated topics but rather in relation to other standards. Making mathematical models is a Standard for Mathematical Practice, and specific modeling standards appear throughout the high school standards indicated by a star symbol (★).

# Mathematics Model Curriculum

## with Instructional Supports

# Math 1 Course

| STANDARDS | MODEL CURRICULUM (N.Q.1-3) |
|---|---|
| **Number and Quantity**<br>**QUANTITIES**<br>**Reason quantitatively and use units to solve problems.**<br>**N.Q.1** Use units as a way to understand problems and to guide the solution of multi-step problems; choose and interpret units consistently in formulas; choose and interpret the scale and the origin in graphs and data displays. ★<br>**N.Q.2** Define appropriate quantities for the purpose of descriptive modeling.★<br>**N.Q.3** Choose a level of accuracy appropriate to limitations on measurement when reporting quantities.★ | **Expectations for Learning**<br>In elementary grades, students use units for distance, time, money, mass, etc. In grades 6, 7, and 8, students work with rates, especially speed, as a quotient of measurements. In this cluster, students extend the use of units to more complicated applications including rates, formulas, interpretation of scale and origin in graphs, data displays, and related applications. Next, students will apply modeling within the context of the algebra concepts studied and begin to develop strategies to solve more complicated mathematical problems.<br><br>**ESSENTIAL UNDERSTANDINGS**<br>• Units are necessary when representing quantities in a modeling situation to make sense of the problem in context.<br>• A particular quantity can be represented with units from multiple systems of measurement.<br>• Quantities in different units of measure can be compared using equivalent units.<br>• Derived quantities are calculated by multiplying or dividing known quantities, along with their units, e.g., 40 miles in 8 hours is 5 miles per hour.<br>• Quantities can be converted within a system of units, e.g., feet to inches, and between two systems of units, e.g., feet to meters.<br>• There are some contexts in which the origin of a graph or data display is essential to show, and other contexts in which the origin of a graph or data display where it is common to omit the origin, e.g., stock prices over time.<br>*Continued on next page* |

| STANDARDS | MODEL CURRICULUM (N.Q.1-3) |
|---|---|
| | **Expectations for Learning, continued**<br>**MATHEMATICAL THINKING**<br>• Determine reasonableness of results.<br>• Attend to the meaning of quantities.<br>• Consider mathematical units involved in a problem.<br><br>**INSTRUCTIONAL FOCUS**<br>• When modeling, consider the scale when choosing or deriving suitable units.<br>• Choose a level of accuracy appropriate for the given context.<br>• Convert measurements within a system of units, e.g., convert 4.6 feet to inches and feet, or between a two systems of units, e.g., convert 4.6 feet to meters.<br><br>**Content Elaborations**<br>**OHIO'S HIGH SCHOOL CRITICAL AREAS OF FOCUS**<br>• Math 1, Number 1, page 3<br><br>**CONNECTIONS ACROSS STANDARDS**<br>Create equations that describe numbers or relationships (A.CED.1-4). |

# INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (N.Q.1-3)

## Instructional Strategies

*Note: The model curriculum is not a collection of lessons nor a full curriculum; it does not suggest pace, sequence, or amount of time spent on topics. It provides information about a topic related to the standards including ideas for examples, strategies for teaching, common misconceptions, and possible connections between topics. In addition, the emphasis of the model curriculum is on instruction not on assessment; therefore, the examples in the Instructional Strategies section are written for instructional purposes not necessarily as assessment items. Examples may need to be rewritten to accommodate the needs of each individual classroom.*

***Standards for Mathematical Practice***
*This cluster focuses on but is not limited to the following practices:*
***MP.2*** *Reason abstractly and quantitatively.*
***MP.4*** *Model with mathematics.*
***MP.6*** *Attend to precision.*

Units are central to applied mathematics and everyday life. In real-world situations, answers are usually represented by numbers associated with units, for example, acceleration, currency, people-hours, energy, power, concentration, and density. Therefore this cluster should not be taught in isolation, but should be combined with other standards that use units and graphs in real-world situations.

### QUANTITIES

Numerical values without units are not quantities. To be added or subtracted, quantities must be measurements of the same attribute (length, area, speed, etc.) and expressed in the same units. Converting quantities expressed in different units to having the same unit is like converting fractions to have a common denominator before adding or subtracting. "Depending on context, quantities are called by different names, such as 'measure' (e.g., productivity measure) or 'index' (e.g., Consumer Price Index). In situations where quantities are represented as variables, quantities are often referred to as 'variables'" (Arizona High School Progression on Quantity, page 4).

Reasoning quantitatively includes the following:

- knowing when and how to convert units;
- analyzing the units in a calculation;
- using units to check work;
- making sound choices for the scale and origin of a graph or a display;
- choosing an appropriate level of accuracy when reporting quantities;
- specifying units when defining variables; and
- attending to units when writing expressions or equations.

To accomplish this goal, teachers should provide students with a broad range of contextual problems that offer opportunities for performing operations with quantities. These should involve units where students develop math reasoning skills to make judgements about the correctness of their answers. These problems should be connected to science, engineering, economics, finance, medicine, etc.

# INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (N.Q.1-3)

**TIP!**

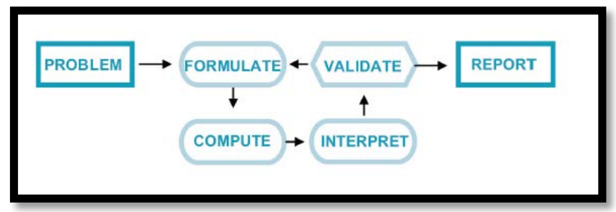
The seven units in the International System of Units (S.I.) were defined by actual material objects locked up in the International Bureau of Weights and Measures in Paris. The kilogram was defined by a cylinder of platinum-iridium locked in a vault in Paris. The meter was defined by the distance between two scratched on a hallowed bar of platinum-iridium metal. Copies would be distributed to various countries to keep the standardization of the units consistent. The problem is that physical objects could be broken, scratched, or chipped, so scientists have been working on redefining the units in terms of fundamental concepts of nature. Therefore in 1983 the meter was redefined as the distance traveled by the speed of light in a vacuum in $\frac{1}{299,792,458}$ of a second. The second has been defined as the amount of time it takes an atom of cesium-133 to vibrate 9,192,631,770 times. On May 20, 2019 the kilogram, ampere, kelvin, and mole will also be redefined. Discuss with students the importance of standardization with respect to units of measure. For more information https://www.nytimes.com/2018/11/16/science/kilogram-physics-measurement.html?smid=fb-nytimes&smtyp=cur&fbclid=IwAR0pSkhwRVla9MkkUR10B0ggQZKFyEjnXKPJ4LsRQWFdqAr9ec5n8124388

Some measures where the units are not specified have to be understood in the way the measure is defined. For example, the S&P 500 stock index is a measure derived from the quotient of the value of 500 companies now and in 1940–42.

## MODELING

Throughout the modeling process, units are critical for several reasons:

- Units guide calculations.
- Units communicate the results of a model since real-world examples are usually quantities (numbers with units).
- Units help make or evaluate assumptions.
- Units show the reasonableness of an answer.
- Units help define the quantity.




Sometimes in the modeling process students will need to make their own decisions about units. See page 13 for more information about modeling.

## MISLEADING GRAPHS

Graphical representations serve as visual models for understanding phenomena that take place in our daily surroundings. The use of different kinds of graphical representations along with their units, labels, and titles demonstrates the level of students' understanding and fosters the ability to reason, prove, self-check, examine relationships and establish the validity of arguments. Students can benefit from examples of misleading graphs where they must analyze data from the graph. It may be helpful to initially choose uniform units and scales in order to create a correct representation and make correct conclusions of a situation. This should also be applied to line plots, histograms, and box plots in S.ID.1-3 and S.ID.6.

# INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (N.Q.1-3)

## PRECISION VS ACCURACY

Measurement in mathematics and science involves both precision and accuracy. *Precision* refers to the closeness of two or more values to each other. *Accuracy* refers to the closeness of the measured value to a standard or known value. Values can be accurate but imprecise or precise but inaccurate; however it is better to be precise and accurate. Accuracy depends on the instrument being used.

### Intermediate Rounding

Teachers at the high school level should discourage intermediate rounding when solving problems. Students should be encouraged to use the $\pi$ button on the calculator whenever possible for greater precision. Give students a problem where intermediate rounding provides two very different answers and have a discussion on which answer is more precise and the consequences of less precision.

## UNIT CONVERSION

Measurement involves units and often requires a unit conversion. Some contextual problems may require an understanding of derived measurements and capability in unit analysis.

### *EXAMPLE*

While driving in the United Kingdom (UK), a U.S. tourist puts 60 liters of gasoline in his car. The gasoline cost is £1.28 per liter. The current exchange rate is £ 0.62978 for each $1.00. The price for a gallon of a gasoline in the United States is $3.05. Compare the costs for the same amount and the same type of gasoline when he/she pays in UK pounds.

*Discussion*:

- Making reasonable estimates should be encouraged prior to solving this problem. If the current exchange rate has inflated the UK pound to more than the U.S. dollar, the driver will pay more for the same amount of gasoline. By dividing $3.05 by 3.79L (the number of liters in one gallon), students can see that 80.47 cents per liter of gasoline in US is less expensive than £1.28 or $ 2.03 per liter of the same type of gasoline in the UK when paid in U.S. dollars. The cost of 60 liters of gasoline in UK is $£\,76.8$ $\left(\frac{£1.28}{1\,L} \times 60L = UK\,£\,76.8\right)$.
- In order to compute the cost of the same quantity of gasoline in the United States in UK currency, it is necessary to convert between both monetary systems and units of volume. Based on UK pounds, the cost of 60 liters of gasoline in the U.S. is $£30.41$ $\left(\frac{US\$3.05}{1\,gal} \times \frac{1\,gal}{3.79\,L} \times 60L \times \frac{UK\,£0.62978}{US\,\$1.00} = UK\,£30.41\right)$.
- The computation shows that the gasoline is less expensive in the United States and shows how an analysis can be helpful in keeping track of unit conversations. Students should be able to correctly identify the degree of precision of the answers, understanding that the degree of precision should not be greater than the actual accuracy of measurement. The use of significant digits and unit analysis is an emphasis in high school science classes, and efforts could be made to collaborate with science teachers.

# INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (N.Q.1-3)

Students will compare the cost of gasoline across two countries with different monetary systems and units of measure. Host a career speaker in the classroom, where students can ask questions of related to the work-based applications of these concepts (e.g., economics, engineering, finance, health). See also "Corn and Oats" and "Dental Impressions" lessons in the Instructional Resources section.

Students may not realize the importance of the units' conversions in conjunction with the computation when solving problems involving measurements. Since today's calculating devices often display 8 to 10 decimal places, students frequently express answers to a much greater degree of precision than required or reasonable for the given problem situation.

TIP!

## Instructional Tools/Resources

*These tools and resources may be helpful for instruction. Before use, they should be evaluated at the local level to determine their appropriateness for instruction.*

**Understanding Units**

- Scale of the Universe: Putting the Universe into Perspective is an applet that shows the perspective of the universe. Teachers can use this as an introduction to talk about units.
- Leaky Faucet is lesson by Dan Meyer that aligns to 6.RP.3, but could be used at the high school level to start a discussion about units and why they are necessary to solve problems.

**Converting Units**

- Dental Impressions is a lesson connecting math standards to CTE by Achieve the Core. This lesson has students converting units to make a stone model from dental impressions.
- Corn and Oats is a lesson connecting math standards to CTE by Achieve the Core. The lesson has the students assist Producer Bob with management tasks regarding the planting of corn and oats.
- Dimensional Analysis is a lesson by Rachel Meisner on dimension analysis with a video, PowerPoint, and worksheet using dimensional analysis. In addition to some cute animation in the video, it ties in real-world problems and significant figures.
- Dimensional Analysis is a lesson by Leslie Gushwa that includes three resources on dimensional analysis. The first two Word documents could be used in a math classroom. The third resource deals with moles in conjunction with science.

*Continued on next page*

# INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (N.Q.1-3)

**Precision and Accuracy**

- Precision and Accuracy is a lesson on the difference between precision and accuracy by Annenberg Learner. It includes a video, some problems, and an interactive applet. It also has some homework questions.
- Measurement, Accuracy, and Precision is an activity that discusses accuracy and precision with respect to measurement and discusses the importance of accuracy. The first activity has students weigh Mars Bars and the graph the results. The second lesson shows how equipment can affect accuracy and precision. The third lesson discusses the importance of accuracy.

**General Resources**

- Arizona High School Progression on Number and Quantity is an informational document for teachers. This cluster is addressed on pages 2-6
- Arizona High School Progression on Modeling is an informational document for teachers. This cluster is addressed on pages 7-10.
- High School Coherence Map by UnboundED is a map of the relationships among high school mathematics standards. Please note: This resource is aligned to the Common Core not Ohio's Learning Standards. Although the tool is useful, it needs to be checked with respect to Ohio's Learning Standards.

**References**


- Common Core Standards Writing Team. (2013, July 4). *Progressions for the Common Core State Standards in Mathematics (draft). High School, Modeling*. Tucson, AZ: Institute for Mathematics and Education, University of Arizona.
- Common Core Standards Writing Team. (2013, July 4). *Progressions for the Common Core State Standards in Mathematics (draft). Grades 6–8, The Number System; High School, Number*. Tucson, AZ: Institute for Mathematics and Education, University of Arizona.
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| STANDARDS | MODEL CURRICULUM (A.SSE.1) |
|---|---|
| **Algebra**<br>**SEEING STRUCTURE IN EXPRESSIONS**<br>**Interpret the structure of expressions.**<br>**A.SSE.1**. Interpret expressions that represent a quantity in terms of its context. ★<br>**a.** Interpret parts of an expression, such as terms, factors, and coefficients.<br>**b.** Interpret complicated expressions by viewing one or more of their parts as a single entity. | **Expectations for Learning**<br>Students build expressions in grades K-5 with arithmetic operations. As they move into the middle grades and progress through high school, students build expressions with algebraic components, beginning with linear and exponential expression. Then in Math 2 quadratic expressions. In later courses, they build algebraic expressions with polynomial, rational, radical, and trigonometric expressions. In this cluster, they focus on interpreting the components of linear and exponential expressions and their meaning in mathematical and real-world contexts. They also determine when rewriting or manipulating expressions is helpful in order to reveal different insights into a mathematical or real-world context.<br><br>**ESSENTIAL UNDERSTANDINGS**<br>• An expression is a collection of terms separated by addition or subtraction.<br>• A term is a product of a number and a variable raised to a nonnegative integer exponent.<br>• Components of an expression or expressions within an equation may have meaning in a mathematical context, e.g., $y = mx + b$, $b$ represents the $y$-intercept.<br>• Components of an expression may have meaning in a real-world context, e.g., in data surcharges, $60 + 0.05x$, the 60 represents the fixed costs and the 0.05 represents the cost per unit of data.<br>• Expressions may potentially be rearranged or manipulated in ways to reveal different insights into the mathematical or real-world context.<br><br>**MATHEMATICAL THINKING**<br>• Attend to the meaning of quantities.<br>• Use precise mathematical language.<br>• Apply grade-level concepts, terms, and properties.<br>• Look for and make use of structure.<br>*Continued on next page* |

| STANDARDS | MODEL CURRICULUM (A.SSE.1) |
|---|---|
| | **Expectations for Learning, continued**<br>**INSTRUCTIONAL FOCUS**<br>• Identify the components, such as terms, factors, or coefficients, of an expression and interpret their meaning in terms of a mathematical or real- world context.<br>• Explain the meaning of each part of an expression, including linear and simple exponential expressions in a mathematical or real-world context.<br>• Analyze an expression and recognize that it can be rewritten in different ways.<br><br>**Content Elaborations**<br>**OHIO'S HIGH SCHOOL CRITICAL AREA OF FOCUS**<br>• Math 1, Number 1, page 3<br><br>**CONNECTIONS ACROSS STANDARDS**<br>• Create equations in one or two variables (A.CED.1-2).<br>• Interpret expressions for functions in terms of the situations they model (F.LE.5).<br>• Interpret linear models (S.ID.7). |

# INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (A.SSE.1)

## Instructional Strategies

*Note: The model curriculum is not a collection of lessons nor a full curriculum; it does not suggest pace, sequence, or amount of time spent on topics. It provides information about a topic related to the standards including ideas for examples, strategies for teaching, common misconceptions, and possible connections between topics. In addition, the emphasis of the model curriculum is on instruction not on assessment; therefore, the examples in the Instructional Strategies section are written for instructional purposes not necessarily as assessment items. Examples may need to be rewritten to accommodate the needs of each individual classroom.*

***Standards for Mathematical Practice***
*This cluster focuses on but is not limited to the following practices:*
***MP.1*** *Make sense of problems and persevere in solving them.*
***MP.4*** *Model with mathematics.*
***MP.7*** *Look for and make use of structure.*

The focus of this cluster is in "seeing" how each part of an algebraic expression interplays with the other parts of the expression to create meaning. Since the goal is on the interpretation of the components, the focus should not be simplifying expressions or solving equations. As the expressions become more complex, students should be able to see them built out of basic operations such as sums of terms or products of factors.

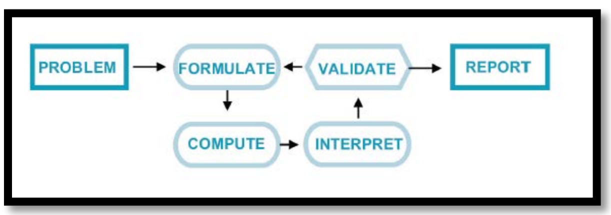
Development and proper use of mathematical language is an important building block for future content. For example, a student should recognize that in the expression $2x + 1$, "$2x$" and "1" are *terms* of the binomial, "2" is the coefficient, "2" and "$x$" are factors, and "1" is a constant.

A student should also be able to see that more complicated expressions can be built from simpler ones. For example, the expression $3 + (y – 2)^2$ can be viewed as the sum of the constant term 3 and the squared term $(y – 2)^2$; therefore viewing the expression in this manner allows a student to recognize that it is always greater than or equal to 3, because $(y – 2)^2$ is nonnegative and the sum of 3 and a nonnegative number is greater than 0. A student can also recognize that inside the squared term is the expression $y – 2$, so the square term is 0 when $y = 2$.

Students may believe that the use of algebraic expressions is merely the abstract manipulation of symbols. To counter this idea, the use of real-world examples is very helpful. Students can be asked to explain the meaning of the parts of algebraic expression that represent the situation, and provide a rationale for why one form of the expression is more beneficial than another.

TIP!

Math 2 will focus on quadratics and more complicated exponential expressions.




### MODELING

A.SSE.1 is also a modeling standard. See page 13 for more information about modeling.

# INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (A.SSE.1)

## LINEAR, EXPONENTIAL, AND OTHER EXPRESSIONS AND EQUATIONS

Building upon prior knowledge of properties and structure of expressions, students can apply mental manipulations to solve problems. *Note: although some of the following examples involve quadratics, they are appropriate for Math 1 because the emphasis is on reasoning using properties and knowledge of quadratics is not necessary to reason through the problem.*

### *EXAMPLE*

For which values of $m$ are the following inequalities true?

**a.** $m^2 + m < m^2 + m + 3$

**b.** $m^4 - m + 9 \geq -m + m^2 + m^4$

*Discussion:* For Part **a.**, students should realize that $m^2 + m$ appears on both sides of the inequality. Therefore, the inequality is equivalent to $0 < 3$, which is a true statement, so any real number will make the inequality true. For Part **b.**, students should realize that $m^4 - m$ is present on both sides of the inequality; therefore the inequality is equivalent to $9 \geq m^2$ and $-3 \leq m \leq 3$.

### *EXAMPLE*

Evaluate.

$$6.7 \times \frac{2}{3}\left(\frac{5.2 \times 4.3}{9.6^2}\right)(0)$$

*Discussion:* Students should recognize that the actual multiplication is pointless because the Zero Product Property can be applied. Therefore, the expression is equal to 0.

### *EXAMPLE*

Solve for $b$.

**a.** $\frac{3}{5}(y-7)(y+2) = b(y-7)(y+2)$

**b.** $3(x-2) - b = 3x - 6$

*Discussion:* In Part **a.**, students should recognize that $(y-7)(y+2)$ appears on both side of the equation, so $b = \frac{3}{5}$. In Part **b.**, students should realize that $3(x-2) = 3x - 6$ by the Distributive Property, so $b = 0$.

Explore the nature of algebraic equations and systems of algebraic equations using mathematical and real-world contexts.

# INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (A.SSE.1)

***EXAMPLE***

The equations below represent Mario's trip to the store to buy school supplies, where *m* represent binders and *n* represents folders.

$$m + n = 15$$
$$0.50n + 4.69m + 2.28 = 34.92$$

**a.** Determine how much each binder costs.
**b.** What could 2.28 represent in the second equation?
**c.** What does $0.50n + 4.69m + 2.28$ in the second equation represent?
**d.** How many different items did he buy?
**e.** What does $0.50n$ represent?

***EXAMPLE***

Create two equivalent expressions. The first expression should have two terms, and the second expression should have three terms. Then, write an appropriate context for each equivalent expression and explain why the context best represents that form.

***EXAMPLE***

What information is true about the expression?

$$7 + (a - 1)^2$$

**a.** It is always less than or equal to 1.
**b.** It is always greater than or equal to 7.
**c.** There will be two numbers for ***a*** that will make equivalent expressions.
**d.** There will be no numbers for ***a*** that will make equivalent expressions.
**e.** There will be three numbers for ***a*** that will make equivalent expressions.

Offer multiple real-world examples of exponential functions. For instance, students have to recognize that in an equation representing automobile cost $C(t) = 20{,}000(0.75)^t$, since the base of the exponential factor, 0.75, is positive and smaller than 1, it represents an exponential decay or a yearly 25% ($1 - 0.75 = 0.25$) depreciation of the initial \$20,000 value of automobile over the course of *t* years. On the contrary, in an exponential equation representing the amount of investment $A(t) = 10{,}000(1.03)^t$, over *t* years, since the base of exponential factor, 1.03, is greater than 1, it represents exponential growth or a yearly 3% increase of the initial investment of \$10,000 over the course of *t* years.

## INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (A.SSE.1)

***EXAMPLE***

The equation below represents the amount of money Maria deposited in the bank at a fixed annual interest rate that is compounded monthly.

$$P = 2500\left(1 + \frac{0.06}{12}\right)^{12t}$$

- **a.** What does the "1 +" in the equation indicate? How would the equation change if it was "1 – "?
- **b.** What does 2500 represent in terms of the context of the situation?
- **c.** What does 0.06 represent in terms of the context of the situation?
- **d.** What does $\left(1 + \frac{0.06}{12}\right)$ represent in terms of the context of the situation?
- **e.** What does $12t$ represent in terms of the context of the situation?
- **f.** What is the contextual meaning of $\frac{0.06}{12}$?

## Instructional Tools/Resources

*These tools and resources may be helpful for instruction. Before use, they should be evaluated at the local level to determine their appropriateness for instruction.*

**Manipulatives/Technology**

- Hands-on materials, such as algebra tiles, can be used to establish a visual understanding of algebraic expressions and the meaning of terms, factors and coefficients.
- Desmos is a free graphing calculator that is available to students as website or an app.
- Geogebra is a free graphing calculator that is available to students as website.
- Wolframalpha is dynamic computing tool.

**Interpreting Linear or Exponential Expressions and/or Equations**

- Kitchen Floor Tiles is an Illustrative Mathematics task that has students interpret algebraic expressions in connection with geometric patterns. This task integrates modeling.
- Animal Populations is an Illustrative Mathematics task that has students reason which expression is larger based on its structure.
- Delivery Trucks is an Illustrative Mathematics task that has students relate structure to a context without any algebraic manipulation.
- Exponential Parameters is an Illustrative Mathematics task that has students interpret the parameter of an exponential function.
- Mixing Candies is an Illustrative Mathematics task that has students interpret a system of equations.
- The Bank Account is an Illustrative Mathematics task that explores the structure of a real-world exponential equation dealing with interest rates.

# INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (A.SSE.1)

**Interpreting Expressions and/or Equations**

- Mixing Fertilizer is an Illustrative Mathematics task that applies ratio and proportions to mixture problems.
- The Physics Professor is an Illustrative Mathematics task has students drawing conclusions about expressions using information they already know.

**Curriculum and Lessons from Other Sources**

- EngageNY, Algebra 1, Module 1, Topic B, Lesson 6: Algebraic Expressions—The Distributive Property and Lesson 7: Algebraic Expressions—The Commutative and Associative Property are lessons that pertain to this cluster.
- EngageNY, Algebra 1, Module 1, Topic D, Lesson 25: Solving Problems in Two Ways—Rates and Algebra is a lesson that pertains to this cluster.

**General Resources**

- Arizona High School Progressions on Algebra is an informational document for teachers. This cluster is addressed on pages 4-6 and pages 11-12.
- Arizona High School Progression on Modeling is an informational document for teachers.
- High School Coherence Map by UnboundED is a map of the relationships among high school mathematics standards. Please note: This resource is aligned to the Common Core not Ohio's Learning Standards. Although the tool is useful, it needs to be checked with respect to Ohio's Learning Standards.

**References**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| STANDARDS | MODEL CURRICULUM (A.SSE.3) |
| --- | --- |
| **Algebra**<br>**SEEING STRUCTURE IN EXPRESSIONS**<br>**Write expressions in equivalent forms to solve problems.**<br>**A.SSE.3** Choose and produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression.★<br>**c.** Use the properties of exponents to transform expressions for exponential functions. *For example, $8^t$ can be written as $2^{3t}$.* | **Expectations for Learning**<br>In middle school, students explore the properties of exponents informally using patterns. In Math 1, students are expected to formally know the properties of exponents and rewrite exponential expressions with integer exponents using properties of exponents. In Math 3, students expand their skills and knowledge to situations involving rational exponents.<br><br>**ESSENTIAL UNDERSTANDINGS**<br>• Expressions may potentially be rearranged or manipulated in ways to reveal different insights into the mathematical or real-world context.<br>• Understanding the properties of exponents is essential for rewriting exponential expressions.<br><br>**MATHEMATICAL THINKING**<br>• Plan a solution pathway.<br>• Determine the appropriate form of an expression in context.<br><br>**INSTRUCTIONAL FOCUS**<br>**Limit exponential expression to expression with integer exponents.*<br>• Determine the appropriate equivalent form of an expression for a given purpose.<br>• Rewrite exponential expressions by using properties of exponents.<br><br>**Content Elaborations**<br>**OHIO'S HIGH SCHOOL CRITICAL AREA OF FOCUS**<br>• Math 1, Number 1, page 3<br><br>**CONNECTIONS ACROSS STANDARDS**<br>• Interpret key features of graphs (F.IF.4).<br>• Interpret the structure of expressions (A.SSE.1).<br>• Analyze functions using different representations (F.IF.8). |

# INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (A.SSE.3)

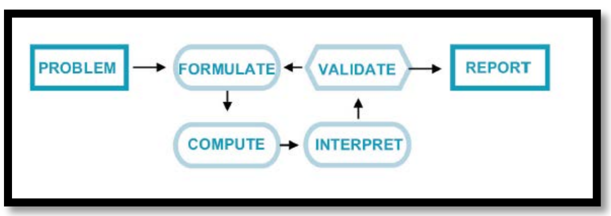
## Instructional Strategies

*Note: The model curriculum is not a collection of lessons nor a full curriculum; it does not suggest pace, sequence, or amount of time spent on topics. It provides information about a topic related to the standards including ideas for examples, strategies for teaching, common misconceptions, and possible connections between topics. In addition, the emphasis of the model curriculum is on instruction not on assessment; therefore, the examples in the Instructional Strategies section are written for instructional purposes not necessarily as assessment items. Examples may need to be rewritten to accommodate the needs of each individual classroom.*

***Standards for Mathematical Practice***
*This cluster focuses on but is not limited to the following practices:*
***MP.4** Model with mathematics.*
***MP.7** Look for and make use of structure.*
***MP.8** Look for and express regularity in repeated reasoning.*

This cluster extends the previous cluster (A.SSE.1-2) from analyzing structure to now using the most efficient equivalent form to solve a problem. It focuses specifically on transforming exponential equations using properties and rewriting quadratics in factored form and vertex form. Students are also connecting the various forms of an expression to the context of the problem and to the analysis of its corresponding graph.

Students must understand the idea that changing the forms of expressions, such as factoring, completing the square, or transforming expressions from one quadratic form to another are not independent algorithms that are learned for the sake of symbolic manipulations. They are processes that are guided by goals (e.g., investigating properties of families of functions, solving contextual problems, finding roots, and identifying maximum or minimum values). An expression can be written in many forms that may look different but are in fact equivalent, as each form still represents the given expression. Rewriting an expression in simplified form may not always be the best form for all situations. It is much more advantageous for students to think about which equivalent form would be the most useful for a particular context instead of always immediately simplifying.




### MODELING

A.SSE.3 is a modeling standard. See page 13 for more information about modeling.

# INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (A.SSE.3)

## PROPERTIES OF EXPONENTS

This is the first time that students are expected to formally know the properties of exponents. In Grade 8 students informally gained experience with the properties of exponents and applied exponent reasoning to scientific notation. They explored some of the properties of exponents using patterns and structure involving negative integer and zero exponents, but were limited to numerical bases. Now they need to extend their understanding of the properties of powers to exponential expressions that have variables as bases. It may be helpful for students to do explorations that revisit the patterns of exponents with the purpose of deriving the more formal properties of exponents. Building on this, students can create and manipulate more advanced exponential expressions and equations. An activity such as Properties of Exponents from S$^2$TEM Centers might be a helpful bridge. Students should continue to write out the expanded form when simplifying exponents until they are able to internalize the properties.

**Properties of Integer Exponents**

For any nonzero real numbers *a* and *b* and integers *n* and *m*:

**1.** $a^0 = 1$

**2.** $a^{-n} = 1/a^n$

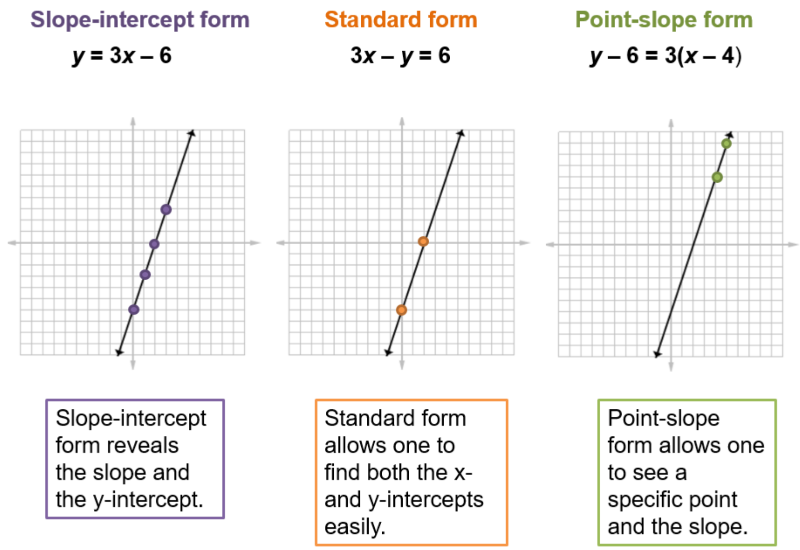
**3.** $a^n a^m = a^{n+m}$

**4.** $(a^n)^m = a^{nm}$

**5.** $a^n b^n = (ab)^n$

**6.** $\frac{a^n}{a^m} = a^{n-m}$

## LINEAR EXPRESSIONS AS PARTS OF A LINEAR EQUATION




Students should be able to write an equation in equivalent forms to reveal and explain different properties of the equation. Writing an equation in different ways can reveal different features of the graph of a function. Students should be given the opportunity to come up with equivalent forms of the same equation, and then explore why the functions are the same both algebraically and graphically. The process of rewriting equations to reveal key features should be used to explain/reveal features in the context of real-world scenarios.

Students should be able to produce equivalent forms for the equation to reveal and explain different features of the graph represented by this equation. After students are provided with the opportunity to explore equivalent forms of the same equation, they will realize that those forms represent the same equation both algebraically and graphically.

# INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (A.SSE.3)

## Instructional Tools/Resources

*These tools and resources may be helpful for instruction. Before use, they should be evaluated at the local level to determine their appropriateness for instruction.*

**Manipulatives/Technology**

- Graphing utilities to explore the effects of parameter changes on a graph
- Desmos is a free graphing calculator that is available to students as website or an app.
- Geogebra is a free graphing calculator that is available to students as website.
- Wolframalpha is dynamic computing tool.

**Different Forms of Functions**

- Forms of Exponential Expressions is an Illustrative Mathematics Task that investigates usefulness of different exponential expressions.

**Curriculum and Lessons from Other Sources**

- EngageNY, Algebra 1, Module 3, Topic D, Lesson 22: Modeling an Invasive Species Population and Lesson 23: Newton's Law of Cooling are lessons that pertain to this cluster.

**General Resources**

- Arizona High School Progression on Algebra is an informational document for teachers. This cluster is address on pages 4-6 and pages 11-12.
- Arizona High School Progression on Modeling is an informational document for teachers. This cluster is addressed on pages 6-7 and 13.
- High School Coherence Map by UnboundED is a map of the relationships among high school mathematics standards. Please note: This resource is aligned to the Common Core not Ohio's Learning Standards. Although the tool is useful, it needs to be checked with respect to Ohio's Learning Standards.

**References**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- Common Core Standards Writing Team. (2013, July 4). *Progressions for the Common Core State Standards in Mathematics (draft). High School, Modeling*. Tucson, AZ: Institute for Mathematics and Education, University of Arizona.
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| STANDARDS | MODEL CURRICULUM (A.CED.1-4) |
| --- | --- |
| ## Algebra<br>**CREATING EQUATIONS**<br>**Create equations that describe numbers or relationships.**<br>**A.CED.1** Create equations and inequalities in one variable and use them to solve problems. *Include equations and inequalities arising from linear, quadratic, simple rational, and exponential functions.* ★<br>**a.** Focus on applying linear and simple exponential expressions. (A1, M1)<br>**A.CED.2** Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales.★<br>**a.** Focus on applying linear and simple exponential expressions. (A1, M1)<br>**A.CED.3** Represent constraints by equations or inequalities, and by systems of equations and/or inequalities, and interpret solutions as viable or non-viable options in a modeling context. *For example, represent inequalities describing nutritional and cost constraints on combinations of different foods.*★ (A1, M1)<br>*Continued on next page* | ## Expectations for Learning<br>In middle school, students create simple equations and simple inequalities and use them to solve problems. In this cluster, students extend this knowledge to write equations and inequalities for more complicated situations, focusing on linear and simple exponential equations. Students also rearrange formulas to highlight a particular variable. In Math 2, students model situations that include quadratic equations.<br><br>*Note: Simple exponential functions include integer exponents only.*<br><br>**ESSENTIAL UNDERSTANDINGS**<br>• Regularity in repeated reasoning can be used to create equations that model mathematical or real-world contexts.<br>• The graphical solution of a system of equations or inequalities is the intersection of the graphs of the equations or inequalities.<br>• Solutions to an equation, inequality, or system may or may not be viable, depending on the scenario given.<br>• A formula relating two or more variables can be solved for one of those variables (the variable of interest) as a shortcut for repeated calculations.<br><br>**MATHEMATICAL THINKING**<br>• Create a model to make sense of a problem.<br>• Represent the concept symbolically.<br>• Plan a solution pathway.<br>• Determine the reasonableness of results.<br>• Consider mathematical units and scale when graphing.<br>*Continued on next page* |

| STANDARDS | MODEL CURRICULUM (A.CED.1-4) |
|---|---|
| **A.CED.4** Rearrange formulas to highlight a quantity of interest, using the same reasoning as in solving equations.★<br>**b.** Focus on formulas in which the variable of interest is linear. *For example, rearrange Ohm's law* $V = IR$ *to highlight resistance R.* (M1) | **Expectations for Learning, continued**<br>**INSTRUCTIONAL FOCUS**<br>• Given a mathematical or real-world context, express the relationship between quantities by writing an equation or inequality that must be true for the given relationship. Focus on situations where the equations will be linear and exponential.<br>• For equations or inequalities relating two variables, graph the relationships on coordinate axes with proper labels and scales. Focus on situations where the equations will be linear and exponential.<br>• Identify the constraints implied by the scenario, and represent them with equations or inequalities.<br>• Determine the feasibility (possibility) of a solution based upon the constraints implied by the scenario.<br>• Solve formulas for a given variable.<br><br>**Content Elaborations**<br>**OHIO'S HIGH SCHOOL CRITICAL AREA OF FOCUS**<br>• Math 1, Number 1, page 3<br><br>**CONNECTIONS ACROSS STANDARDS**<br>• Interpret the structure of expressions (A.SSE.1).<br>• Solve equations and inequalities in one variable (A.REI.3).<br>• Interpret parameters of linear or exponential functions (F.LE.5).<br>• Represent and interpret equations and inequalities (including systems) with two variables graphically (A. REI.10).<br>• Build a function that models a linear or exponential relationship between two quantities (F.BF.1).<br>• Interpret the slope and intercept of a linear model (S.ID.7).<br>• Solve systems of equations (A.REI.6).<br>• Construct and compare linear and exponential models, and solve problems (F.LE.1). |

# INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (A.CED.1-4)

## Instructional Strategies

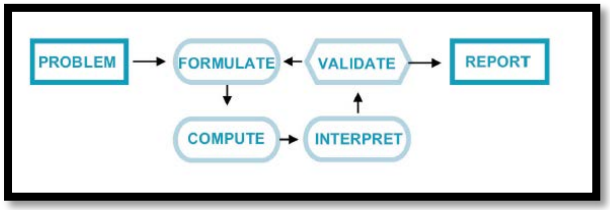
*Note: The model curriculum is not a collection of lessons nor a full curriculum; it does not suggest pace, sequence, or amount of time spent on topics. It provides information about a topic related to the standards including ideas for examples, strategies for teaching, common misconceptions, and possible connections between topics. In addition, the emphasis of the model curriculum is on instruction not on assessment; therefore, the examples in the Instructional Strategies section are written for instructional purposes not necessarily as assessment items. Examples may need to be rewritten to accommodate the needs of each individual classroom.*

***Standards for Mathematical Practice***
*This cluster focuses on but is not limited to the following practices:*
***MP.2*** *Reason abstractly and quantitatively.*
***MP.4*** *Model with mathematics.*
***MP.5*** *Use appropriate tools strategically.*
***MP.6*** *Attend to precision.*
***MP.7*** *Look for and make use of structure.*

### MODELING

A.CED.1-4 is included in the modeling standards. See page 13 for more information about modeling.

The Arizona High School Progression on Algebra has a helpful statement with respect to modeling: "In high school, there is again a difference between directly representing the situation and finding a solution. The formulation of the equation tracks the text of the problem fairly closely, but requires more than a direct representation. The Compute node of the modeling cycle is dealt with in the next section, on solving equations. The Interpret node also becomes more complex. Equations in high school are also more likely to contain parameters than equations in earlier grades, and so interpreting a solution to an equation might involve more than consideration of a numerical value, but consideration of how the solution behaves as the parameters are varied. The Validate node of the modeling cycle pulls together many of the standards for mathematical practice, including the modeling standard itself." (Common Core Standards Writing Team, 2013)




### CREATING EQUATIONS AND INEQUALITIES

Provide examples of real-world problems that can be modeled by writing an equation or an inequality. Students may believe that equations of linear, exponential, and other functions are abstract and exist only "in a math book," without seeing the usefulness of these equations as modeling real-world phenomena. For example, use familiar contexts such as car depreciation to highlight linear and exponential equations. Begin with simple equations and inequalities and build up to more complex equations in two or more variables that may involve exponential functions.

Make sure students have experience writing an equation of a line given two-points or given the slope and a point.

# INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (A.CED.1-4)

## Lack of Fractional Knowledge

Research has shown a link between students' knowledge of fractions and their ability to write equations. Although there has been some work done at the middle school level, some additional work may be needed for scaffolding at the high school level (See Grade 6 Model Curriculum 6.EE.5-8 and 6.EE.9 and Grade 7 Model Curriculum 7.EE.3-4 for scaffolding ideas.)

## Key Words Strategy

Another issue in writing equations is that students are over reliant on key words in creating equations as they use key words in place of reasoning. This approach fails when problems become more complex and when there are several relationships between quantities. One of the problems with the Key Word approach is that it relies too heavily on numbers or values in the problem instead of the relationship between quantities, which ties into understanding the structure of an expression (A.SSE.1-2). To combat the misconceptions involving using key words, intentionally give students situations where aligning key words does not lend itself to writing equations that represents the situation. Drawing diagrams is one way to help students understand the structure of a problem. Another strategy is directing students to make sense of the situation by asking questions. Also, having students discuss their thinking in terms of quantities and relationships instead of values, calculations, and operations is another strategy for student success. Encourage students to explain "why" they did something in contrast to explaining "what" they did. It may be helpful for some students to write down all the quantities in the problem and to state the important aspects of a problem.

## Writing Comparison Equations

Students also have a difficult time writing equations where the context is reversed such as "There are 8 times as many football players as cheerleaders." Many students incorrectly write $8F = C$ instead of correctly writing $F = 8C$. Studies show that this problem is not just limited to misunderstanding key words, but rather—

- incorrectly matching the order of the words in the situation to the equation;
- thinking that the larger number is placed next to the variable defining the larger group;
- treating variables as labels;
- treating variables as a fixed unknown rather than as a variable quantity; and/or
- treating the equal sign as representing equivalence but more like an association.

Students who are able to represent these problems correctly invent operations to establish equivalence thereby forcing unequal groups to be equal. They also look at the situation in terms of a function context instead of as two unrelated quantities. See Model Curriculum 8.F.4-5 for more information on writing comparison equations.

TIP!

Students should always identify variables when creating equations. A reader should never have to assume anything. Students should be precise when identify variables rather than just stating $x$ = apples, they should state $x$ = number of apples or $x$ = price per apple.

# INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (A.CED.1-4)

***EXAMPLE***

There are four times the number of pizza places in Centerville as there are Chinese restaurants. Write an equation to represent the number of pizza places and the number of Chinese restaurants. Make sure to define the variables.

***EXAMPLE***

There are twenty-seven more students in choir than in band. Write an equation to represent the number of students in choir and the number of students in band. Make sure to define the variables.

***EXAMPLE***

At the local steakhouse, for every 5 people who order steak, two order chicken. Write an equation to represent the number of steak entrees and the number of chicken entrees. Make sure to define the variables.

**Guess and Check Strategy**

Students often have difficulty writing equations and inequalities for given situations. Consider a strategy using guess and check as a process for writing an equation that must be true as described in Al Cuoco's blog entry in *Mathematical Musings*, "Teachers know that building is much harder for students than checking. The same practice of abstracting from numerical examples is useful here, too."

***EXAMPLE***

"Emilio drives from Tucson to Phoenix at an an average speed of 60 MPH and returns at an average speed of 50 MPH. If the total time on the road is 4.4 hours, how far is Tucson from Phoenix?"

*Discussion:* "The practice of abstracting regularity from repeated actions can be used to build an equation whose solution is the answer to the problem: One takes several guesses (for the distance) and checks them, focusing on the steps that are common to each of the checks. The goal isn't to stumble on (or approximate) an answer by "guess and check;" the goal is to come up with a general "guess checker" expressed as an algebraic equation: $\frac{guess}{60}+\frac{guess}{50}=4.4$...Coherence comes from the fact that exactly the same mathematical practice is used to find an algebraic equation whose solution solves the problem." (Cuoco, 2017).

**Using Arithmetic to Write a Generalized Equation Strategy**

Students can use arithmetic to write an equation from a real-world context as a strategy for writing equations. They can start with an example using numbers and move towards a more general equation that is true.

# INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (A.CED.1-4)

***EXAMPLE***

Suppose a friend tells you she paid a total of $16,368 for a car, and you would like to know the car's list price (the price before taxes) so that you can compare prices at various dealers. Find the list price of the car if your friend bought the car in—

- **a.** Arizona, where the sales tax is 6.6%.
- **b.** New York, where the sales tax is 8.25%.
- **c.** A state where the sales tax is *r*.

Taken from Buying a Car by Illustrated Mathematics.

*Discussion:* Students progress from two equations that involve an actual value of sales tax to an equation that uses sales tax as a parameter. The goal is for students to start to notice regularity in solution procedures.

**Using Tables to Write Equations**

Students can use tables to help them notice patterns and write equations. Use contexts that lend themselves to tabular representations. They can create tables that represent this relationship simply by counting and use this table to write equations.

## REPRESENTING CONSTRAINTS

Creating constraints involves interpreting the equation that represents the context accurately. There are different types of constraints that can be represented by equations or inequalities. A context can require a constraint to just be in one-variable $x > 3$ or in both variables $x > 3$ and $y \leq 20$. Constraints can also be written as an equation or inequality in two-variables such as $y < -2x - 40$ or even as a system of equations and/or inequalities depending on the context. They can also include identifying the number sets such as whole numbers.

Desmos Marbleslides may be useful tool to help students represent constraints.

Examine real-world graphs in terms of constraints that are necessary to balance a mathematical model with the real-world context. For example, when a student writes a linear equation modeling equal monthly deposits into a bank account such as $y = 1{,}500x + 7{,}000$, where $y$ represents the total amount of money in the bank account with no monthly withdrawals (and not factoring monthly interest rates), and $x$ represents the month's number given that January is month number 1, they should recognize that situation only makes sense when the domain is a whole number since months are whole numbers.

## REARRANGING FORMULAS

Explore examples illustrating when it is useful to rewrite a formula by solving for one of the variables in the formula. For example, the formula for the area of a trapezoid, $A = \frac{1}{2}h(b_1 + b_2)$, can be solved for *h* if the area and lengths of the bases are known, but the height needs to be calculated. This strategy of selecting a different representation has many applications in science and business when using formulas. *Note: In Math 1 exponential equations will not be rearranged, because this would introduce logarithms.*

# INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (A.CED.1-4)

TIP!

"Variable of interest" means the variable or quantity a person is interested in solving for or looking for a solution or relationship. It is not interest in the sense of banks and rate of growth.

Provide examples of real-world problems that can be solved by writing an equation, and have students explore the graphs of the equations on a graphing calculator to determine which parts of the graph are relevant to the problem context.

Use a graphing calculator to demonstrate how dramatically the shape of a curve can change when the scale of the graph is altered for one or both variables.

Give students geometric formulas such as area and volume or formulas from science or business contexts, and have students solve the equations for each of the different variables in the formula.

Solving equations for the specified letter with coefficients represented by letters (e.g., $A = \frac{1}{2}h(b_1 + b_2)$ when solving for $b_1$) is similar to solving an equation with one variable. Provide students with an opportunity to apply the same kind of manipulations to formulas as they did to equations.

***EXAMPLE***

Solve for *y*.

$\frac{a}{x} = \frac{b}{y}$

TIP!

Letters can be referred to as "variables," "parameters," or "constants," which can be helpful if they are used consistently as it may give insight into how students view a problem. However, for formulas such as Ohm's Law it may be best to avoid using those terms all together when working with this formula, because there are six different ways it can be viewed as defining one quantity as a function of the other with a third held constant (Common Standards Core Writing Team, March 2013).

Students may believe that formulas are static, but formulas that are models may sometimes be readily transformed into functions that are models. For example, the formula for the volume of a cylinder can be viewed as giving volume as a function of area of the base and the height, or, rearranging, giving the area of the base as a function of the volume and height. Similarly, Ohm's law can be viewed as giving voltage as a function of current and resistance.

# INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (A.CED.1-4)

In Grade 7 students learned about proportional relationships and constants of proportionality: "7.RP.2 Recognize and represent proportional relationships between quantities." These concepts surface often in high school modeling situations. Students learn that many modeling situations begin with a statement like Ohm's Law or Newton's Second Law. In Ohm's Law, $V = IR$, $V$ is the quantity of interest, $V$ is directly proportional to $R$ where $I$ is the constant. The formula can be also rearranged to highlight a different quantity of interest, $I$, where $I = \frac{V}{R}$. In this case $I$ is inversely proportional to $R$ where $V$ is the constant.

## ADDITIONAL NOTES

To understand the differences among A.CED.1, A.CED.2, and A.CED.3, consider the following problem:

- We have 14 coins (nickels and dimes) and they are worth $0.95. How many of each coin?

For A.CED.1, students can write an equation in one variable (as shown below) and then solve:
Value of nickels + value of dimes = 95 cents
$5n + 10(14 - n) = 95$

Alternatively, for A.CED.2 and A.CED.3, students can write two equations using two variables (A.CED.2), creating a system of equations in two variables (A.CED.3) and then solve:
$n + d = 14$
$5n + 10d = 95$

All students should be able to understand both of these approaches and should be able to use them as appropriate without requiring a particular approach on a given problem.

For A.CED.4, when rewriting the formula for the area of a circle to highlight radius $r$ (for example), first ask students to figure out what the radius would be if the area is 10 square units. Then ask them what the radius would be if the area is 20 square units. Then if the area is 23 square units. Eventually, students should understand the rewritten equation solved for $r$ is a general formula for finding the radius given any area, instead of going through the several steps to find $r$ every time. This process is a way to encourage students look for and express regularity in repeated reasoning (Mathematical Practice 8).

# INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (A.CED.1-4)

## Instructional Tools/Resources

*These tools and resources may be helpful for instruction. Before use, they should be evaluated at the local level to determine their appropriateness for instruction.*

**Manipulatives/Technology**

- Graphing calculators
- Computer software that generate graphs of functions
- Examples of real-world situations that lend themselves to writing equations that model the contexts.
- Desmos is a free graphing calculator that is available to students as website or an app.
- Geogebra is a free graphing calculator that is available to students as website.
- Wolframalpha is dynamic computing tool.
- Visual Patterns is a website that shows pictures of linear, exponential, and quadratic patterns.
- Patterns Posters for Algebra 1 from Finding Ways by Fawn Nguyen is a blog about a class activity where students are given patterns, and they have to make posters from them. She is the creator of the visual patterns link above.

**Creating Equations**

- Dairy Barn is a lesson connecting math standards to CTE by Achieve the Core. The lesson has students create equations in order to establish the amount of fill sand need to fill the barn stalls.
- Ivy Smith Grows Up is a lesson connecting math standards to CTE by Achieve the Core. The lesson has students evaluate the growth of newborns and infants. They write equations to model the situation.
- New and Improved Thinking Blocks by Math Playground has videos, a game, and an applet to use thinking blocks to solve system of equation problems. Thinking blocks are similar to tape diagrams.
- Buying a Car is a task from Illustrative Mathematics where students create equations that involve different values for sales tax but move towards representing sales tax as a parameter.
- Dirt Bike Dilemma is a two-part lesson by NCTM Illuminations where students write algorithms for linear programming. *NCTM now requires a membership to view their lessons.*
- Interpreting Algebraic Expressions from Mathematics Assessment Project is a task where students translate algebraic expressions.

**Creating Inequalities**

- How much Folate? is a task by Illustrative Mathematics that could be used as an introduction to writing and graphing linear inequalities.
- Rabbit Food is a lesson connecting math standards to CTE by Achieve the Core. The lesson has students write inequalities, equations, and applying constraints to the situation. This lesson aligns with A.CED.2-3.

# INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (A.CED.1-4)

**Rearranging Formulas**

- Forget the Formula is a task from Georgia Standards of Excellence Framework pages 49-55. In this task students will develop a formula to convert temperatures from Celsius to Fahrenheit, and then they will rearrange the formula. This aligns to A.CED.2-4.

**Representing Constraints**

- Cara's Candles Revisited is a task from Georgia Standards of Excellence Framework, Algebra 1, Unit 2: Reasoning with Linear Equations and Inequalities, pages 63-70. In this task students will create equations with constraints and then graph the equations.

**Curriculum and Lessons from Other Sources**

- EngageNY, Algebra 1, Module 1, Topic C, Lesson 19: Rearranging Formulas and Lesson 24: Applications of Systems of Equations and Inequalities are lessons that pertain to this cluster.
- EngageNY, Algebra 1, Module 1, Topic D, Lesson 25: Solving Problems in Two Ways—Rates and Algebra and Lesson 28: Federal Income Tax are lessons that pertain to this cluster.
- Mathematics Vision Project , Algebra 1, Module 4: Solving Equations has lessons that pertain to this cluster.
- Exporing Symbols by Burrill, Clifford, Scheaffer is the teacher's edition of a textbook in the Data-Driven Mathematics series published by Dale Seymour Publications. There are several lessons that pertain to this cluster. The student edition can be found here.

**General Resources**

- Arizona High School Progression on Algebra is an informational document for teachers. This cluster is addressed on the last paragraph of pages 10-12.
- Arizona High School Progression on Modeling is an informational document for teachers. This cluster is addressed on the last paragraph of page 13 which continues on page 14 and is addressed under the Formulas as Models section on pages 16-17.
- High School Coherence Map by UnboundED is a map of the relationships among high school mathematics standards. Please note: This resource is aligned to the Common Core not Ohio's Learning Standards. Although the tool is useful, it needs to be checked with respect to Ohio's Learning Standards.

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# INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (A.CED.1-4)

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| STANDARDS | MODEL CURRICULUM (A.REI.1) |
| --- | --- |
| **Algebra**<br>**REASONING WITH EQUATIONS AND INEQUALITIES**<br>**Understand solving equations as a process of reasoning and explain the reasoning.**<br>**A.REI.1** Explain each step in solving a simple equation as following from the equality of numbers asserted at the previous step, starting from the assumption that the original equation has a solution. Construct a viable argument to justify a solution method. | **Expectations for Learning**<br>In previous courses, students solve simple equations using a variety of methods and investigate whether a linear equation (8.EE.7) or a system of linear equations (8.EE.8) has one solution, infinitely many solutions, or no solutions. In this cluster, students explain the process for finding a solution for any type of simple equation. Similar to proofs in Math 2, students provide reasons for the steps they follow to solve an equation. In Math 3, students solve simple rational and radical equations and explain why extraneous solutions may arise.<br><br>**ESSENTIAL UNDERSTANDINGS**<br>• Solving equations is a process of reasoning based on properties of operations and properties of equality.<br>• Assuming no errors in the equation-solving process,<br>  o A result that is false (e.g., 0 = 1) indicates the initial equation must have had no solutions; and<br>  o A result that is always true (e.g., 0 = 0) indicates the initial equation must have been an identity.<br>• Adding or subtracting the same value or expression to both sides of an equation results in an equivalent equation.<br>• Multiplying or dividing both sides by the same value or expression (except by 0) results in an equivalent equation.<br>• The Addition Property of Equality and Subtraction Property of Equality can be used interchangeably, since subtracting a number is the same as adding its opposite.<br>• The Multiplication Property of Equality and the Division Property of Equality can be used interchangeably (except when multiplying by 0), since dividing a number is the same as multiplying the number by its inverse.<br><br>**MATHEMATICAL THINKING**<br>• Explain mathematical reasoning.<br>• Plan a solution pathway.<br>*Continued on next page* |

| STANDARDS | MODEL CURRICULUM (A.REI.1) |
|---|---|
| | **Expectations for Learning, continued**<br>**INSTRUCTIONAL FOCUS**<br>*Note: Although, rote memorization of the names of the properties is not encouraged, it is expected for teachers to use formal language so that students gain familiarity and are able to recognize and apply the correct terminology.*<br>• Justify the steps in solving an equation.<br>• Solve equations in which there is one solution; equations in which there is no solution; and equations in which there are infinitely many solutions.<br><br>**Content Elaborations**<br>**OHIO'S HIGH SCHOOL CRITICAL AREA OF FOCUS**<br>• Math 1, Number 3, page 7<br><br>**CONNECTIONS ACROSS STANDARDS**<br>• Solve linear equations and inequalities in one variable (A.REI.3).<br>• Create equations that describe numbers or relationship (A.CED.1). |

# INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (A.REI.1)

## Instructional Strategies

*Note: The model curriculum is not a collection of lessons nor a full curriculum; it does not suggest pace, sequence, or amount of time spent on topics. It provides information about a topic related to the standards including ideas for examples, strategies for teaching, common misconceptions, and possible connections between topics. In addition, the emphasis of the model curriculum is on instruction not on assessment; therefore, the examples in the Instructional Strategies section are written for instructional purposes not necessarily as assessment items. Examples may need to be rewritten to accommodate the needs of each individual classroom.*

***Standards for Mathematical Practice***

*This cluster focuses on but is not limited to the following practices:*

***MP.1** Make sense of problems and persevere in solving them.*

***MP.2** Reason abstractly and quantitatively.*

***MP.3** Construct viable arguments and critique the reasoning of others.*

***MP.6** Attend to precision.*

### THE PROCESS OF REASONING IN EQUATION SOLVING

Solving equations is a process of reasoning. Each manipulation in an equation is based on a property that is known to be true. Understanding the reasoning behind each step in the manipulation of an equation helps students prevent errors or make up their own "rules."

The properties of operations (commutative, associative, distributive) should already be familiar and known to students. While the properties of equality will be used to solve equations, requiring students to use the formal names of these properties is not necessary; although students should be able to apply and recognize them. This is similar to Euclid's Common Notions where students demonstrate understanding without using formal names, but use appropriate informality. For example, Common Notion 1 states "Things which equal the same thing also equal one another" (Transitive and Symmetric Properties of Equality), and Common Notion 2 states "If equals are added to equals, then the wholes are equal" (Addition Property of Equality). Similarly, students do not need to know the formal names for the properties of equality, but should be able to recognize and apply them.

*Note: The Distributive Property applied to the multiplication of two binomials should not be referred to as FOIL. Also, collecting like terms is an application of the Distributive Property (see A.APR.1). In addition, the Subtraction Property of Equality and Addition Property of Equality can be interchanged because subtraction is the same thing as adding the opposite. This also applies to the Multiplication and Division Properties of Equality.*

Begin with simple, one-step equations and require students to write out a justification for each step used to solve the equation. Since students are not required to know the formal names of properties but only recognize them, provide a pool of properties for students to choose from. This type of reasoning will set the stage for proofs in Geometry Conceptual Category

## INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (A.REI.1)

***EXAMPLE***

| Step | Justification |
|---|---|
| $a + 4 = 12$ | Given |
| $a + 4 - 4 = 12 - 4$ | |
| $a + 0 = 12 - 4$ | |
| $a = 12 - 4$ | |
| $a = 8$ | |

*Discussion:* It may be helpful to give students a list of properties to choose from. In line 2 students could write the Addition or Subtraction Property of Equality or another sufficient explanation. Draw attention to the fact that the Addition and Subtraction Properties of Equality can be used interchangeably. In line 3 students could write the Additive Inverse Property or another sufficient explanation. In line 4 students could write the Additive Identity Property or other sufficient explanation. In line 5 students may write simplification or other sufficient explanation. Although students may not use the formal vocabulary, teachers should be modeling and using the correct vocabulary and encouraging it during discussion.

***EXAMPLE***

$\frac{1}{3}(4 - m)6 = \frac{4}{5}m - 2$

- **a.** Consider the equation and apply the Commutative Property of Multiplication to rearrange the equation having the same solution set.
- **b.** Then add a value to both sides of the equation and explain why the new equation has the same solution set as part **a.**
- **c.** Then multiply each side of the equation by a value. Discuss what would happen if you only multiplied part of one side of the equation by the value.

*Discussion:* The purpose of the above example is to reinforce the understanding of properties and equivalence. Draw students' attention to the fact that regardless of which property they apply, the original equation and the resulting equations have the same solution set. This example can be changed by having students apply different properties in part **a.** such as the Commutative Property of Addition to one side of the equation or applying the Distributive Property etc.

***EXAMPLE***

Do not solve! Determine which of the following equations have the same solution set by recognizing the properties.

- **a.** $6(3y + 2) = 5y - 4$
- **b.** $18y + 12 + 6y = 5y - 4 + 6y$
- **c.** $12(6y + 4) = 10y - 8$
- **d.** $18y + 12 = -4 + 5y$
- **e.** $(3y + 2)6 = 5y - 4$
- **f.** $18y + 2 = -4 + 5y$
- **g.** $12(3y + 2) = 10y - 8$
- ***h.*** $(3y + 2)6 = 5y - 4 + 6y$

# INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (A.REI.1)

Challenge students to justify each step of solving an equation. Transforming $2x - 5 = 7$ to $2x = 12$ is possible because $5 = 5$, so adding the same quantity to both sides of an equation makes the resulting equation true. Each step of solving an equation can be defended, much like providing evidence for steps of a geometric proof.

Provide examples for how the same equation might be solved in a variety of ways as long as equivalent quantities are added or subtracted to both sides of the equation. The order of steps taken will not matter.

***EXAMPLE***

Have students work in groups. Instruct Group A to subtract 2 from both sides first. Instruct Group B to add 10 to both sides first. Instruct Group C to subtract *n* from both sides first. Then have them solve the equation. Have them compare solutions with your classmates.

**Group A**

$$\begin{array}{rcl} 3n + 2 &=& n - 10 \\ \underline{\quad - 2} &\underline{=}& \underline{\quad - 2} \\ 3n &=& n - 12 \\ \underline{-n} &\underline{=}& \underline{-n} \\ 2n &=& -12 \\ n &=& -6 \end{array}$$

**Group B**

$$\begin{array}{rcl} 3n + 2 &=& n - 10 \\ \underline{+ 10} &\underline{=}& \underline{+ 10} \\ 3n + 12 &=& n \\ \underline{-3n} &\underline{=}& \underline{-3n} \\ 12 &=& -2n \\ -6 &=& n \end{array}$$

**Group C**

$$\begin{array}{rcl} 3n + 2 &=& n - 10 \\ \underline{-n} &\underline{=}& \underline{-n} \\ 2n + 2 &=& -10 \\ \underline{-2} &\underline{=}& \underline{-2} \\ 2n &=& -12 \\ n &=& -6 \end{array}$$

*Discussion:* Use an example similar to this one to launch a discussion about equivalence and the properties of equality. Students should come to the conclusion that all groups arrived at the same solution regardless of the method. Draw attention to the fact that in certain situations some methods are easier to solve than others. See A.REI.3-4 for more information about efficiency and solving equations.

**TIP!**

Discuss the difference between mathematical errors and strategic errors. Strategic errors have mathematically correct steps but are not efficient strategies for solving an equation.

Students may believe that solving an equation such as $3x + 1 = 7$ involves "only removing the 1," failing to realize that the equation $1 = 1$ is being subtracted to produce the next step.

# INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (A.REI.1)

## Instructional Tools/Resources

*These tools and resources may be helpful for instruction. Before use, they should be evaluated at the local level to determine their appropriateness for instruction.*

**Reasoning about Equations**

- Reasoning with Linear Inequalities is an Illustrative Mathematics task that has students identify errors in mathematical reasoning.
- Same Solutions? is an Illustrative Mathematics task that has students evaluate several equations and determine which ones have the same solutions.
- How Does the Solutions Change? is an Illustrative Mathematics task that has students reason about equations without explicitly solving them.

**Curriculum and Lessons from Other Sources**

- EngageNY, Geometry, Module 4, Topic A, Lesson 2: Finding Systems of Inequalities That Describe Triangular and Rectangular Regions is a lesson that pertains to this cluster.
- EngageNY, Algebra 1, Module 1, Topic C, Lesson 11: Solution Sets for Equations and Inequalities, Lesson 12: Solving Equations, and Lesson 13: Some Potential Dangers When Solving Equations are lessons that pertain to this cluster. Students are not required to know set notation, but it could be used as an extension for more advanced students.
- Jaden's Phone Plan from Georgie Standards of Excellence Curriculum Frameworks, Algebra 1, Unit 2: Reasoning with Linear Equations and Inequalities is a lesson that pertains to this cluster. This lesson is found on pages 39-46.
- Mathematics Vision Project, Algebra 1, Module 4: Equations and Inequalities has several lessons that pertain to this cluster.

**General Resources**

- Arizona High School Progression on Algebra is an informational document for teachers. This cluster is addressed on page 13, paragraphs 1 and 2.
- High School Coherence Map by UnboundED is a map of the relationships among high school mathematics standards. Please note: This resource is aligned to the Common Core not Ohio's Learning Standards. Although the tool is useful, it needs to be checked with respect to Ohio's Learning Standards.

**References**


- Common Core Standards Writing Team. (2013, March 1). *Progressions for the Common Core State Standards in Mathematics (draft). High School, Algebra*. Tucson, AZ: Institute for Mathematics and Education, University of Arizona.

| STANDARDS | MODEL CURRICULUM (A.REI.3) |
|---|---|
| **Algebra**<br>**REASONING WITH EQUATIONS AND INEQUALITIES**<br>**Solve equations and inequalities in one variable.**<br>**A.REI.3** Solve linear equations and inequalities in one variable, including equations with coefficients represented by letters. | **Expectations for Learning**<br>In previous courses, students solve linear equations and inequalities. In this cluster, students extend this knowledge to solve equations with numeric and letter coefficients. In Math 2, students solve quadratic equations (with real solutions) using a variety of methods. In Math 3, students use these skills to solve more complicated equations.<br><br>**ESSENTIAL UNDERSTANDINGS**<br>• An appropriate solution path can be determined when the equation is linear in the variable of interest.<br>• When the coefficients of the variable of interest are letters, the solving process is the same as when the coefficients are numbers.<br><br>**MATHEMATICAL THINKING**<br>• Generalize concepts based on properties of equality.<br>• Solve routine and straightforward problems accurately.<br>• Plan a solution pathway.<br>• Solve math problems using appropriate strategies.<br>• Solve multi-step problems accurately.<br>• (+) Use formal reasoning with symbolic representation.<br><br>**INSTRUCTIONAL FOCUS**<br>• Recognize when an equation or inequality is linear in one variable, and plan a solution strategy.<br>• Solve linear equations and inequalities with coefficients represented by letters.<br>    o For inequalities, graph solutions sets on a number line.<br>• Solve compound linear inequalities in one-variable.<br>    o Graph solution sets on a number line.<br>*Continued on next page* |

| STANDARDS | MODEL CURRICULUM (A.REI.3) |
|---|---|
| | **Content Elaborations**<br>**OHIO'S HIGH SCHOOL CRITICAL AREA OF FOCUS**<br>• Math 1, Number 3, page 7<br><br>**CONNECTIONS ACROSS STANDARDS**<br>• Understand solving equations as a process of reasoning (A.REI.1).<br>• Rearrange formulas to highlight a quantity of interest (A.CED.4).<br>• Interpret the structure of expression (A.SSE.1).<br>• Create equations in one variable (A.CED.1).<br>• Graph the solution of an inequality in two variables (A.REI.12). |

# INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (A.REI.3)

## Instructional Strategies

*Note: The model curriculum is not a collection of lessons nor a full curriculum; it does not suggest pace, sequence, or amount of time spent on topics. It provides information about a topic related to the standards including ideas for examples, strategies for teaching, common misconceptions, and possible connections between topics. In addition, the emphasis of the model curriculum is on instruction not on assessment; therefore, the examples in the Instructional Strategies section are written for instructional purposes not necessarily as assessment items. Examples may need to be rewritten to accommodate the needs of each individual classroom.*

This cluster should not be taught in isolation; it should be joined with A.CED.1-4 where students create real-world problems and parameters for equations in context.

***Standards for Mathematical Practice***
*This cluster focuses on but is not limited to the following practices:*
***MP.1*** *Make sense of problems and persevere in solving them.*
***MP.2*** *Reason abstractly and quantitatively.*
***MP.6*** *Attend to precision.*
***MP.7*** *Look for and make use of structure.*

### LINEAR EQUATIONS

Since students were required to solve linear equations in Grade 8 (8.EE.7), the focus in Math 1 should be on more complicated equations especially with respect to rational numbers. See Grade 8 Model Curriculum 8.EE.7-8 for ideas for scaffolding.

### Equations

Students especially have difficulty with equations involving fractions. Discuss alternative methods to solving equations.

***EXAMPLE***

Solve for *h*.

$$\frac{3}{4}h+\frac{5}{12}=\frac{2}{3}$$

**Method 1**

$$\frac{3}{4}h+\frac{5}{12}-\frac{5}{12}=\frac{2}{3}-\frac{5}{12}$$
$$\frac{3}{4}h=\frac{8}{12}-\frac{5}{12}$$
$$\frac{3}{4}h=\frac{3}{12}$$
$$\left(\frac{4}{3}\right)\frac{3}{4}h=\frac{3}{12}\left(\frac{4}{3}\right)$$
$$h=\frac{1}{3}$$

**Method 2**

Clear the fractions by using the multiplication property of equality, and multiply each side by the common denominator of 12.

$$\frac{12}{1}\left(\frac{3}{4}h+\frac{5}{12}\right)=\frac{2}{3}\left(\frac{12}{1}\right)$$
$$9h+5=8$$
$$9h=3$$
$$h=\frac{1}{3}$$

# INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (A.REI.3)

TIP!

Make sure students have a clear understanding of the Multiplication Property of Equality. Many students misapply the strategy of clearing fractions in situations involving the Distributive Property. For example, in the equation $\frac{2}{3}\left(a-\frac{1}{2}\right)=\frac{5}{6}$, students may have difficulty in viewing $\frac{2}{3}\left(a-\frac{1}{2}\right)$ as a term, so they incorrectly multiply $\frac{2}{3}a, -\frac{1}{2}$, and $\frac{5}{6}$ by the common denominator of 6 instead of just $\frac{2}{3}$ and $\frac{5}{6}$. Students may benefit by distributing first before clearing fractions.

***EXAMPLE***

$$\frac{5}{6}\left(2q-\frac{1}{2}\right)=\frac{3}{4}$$

TIP!

Emphasize the multiplication identity $1 \cdot a = a$ in equations that do not appear to have a coefficient. Have your struggling students actually write out the coefficient $1$, so they combine like terms effectively.

Regardless of the type of numbers or variables in the equation or inequality, students have to examine the validity of each step in the solution process. This is a connection to A.REI.1. Continually ask students for justifications when performing each step. This should help prevent students from making up their own rules. Students must be aware of what it means to check the solution of an equation of inequality.

## Variables

Draw students' attention to equations containing several letters or variables with subscripts. The same variables with different subscripts (e.g., $x_1$ and $x_2$) should be viewed as different variables that cannot be combined as like terms. A variable with a variable subscript, such as $a_n$, must be treated as a single variable—the *n*th term, where letters *a* and *n* have different meaning.

Some students may incorrectly believe that subscripts can be combined as $b_1 + b_2 = b_3$ and that the capitalized and lowercase version of the same letter as *d* and D is 2D (*d* + D = 2D) can also be combined.

TIP!

Letters do not always represent variables. Letters can be unknowns, coefficients, parameters, names of functions, etc. Try not to use the word variable as a synonym of letter.

## LINEAR INEQUALITIES

Solving inequalities can be taught either at the same time or after solving equations. If inequalities are taught after equations, it is important to highlight the similarities in the solving process. Examine the validity of each step in the solution process (A.REI.1).

# INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (A.REI.3)

***EXAMPLE***

Consider the values {–100, –1, 3, 4, 4½, 4.9999, 5, 5.00001, 6, 7, 101} to determine the solution set for the two inequalities. How do their solutions sets differ? Explain.

- $3x – 3 > x + 7$ *vs* $3x – 3 \geq x + 7$

*Discussion:* Consider the values {–100, –1, 3, 4, 4½, 4.9999, 5, 5.00001, 6, 7, 101}, have students determine which values are members of the solution set and how the presence of the symbol ≥ versus > signs changes the solution set. Have students graph the solutions on a number line and compare the graphs. Discuss why the > and < have an open circle and the ≥ and ≤ signs have a closed circle.

**Flipping the Inequality Sign**

Have each student choose two rational numbers and plot them on a number line. Then have each student write an inequality statement to compare the numbers. Write each students' statements on the board. Then have the students multiply each side of their inequality by –1 and plot it on the same number line they used initially. Then have them write a new inequality statement. Discuss why the inequality sign flips. Have them try multiplying by other negative numbers besides –1. Ask if flipping the inequality sign also is necessary for addition. Continue this process by having students write several numerical inequality expressions. Eventually extend to algebraic inequality expressions. Discuss the Multiplication Property of Inequality and why it always works. Here is a YouTube video that models a similar idea. See Grade 7 Model Curriculum 7.EE.3-4 for more ideas about scaffolding.

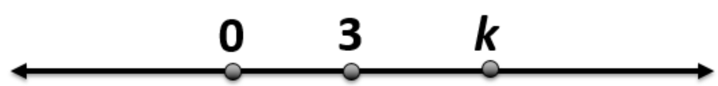
## INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (A.REI.3)

***EXAMPLE***

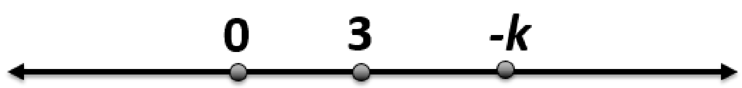
**Part 1**

- Given that $k$ is a number greater than 3, draw a number line with the points 0, $k$, and 3 that represent a possible situation. *(Shown in Line 1.)*
- What type of number is $k$? Positive? Negative? Rational? Integer? Explain.
- Write an inequality describing the relationship between 3 and $k$. *(Shown in Line 1.)*
- Write an inequality describing the relationship between 3 and $-k$. *(Still using Line 1.)*
- Given that $-k$ is a number greater than 3, draw a number line with the points 0, $-k$, and 3 that represent a possible situation. *(Shown in Line 2.)*
- What type of number is $k$? Positive? Negative? Rational? Integer? Explain.
- Write an inequality describing the relationship between 3 and $-k$. *(Shown in Line 2.)*
- Write an inequality describing the relationship between 3 and $k$. *(Still shown in Line 2.)*

Line 1




Line 2



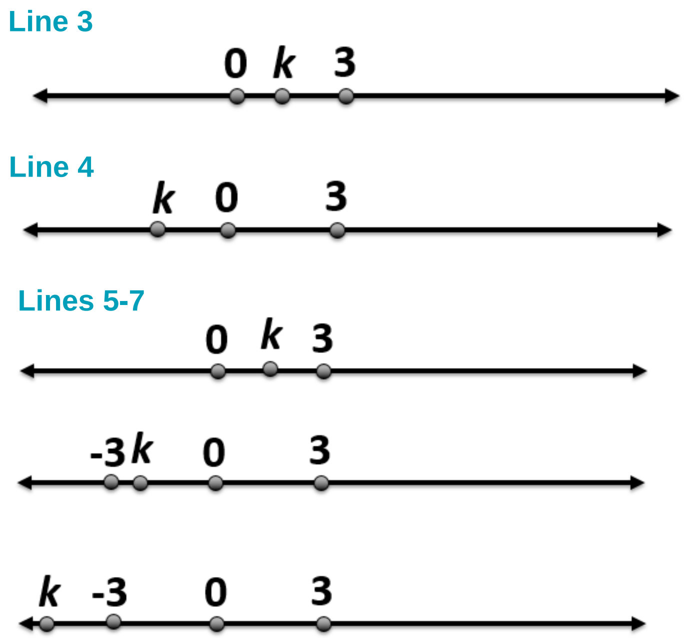

*Example continued on next page*

# INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (A.REI.3)

**Part 2**

Given that 3 > *k*, draw a number line with the points 0, *k*, and 3 that represent a possible situation.

- What types of numbers could *k* be? Positive? Negative? Rational? Integer? Explain.
- If *k* is a positive number, write an inequality comparing the situation. *(Shown in Line 3.)*
- If *k* is a negative number write an inequality comparing the situation. *(Shown in Line 4.)*
- Does it matter in this case if the value of *k* is positive or negative?
- Plot –*k* in Lines 5-7. What do you notice about –*k*?
- If we still hold to 3 > *k*, can we write an inequality comparing 3 to –*k*? Explain.
- However, if we still hold to 3 > *k*, what do we know for sure about the value of –*k* value? Explain.
- Write an equivalent inequality to 3 > *k* using –*k*. Explain why it is true.
- Using what you learned, write an equivalent inequality using –*p* for the following:
  - o $p < 4$
  - o $2.5 > p$
  - o $-3 < p$
  - o $p > -\frac{1}{4}$
  - o $p \geq 7$
- Using what you learned write an equivalent inequality using *b* for the following:
  - o $-b > 5$
  - o $-b < 2.4$
  - o $-3 > -b$
  - o $\frac{5}{6} < -b$
  - o $-b < -6.1$
- Write a rule that explains to a friend how to change any inequality with a negative variable to an equivalent one with a positive variable. Make sure to explain why it works.



## INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (A.REI.3)

**Parameters**
Give students real-world contexts modeled by equations or inequalities where students need to incorporate parameters.

***EXAMPLE***
**Write and solve an inequality to represent the following situation:**
Dee is the school treasurer. She needs to buy a large number of pencils for a fundraiser at school, so she decides to join Costco. A Costco membership costs $60 and sells 96 pencils in a pack for $11.49. If she has $300 in the account, how many packs of pencils can she buy?

*Discussion:* The solution to this inequality is $p < 21$, where $p$ is the set of integers because one cannot buy part of a pencil package. Also the number of packages cannot be negative which makes the solution a compound inequality $0 < p < 21$ that contains only natural numbers between 0 and 21.

**Compound Inequalities**
Compound inequalities need to be addressed in contextual situations to aid student understanding. It may be helpful to give students the opportunity when given an inequality to create their own conjunction (and) and disjunction (or) word problem situations. Begin by developing the concept that an "and" statement is an overlap an that an "or" statement is a "union" by using familiar categorical contexts. Then move towards using numbers and graphing compound inequalities on a number line. After students become familiar with the contexts of "and" and "or" move to real-world examples using numbers that can be graphed using compound inequalities.

***EXAMPLE***
**Part 1**
Have two students list their favorite fruits.

**Beth:**

**Jennifer:**

**a.** What are the favorite fruits of Beth *and* Jennifer?
**b.** What are the favorite fruits of Beth *or* Jennifer?

*Discussion:* Draw attention to the fact that the word "and" refers to the overlap between the two sets of fruit. Therefore the solution to part **a.** is bananas and oranges since both girls list those fruits as one of their favorites. The solution in part **b.** is each fruit listed either by Beth *or* Jennifer which would be strawberries, oranges, grapes, apples, bananas, cherries, pears, lemons. Notice that repeated fruit is only listed once, not twice.
*Example continued on next page*

## INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (A.REI.3)

**Part 2**

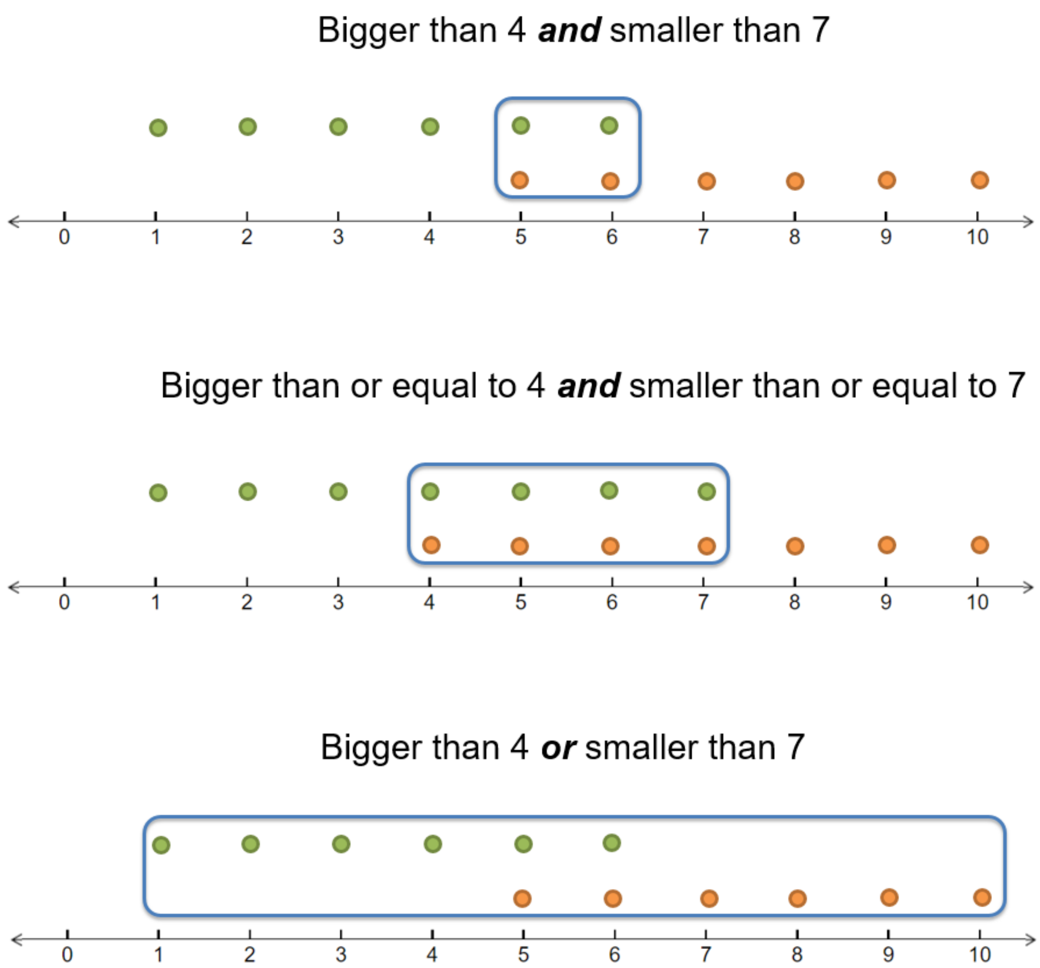
**a.** Use a number line to show the whole numbers between 1-10 that are bigger than 4 *and* smaller than 7?

**b.** Use a number line to show the whole numbers between 1-10 that are bigger than or equal to 4 *and* smaller than or equal to 7?

**c.** Use a number line to show the whole numbers between 1-10 that are bigger than 4 *or* smaller than 7?

*Discussion:* Students should build off the idea behind "and" and "or" in the previous fruit context. Draw attention to the fact that "and" refers to the overlap of the solutions between the two sets and "or" refers to each number listed in either set. So the solution to part **a.** is 5 and 6. The solution to part **b.** is 4, 5, 6, and 7. The solution to part **c.** is 1, 2, 3, 4, 5, 6, 7, 8, 9, and 10. Notice that the numbers are listed only listed once, not twice.

*Example continued on next page*



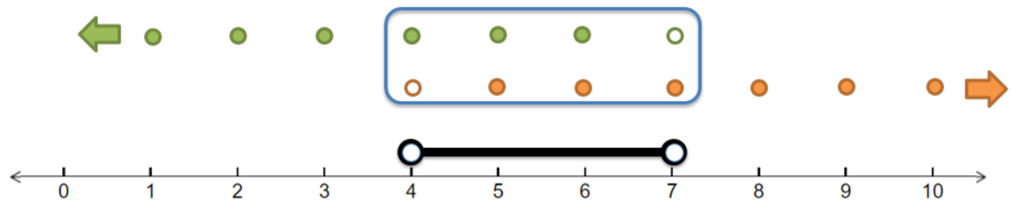
## INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (A.REI.3)

**Part 3**

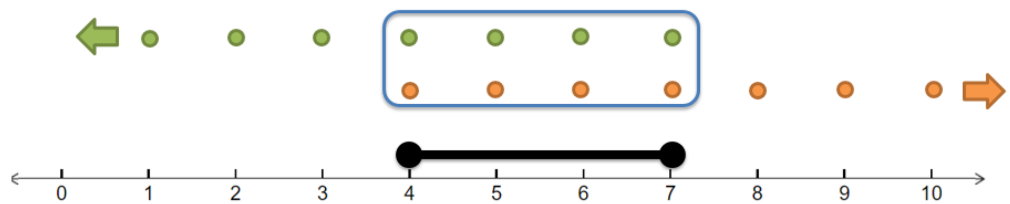
- **a.** Use a number line to show $x > 4\ and\ x < 7$.
- **b.** Use a number line to show $x \geq 4\ and\ x \leq 7$.
- **c.** Use a number line to show $x > 4\ or\ x < 7$.

*Discussion:* Connect the inequalities to the previous two examples. Draw attention to the fact that "and" still mean an overlap or intersection of the two statements. The solution to part **a.** is $4 < x < 7$, and the solution to part **b.** is $4 \leq x \leq 7$. The solution to part **c.** is all real numbers since the graph covers the entire number line.

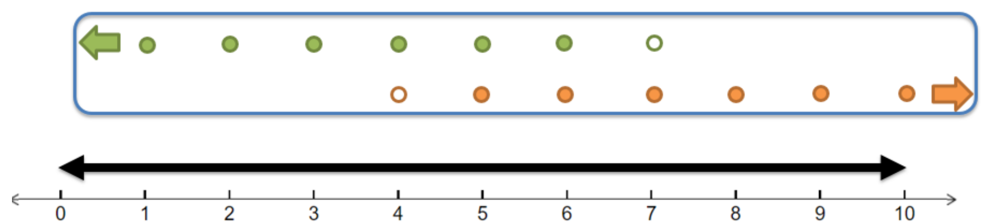
Greater than 4 ***and*** less than 7




Greater than or equal 4 ***and*** less than or equal to 7




Greater than 4 ***or*** *less* than 7



# INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (A.REI.3)

## Instructional Tools/Resources

*These tools and resources may be helpful for instruction. Before use, they should be evaluated at the local level to determine their appropriateness for instruction.*

**Manipulatives/Technology**

- Desmos is a free graphing calculator that is available to students as website or an app.
- Geogebra is a free graphing calculator that is available to students as website.
- Wolframalpha is dynamic computing tool.

**Algebra Tiles**

- Algebra Tiles Applet by NCTM Illuminations is a link to a virtual algebra tiles applet.
- Virtual Algebra Tiles is an applet from Michigan Virtual University that allows students to use algebra tiles. This applet allows for positive and negative representation of the tiles.
- CPM Tiles is an applet from the CPM Educational Program that allows students to use algebra tiles. The benefit of this applet is that students can change the dimensionality of *x* and *y*. However, it is limited by not allowing for a negative representation of the tiles.
- Algebra tile templates on the SMART Exchange has a variety of useful models that teacher can use if they have access to a SMART Board.

**Inequalities**

- Compound Inequalities on the Number Line is a Desmos activity that introduces compound inequalities.
- Best Buy Tickets by Mathematics Assessment Project is a task where students write and solve inequalities.

**Curriculum and Lessons from Other Sources**

- EngageNY, Module 1, Topic C, Lesson 10: True and False Equations, Lesson 11: Solution Sets for Equations and Inequalities, Lesson 12: Solving Equations, Lesson 13: Some Potential Dangers when Solving Equations, Lesson 14: Solving Inequalities, Lesson 15: Solution Sets of Two or More Equations (or Inequalities) Joined by "And" or "Or," Lesson 16: Solving and Graphing Inequalities Joined by "And" or "Or", Lesson 17: Equations Involving Factored Expressions are lessons that pertain to his cluster.
- Mathematics Vision Project, Algebra 1, Module 4: Equations and Inequalities has several tasks that pertain to this cluster.

## INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (A.REI.3)

**General Resources**

- Arizona High School Progressions on Algebra is an informational document for teachers. This cluster is addressed on page 13, paragraph 3.
- High School Coherence Map by UnboundED is a map of the relationships among high school mathematics standards. Please note: This resource is aligned to the Common Core not Ohio's Learning Standards. Although the tool is useful, it needs to be checked with respect to Ohio's Learning Standards.

**Research**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- Ceyanes, B., Lockwood, P., Gill, K. (December 2014). Three lessons on parabolas—what, where, why. T*he Mathematics Teacher*. *108(5),* 368-375.
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<table>
<tr><th>STANDARDS</th><th>MODEL CURRICULUM (A.REI.5-6)</th></tr>
<tr><td>

## Algebra

**REASONING WITH EQUATIONS AND INEQUALITIES**

**Solve systems of equations.**

**A.REI.5** Verify that, given a system of two equations in two variables, replacing one equation by the sum of that equation and a multiple of the other produces a system with the same solutions.

**A.REI.6** Solve systems of linear equations algebraically and graphically.

**a.** Limit to pairs of linear equations in two variables. (A1, M1)

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## Expectations for Learning

In previous courses, students solve systems of linear equations graphically with an emphasis on the meaning of the solution. In this cluster, students solve systems of linear equations in two variables graphically and algebraically, with a focus on the meaning of a solution to a system of equations. In Math 2, students extend this knowledge to solve systems of linear and quadratic equations in two variables.

### ESSENTIAL UNDERSTANDINGS

- The graph of a linear equation is the set of ordered pairs that make the equation true. Therefore, multiplying that equation by a non-zero constant produces an equivalent equation, which has the same set of ordered pairs that make the equation true.
- If a system of equations in two variables has a unique solution, then the sum of one equation and a (non-zero) multiple of the other equation also has that same solution.
- The graphical solution to a system of equations in two variables is the intersection of the equations when graphed.
- The solution to a system of equations in two variables is the set of ordered pairs that satisfies both equations.
- A system of two linear equations can have no solutions, one solution, or infinitely many solutions.

### MATHEMATICAL THINKING

- Determine reasonableness of results using informal reasoning.
- Solve multi-step problems accurately.
- Plan a solution pathway.
- Use technology strategically to deepen the understanding.

### INSTRUCTIONAL FOCUS

- Substitute a solution into the original system and a manipulation of the system to show solutions are the same.
- Solve a system of linear equations in two variables algebraically using substitution, algebraically using elimination, and by graphing.
- Discuss the efficiency and effectiveness of various methods of solving systems of equations.

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| STANDARDS | MODEL CURRICULUM (A.REI.5-6) |
|---|---|
| | **Content Elaborations**<br>**OHIO'S HIGH SCHOOL CRITICAL AREA OF FOCUS**<br>• Math 1, Number 3, page 7<br><br>**CONNECTIONS ACROSS STANDARDS**<br>• Solve linear equations in one variable (A.REI.3).<br>• Graph linear models (F.IF.4, 7).<br>• Rearrange formulas (A.CED.4).<br>• Solve systems of equations and inequalities graphically (A. REI.10-12). |

# INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (A.REI.5-6)

## Instructional Strategies

*Note: The model curriculum is not a collection of lessons nor a full curriculum; it does not suggest pace, sequence, or amount of time spent on topics. It provides information about a topic related to the standards including ideas for examples, strategies for teaching, common misconceptions, and possible connections between topics. In addition, the emphasis of the model curriculum is on instruction not on assessment; therefore, the examples in the Instructional Strategies section are written for instructional purposes not necessarily as assessment items. Examples may need to be rewritten to accommodate the needs of each individual classroom.*

***Standards for Mathematical Practice***
*This cluster focuses on but is not limited to the following practices:*
***MP.1*** *Make sense of problems and persevere in solving them.*
***MP.2*** *Reason abstractly and quantitatively.*
***MP.4*** *Model with mathematics.*
***MP.6*** *Attend to precision.*

### SYSTEMS OF LINEAR EQUATIONS

A linear system in two variables is a set of two equations that are joined by the word "and." There are several ways to present a system:

$y = 5x + 40$
$y = 9x$

$$\begin{cases} y = 5x + 40 \\ y = 9x \end{cases}$$

$y = 5x + 40$ and $y = 9x$

Since the meaning of "and" is intersection or overlap, then the solution set of a system is the intersection or overlap of the solution sets of two individual sentences (equations in this case). Therefore, a solution to the system of two linear equations in two variables is the set of ordered pair(s) that satisfies both equations. Students need to be aware that for a solution to be true for the set, it must be true for all equations in the system simultaneously.

Begin by presenting systems of equations in the context of real-world problems. It is preferable to pose problems before presenting procedures.

In Grade 8 students solved systems of equations graphically only. This is the first time they are expected to solve systems of equations algebraically. Point out that there is a need for algebraic methods because it is often difficult to get an exact solution using a graphical representation. Connections should still be made between the graphical representations of a system of equations and the algebra methodology.

From middle school, students should be familiar with the idea that a system of linear equations can have one solution, no solution, or infinitely many solutions. (Although not required, students should become familiar with terminology such as consistent, inconsistent, dependent, and independent.) They still need to be presented with problems that result in these types of systems algebraically. Students should be able understand why $a = a$ has infinitely many solutions, why $a = b$ has no solutions and should make connections to the corresponding graphical representation.

Students need to understand if a system has an infinite number of solutions, it does not mean that all real numbers make the system true, but only those that lie on the line of $y = mx + b$. Math 1 students should also understand that some systems that appear to have infinitely many solutions really have a finite number of solutions when there is a constraint on the function.

## INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (A.REI.5-6)

***EXAMPLE***

Eduardo had 102 shirts and sold 3 shirts every 3 hours. Blake also had 102 shirts, but sold 6 shirts every 6 hours. When did Blake and Eduardo sell the same number of shirts?

*Discussion:* Blake and Eduardo's situation can be modeled by a system of equivalent (dependent) equations that has a finite (not an infinite) number of solutions. The graphs of the equations modeling both situations are discrete linear graphs that are alike, yet slightly different. Eduardo's graph consists of points graphed every 3 hours and Blake's graph consists of points graphed every 6 hours. It is difficult to tell what happens between the hours, but the amount of shirts they sell overlaps at the end of every 6 hours which means that Blake and Eduardo have sold the same number of shirts at the end of 6th hour, 12th hour, 18th hour, 24th hour and so on. They stopped selling shirts after 102 hours when they run out of shirts. Therefore, the graphs share 18 distinct data points that represents a finite number of solutions.

When solving systems of equation, students should be encouraged to always verify their solution by substituting their solutions into the original equations.

Systems of equations in two variables should include linear equations that are not in slope-intercept form. Encourage standard form in particular. In addition, they should be also able to solve a system given a table of values. Math 1 students should be limited to solving systems of equations in two variables.

**Continuous and Discrete**

Students should have practice with both continuous and discrete functions. They should be provided with some examples that appear to have a solution if the graphs were continuous that do not have a solution when correctly graphed as a discrete function according to context.

**Inspection**

Develop students' understanding of structure by encouraging them to "look" at the system before solving. For example, in a system which includes the equation $y = \frac{3}{5}x + 7$ and $y = \frac{3}{5}x - 2$, students should notice the equations have the same slope and different *y*-intercepts indicating that the lines are parallel and therefore the system has no solution.

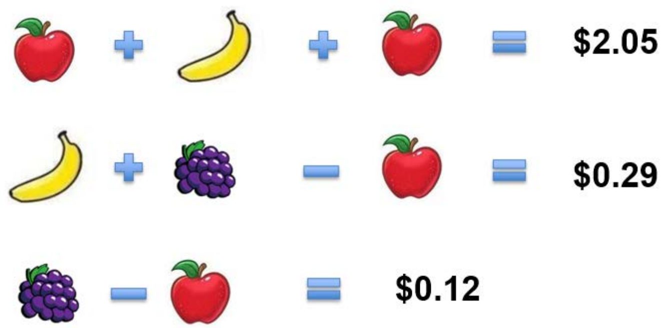
**Substitution**

Before solving systems of equations, pique students' interest with a system involving symbols such as they would see on social media.

# INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (A.REI.5-6)

***EXAMPLE***

Find the value of each fruit.




*Discussion:* Discuss with students how they got the answer. They will most likely describe to you a strategy involving substitution. Capitalize on this discussion to launch into the substitution property by having students solve the same system with variables using substitution.

$2A + B = 2.45$

$B + G – A = 1.63$

$G – A = 0.12$

**TIP!** For kinesthetic learners have students physically cut out the equivalent expression from one side of the equal side of equation A and physically place it over the variable in equation B. This kinesthetic process allows students to make sense of substitution from the concrete to the abstract.

**Elimination/Linear Combination**

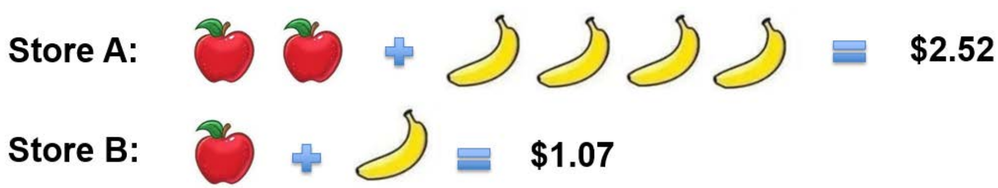
Before learning elimination techniques, students should be given the opportunity to tinker with a single equation and some ordered pairs that satisfy the equation. Then multiply the equation by a constant to build the concept that the same ordered pairs satisfy the new and old equations.

# INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (A.REI.5-6)

Another approach is to explore the Multiplication Property of Equality using graphs. Have students graph equations such as $2x + 4y = 8$, $3x + 6y = 12$, $4x + 8y = 16$, $x + 2y = 4$, $0.5x + y = 2$, and $-2x - 2y = -8$. Draw connections between the Multiplication Property of Equality and the graphical representations of the equivalent equations. Have students verify their equality by using ordered pairs that satisfy the equations.

Students have experience with the Addition Property of Equality in the form if $a = b$, then $a + c = b + c$. Use a variation of the fruit problem to promote a discussion of the extension of the Addition Property of Equality to if $a = b$ and $b = c$, then $a + b = b + c$. This will lay the ground work for students to understand solving systems by elimination.

***EXAMPLE***
Mary bought apples and bananas at two different stores. John only needs to buy apples to make a pie. Which store should he purchase his apples from? How much will it cost him per apple?




*Discussion:* Students will most likely solve this problem in a variety of ways. Some students may subtract equation B (an apple, banana, and \$1.07) from equation A. Discuss why this is allowed, and connect it with the property if $a = b$ and $b = c$, then $a + b = b + c$. (It may be wise to use variables other than $a$ and $b$ when discussing this property since the problem has apples and bananas in it.) Some students may double the apple and bananas in equation B and then subtract it from Equation A. Discuss why both strategies work and connect to elimination. Other students may use a substitution method. Discuss why substitution also works. Challenge students to graph the equations, and verify whether they get the same results.

Stress to students that efficiency and strategizing is important in solving systems of equations. This is also especially with respect to the elimination method. Have discussions with students about which term would be the most efficient to "eliminate" and why.

**Choosing the Most Efficient Method**
Before beginning solving systems of equations, students should analyze the structure of the equation and decide on the most efficient way to solve the system: graphing, substitution, inspection, or combination (elimination).

# INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (A.REI.5-6)

***EXAMPLE***

Fill in the table by choosing pairs of the equations that would highlight the efficiency of each method: inspection, graphing, substitution, elimination. An equation may be used more than once. Provide an explanation for your choice.

| | | | | |
|---|---|---|---|---|
| $y = \frac{2}{3}x - 4$ | $3y = 3$ | $3x + 4y = 8$ | $y = \frac{2}{3}x + 7$ | $4x = -2$ |
| $y = \frac{3}{4}x + 1$ | $x = 4y$ | $y = 3x$ | $6x + 8y = 16$ | $3x + 4y = 9$ |
| $x - y = 2$ | $-3x + 2y = 20$ | $21x - 7y = 14$ | $0.4y = 2.8x - 8$ | $3.5x = 24 + y$ |

| | Inspection | Graphing | Substitution | Elimination |
|---|---|---|---|---|
| Pairs of Equations | | | | |
| Explanation | | | | |
| Pairs of Equations | | | | |
| Explanation | | | | |
| Pairs of Equations | | | | |
| Explanation | | | | |

**Real-world problems**

Provide students opportunities to practice linear vs. non-linear systems; consistent vs. inconsistent systems; pure computational vs. real-world contextual problems (e.g., chemistry and physics applications encountered in science classes). A rich variety of examples can lead to discussions of the relationships between coefficients and consistency that can be extended to graphing and later to determinants and matrices.

***EXAMPLE***

Gym A had a joining fee of \$99 and a single-club membership of \$24.99 per month. Gym B has a flat yearly fee of \$350. Write a system of equations to represent the situation.

- **a.** After how many months would the memberships be of equal value?
- **b.** What are the constraints on these functions?
- **c.** After how many months would Gym A be cheaper?
- **d.** After how many months would Gym B be cheaper?
- **e.** How would the problem change if Gym A only had a \$50 joining fee?
- **f.** How would the problem change if Gym B kept the same rate, but did not lock you into a yearly contract?

Include a variety of real-world applications of systems such as mixture problems, motion problems, break even problems, and area models.

# INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (A.REI.5-6)

## Instructional Tools/Resources

*These tools and resources may be helpful for instruction. Before use, they should be evaluated at the local level to determine their appropriateness for instruction.*

**Manipulatives/Technology**

- Desmos is a free graphing calculator that is available to students as website or an app.
- Geogebra is a free graphing calculator that is available to students as website.
- Wolframalpha is dynamic computing tool.

**Systems of Equations**

- Suit Yourself is a lesson designed by NASA where students use systems of equation to evaluate the oxygen use of two astronauts.
- Ground Beef is a lesson by Achieve the Core that ties in CTE. Students use systems of equations and Pearson's square to determine the profit with respect to selling meat.
- Those Horrible Coin Problems (And What We Can Do About Them) is a task from Dan Meyer on making value problems more interesting by showing the need for systems in computation.
- Quinoa Pasta 2 is a task from Illustrative Mathematics that integrates modeling and systems of equations in the context of a pasta made of both quinoa and corn. It continues with Quinoa Pasta 3.
- Find a System is a task from Illustrative Mathematics that gives two points and has students create a system given two points. It encourages critical thinking by reversing the typical process.
- Mix It Up and Don't Freeze the Engine are lessons from NCTM's Illuminations that have students write equations in the context of a concentration. "Mix It Up" is a tactile lesson where students develop and use a formula to determine the final percent mix from two mixtures. In "Don't Freeze the Engine" the students use systems to analyze the antifreeze in a particular cooling system. *NCTM now requires a membership to view their lessons.*
- Algebra 1-Mixture Problems is a teaching channel video on teaching mixture problems. It ties an informal understanding of a fulcrum and inverse variation which she calls the see-saw method. This could be contrasted with solving the same problem using a system of equations.
- Students use matrices and technology to solve the Meadows or Malls problem, a linear programming problem with six variables.
- Piling Up Systems of Linear Equations: How Much Does Each Weigh? by Tap Into Teen Minds is a 3-act task that has students write systems of linear equations using the weight of office supplies.
- [Makeover] Systems of Equations from Dan Meyer's Blog discusses how to make systems of equation word problems more meaningful.
- A Linear and Quadratic System by Illustrative Mathematics is a task where students make connections between equations and the geometry of their graphs to find the intersection point of a line and a parabola.
- A Mixture of Problems by Laurie Riggs, et. al has a variety of problems including mixture problems using different conceptual methods to solve equations.

# INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (A.REI.5-6)

**Curriculum and Lessons from Other Sources**

- EngageNY, Algebra 1, Module 1, Topic C," Lesson 20: Solution Sets to Equations with Two Variables, Lesson 22: Solution Sets to Simultaneous Equation, Lesson 23: Solution Sets to Simultaneous Equation, and Lesson 24: Applications of Systems of Equations and Inequalities are lessons that pertain to this cluster.
- Mathematics Vision Project, Algebra 1, Module 5: Systems of Equations and Inequalities has many tasks that pertain to this cluster.
- Exploring Systems of Inequalities by Burrill and Hopfensperger is a pdf of the teacher's edition of the series Data-Driven Mathematics published by Dale Seymour Publications. It has many lessons that pertain to this cluster. The student edition can be found here.

**General Resources**

- Arizona High School Progression on Algebra is an informational document for teachers. This cluster is addressed on page 14 in paragraphs 3-6.
- High School Coherence Map by UnboundED is a map of the relationships among high school mathematics standards. Please note: This resource is aligned to the Common Core not Ohio's Learning Standards. Although the tool is useful, it needs to be checked with respect to Ohio's Learning Standards.

**References**


- Common Core Standards Writing Team. (2013, March 1). *Progressions for the Common Core State Standards in Mathematics (draft). High School, Algebra*. Tucson, AZ: Institute for Mathematics and Education, University of Arizona.
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| STANDARDS | MODEL CURRICULUM (A.REI.10-12) |
|---|---|
| **Algebra**<br>**REASONING WITH EQUATIONS AND INEQUALITIES**<br>**Represent and solve equations and inequalities graphically.**<br>**A.REI.10** Understand that the graph of an equation in two variables is the set of all its solutions plotted in the coordinate plane, often forming a curve (which could be a line).<br>**A.REI.11** Explain why the *x*-coordinates of the points where the graphs of the equation $y = f(x)$ and $y = g(x)$ intersect are the solutions of the equation $f(x) = g(x)$; find the solutions approximately, e.g., using technology to graph the functions, making tables of values, or finding successive approximations.<br>**A.REI.12** Graph the solutions to a linear inequality in two variables as a half-plane (excluding the boundary in the case of a strict inequality), and graph the solution set to a system of linear inequalities in two variables as the intersection of the corresponding half-planes. | **Expectations for Learning**<br>In prior courses, students graph linear relationships and identify slope and intercepts. In this cluster, students extend this knowledge to include the idea that a graph represents all of the solutions of an equation. Students use graphs and tables of equations in two variables to approximate solutions to equations in one variable. They also graph solutions to linear inequalities in two variables. In Math 3, students similarly study the relationship between the graph and the solutions of rational, radical, absolute value, polynomial, and exponential equations.<br><br>**ESSENTIAL UNDERSTANDINGS**<br>• A point of intersection of any two graphs represents a solution of the two equations that define the two graphs.<br>• An equation in one variable can be rewritten as a system of two equations in two variables, by thinking of each side of the equation as a function, i.e., writing $y$ = left hand side and $y$ = right hand side.<br>    o The approximate solution(s) to an equation in one variable is the *x*-value(s) of the intersection(s) of the graphs of the two functions.<br>    o Two-variable graphical and numerical (tabular) techniques to solve an equation with one variable always work and are particularly useful when algebraic methods are not applicable, e.g., $3x + 4 = 2^x$.<br>• A half plane represents the solutions of a linear inequality in two variables.<br>• The intersection of two half planes represents the solution set to two inequalities in two variables.<br><br>**MATHEMATICAL THINKING**<br>• Use technology strategically to deepen understanding.<br>• Plan a solution pathway.<br>• Create a model to make sense of a problem.<br>Continued on next page |

| STANDARDS | MODEL CURRICULUM (A.REI.10-12) |
|---|---|
| | **Expectations for Learning, continued**<br>**INSTRUCTIONAL FOCUS**<br>• Rewrite a one-variable equation as two separate functions and use the *x*-coordinate of their intersection point to determine the solution of the original equation.<br>• Approximate intersections of graphs of two equations using technology, tables of values, or successive approximations (focus on equations with linear and exponential expressions).<br>• Graph the solution set of a linear inequality in two variables.<br>• Graph the solution set of a system of linear inequalities in two variables.<br><br>**Content Elaborations**<br>**OHIO'S HIGH SCHOOL CRITICAL AREA OF FOCUS**<br>• Math 1, Number 2, pages 4-6<br><br>**CONNECTIONS ACROSS STANDARDS**<br>• Solve equations in one variable (A.REI.3).<br>• Create equations in two variables (A.CED.2).<br>• Graph functions expressed symbolically (F.IF.7).<br>• Analyze functions using different representations (F.IF.9)<br>• Solve systems of equations graphically. (A.REI.6a) |

# INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (A.REI.10-12)

## Instructional Strategies

*Note: The model curriculum is not a collection of lessons nor a full curriculum; it does not suggest pace, sequence, or amount of time spent on topics. It provides information about a topic related to the standards including ideas for examples, strategies for teaching, common misconceptions, and possible connections between topics. In addition, the emphasis of the model curriculum is on instruction not on assessment; therefore, the examples in the Instructional Strategies section are written for instructional purposes not necessarily as assessment items. Examples may need to be rewritten to accommodate the needs of each individual classroom.*

***Standards for Mathematical Practice***
*This cluster focuses on but is not limited to the following practices:*
***MP.2*** *Reason abstractly and quantitatively.*
***MP.4*** *Model with mathematics.*
***MP.5*** *Use appropriate tools strategically.*
***MP.6*** *Attend to precision.*

### REPRESENTING SOLUTIONS OF EQUATIONS GRAPHICALLY

Begin with solving simple linear equations by tracing graphs and using tables on a graphing calculator. Then, advance students to nonlinear situations, so they can see that even complex equations that might involve quadratics, absolute value, or rational functions can also be solved fairly easily using this same strategy. While a standard graphing calculator does not simply solve an equation for the user, it can be used as a tool to approximate solutions. *Note: Although Math 1 students, typically do not do quadratics, they should be able to solve complex equations including quadratics by converting an equation to two equations and graph them and find a solution.*

***EXAMPLE***

Help students to recognize a graph as a set of solutions to an equation. If the equation $y = 6x + 5$ represents the amount of money paid to a babysitter (for example, \$5 for gas to drive to the job and \$6/hour to do the work), then every point on the line represents an amount of money paid, given the amount of time worked.

Using technology, have students graph a function and use the trace function to move the cursor along the curve. Discuss the meaning of the ordered pairs that appear at the bottom of the calculator, emphasizing that every point on the curve represents a solution to the equation.

Students may incorrectly believe that the graph of a function is simply a line or curve "connecting the dots" without recognizing that the graph represents all solutions to the equation.

**Converting an Equation to Two Equations**

An equation in one variable such as $2x + 3 = x – 7$ can be solved by converting an equation to a system of two equations in two variables: $y = 2x + 3$ and $y = x – 7$ and then graphing the functions $y = 2x + 3$ and $y = x – 7$. They should recognize that the intersection point of the lines is at $(–10, –17)$. They should be able to verbalize that the intersection point means that when $x = –10$ is substituted into both sides of the equation, each side simplifies to a value of $–17$. Therefore, $–10$ is the solution to the equation. This same approach can be used whether the functions in the original equation are linear, nonlinear, or both.



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## INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (A.REI.10-12)

***EXAMPLE***

Solve the equation $x^2 - 3x + 2 = 5$.

*Discussion:* In Math 2, students will learn various strategies to solve this equation algebraically (factor, complete the square, quadratic formula). An approach to solve this in Math 1 is to set the equation equal to 0 to get $x^2 - 3x - 3 = 0$, and then have students graph the equations $y = x^2 - 3x - 3$ and $y = 0$ and connect the intersection points to the solutions (*x*-intercepts) of the quadratic.

***EXAMPLE***

Solve the equation $x^2 - 3x + 2 = 2^x$.

*Discussion:* Algebraic techniques for solving equations require that equations can be manipulated into particular forms. Some equations like the example cannot be manipulated into forms that can be solved with algebraic techniques. The advantage of using the technique of setting a one-variable equation into two equations in two variables and finding the intersection points is that always works, even for equations like $x^2 - 3x + 2 = 2^x$, for which there are no algebraic techniques.

***EXAMPLE***

Compare the graphs of $x^2 - 3x + 2 = 2^x$ to $x^2 - 3x + 2 = 1.1^x$.

Students graphed the first equation in the previous example. Now graphing the second equation as a system of two equations $y = x^2 - 3x + 2$ and $y = 1.1^x$ appears to have two solutions in a traditional window. However since the exponential function will eventually exceed the quadratic function, there is a third solution if an appropriate window is found.

**Tables**

Use the table function on a graphing calculator to solve equations. For example, to solve the equation $x^2 = x + 12$, students can examine the equations $y = x^2$ and $y = x + 12$ and determine that they intersect when $x = 4$ and when $x = –3$ by examining the table to find where the *y*-values are the same.

Students who make a table of values to find the solution to a system may start with evaluating each function at integer values to determine an approximate solution. Using technology, students can then zoom-in on a smaller window (more precise) of values that would include a solution of the system, and make a zoomed-in table of values. They can continue this process, recognizing when the solution is exact, and when the solution is approximate.

# INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (A.REI.10-12)

## INEQUALITIES

Inequalities should be taught in the context of real-world examples, so students can create meaning.

***EXAMPLE***

Before teaching students how to shade a graph with respect to inequalities, explore what the graph would look like for money earned when a person earns *at least* \$6 per hour compared to a person who earns *exactly* \$6/hour. Have students plot points for all the correct combinations of solutions. Using technology or a transparency, have students combine their dots (solutions) onto one graph.

*Discussion:* The graph for a person earning *exactly* \$6/hour would be a linear function, while the graph for a person earning at least \$6/hour would be a half-plane including the line and all points above it. Then compare the graphs to the graph of a person who earns *more than* \$6 per hour.

***EXAMPLE***

Before teaching the graphical representation of solid and dotted lines, put students into pairs. Have one student write an inequality and plot solutions on a coordinate plane to represent a woman who deposits \$100 in the bank with *at least* a 0.05% simple interest rate? Have one student write an inequality and plot solutions on a coordinate plane to represent a man who deposits \$100 in the bank with a simple interest rate *greater than* 0.05%. Then have the students compare graphs.

*Discussion:* The students should realize that the woman who earns at least a 0.05% interest rate could earn exactly that, so her solutions would fall on the line. Use this as a discussion point for the need of representing the lines differently. Explain that mathematicians came up with the convention of using solid and dotted lines.

***EXAMPLE***

Before formally teaching systems of inequalities, put students into pairs. Have one student write an inequality and plot solutions on a coordinate plane to represent part **a.** of the example. Then have the second student do part **b.** of the example.

- Julia has two jobs. She earns \$8 an hour babysitting and \$11 an hour walking the neighbor's dog. She must make at least \$100 a week to save up for her class trip. Explain which solutions make your inequality true.
- Julia both babysits and walks her neighbor's dogs to earn money. Julia's mom will not let her work more than 15 hours a week. Explain which solutions make your inequality true.

*Discussion:* Then put both students back together, and have them use transparencies, tracing paper, or technology to overlay the second student's inequality on top of the first student's. Have students discuss the meaning behind the overlapping section. Use this as a launching pad to discuss systems of inequalities.

# INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (A.REI.10-12)

**Optimization Problems and Linear Programming**
The ability to solve systems of equations is useful for solving linear programming problems.
Applications such as linear programming can help students recognize how businesses use constraints to maximize profit while minimizing the use of resources. These situations often involve the use of systems of two variable inequalities. Have students explore and discover on their own that the optimization point (where profits are maximized and costs are minimized) occurs at the corners of the feasible region. See the Instructional Tools/Resources section for examples of linear programming problems.

## Instructional Tools/Resources

*These tools and resources may be helpful for instruction. Before use, they should be evaluated at the local level to determine their appropriateness for instruction.*

**Manipulatives/Technology**

- Desmos is a free graphing calculator that is available to students as website or an app.
- Geogebra is a free graphing calculator that is available to students as website.
- Wolframalpha is dynamic computing tool.

**Solutions on a Graph**

- Collinear Points by Illustrative Mathematics is a task where students connect Algebra and Geometry in exploring solutions on a graph.

**Inequalities**

- Rabbit Food by Achieve the Core is a CTE lesson where students use linear inequalities to analyze rabbit food.
- Maximizing Profit: Selling Boomerangs by Mathematics Assessment Project that has students explore an optimization problem.
- Solution Sets by Illustrative Mathematics is a task that gives students solution sets and asks them to create a corresponding system.

**Linear Programming**

- Hassan's Pictures-Linear Programming and Profit Lines is an Annenberg Learner lesson where students find the feasible region in a linear programming problem and find the optimum point.
- Linear Programming is a Desmos activity that introduces linear programming.
- Building Cat Furniture: An Introduction to Linear Programming is a lesson by Tim Marley that uses Legos as an introduction to linear programming.

# INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (A.REI.10-12)

**Curriculum and Lessons from Other Sources**

- Georgia Standards of Excellence Curriculum Frameworks, Algebra 1, Unit 2: Reasoning with Linear Equations and Inequalities has several lessons that pertain to this cluster. These can be found on pages 63-70, 95-98, 103-127, 130-146.
- Mathematics Vision Project, Algebra 1, Module 5: Systems of Equations and Inequalities has tasks that pertain to this cluster.
- EngageNY, Algebra 1, Module 1, Topic C, Lesson 21: Solution Sets to Inequalities with Two Variables and Lesson 24: Applications of Systems of Equations and Inequalities are lessons that pertain to this cluster.
- EngageNY, Algebra 1, Module 3, Topic C, Lesson 16: Graphs Can Solve Equations Too is a lesson that pertains to this cluster. This lesson extends A.REI.11 to absolute value equations which can be used as an extension but is not required in Ohio.
- Exploring Systems of Inequalities by Burrill and Hopfensperger is a pdf of the teacher's edition of the series Data-Driven Mathematics published by Dale Seymour Publications. It has many lessons that pertain to this cluster. The student edition can be found here.

**General Resources**

- Arizona High School Progression on Algebra is an informational document for teachers. This cluster is addressed on page 15.
- High School Coherence Map by UnboundED is a map of the relationships among high school mathematics standards. Please note: This resource is aligned to the Common Core not Ohio's Learning Standards. Although the tool is useful, it needs to be checked with respect to Ohio's Learning Standards.

**References**


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| STANDARDS | MODEL CURRICULUM (F.IF.1-3) |
|---|---|
| **Functions**<br>**INTERPRETING FUNCTIONS**<br>**Understand the concept of a function, and use function notation.**<br>**F.IF.1** Understand that a function from one set (called the domain) to another set (called the range) assigns to each element of the domain exactly one element of the range. If $f$ is a function and $x$ is an element of its domain, then $f(x)$ denotes the output of $f$ corresponding to the input $x$. The graph of $f$ is the graph of the equation $y = f(x)$.<br>**F.IF.2** Use function notation, evaluate functions for inputs in their domains, and interpret statements that use function notation in terms of a context.<br>**F.IF.3** Recognize that sequences are functions, sometimes defined recursively, whose domain is a subset of the integers. *For example, the Fibonacci sequence is defined recursively by* $f(0) = f(1) = 1$, $f(n + 1) = f(n) + f(n - 1)$ *for* $n \geq 1$. | **Expectations for Learning**<br>In the eighth grade, students have learned a semi-formal definition of a function and know that a function pairs an input value with an output value. Eighth grade students do not use function notation or the terms domain and range.<br><br>In this cluster, students will now expand their understanding of functions to include function notation and the terms domain and range. Also, students will evaluate and interpret functions, including sequences as functions. Distinguishing between relations and functions is not a primary focus.<br><br>This cluster is the foundation for all future work with functions.<br><br>**ESSENTIAL UNDERSTANDINGS**<br>• Function notation illustrates a formal connection between inputs and outputs.<br>• Functions can be tied to real-world scenarios given by tables, graphs, equations, or verbal descriptions.<br>• Function notation $f(x)$ is shorthand for the output of $f$ when the input is $x$.<br>• Function notation, $f(x)$, is a new representation for students and is articulated as "$f$ of $x$", and it is not related to multiplication.<br>• Sequences are functions whose domain is a subset of the integers, paying careful attention to how a sequence is indexed. For example, the sequence may be indexed from 0 to $n$, from 1 to $n - 1$, or something else.<br>• An arithmetic sequence is a linear function, and a geometric sequence is an exponential function.<br>**MATHEMATICAL THINKING**<br>• Use accurate mathematical vocabulary to describe mathematical reasoning.<br>• Represent a concept symbolically.<br>• Determine reasonableness of results.<br>• Make connections between concepts, terms, and properties within the grade level and with previous grade levels.<br>*Continued on next page* |

| STANDARDS | MODEL CURRICULUM (F.IF.1-3) |
|---|---|
| | **Expectations for Learning, continued**<br>**INSTRUCTIONAL FOCUS**<br>• Make connections among different representations (tables, graphs, symbols, and verbal descriptions) of functions, focusing on linear and exponential functions.<br>• Solve problems with functions represented in tables, graphs, symbols, and verbal descriptions.<br>• Explain function notation in a real-world context. For example, if *f(x)* represents the height of particle at *x* seconds, then *f*(1) represents the height of the particle at 1 second.<br>• Interpret number patterns as sequences and their graphs as discrete points. When the number pattern arises from a context, consider whether it is appropriate to "connect the dots."<br>• Use function notation to specify sequences, both explicitly and recursively. (Subscript notation is not required.)<br>• Relate linear functions to arithmetic sequences and relate exponential functions to geometric sequences.<br><br>**Content Elaborations**<br>**OHIO'S HIGH SCHOOL CRITICAL AREAS OF FOCUS**<br>• Math 1, Number 2, pages 4-6<br><br>**CONNECTIONS ACROSS STANDARDS**<br>• Build a function that models a relationship between two quantities (F.BF.1a, 2).<br>• Build new functions from existing functions (F.BF.4a).<br>• Interpret expressions for functions in terms of the situation they model (F.LE.5).<br>• Construct and compare linear and exponential models, and solve problems (F.LE.2).<br>• Represent and solve equations and inequalities graphically (A.REI.10). |

# INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (F.IF.1-3)

## Instructional Strategies

*Note: The model curriculum is not a collection of lessons nor a full curriculum; it does not suggest pace, sequence, or amount of time spent on topics. It provides information about a topic related to the standards including ideas for examples, strategies for teaching, common misconceptions, and possible connections between topics. In addition, the emphasis of the model curriculum is on instruction not on assessment; therefore, the examples in the Instructional Strategies section are written for instructional purposes not necessarily as assessment items. Examples may need to be rewritten to accommodate the needs of each individual classroom.*

***Standards for Mathematical Practice***
*This cluster focuses on but is not limited to the following practices:*
***MP.6*** *Attend to precision.*
***MP.7*** *Look for and make use of structure.*

The focus of this cluster is on understanding the concept of a function and reading and writing functions written in function notation. It also focuses on both discrete and non-discrete (continuous) functions in the form of linear, quadratic, and exponential functions.

### THE CONCEPT OF A FUNCTION

The concept of functions, although central to high school mathematics, is one of the most challenging concepts for students. A function is a mathematical concept that describes how two values relate to one another. In a function every element in the domain must be mapped to one and exactly one element in the range.

A function can be thought of as—

- having two sets;
- having a correspondence between the two groups;
- meeting a special requirement where every input is matched or assigned to one and only one output.

Students may incorrectly believe that all relationships having an input and an output are functions, and therefore, misuse the function terminology.

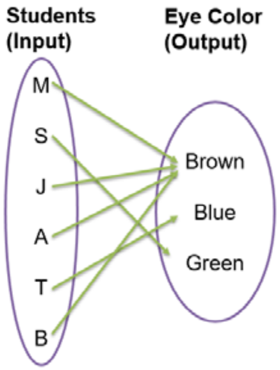
Distinguish between relationships that are not functions and those that are functions (e.g., present a table or diagram in which one of the input values results in multiple outputs and contrast that with a functional relationship). See Model Curriculum 8.F.1-3 for scaffolding ideas about functions.

Students may incorrectly think that functions are always equations because that is usually how students see functions represented in an academic setting. However, some functions may have no algebraic representation at all. To prevent this misconception, it may be helpful to introduce the concept functions using non-algebraic contexts.

# INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (F.IF.1-3)

***EXAMPLE***

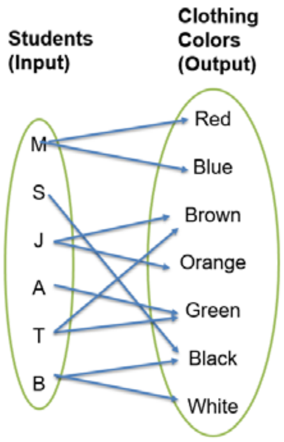
*In this Function Wall activity students are the inputs and the outputs are other categories assigned to spaces on the wall where students can physically go (or be assigned). This activity is adapted from Putting Essential Understanding of Functions Into Practice 9-12.*




**a.** Put up signs on the wall of different eye colors (assuming no one in your class has two different colored eyes). Write a mapping diagram on the board, and then have students go to the appropriate place along the wall that represent each student's eye color. Complete the mapping diagram to represent their movement.

| **Students** | **Eye Color (signs along the wall)** |
|---|---|
| Marsha | Brown |
| Shaqueesha | Blue |
| Jose | Green |
| Ashley | |
| Tim | |
| Brendan | |

*Discussion:* Point out how each person can only go to one and exactly one place.




**b.** Put up signs on the wall to represent different clothing colors. Write a mapping diagram on the board, and then have students go to the appropriate place along the wall that represents the color of clothing each student is wearing.

*Discussion:* Discuss the difference between the two situations. Students should come to the realization that in part **a**., each student had only and exactly one place to go, but in part *b*. if a student was wearing more than one color, he or she had no clear cut place to go. Therefore part **a.** is a function, but part **b.** is just a correspondence (relation). Discuss how even if even just one student was wearing two colors, then the situations would not represent a function. After the discussion make a connection that the names of the students are the domain or input, and the clothing color is the range, or output. Also, discuss how limiting the domain of a set can affect whether a correspondence is a function or not. For example, if the class is limited to only students wearing monochromatic colors, then the situation would be a function.

**c.** Put up signs on the wall using only clothing colors black and white. Have students go to the appropriate place along the wall that represents the color of clothing each student is wearing.

*Discussion:* Students should come to the conclusion that many students will have no place to go, so the situation does not represent a function.

*Example continued on next page*

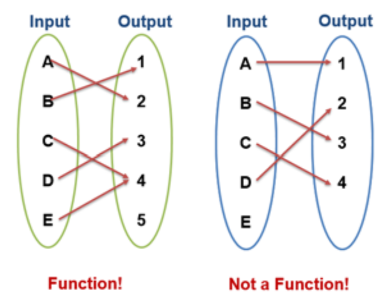
## INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (F.IF.1-3)

**d.** Put up signs on the wall to represent different planets in the solar system. Have students go to the appropriate place along the wall on the planet that each student lives.

*Discussion:* Point out that even though no one in the class lives on Mars and Jupiter, it is still a function. This breaks the misconception that every output must have an input.

**e.** Reverse the situations in parts **a.-d.** Define the wall signs as the inputs and have several students stand along the wall as outputs. Move the signs to its assigned student.

*Discussion:* Discuss how determining which set is the input or output can determine whether a correspondence is a function or not. Explain that is some contexts the input (independent variable) and output (dependent variable) is clearly defined, but sometimes it is arbitrary. This activity is adapted from Ronau, et al., 2014




**TIP!**

In a function, all the elements from the input set must be assigned or matched to one and exactly one element from the output set. In other words, there can be no inputs left unassigned. However, the reverse is not necessarily true; there can be outputs that exist without any assigned inputs.

Students may also incorrectly believe a mapping is not a function when multiple input values are paired with same output value. Exposure to real-world examples such as the mapping the amount of data used on an unlimited data plan to its corresponding monthly bill.

Functions should be explored in applied contexts. For example, examine the amount of money earned when given the number of hours worked at a fast food job, and contrast this with a situation such as riding an Uber where a single fee is paid by the "carload" of people, regardless of whether 1, 2, or 3 people ride.

Students may have the following misconceptions:

- They may incorrectly think a function consists of a single rule, so a split function is not a function.
- They may incorrectly think the graph of a function needs to be continuous.
- They may incorrectly think that the range must map back onto the domain creating a one-to-one correspondence.

Give students problems to confront their misconceptions.

Function Carnival by Desmos can be used to emphasize that in a function one element in the domain corresponds (or maps) to exactly one element in the range.

# INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (F.IF.1-3)

### Challenges Due to the Use of the Vertical Line Test

The vertical line test is not a universal tool and should be used very carefully or not at all. The emphasis should be on the definition of what a function is instead of just using the vertical line test. Instead of the vertical line test emphasize the concept of a function: For a given relation, does each input have a single output? In other words, does the input determine the output?" If the answer to the question is no, then it is not a function. Therefore the emphasis should be on determining whether a relation is a function by analyzing the correspondence between domain and range instead of using the vertical line test. (The vertical line test also makes it difficult to discuss functionality when *x* is a function of *y* if students interchange the *x*-axis as the output and the *y*-axis as the input.) In addition the vertical line test may not always work for inverse functions, polar functions, and parametric functions.

A caterpillar is crawling around on a piece of graph paper, as shown below. If we wished to determine the creature's location on the paper with respect to time, would this location be a function of time? Why or why not?

Fig. 1.6. Is a caterpillar's location a function of time? (Clement 2001, p. 745)

One reason to caution against the vertical line test is that even more advanced students struggle applying the vertical line test correctly. In one study over 60% of high school precalculus students misapplied the vertical line test to the Caterpillar Problem (shown on the left) where students were to determine whether the situation was a function. They confused the path of the caterpillar with a graph showing the correspondence between time and location. If you choose to use the vertical line test with your students, give the caterpillar students the problem after some exposure to the vertical line test, and see how they solve it. After discussion encourage them to graph it using time and distance. After students spend some time plotting the relation, they should see that it is indeed a function.

### Function Machines

Use diagrams to help students visualize the idea of a function machine. Students can examine several pairs of input and output values and try to determine a simple rule for the function. Make connections of input and output values to independent and dependent variables which are used in science classes.

## DOMAIN AND RANGE

Help students to understand that the word "domain" means the set of all possible input values and that "range" means the set of all possible output values.

Students need to understand that restricting the domain can create a function. For example, the equation $x = y^2$, where *x* is the input and *y* is the output, can be rewritten as $y = \pm\sqrt{x}$, and is therefore not a function. However, if the domain is restricted to $x \geq 0$, then the relation is a function. Items can also be added to the range of a relation to produce a function if an input is lacking a correspondence.

# INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (F.IF.1-3)

The understanding of function and domain and range is helpful for people who design web surveys.

***EXAMPLE***

How long have you been working for the company?

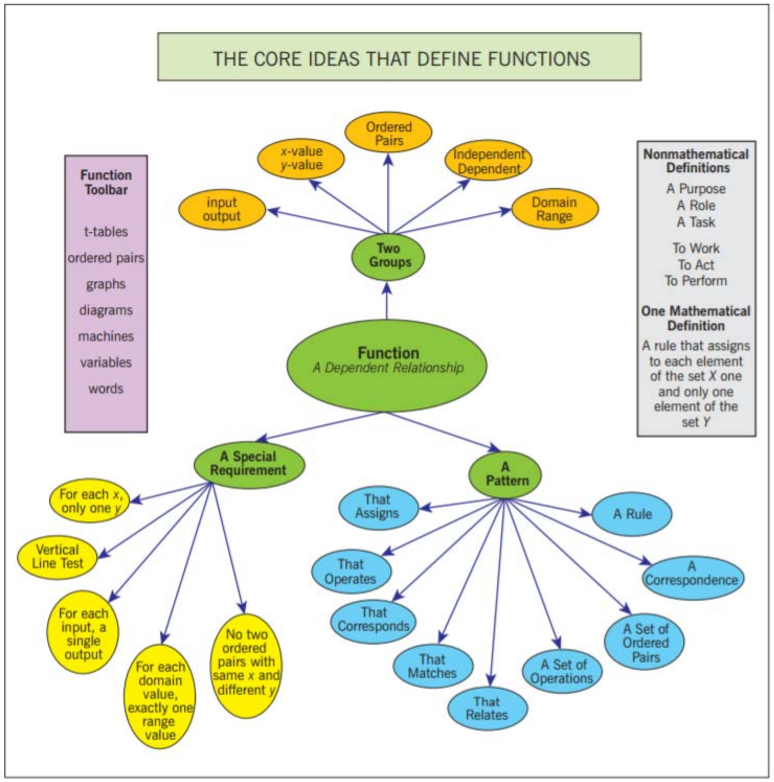
- More than 20 years
- More than 15 years
- More than 5 years
- More than 2 years
- More than 1 year
- Less than 6 months

*Discussion:* Students should recognize that a person who has worked at the company for 23 years should be able to select many of the statements on the survey and still answer truthfully. They also might notice that there can exist a person, such as an 8-month employee, who has nothing to select. Therefore it is important to carefully think about the wording of survey to ensure that the range is restricted so that the data will create a function.

Give students graphs and real-life situations that can be modeled by of various advanced relations such as circles, piecewise graphs, trigonometry curves etc., and have students write the domain and range.

Have students also consider what values could be added to the domain or range that would "break" the function by causing the relation to no longer be a function.

Connect to F.BF.1-2 which includes discrete vs continuous domains.




From Gregorio A. Ponce, *Mathematics Teacher*, 2007, "Critical Juncture Ahead"

*Note: This graphic at this grade level is for teacher use not student use. Also, note that even though the vertical line test is shown in the graphic, using it is discouraged.*

# INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (F.IF.1-3)

## FUNCTION NOTATION

The notation $f(x)$ (pronounced $f$ of $x$) represents the output of the function when $x$ is in the input. While it is often correct to say $f(x)$ is the same as $y$, students need to understand the significance of using function notation. One advantage is that it is a short way of saying that $h(3)$ means "the output of the function $h$ when the input value is 3". And more complicatedly, $h(3) - h(1)$ is "the difference between the output values of function $h$ from the corresponding input values of 3 and 1", which is a vertical distance between output values along the $y$-axis. Make sure that $f(x)$ is not the only notation for functions used in the classroom. For example, use $g(x)$, $k(x)$, $r(t)$, $h(t)$, $v(t)$, etc.

Students may incorrectly believe that the notation $f(x)$ means to multiply some value $f$ times another value $x$. The notation alone can be confusing and needs careful development. For example, $f(2)$ means the output value of the function $f$ when the input value is 2.

Students may incorrectly believe that $f(2)$ within the example $f(2) = 3$ is a command to do a computation or a process given by a formula instead of imagining that $f(2)$ is a specific number (in this case 3). When asked "Where is f(2)?", students who are exposed to mapping diagrams or function machines and have this misconception may refer to $f(2)$ as "in-between," "on the way to 3," or "in process."

Students may incorrectly believe that $f(x)$ refers to the entire relation, when it actually refers to the output value when the input value is $x$. It is better to use $f$ to refer to the entire function.

The use of function notation is beneficial when situations have more than one function. Changing the independent variable that represents the same set of domain values, does not mean that the function is changed. For example, $f(x) = x^2 + 3$ is the same as $f(t) = t^2 + 3$. However, if two functions expressed by the same formula have different domains, they are not the same function. Also, it is important to have students interpret function notation within a context.

Function notation can be utilized for functions of two variables. For example, $A(b, h) = bh$ is a function that identifies or determines the area of a rectangle given its base and height. Since the base and height can vary, it requires two input values. Also to be noted, $T(x, y) = (x + 3, y + 2)$ is an algebraic way of representing a translation of the plane. Function notation $f(0)$ is an alternative way of saying to evaluate the value of function $f$ at $x = 0$ or to find the $y$-intercept of the graph $f(x)$. Similarly function notation such as $f(x) = 0$ represents a condensed way to communicate the idea of finding the zeros of the function $f(x)$ or the $x$-intercepts of the graph of $f(x)$.

# INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (F.IF.1-3)

***EXAMPLE***

Given the function $g(b) = 2b + 5$, compare $g(3 + 4)$ to $g(3) + g(4)$.

*Discussion:* Students will most likely use trial and error and substitution to realize that the expression $g(3 + 4)$ is not equal to $g(3) + g(4)$. Some students may incorrectly use the Distributive Property when solving. Clarify that the Distributive Property applies to products of numbers or variables. Although function notation looks similar to multiplication, it is read "*g* of a sum of 3 and 4", so it means the output of the function when $b = 7$. Use this to launch a discussion of how $f(a + b)$ is usually not the same as $f(a) + f(b)$.

## SEQUENCES

This cluster is recommended to be taught in conjunction with F.BF.1-2 and F.LE.1-3. The main difference between F.BF.1-2 and this cluster is that in this cluster the emphasis is on sequences being functions with integer domains, whereas F.BF.1-2 focuses on writing functions recursively and explicitly, and F.LE.1-3 is comparing the different types of functions including those formed by sequences and using the appropriate type to solve problems.

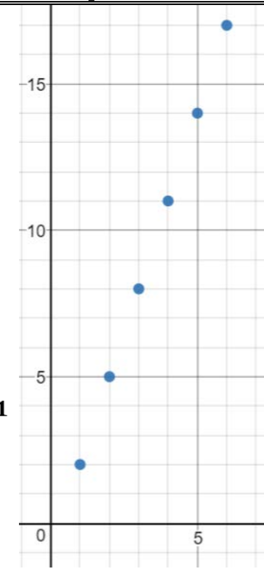
A sequence is a function whose domain is a subset of integers. (In Math 1, the domain is often counting numbers and whole numbers; in some mathematical contexts negative integers may be allowed in the domain.) A sequence can be thought of as ordered list of elements where each element in the list is called a term. A sequence is defined by a function *f* from a domain of a subset of integers to a range consisting of real numbers. Save sequence notation until more advanced courses. The use of subscript notation is not encouraged but can be used as an extension or saved for advanced courses. Pay careful attention to how a sequence is indexed. For example, the sequence may be indexed from 0 to $n$, from 1 to $n - 1$, or something else.

There are two main types of sequences students encounter in Math 1: an *arithmetic sequence* which has a *common difference* and a *geometric sequence* which has a *common ratio.* (Although most sequences are neither, and Math 1 students should have exposure to sequences that are neither arithmetic and geometric.) The goal is that at the end of Math 1 a student should know that an arithmetic sequence is a linear function, and a geometric sequence is an exponential function. Note that the converse of each statement is not always true. A linear function is an arithmetic sequence only when the domain is restricted to a subset of the integers. An exponential function is a geometric sequence only when

# INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (F.IF.1-3)

the domain is restricted to a subset of the integers. Help students understand this concept by emphasizing the difference between the graph of a linear function that is solid and that the graph of a sequence must be is discrete. An *explicit form* of an equation representing a sequence allows direct computation of any term in a sequence. A *recursive form* requires the preceding term to define the next term in the sequence. The recursive formula requires a starting value and a rule for computing the next terms; it should also include the parameters of the domain as part of the description. Sequences can be easily connected to the patterns that students learned in elementary school. For example, given the sequence 2, 5, 8, 11…, they will quickly recognize the pattern as plus 3. They should be able connect the pattern to the arithmetic sequence in order to make a transition to the recursive form *f*(1) = 2, *f*(*n* + 1) = *f*(*n*) + 3. Difficulty may lay in finding the explicit form. To do so, student can find the 0th term or the *y*-intercept, which may or may not be part of the domain of the sequence, and use that to figure out the rest of the equation in explicit form. They should be able to connect the common difference to the coefficient the independent variable in explicit form. Likewise, the connection can be established between the terms of geometric sequence, a common ratio, and the recursive form. Draw attention to the fact that the graph of a sequence has discrete points, because it is unknown what happens "between the dots." Have students discuss the advantage and disadvantage of each form. *Note: If indexes are defined differently, the resulting formulas may be different, but the unindexed sequence would be the same.*

| Term (*n*) | *f(n)* | |
|---|---|---|
| 0 | -1 | +3 |
| 1 | 2 | +3 |
| 2 | 5 | +3 |
| 3 | 8 | +3 |
| 4 | 11 | +3 |



**Recursive Form**

$f(1) = 2, f(n+1) = f(n) + 3$, for integers $n \geq 1$

$f(3) = f(2) + 3 = 8$

5

Common difference

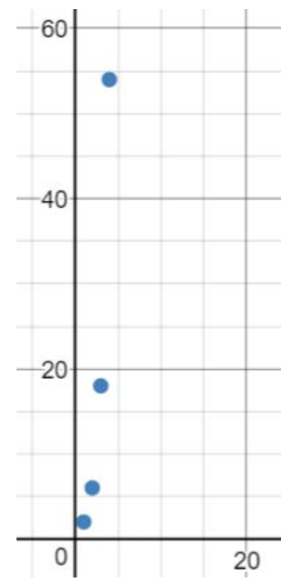
**Explicit Form**

$f(n) = 3n - 1$ for integers $n \geq 1$

Common difference

0th term

| Term (*n*) | *f(n)* | |
|---|---|---|
| 0 | $\frac{2}{3}$ | ×3 |
| 1 | 2 | ×3 |
| 2 | 6 | ×3 |
| 3 | 18 | ×3 |
| 4 | 54 | ×3 |



**Recursive Form**

$f(1) = 2, f(n+1) = f(n) \times 3$, for integers $n \geq 1$

$f(3) = f(2) \times 3 = 18$

6

Common ratio

**Explicit Form**

$f(n) = \frac{2}{3} \cdot 3^n$, for integers $n \geq 1$

0th term

Common ratio

# INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (F.IF.1-3)

An *index* refers to the part of the sequence that is being discussed and serves as the input value. Students may wish to start their explicit formula with an index of $n = 1$ or $n = 0$.

Rewrite sequences of numbers in tabular form, where the input represents the term number (the position or index) in the sequence, and the output represents the number in the sequence.

***EXAMPLE***

**a.** Have students write an explicit form of an equation in function notation for the sequence 5, 8, 11, 14…

***b.*** Then have the students graph the function.

*Discussion:*

1.) Students may have written an equation in explicit form for the sequence as $f(n) = 3n + 2$ or $f(n) = 3(n - 1) + 5$ when $n = 1$. Discuss the benefits and drawbacks of the equation in the form of $f(n) = 3n + 2$ versus $f(n) = 3(n - 1) + 5$ that arises from the formula for the $n$th term of the arithmetic sequence. The benefit of the first form, $f(n) = 3n + 2$, is that it connects the sequence to the graph of linear equation and writing linear equations is familiar to students. Using this form also avoids the $(n - 1)$ notation in the equation which may be difficult for some students; its drawback is that the first term of the sequence is not evident. Since traditionally the sequence begins with the 1st term, to find this equation students would need to find the 0th term ($n = 0$), or the $y$-intercept of the graph, before writing the equation in slope-intercept form. Remind students that the graph represents the entire set of infinite solutions of the equation $f(n) = 3n + 2$ but only particular points using the first coordinates 1, 2, 3, etc. would represent terms of the sequence. Point out that the graphs of sequences must be discrete. The drawback of the second form, $f(n) = 3(n - 1) + 5$ is that it is not useful for visualizing the graphs and some students have difficulty understanding that $(n - 1)$ refers to the preceding term. The benefit of using the second version of the explicit form is that it allows the first term in the sequence to be easily identified. Note: Although the form $f(n) = 3(n - 1) + 5$ makes sense to students who have been recently exposed to the formula for finding the $n$th term of an arithmetic sequences, the form $f(n) = 3n + 2$ is more intuitive arising from students' ideas of functions. Having an understanding of denoting the preceding term by $(n - 1)$ is especially beneficial for students intending to take advanced mathematics courses.

2.) Students can choose an index indicating where the sequence begins. Students may have written an equation in explicit form for the sequence as $f(n) = 3n + 2$ or $f(n) = 3n + 5$ depending if they choose an index of 1 or of 0 (or they could have chosen another index and would therefore have a different equation) for the beginning of the sequence. This forces them to make a decision about how they want to view the input-output relationship when viewing the sequence as a function. For example, for the explicit form $f(n) = 3n + 2$, for $n \geq 1$, when students choose $n = 1$, the input-output table will look like this:

| $n$ | 1 | 2 | 3 | 4 | … |
|---|---|---|---|---|---|
| $f(n)$ | 5 | 8 | 11 | 14 | |

When students choose $n = 0$, they will get the explicit form $f(n) = 3n + 5$, for $n \geq 0$, and the input-output table will look like this:

| $n$ | 0 | 1 | 2 | 3 | … |
|---|---|---|---|---|---|
| $f(n)$ | 5 | 8 | 11 | 14 | |

# INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (F.IF.1-3)

***EXAMPLE***

Describe the pattern for numbers 2, 3, 5, 8, …

*Discussion:* To encourage caution when guessing with patterns, give the students the following sequence 2, 3, 5, 8, … Most students will state the next number to be 12, by seeing the pattern +1, +2, +3, but it could also be 13 if the pattern really is adding the two previous terms. Another example illustrating a similar point is 1, 2, 4, 8, 16,… because both 31 or 32 can be legitimate numbers next in the sequence. See https://epicmath.org/2013/02/13/7-very-misleading-sequences/ for more misleading sequences.

***EXAMPLE***

Describe the pattern for numbers 1, 1, 2, 1, 1, 1, 2, 1, 1, 1, 1, 2, …..

*Discussion:* Students should be aware that many sequences in both mathematics and real-life are neither arithmetic nor geometric sequences.

## Instructional Tools/Resources

*These tools and resources may be helpful for instruction. Before use, they should be evaluated at the local level to determine their appropriateness for instruction.*

**Manipulatives/Technology**

- Diagrams or drawings of function machines, as well as tables and graphs.
- Function Machine virtual manipulatives
- Desmos is a free graphing calculator that is available to students as website or an app.
- Geogebra is a free graphing calculator that is available to students as website.
- Wolframalpha is dynamic computing tool.
- Visual Patterns is a website that shows pictures of linear, exponential, and quadratic patterns.
- Patterns Posters for Algebra 1 from Finding Ways by Fawn Nguyen is a blog about a class activity where students are given patterns, and they have to make posters from them. She is the creator of the visual patterns link above.

# INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (F.IF.1-3)

**The Concept of a Function**

- Card Sort Function by Desmos is an activity in which students sort graphs, equations, and contexts according to whether each one represents a function.
- Polygraph: Functions and Relations by Sam Wright from Desmos is an activity that is designed to spark vocabulary-rich conversations about discrete and continuous functions and relations. Key vocabulary that may appear in student questions includes: function, non-function, relation, discrete, continuous, input, output, $x$-value, and $y$-value.
- Points on a Graph is a task from Illustrative Mathematics that addresses a common confusion between independent and dependent variables.
- Domains is a task from Illustrative Mathematics that helps students consider the domain in terms of values for which each operation is invalid.
- Yam in the Oven is a task from Illustrative Mathematics that gives students practice interpreting statements in function notation. A similar task is Cell Phones.

**Function Machines**

- Function Machine is an applet by Math Playground of a function machine.
- Function Machine is an applet by Shodor of a function machine.
- Function Rules is a task by Illustrative Mathematics where students use a function machine to create a rule.

**Function Notation**

- Functioning Well is a lesson published in the Georgia Standards of Excellence Framework that is an introduction to functions and function notation. This lesson can be found on pages 147-155.
- Using Function Notation I is a task from Illustrative Mathematics that addresses a common misconception with respect to function notation.
- The task Interpreting the Graph connects interpreting function notation to a graph.

**Sequences**

- The Devil and Daniel Webster by NCTM Illuminations has students use recursive forms of a function to represent relationships. *NCTM now requires a membership to view their lessons.*
- Snake on a Plane is a task from Illustrative Mathematics that has students approach a function using a recursive and algebraic definition.

# INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (F.IF.1-3)

**Curriculum and Lessons from Other Sources**

- EngageNY, Algebra 1, Module 3, Topic B, Lesson 8: Why Stay with Whole Numbers?, Lesson 9: Representing, Naming, and Evaluating Functions, Lesson 10: Representing, Naming, and Evaluating Functions are lessons that pertain to this cluster. *Note: Ohio does not require the use of sequence notation.*
- Mathematics Vision Project, Algebra 1, Module 1: Sequences has many lessons that pertain to this cluster.
- Mathematics Vision Project, Algebra 1, Module 2: Linear and Exponential Functions has many lessons that pertain to this cluster.
- Unit 3: Functions from eMATHinstruction has materials that could be used for intervention. These documents can be used for individual students or for the entire class.
- Exporing Symbols by Burrill, Clifford, Scheaffer is the teacher's edition of a textbook in the Data-Driven Mathematics series published by Dale Seymour Publications. There are several lessons that pertain to this cluster. The student edition can be found here.

**General Resources**

- Arizona High School Progression on Functions is an informational document for teachers. This cluster is addressed on pages 7-8.
- High School Coherence Map by UnboundED is a map of the relationships among high school mathematics standards. Please note: This resource is aligned to the Common Core not Ohio's Learning Standards. Although the tool is useful, it needs to be checked with respect to Ohio's Learning Standards.

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# INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (F.IF.1-3)

**References, continued**


- Ponce, G. (September 2007) Critical juncture ahead! Proceed with caution to introduce the concept of function. *Mathematics Teacher*, *101(2),* 136-144.
- Ronau, R., Meyer, D., Crites, T., & Doughtery, B. (2014). *Putting Essential Understanding of Functions into Practice in Grades 9-12*. Reston, VA: National Council of Teacher of Mathematics.
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| STANDARDS | MODEL CURRICULUM (F.IF.4-5) |
|---|---|
| **Functions**<br>**INTERPRETING FUNCTIONS**<br>**Interpret functions that arise in applications in terms of the context.**<br>**F.IF.4** For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship. *Key features include the following: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; and periodicity.★* (A2, M3)<br>**a.** Focus on linear and exponential functions. (M1)<br>**F.IF.5** Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes. *For example, if the function h(n) gives the number of person-hours it takes to assemble n engines in a factory, then the positive integers would be an appropriate domain for the function.★*<br>**a.** Focus on linear and exponential functions. (M1) | **Expectations for Learning**<br>In eighth grade, students model linear relations between two quantities; analyze graphs to determine where they are increasing and decreasing; and determine if relations are linear or non-linear.<br><br>In this cluster, students interpret additional key features of the graphs and tables of linear and exponential functions only. They also determine the domain of a function by looking at a graph or table. In a real-life scenario students can find the restrictions on the domain.<br><br>In Math 2, students apply these concepts to quadratic functions.<br><br>*Note on differences between standards: In F.IF.4 and F.IF.5, the emphasis is on the context of the problem and on making connections among graphs, tables, and the context. In F.IF.7, the emphasis is on creating a graph given a symbolic representation, and then identifying the key features of the graph and connecting the key features to the symbols.*<br><br>**ESSENTIAL UNDERSTANDINGS**<br>• Key features (as listed in the standard) of a function can be illustrated graphically and interpreted in the context of the problem.<br>• The sensible domain for a real-world context should be accurately represented in graphs, tables, and symbols.<br>• Functions can have continuous or discrete domains.<br><br>**MATHEMATICAL THINKING**<br>• Connect mathematical relationships to contextual scenarios.<br>• Attend to meaning of quantities.<br>• Determine reasonableness of results.<br>*Continued on next page* |

| STANDARDS | MODEL CURRICULUM (F.IF.4-5) |
|---|---|
| | **Expectations for Learning, continued**<br>**INSTRUCTIONAL FOCUS**<br>*Remember, in this course, for exponential functions, assessments should focus on integer exponents only.*<br>• For linear functions, represented as tables, graphs, or verbal descriptions, interpret intercepts and rates of change in the contexts of the problems, given tables, graphs, and verbal descriptions.<br>• For exponential functions, interpret intercepts, growth/decay rates, and end behaviors in the contexts of the problems, given tables, graphs, and verbal descriptions.<br>• Use written descriptions or inequalities to describe intervals on which a function is increasing/decreasing and/or positive/negative (neither interval notation nor set builder notation are required).<br>• Determine whether to connect points on a graph based on the context (continuous vs. discrete domain).<br>• Demonstrate understanding of domain in the context of a real-world problem.<br><br>**Content Elaborations**<br>**OHIO'S HIGH SCHOOL CRITICAL AREAS OF FOCUS**<br>• Math 1, Number 2, pages 4-6<br><br>**CONNECTIONS ACROSS STANDARDS**<br>• Create equations that describe numbers or relationships (A.CED.2a, 3).<br>• Represent and solve equations and equations and inequalities graphically (A.REI.10).<br>• Understand the concept of a function, and use function notation (F.IF.1-3).<br>• Graph linear functions and indicate intercepts (F.IF.7a).<br>• Graph simple exponential functions, indicating intercepts, and end behavior (F.IF.7e).<br>• Interpret expressions for functions in terms of the situation they model (F.LE.5).<br>• Interpret linear models (S.ID.7). |

# INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (F.IF.4-5)

## Instructional Strategies

*Note: The model curriculum is not a collection of lessons nor a full curriculum; it does not suggest pace, sequence, or amount of time spent on topics. It provides information about a topic related to the standards including ideas for examples, strategies for teaching, common misconceptions, and possible connections between topics. In addition, the emphasis of the model curriculum is on instruction not on assessment; therefore, the examples in the Instructional Strategies section are written for instructional purposes not necessarily as assessment items. Examples may need to be rewritten to accommodate the needs of each individual classroom.*

**Standards for Mathematical Practice**

*This cluster focuses on but is not limited to the following practices:*

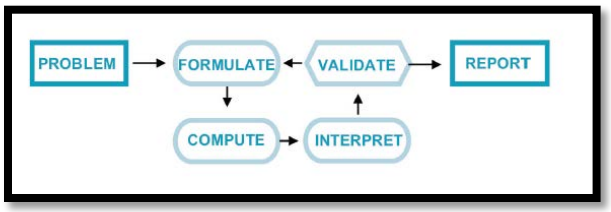
***MP.1** Make sense of problems and persevere in solving them.*

***MP.2** Reason abstractly and quantitatively.*

***MP.4** Model with mathematics.*

Functions are often described in terms of their using key features. Graphs allow the behavior of the function to be more apparent.

### MODELING



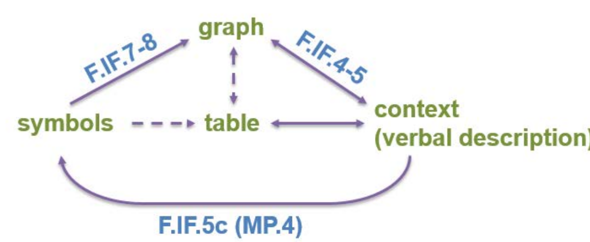

This cluster is included in the modeling standards. See page 13 for more information about modeling.

Begin instruction from a modeling standpoint. Start with a context and ask "Do one of the functions fit the behavior seen in the graph?" The answer sometimes needs to be "no," so that other function types can be explored within the context of the problem. For example, although not necessarily in Math 1, students should be aware that some scenarios are modeled with a periodic phenomenon that have graphs that repeat themselves after the particular interval along $x$ – axis. Other situations are modeled by graphs with "wiggles" that are called polynomial functions.

Students should be given a formula that can be graphed using Desmos or other graphing technology and they should be able to reason about the graph after they can see it. Given a table of values students could then create a scatterplot, possibly fit a curve to it, and reason about it in the same way they reason about formulas. *Note: Draw attention to the fact that sometimes the function may not be able to be described by a formula; sometimes the best we can do to describe a function is by a graph or a table.*

### INTERPRETING FUNCTIONS

Standards F.IF.4, F.IF.5, and F.IF.7 should be approached individually and in conjunction with one another. On a particular day, the focus could be on F.IF.4, F.IF.5, or F.IF.7 individually, but over the course of a few days, the standards should be woven together. See cluster F.IF.7-9 for an introductory activity about graphing stories. A follow-up activity should be connecting graphs to stories. Students should also be encouraged to write their own stories and then graph them. Then they could share their graphs with their classmates. Focus on graphs that are neither linear, quadratic, or exponential including piecewise scenarios.



# INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (F.IF.4-5)

***EXAMPLE***

Put students into groups. Give each student a role such as walker, measurer, marker, recorder, and timer. To prepare give students sticky notes and have them label each sticky note in increments of 5 seconds from 0 seconds to 45 seconds. Have them make 4 sets of these sticky notes in advance. Place the 0 second sticky note on the ground as the starting point. Have one student walk the scenario below. Another student should time the students calling out the time out loud. Another student should place a sticky at the foot of the walker every time the timer calls out a multiple of 5 seconds. The measurer should measure the distance from the beginning to each sticky note, and the recorder should record the information in a distance/time table. Before doing the experiment, have students do a quick sketch of what they think the graph will look like. Many students will draw graphs that are not functions.

- **a.** Walk forward slowly for 10 seconds. Stop for 10 seconds. Walk quickly for 5 seconds. Walk backwards for 15 seconds. Stop for 5 seconds.
- **b.** Walk forward quickly for 10 seconds. Walk backwards slowly for 10 seconds. Stop for 5 seconds. Walk backwards quickly for 5 seconds. Stop for 10 seconds. Walk forward slowly for 5 seconds.
- **c.** Walk forward quickly for 5 seconds. Stop for 15 seconds. Walk backwards slowly for 5 seconds. Stop for 5 seconds. Walk forward slowly for 15 seconds.

*Discussion:* Have each student graph the three different situations. Then have a class discussion about the graphs. Some students will have a difficult time realizing that stopping creates a horizontal line and that going backwards is graphed by a line with a negative slope (Many will assume that the line actually goes backwards and is therefore not a function.) Then have a discussion with students about how a speed/time graph would differ from a distance/time graph given the same situation. It may also be helpful to have students jump vertically and measure the height along the wall, so they realize that time still moves forward and forms a parabola even though their distance jumped is vertical.

***EXAMPLE***

Have students do the activity found here about matching graphs to their stories. See the Instructional Tools/Resources section for more resources on matching graphs to their stories.

Desmos Function Bundle has more activities involving graphing stories.

Investigate real-world data in which—

- several outputs may be paired with one input, like height vs. arm span (two people may have the same height, but different arm spans), or
- one output is paired with one input, like city population over time.

Students should be able to reason about trends in the data.

## INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (F.IF.4-5)

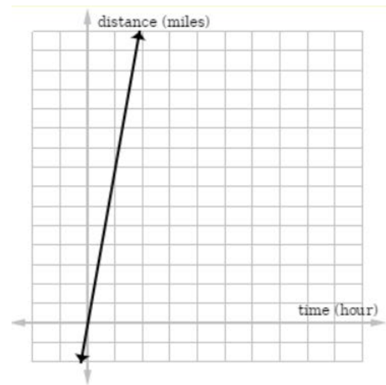
Have students flexibly move from examining a graph and describing its characteristics (e.g., intercepts, relative maximums, etc.) to using a set of given characteristics to sketch the graph of a function.

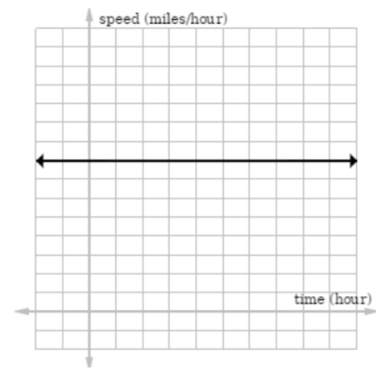
***EXAMPLE***

The teacher asked students to graph the following situation: Monica ran 8 miles per hour.

- Lawrence illustrated this situation by drawing the graph on the left.
- Michelle illustrated this situation by drawing the graph on the right.

Whose graph correctly describes the situation? Explain.




*Discussion:* Students should come to the realization that both graphs are correct. The first graph illustrates a distance/time graph and the second graph illustrates a speed/time graph. Since the distance increases 8 mile for every hour, the line in Lawrence's graph has a positive slope of 8. Since the speed is constant, the line in Michelle's graph is horizontal line through $y = 8$.

Emphasize that, for all functions, the $x$-intercepts of the graph of $f$ are the solutions of the equation $f(x) = 0$. *Note: For quadratics, the methods of solving equations in A.REI.4 can be referenced.*

Examine a table of a function and identify its key features from the table, such as intervals on which the function increases, decreases, or exhibits periodic behavior.

***EXAMPLE***

The table represents a continuous function defined on the interval $-2 < x < 3$, where just some integer inputs being used are displayed. Identify the key features of a graph.

| $x$ | $f(x)$ |
|---|---|
| -2 | $\frac{2}{9}$ |
| -1 | $\frac{2}{3}$ |
| 0 | 2 |
| 1 | 6 |
| 2 | 18 |
| 3 | 54 |

# INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (F.IF.4-5)

## KEY FEATURES OF FUNCTIONS

In Math 1, the majority of graphs representing functions may have the following key features:

- increasing intervals,
- decreasing intervals, and
- *x*- and *y*-intercepts

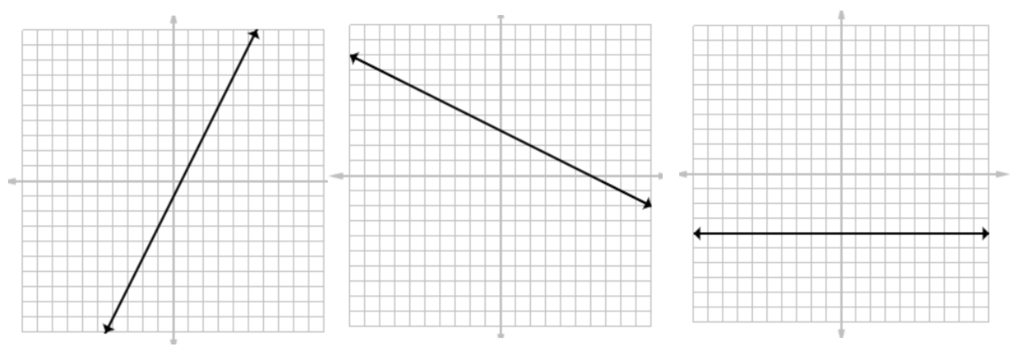
There are limits to the key features depending on the function type. If the domain is restricted, additional key features are possible. Although the focus of Math 1 is on linear and exponential functions, students should be exposed to other function types and informally discuss their key features such as relative maximums and minimums and symmetries. Connect with concepts of parent functions and function families in F.IF.7-9. It might be helpful to start with a non-formula graph such as temperature over time.

When discussing intervals, informal descriptions of end behaviors are acceptable. For example, "To the right, the graph goes to infinity and to the left, it is 'leveling off.'" (Interval notation and set builder notation are not necessary.) Also, written descriptions or inequalities are acceptable. For example,

- all x-values greater than 3 or $x > 3$;
- the function is increasing between 3 and 7, or is increasing $3 < x < 7$.

### Linear Functions

A graph of a linear function with an unrestricted domain may have *x*-intercept or *y*-intercept, or both. The graph may be increasing or decreasing, or neither. Regardless how the graph of a linear function looks, it does not have a minimum or maximum. See 8th Grade Model Curriculum cluster 8.F.4-5 for ideas about scaffolding with linear functions.



### Rate of Change/Slope

Although F.IF.6 has been moved to a later course, slope should still be discussed with respect to linear functions as it was a concept introduced in 8th grade. (See Grade 8 Model Curriculum 8.F.1-3 for scaffolding ideas.) Given a table of values, such as the height of a plant over time, students can estimate the rate of plant growth. Also, if the relationship between time and height is expressed as a linear equation, students should explain the meaning of the slope of the line in the context of the situation. In addition students should review the slope formula.

Students may believe that the slope of a linear function is merely a number used to sketch the graph of the line. In reality, slopes have real-world meaning, and the idea of a rate of change or related quantities is fundamental to understanding major concepts in mathematics.

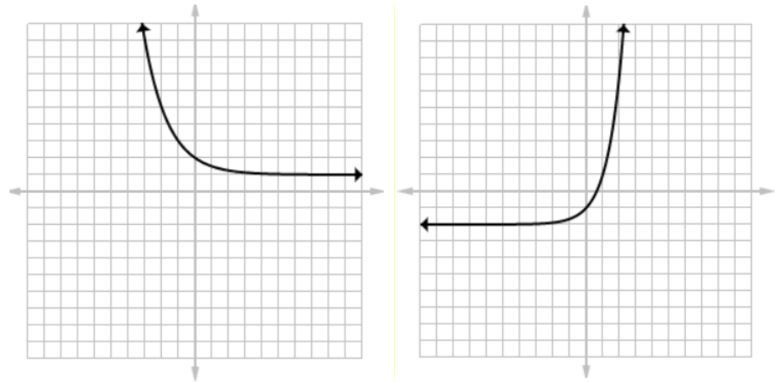
## INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (F.IF.4-5)

Some students perceive slope as the tilt of a line instead of a relationship between two quantities on a graph. Correct this misconception by changing the scale so that a large slope appears "flat" or a fractional slope (between 0 and 1) appears "steep."

There are some students who incorrectly think that the first term, *mx*, of the linear equation $y = mx + b$ represents a slope. Instead point out that the slope, *m*, represents the covariance between *y* and *x*.

**Exponential Functions**

A graph of an exponential function may have an *x*-intercept or both *x*- and *y*-intercepts. If the definition of an exponential functions is that it is a function in which the values of the domain are exponents, then adding or subtracting a constant can make different *x*-intercepts possible. For example, a function such as $y = 2^x - 8$ has an *x*-intercept at 3 since the point (3, 0) is on the graph. The graph of an exponential function may be increasing or decreasing. It does not have relative maximums, minimums, or symmetry. However, it can be described by its end behavior.



### DOMAIN OF A FUNCTION

When choosing a function family, be sure to ask whether that function family makes sense within the context. Sometimes the answer is no, and other times the answer may be yes over a restricted domain. For example, an entire roller coaster cannot be defined by a single quadratic equation, but one hill may be modeled by one quadratic function, and the next hill could be modeled by a different quadratic function.

Recognize appropriate domains of functions in real-world settings. For example, when determining a weekly salary based on hours worked, the hours (input) could be a rational number, such as 25.5, but not a negative number. Furthermore, there must be a maximum number of hours worked, determined based on reasonable assumptions. If a function relates the number of cans of soda sold in a machine to the money generated, the domain must consist of whole numbers.

Make sure students are exposed to functions in mathematical and real-world contexts that have both continuous and discrete domains.

Students may incorrectly believe that it is reasonable to input any *x*-value into a function, not understanding that context determines the domain. Therefore, they will need to examine multiple situations in which there are various limitations to the domains.

# INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (F.IF.4-5)

## Instructional Tools/Resources

*These tools and resources may be helpful for instruction. Before use, they should be evaluated at the local level to determine their appropriateness for instruction.*

**Manipulatives/Technology**

- Tables, graphs, and equations of real-world functional relationships
- Graphing calculators to generate graphical, tabular, and symbolic representations of the same function for comparison
- Geogebra is a free graphing calculator that is available to students as website.
- Desmos is a free graphing calculator that is available to students as website or an app.
- Wolframalpha is dynamic computing tool.

**Graphing Stories**

- Graphing Stories is an Illustrative Math task that investigates the graphs of relationships between quantities using video clips.
- Stories from Graphs a lesson from Cobb Learning where students match distance/time graphs to stories and then create stories to go with other graphs.
- Graphing Stories is a collaboration between Dan Meyer and Buzz Math that has students graph stories based on a real-life video clip.
- Functions Bundle by Desmos has 7 activities that explore functions.

**Interpreting Functions**

- Warming and Cooling is an Illustrative Mathematics task that could be used as an assessment of reading and interpreting graphs.
- Exponential Graph Characteristics is a lesson from Milwaukee Public Schools surrounding the key features of exponential functions using the Frayer Model.
- Lifespan of a Meme, the Harlem Shake by Yummy Math is an activity where students interpret a graph to explore a viral video.

# INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (F.IF.4-5)

**Domain and Range**

- Oakland Coliseum is a task from Illustrative Mathematics where students are asked to find the domain and range from a given context.
- The Restaurant is a task from Illustrative Mathematics where students are asked to find the domain and range from a given context.
- The Domain and Range Introduction and Finding Domain and Range are lessons by Desmos that explore domain and range.

**Curriculum and Lessons from Other Sources**

- EngageNY, Module 1, Topic A, Lesson 1: Graphs of Piecewise Linear Functions, Lesson 3: Graphs of Exponential Functions, Lesson 4: Analyzing Graphs—Water Usage During a Typical Day at School are lessons that pertain to this cluster.
- EngageNY, Module 3, Topic B, Lesson 13: Interpreting the Graph of a Function, Lesson 14: Linear and Exponential Models—Comparing Growth Rates are lessons that pertain to this cluster.
- EngageNY, Module 5, Topic A, Lesson 1: Analyzing a Graph, Lesson 2: Analyzing a Data Set, Lesson 3: Analyzing a Verbal Description are lessons that pertain to this cluster.
- EngageNY, Module 5, Topic B, Lesson 9: Modeling a Context from a Verbal Description is a lesson that pertains to this cluster.
- Mathematics Vision Project, Algebra 1, Module 2: Linear and Exponential Functions has many lessons that pertain to this cluster.
- Mathematics Vision Project, Algebra 1, Module 3: Features of Functions has many lessons that pertain to this cluster.
- Mathematics Vision Project, Algebra 1, Module 8: More Functions, More Features has many lessons that pertain to this cluster.
- Unit 3: Functions from eMATHinstruction has materials that could be used for intervention. These documents can be used for individual students or for the entire class.

**General Resources**

- Arizona High School Progression on Functions is an informational document for teachers. This cluster is addressed on pages 8-9.
- Arizona's Progression on High School Modeling is an informational document for teachers. This cluster is addressed on page 12.
- High School Coherence Map by UnboundED is a map of the relationships among high school mathematics standards. Please note: This resource is aligned to the Common Core not Ohio's Learning Standards. Although the tool is useful, it needs to be checked with respect to Ohio's Learning Standards.

# INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (F.IF.4-5)

**References**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| STANDARDS | MODEL CURRICULUM (F.IF.7, 9) |
|---|---|
| **Functions**<br>**INTERPRETING FUNCTIONS**<br>**Analyze functions using different representations.**<br>**F.IF.7** Graph functions expressed symbolically and indicate key features of the graph, by hand in simple cases and using technology for more complicated cases. Include applications and how key features relate to characteristics of a situation, making selection of a particular type of function model appropriate.★<br>**a.** Graph linear functions and indicate intercepts. (A1, M1)<br>**e.** Graph simple exponential functions, indicating intercepts and end behavior. (A1, M1)<br>**F.IF.9** Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions). *For example, given a graph of one quadratic function and an algebraic expression for another, say which has the larger maximum.* (A2, M3)<br>**a.** Focus on linear and exponential functions. (M1) | **Expectations for Learning**<br>In eighth grade, students graph and write linear functions, but their knowledge of key features of functions is limited to slope and *y*-intercept. They are exposed to non-linear functions and can distinguish between linear and non-linear functions. In this cluster, students graph linear and exponential functions given a symbolic representation and indicate intercepts and end behavior. They compare linear and exponential functions given various representations. In Math 2, students graph quadratics and indicate key features. They will compare linear, quadratic, and exponential functions given various representations.<br><br>*Note on differences between standards: In F.IF.4 and F.IF.5, the emphasis is on the context of the problem and on making connections among graphs, tables, and the context. In F.IF.7, the emphasis is on creating a graph given a symbolic representation, then identifying the key features of the graph and connecting the key features to the symbols.*<br><br>**ESSENTIAL UNDERSTANDINGS**<br>• The graph of a linear function shows intercepts and rate of change.<br>• The graph of an exponential function shows the *y*-intercept and end behaviors.<br>• Function families have commonalities in shapes and features of their graphs.<br>• Different representations (graphs, tables, symbols, verbal descriptions) illuminate key features of functions and can be used to compare different functions.<br>• More generally, writing a function in different ways can reveal different features of the graph of a function.<br><br>**MATHEMATICAL THINKING**<br>• Make connections between concepts, terms, and properties within the grade level and with previous grade levels.<br>• Analyze a mathematical model.<br>*Continued on next page* |

| STANDARDS | MODEL CURRICULUM (F.IF.7, 9) |
|---|---|
| | **Expectations for Learning, continued**<br>**INSTRUCTIONAL FOCUS**<br>**Remember, in this course, for exponential functions, assessments should focus on integer exponents only.*<br>• Given symbolic representations of linear and exponential functions, create accurate graphs showing all key features.<br>• Compare and contrast linear and exponential functions given by graphs, tables, symbols, or verbal descriptions.<br><br>**Content Elaborations**<br>**OHIO'S HIGH SCHOOL CRITICAL AREAS OF FOCUS**<br>• Math 1, Number 2, pages 4-6<br><br>**CONNECTIONS ACROSS STANDARDS**<br>• Interpret functions that arise in applications in terms of the context (F.IF.4).<br>• Understand that the graph of an equation in two variables is the set of all its solutions plotted in the coordinate plane, often forming a curve (which could be a line) (A.REI.10).<br>• Construct and compare linear and exponential models and solve problems (F.LE.1-2).<br>• Interpret expressions for functions in terms of the situation they model (F.LE.5). |

# INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (F.IF.7, 9)

## Instructional Strategies

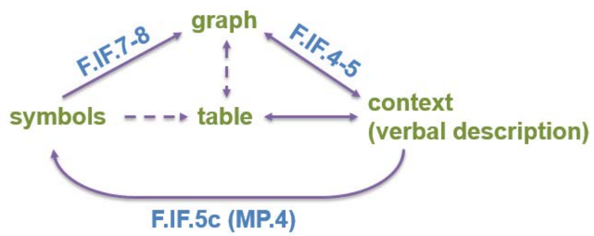
*Note: The model curriculum is not a collection of lessons nor a full curriculum; it does not suggest pace, sequence, or amount of time spent on topics. It provides information about a topic related to the standards including ideas for examples, strategies for teaching, common misconceptions, and possible connections between topics. In addition, the emphasis of the model curriculum is on instruction not on assessment; therefore, the examples in the Instructional Strategies section are written for instructional purposes not necessarily as assessment items. Examples may need to be rewritten to accommodate the needs of each individual classroom.*

***Standards for Mathematical Practice***
*This cluster focuses on but is not limited to the following practices:*
***MP.4*** *Model with mathematics.*
***MP.5*** *Use appropriate tools strategically.*
***MP.7*** *Look for and make use of structure.*

### GRAPHING FUNCTIONS

Standards F.IF.4, F.IF.5, and F.IF.7 should be approached individually and in conjunction with one another. On a particular day, the focus could be on F.IF.4, F.IF.5, or F.IF.7 individually, but over the course of a few days, the standards should be woven together.

Introduce functions by having students create data from the walking and then graph the data to make a connection to distance/time graphs.




Some students may incorrectly believe that a function is a synonym for formula. Point out that some functions do not have formulas at all, and some formulas do not represent functions. For example, recoding the average daily temperature at the Columbus Airport cannot be represented by a formula, but can be represented by a table. There is a formula for the equation of circle, yet a circle is not a function.

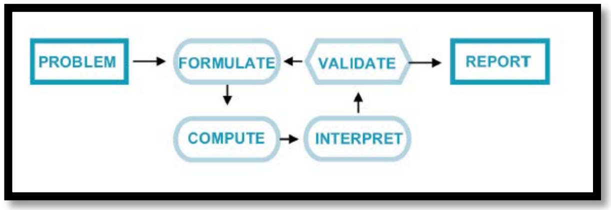
Some students may incorrectly believe a piece-wise function is several different functions because it is represented by several different formulas. Emphasize that it is one function pieced together.

TIP!

Discuss the importance of using appropriate labels and scales on the axes when representing functions with graphs. Some students believe that the labels and scales on a graph are not important and can be assumed by a reader, and that it is always necessary to use the entire graph of a function when solving a problem that uses that function as its model.


Ohio Department of Education

# INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (F.IF.7, 9)




## MODELING

This cluster is included in the modeling standards. See page 13 for more information about modeling.

## FAMILIES OF FUNCTIONS

"Functions are often studied and understood as families, and students should spend time studying functions within a family, varying parameters to develop an understanding of how the parameters affect the graph of a function and formulas and its key features." (Common Core Standards Writing Team, March 2013)

A family of functions is a set of functions that are related by adjustable parameters. Students should develop an understanding of what the parameters in a family do. Explore various families of functions by graphing those families and helping students make connections in terms of the formulas and key features.

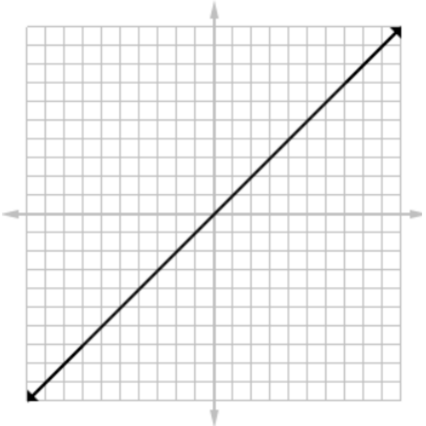
Have students explore and identify the function families: linear, quadratic, exponential, cubic, absolute value, square root, cubed root, sinusoidal among others but strive for fluency on linear and exponential functions with respect to representations and characteristics. However, for the other functions families, focus on shape. Introduce students to non-familiar functions to apply identification of key features. Also some functions, such as piece-wise functions, may not have formulas. This lends to connections with the knowledge about parent functions to model data. Students must be able to differentiate between linear, exponential, and quadratic functions and identify the parent function and interpret its key features. This should be driven by applications for modeling. Use domain and range values that are appropriate to the context.

Students may believe that each family of functions (e.g., quadratic, square root, etc.) have no commonalities, so they may not recognize common aspects across the families of functions and their graphs such as *y*-intercepts and end behavior.

# INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (F.IF.7, 9)

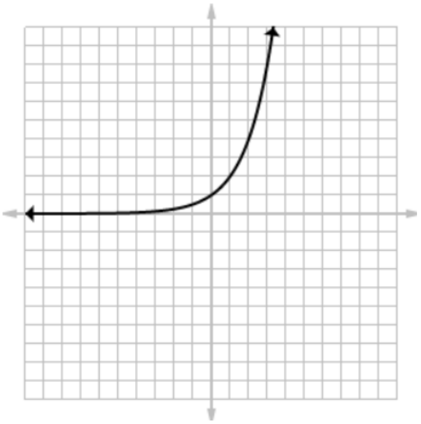
## PARENT FUNCTIONS

### Linear Functions



A linear function is a straight line with a constant rate of change and a *y*-intercept.

### Exponential Functions



An exponential function is a curve that has a *y*-intercept and an end behavior approaching infinity at one end and approaching a number at the other end. It may or may not have an *x*-intercept.

***EXAMPLE***

Using technology have students create card pictures out of different function types. For example, give students a variety of exponential function and determine what all exponential functions have in common. This allows students to connect the graphs of functions with their corresponding algebraic representations.

Students should graph simple cases of functions by hand, but use technology for more complicated graphing done by students. Make connections to algebra work (from A.APR.6) to functions. Sometimes displaying a good graph means getting a good "window."

# INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (F.IF.7, 9)

## FUNCTIONS IN EQUIVALENT FORMS

Writing a function in different ways can reveal different features of the graph of a function. Think of $g(x) = 2x + 6$ as a vertical shift of $y = 2x$, and the equivalent $g(x) = 2(x + 3)$ as a horizontal shift that yields the same graph. Think of $k(x) = 3^{x+2}$ as a horizontal shift of $y = 3^x$ and the equivalent $k(x) = 3^2 \cdot 3^x = 9 \cdot 3^x$ as a vertical stretch. Think of $h(x) = 1.02^{3x}$ as a horizontal shrink of $y = 1.02^x$, and the equivalent $h(x) = (1.02^3)^x = (1.061208)^x$ as a change of base of the exponential, but is not a vertical stretch. Students should be given the opportunity to come up with equivalent forms of the same function, and then explore why the functions are the same both algebraically and graphically. The process of rewriting functions to reveal key features should be used to explain/reveal features in the context of real-world scenarios.

TIP!

Students may believe that the process of rewriting functions into various forms is simply an algebraic symbol manipulation exercise. Focus on the purpose of allowing different features of the function to be exhibited.

Use various representations of the same function to emphasize different characteristics of that function.

## EXPONENTIAL FUNCTIONS

For exponential functions in Math 1, it is acceptable to use continuous graphs as a part of problem solving, even though students will not know what is really going on for the non-integer domain values (if they have evaluated the function only at integer inputs). A continuous graph allows students to see trends and to make claims such as "the city population will reach 1 million between years 7 and 8."

Examine multiple real-world examples of exponential functions so that students recognize that a base between 0 and 1 (such as an equation describing depreciation of an automobile, $f(x) = 15{,}000(0.8)^x$, represents the value of a \$15,000 automobile that depreciates 20% per year over the course of *x* years) results in an exponential decay, while a base greater than 1 (such as the value of an investment over time $f(x) = 15{,}000(1.07)^x$ represents the value of an investment of \$5,000 when increasing in value by 7% per year for *x* years) illustrates growth. Connect to properties of exponents in A.SSE.3.

## COMPARING FUNCTIONS

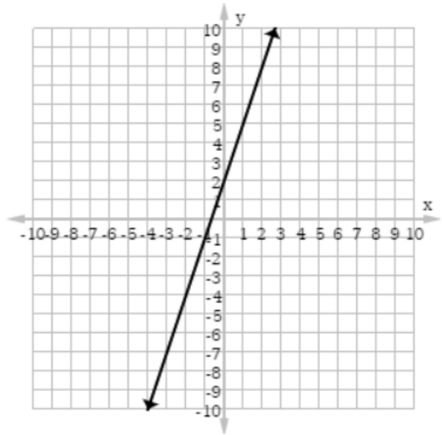
The purpose of F.IF.9 is so that students see key features across different representations of two functions.

# INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (F.IF.7, 9)

***EXAMPLE***
Which function has a greater rate of change?

**a.** $y - 3 = -4(x + 2)$

**b.**



**c.**

| $x$ | $y$ |
|---|---|
| 5 | 19 |
| 8 | 28 |
| 9 | 31 |
| 11 | 37 |

Remind students that using a graphing tool such as a calculator or online applet does not always create an accurate graph. For example, technology may connects points when graphing a function that implies that the graph is continuous when in fact it is not (asymptotes drawn with lines, points of discontinuity are shown as complete points on graph unless traced). Window scale selection is key to show correct shape, features, and end behaviors of graphs.

## Instructional Tools/Resources

*These tools and resources may be helpful for instruction. Before use, they should be evaluated at the local level to determine their appropriateness for instruction.*

**Manipulatives/Technology**

- Graphing utilities on a calculator and/or computer can be used to demonstrate the changes in behavior of a function as various parameters are varied.
- Geogebra is a free graphing calculator that is available to students as website.
- Desmos is a free graphing calculator that is available to students as website or an app.
- Wolframalpha is dynamic computing tool.

# INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (F.IF.7, 9)

**Graphing Functions**

- Graphing Stories by Desmos is a task that has students graph stories using function in the coordinate plane.
- Walking to Class: Modeling Students' Class schedules with Time-Distance Graphs is an NCTM Illuminations lesson where students use their class schedules to create time-distance graphs. *NCTM now requires a membership to view their lessons.*
- Waterline by Desmos is a task that has students watch glasses filling with water and graph functions to uncover misconceptions about graphs.
- Marbleslides by Desmos is a good practice activity for functions and parameters but not the best initial instructional activity as it is not a delivery of content lesson.
- Should I Replace My Toilets? by Yummy Math has students create equations and graphs comparing a pre-1980s toilet and a high efficiency toilet.

**Curriculum and Lessons from Other Sources**

- EngageNY, Module 5, Topic B, Lesson 4: Modeling a Context From a Graph, Lesson 5: Modeling From a Sequence, Lesson 6: Modeling a Context from Data, Lesson 8: Modeling a Context from a Verbal Description, Lesson 9: Modeling a Context from a Verbal Description are lessons that pertain to this cluster.
- Mathematics Vision Project, Algebra 1, Module 2: Linear and Exponential Functions has many lessons that pertain to this cluster.
- Mathematics Vision Project, Algebra 1, Module 3: Features of Functions has many lessons that pertain to this cluster.
- Mathematics Vision Project, Algebra 1, Module 8: More Functions, More Features has many lessons that pertain to this cluster.
- Unit 3: Functions from eMATHinstruction has materials that could be used for intervention. These documents can be used for individual students or for the entire class.

**General Resources**

- Arizona High School Progression on Functions is an informational text for teachers. This cluster is addressed on pages 9-10.
- Arizona's Progression on High School Modeling is an informational text for teachers. This cluster is addressed on page 12.
- High School Coherence Map by UnboundED is a map of the relationships among high school mathematics standards. Please note: This resource is aligned to the Common Core not Ohio's Learning Standards. Although the tool is useful, it needs to be checked with respect to Ohio's Learning Standards.

# INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (F.IF.7, 9)

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| STANDARDS | MODEL CURRICULUM (F.BF.1-2) |
|---|---|
| **Functions**<br>**BUILDING FUNCTIONS**<br>**Build a function that models a relationship between two quantities.**<br>**F.BF.1** Write a function that describes a relationship between two quantities.★<br>**a.** Determine an explicit expression, a recursive process, or steps for calculation from context.<br>**i.** Focus on linear and exponential functions. (A1, M1)<br>**F.BF.2** Write arithmetic and geometric sequences both recursively and with an explicit formula, use them to model situations, and translate between the two forms.★ | **Expectations for Learning**<br>In the eighth grade, students create functions to model relationships between two quantities. In this cluster, students write linear and exponential functions symbolically given the relationship between two quantities. Relationships between quantities could be given as tables, graphs, or within a context. Students also write explicit and recursive rules for arithmetic and geometric sequences. In Math 2, students focus on situations that exhibit exponential or quadratic relationships.<br><br>**ESSENTIAL UNDERSTANDINGS**<br>• Functions can be written as explicit expressions, recursive processes, and in other ways.<br>• An arithmetic sequence (informally, an addition pattern) has a starting term and a common difference between terms.<br>• A geometric sequence (informally, a multiplication pattern) has a starting term and a common ratio between terms.<br>• An arithmetic sequence is a linear function, and a geometric sequence is an exponential function.<br>• Some sequences can be defined recursively or explicitly, while others cannot be defined by a formula.<br>• The relationships between quantities can be modeled with functions that are linear, exponential, or neither of these.<br><br>**MATHEMATICAL THINKING**<br>• Make and modify a model to represent mathematical thinking.<br>• Discern and use a pattern or structure.<br><br>**INSTRUCTIONAL FOCUS**<br>• Model relationships with linear functions, which may be arithmetic sequences using tables, graphs, symbols, and words in context.<br>• Model relationships with exponential functions, which may be geometric sequences using tables, graphs, symbols, and words in context.<br>• Model relationships that are not linear or exponential using tables, graphs, symbols, and words in context.<br>*Continued on next page* |

| STANDARDS | MODEL CURRICULUM (F.BF.1-2) |
|---|---|
| | **Content Elaborations**<br>**OHIO'S HIGH SCHOOL CRITICAL AREAS OF FOCUS**<br>• Math 1, Number 2, pages 4-6<br><br>**CONNECTIONS ACROSS STANDARDS**<br>• Create equations that describe numbers or relationships (A.CED.2).<br>• Fit a linear function for a scatterplot that suggests a linear association (S.ID.6c).<br>• Interpret linear models (S.ID.7).<br>• Construct and compare linear and exponential models, and solve problems (F.LE.1). |

# INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (F.BF.1-2)

## Instructional Strategies

*Note: The model curriculum is not a collection of lessons nor a full curriculum; it does not suggest pace, sequence, or amount of time spent on topics. It provides information about a topic related to the standards including ideas for examples, strategies for teaching, common misconceptions, and possible connections between topics. In addition, the emphasis of the model curriculum is on instruction not on assessment; therefore, the examples in the Instructional Strategies section are written for instructional purposes not necessarily as assessment items. Examples may need to be rewritten to accommodate the needs of each individual classroom.*

***Standards for Mathematical Practice***

*This cluster focuses on but is not limited to the following practices:*

***MP.2*** *Reason abstractly and quantitatively.*

***MP.4*** *Model with mathematics.*

***MP.8*** *Look for and express regularity in repeated reasoning.*

This cluster is recommended to be taught in conjunction with F.IF.3 and F.LE.1-3. The main difference between this cluster and F.IF.3 is that in F.IF.3 the emphasis is on sequences being functions with integer domains. Whereas the focus of this cluster is on writing, building, and interpreting functions recursively and explicitly. The focus of F.LE.1-3 is comparing the different types of functions, including those formed by sequences and using the appropriate type to solve problems. Although, one of the focuses of this cluster is sequences, it is not limited to writing functions that are sequences. Students should be able to write linear, exponential, and quadratic functions in different situations that are both discrete and continuous.

Note that subscript notation is not required. A major goal of Math 1 is that students understand the connection between linear functions and arithmetic sequences as well as between exponential functions and geometric sequences. Later students find a connection between second differences and quadratic functions. This is powerful mathematical connection that may be missed by many students if they are forced to do it while struggling with the subscript notation.

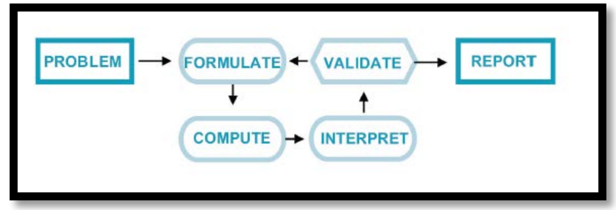
Students should be given pattern tasks where the tasks are—

- able to be modeled recursively and explicitly (flexible tasks);
- more easily modeled explicitly; or
- more easily modeled recursively.

When using flexible tasks students should be asked the following:

- Which rule do you prefer and why?
- Does your preference depend on the situation? Explain.
- What advantages are there for using an explicit rule?
- What advantages are there for using a recursive rule?
- What are the connections between the recursive rule and the explicit rule?
- What are the different ways that slope is represented in the two rules?

# INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (F.BF.1-2)




## MODELING

This cluster is included in the modeling standards. Geometric models that students can build are especially helpful to see patterns. See page 13 for more information about modeling.

## ARITHMETIC AND GEOMETRIC SEQUENCES

An arithmetic sequence is a linear function, and a geometric sequence is an exponential function. Note that the converse of each statement is not quite true. A linear function is an arithmetic sequence only when the domain is restricted to a subset of the integers. An exponential function is a geometric sequence only when the domain is restricted to a subset of the integers.

When teaching sequences, do not emphasize formulas. One reason for not giving or memorizing formulas is that a sequence can be indexed from one to $n$, from 1 to $n-1$, from 0 to $n$, or from 0 to $n-1$, and formulas differ depending on the indexing. (Series are challenging and may be saved for later study.) Instead of using formulas try using words such as "start, next, current" or "next = now + common difference," and make sure that students define the starting term.

## *EXAMPLE*

What are the next 4 numbers in the sequence 3, 6, 9, 12….

*Discussion:* Students may come up with 15, 18, 21, 24, or 3, 6, 9, 12, or 0, 3, 6, 9. Discuss why all of these answers could be true, but then as a class agree to have the rest of the discussion with respect to sequence continuing 15, 18, 21, 24. Have students come up with the $50^{th}$ term in the sequence and then write an expression in terms of $n$ for the $n$th number in the sequence. Discuss how both $3n$ and $3(n-1)$ can both be correct depending on how the starting term is defined. If the starting term is the $0^{th}$ term, $3n$ is correct and if the starting term is the $1^{st}$ term, then $3(n-1)$ is correct. Because of this, tell students it is important to note the starting term: $f(n) = 3n$ starting with $n = 0$ (or for $n \geq 0$) or $f(n) = 3(n-1)$ starting with $n = 1$ (or for $n \geq 1$). Discuss how and why the graphs vary depending on how the starting term is defined. Reinforce to students that $f(n)$ means the $n$th term in the sequence and does not mean $f$ times $n$.

When creating Javascript arrays, computer programmers start with a term that is in the zero place rather than a term that stays in the first place.

## Explicit and Recursive Forms of Functions

An *explicit* rule allows one to take any input and find the corresponding output, whereas a *recursive* rule requires the previous term(s).

**TIP!**

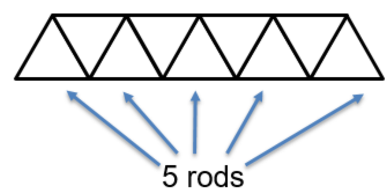
It may take some time for some students to realize that each term in the position *n* is defined by preceding term(s). Give students different sequences such as integers, odd integers, even integers, multiples of 3, etc, and have them pick any term as a starting point of the sequence and define the numbers going forward and backward.

Provide real-world examples (e.g., a table showing how far a car has driven after a given number of minutes, traveling at a uniform speed). If time and distance are column headings, then examine the table by looking "down" the table to describe a relationship recursively, as well as "across" the table to determine an explicit formula to find the distance traveled if the number of minutes is known. (Changing the orientation of the table, swaps the "down" and the "across.")

Start with visual models (e.g., folding a piece of paper in half multiple times to compare the number of folds to the thickness of the paper), to generate sequences of numbers that can be explored and described with both recursive and explicit formulas. Perimeter and area problems that can be modeled with toothpicks or graph paper, could also be useful. As students are already familiar with function tables, use those to help build understanding.

***EXAMPLE***

Beams, which are constructed from rods, are used to support bridges. The length of the beam is determined by the number of rods used to construct the bottom of the beam. Each rod is 1 meter long. An example of a beam is shown in the diagram.




| Beam length $(n)$ | Number of rods $f(n)$ |
|---|---|
| 1 | 3 |
| 2 | 7 |
| 3 | 11 |
| 4 | 14 |
| 20 | 79 |
| 100 | 399 |
| $n$ | $f(n) = 4n - 1$ |

+4 +4 +4

Recursive Rule

Explicit Rule

- How many rods will you need if you need a beam that is 20 meter long? What about 100 meters? For *n* meters?

*Discussion:* Students naturally reason recursively when looking at patterns. Most students will easily see that the recursive pattern is +4, so $f(n)$ = previous term + common difference or $f(n) = f(n - 1) + 4$ starting at the 1st term which equals 3 or $f(n) = f(n - 1) + 4$ for $n \geq 2$ . Discuss how the recursive form could be used to find any number in the sequence, but it might take a really long time. Discuss how the same sequence can be described by an alternative recursive formula such as $f(n + 1) = f(n) + 4$ for $n \geq 1$ and discuss why both forms are equivalent.

Notice that some students may come up with different explicit rules depending on how they view the pattern: $f(n) = n + 2n + (n - 1)$; $f(n) = 4n - 1$; $f(n) = 3 + 4(n - 1)$, $f(n) = 3n + (n - 1)$. Discuss why all these expressions are equivalent and how each lends itself to seeing the structure slightly differently (A.SSE.2)

# INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (F.BF.1-2)

Some students may incorrectly think that *f*(*n* + 1) is *f* times (*n* + 1). To prevent this error have students translate sequences into words, e.g., *f(n)* = *f*(*n* – 1) + 4 is a sequence where the *n*th term is four more than the one before it (*n* – 1), and *f*(*n* + 1) = *f*(*n*) + 4 where the next term (*n* + 1) is the *n*th term plus four.

Tie sequences into the graph of a function. Discuss why working the sequences backwards and finding the $0^{th}$ term (the *y*-intercept), can also help write a explicit rule. Discuss how slope relates to both the recursive and explicit rule. Emphasize that there are times when one form to describe the function is preferred over the other.

TIP! Have students make a 3 or 4 column table, so they can see the patterns more clearly.

| Beam length *n* | Pattern | Number of rods *f(n)* | Shorthand |
|---|---|---|---|
| 1 | Base + Middle + Top<br>1 + 2 | 3 | Base + Middle + Top<br>1 + 2(1) + (1 – 1) |
| 2 | 2 + 4 + 1 | 7 | 2 + 2(2) + (2 – 1) |
| 3 | 3 + 6 + 2 | 11 | 3 + 2(3) + (3 – 1) |
| 4 | 4 + 8 + 3 | 15 | 4 + 2(4) + (4 – 1) |
| 20 | | 79 | |
| 100 | | 399 | |
| n | | *f(n)* =<br>*n* + 2*n* + (*n* – 1) | |

| # of brown tiles |
|---|
| 8 |
| 12 |
| 16 |
| 20 |
| 24 |
| 28 |

***EXAMPLE***

Johanna wants to make a square patio in her garden with brown and white tiles of the same size. The interior tiles are white and the border tiles are brown.

- Write either an explicit or recursive rule to describe the number of brown border tiles needed.
- Make sure to identify the domain of the chosen rule.
- Do you prefer the explicit or recursive rule? Explain.

*Discussion:* The domain of the sequence, {1, 2, 3, 4, …}, is the set of whole numbers representing the position of the term in the sequence. For example, the term 8 is in the position 1; term 12 is in the position 2, etc. Each number in the domain is also equal to the number of white tiles along one side of the patio. A rule in explicit form that describes the situation could be $f(x) = 4x + 4$ for $x \geq 1$ where *f(x)* is the number of brown tiles and *x* is the term in the sequence, and a rule in recursive form for the number of brown tiles could be $f(x) = f(x-1) + 4$ for $x \geq 1$. Discuss with students which form they prefer to write and why.

# INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (F.BF.1-2)

TIP!

Students may believe that the best (or only) way to generalize a table of data is by using a recursive formula. Students naturally tend to look "down" a table to find the pattern but need to realize that finding the $100^{th}$ term requires knowing the $99^{th}$ term unless an explicit formula is developed.

Using the recursive formula has become easier with the use of technology and tables in graphing calculators and spreadsheets.

***EXAMPLE***

Your friend, Dominic, posts a meme to Facebook, and he asks you to not only share it with three people, but also to ask that the three people you share it with also share it with three people, and so on. Write a recursive and explicit rule for the situation. Discuss if this is an example of an arithmetic or geometric sequence and why.

Students may also incorrectly believe that arithmetic and geometric sequences are the same. Students need experiences with both types of sequences (and sequences that are neither) to be able to recognize the difference and more readily develop formulas to describe them.

**Discrete vs Continuous Domains**

When creating graphs of functions within a context, it is important to discuss the usage of a discrete versus a continuous domain. Make sure to present examples where it does not make sense to have a continuous domain and therefore the points on the graph should not be connected. For example, the profit after selling $n$ tickets at $8 each and deducting $100 facility rental fee would have a domain of only whole numbers, so the dots would not be connected. In comparison, if someone is selling fudge at $8 per pound, the dots would be connected because it is possible to sell non-whole number pounds of fudge.

Students may incorrectly believe that they can always "connect the dots" in a graph. Spend time interpreting the ordered pairs in between the dots.Provide examples where the context of the sequence can be modified to make a continuous domain so that students can "connect the dots."

# INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (F.BF.1-2)

***EXAMPLE***

Melissa visits the grand canyon and drops a penny off the cliff. The penny has fallen 16 feet at the first second, 48 feet the next second, and 80 feet the third second continuing at that rate until it hits the ground. What is the total distance the penny will fall in 6 seconds?

- Write a function representing the situation in recursive form.
- Write a function representing the situation in explicit form.
- Graph the function.
- Would you "connect the dots" when graphing the function? Explain.
- What is the total distance the penny will fall in 6 seconds?

*Discussion:* Illustrate through examples that although most sequences are discrete, some situations depending on the context are continuous.

## OTHER TYPES OF FUNCTIONS

Give students examples of other functions represented in symbolic form that are not linear or exponential such as $V(s) = s^3$ or $f(n) = 1.04 \cdot f(n-1) + 500, f(0) = 500$.

## Instructional Tools/Resources

*These tools and resources may be helpful for instruction. Before use, they should be evaluated at the local level to determine their appropriateness for instruction.*

**Manipulatives/Technology**

- Hands-on materials (e.g., paper folding, building progressively larger shapes using pattern blocks, etc.) can be used as a visual source to build numerical tables for examination.
- Desmos is a free graphing calculator that is available to students as website or an app.
- Geogebra is a free graphing calculator that is available to students as website.
- Wolframalpha is dynamic computing tool.
- Visual Patterns is a website that shows pictures of linear, exponential, and quadratic patterns.
- Patterns Posters for Algebra 1 from Finding Ways by Fawn Nguyen is a blog about a class activity where students are given patterns and they have to make posters from them. She is the creator of the visual patterns link above.

# INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (F.BF.1-2)

**Continuous and Discrete Functions**

- Continuous and Discrete Functions is a page by MathBitsNotebook that compare continuous and discrete functions.

**Exponential Functions**

- Game, Set, Flat by Desmos is an activity that helps students understand an exponential relationship to describe a "good" tennis ball. They will also construct an exponential equation.
- Lake Algae is a task from Illustrative Mathematics that introduces students to exponential growth.
- Compound with 100% Interest and Compounding with a 5% Interest from Illustrative Mathematics helps students develop the formulas for compound interest.
- To Fret or Not to Fret is an NCTM Illuminations two-part lesson where students explore geometric sequences and exponential functions by considering the placement of frets on stringed instruments. *NCTM now requires a membership to view their lessons.*

**Recursive Reasoning**

- Susita's Account is a task from Illustrative Mathematics that asks students to determine a recursive process from a context.
- Snake on a Plane is a task from Illustrative Mathematics that approaches a function recursively and by algebraic definition.
- The Devil and Daniel Webster is an NCTM Illuminations lesson that allows students to examine a recursive sequence. *NCTM now requires a membership to view their lessons.*
- Counting the Trains is an NCTM Illuminations three-part lesson where students investigate a relationship between recursive exponential functions. *NCTM now requires a membership to view their lessons.*

**Curriculum and Lessons from Other Sources**

- Although EnageNY, Algebra 1, Module 3, Topic A has good problems in Lessons 1 and 2 that might be used, it emphasizes subscript notation which Ohio does not.
- The Mathematics Vision Project, Algebra 1, Module 1: Sequences. Sections 1.3-1.8 align to this cluster.

**General Resources**

- Arizona High School Progressions on Functions is an informational document for teachers. This cluster is addressed on pages 11-12.
- Arizona High School Progressions on Modeling is an informational document for teachers. This cluster is addressed on pages 3 and 13.
- High School Coherence Map by UnboundED is a map of the relationships among high school mathematics standards. Please note: This resource is aligned to the Common Core not Ohio's Learning Standards. Although the tool is useful, it needs to be checked with respect to Ohio's Learning Standards.

## INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (F.BF.1-2)

**References**


- Burton, L. (March 2017). Discovering linear equations in explicit tables, *Mathematics Teaching in the Middle School*, *22(7),* 398-405.
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- Yarma, C. & Sampson, J. (October 2001). Just say "charge it!" *Mathematics Teacher*, *94(7),* 558-564.

| STANDARDS | MODEL CURRICULUM (F.BF.4) |
|---|---|
| **Functions**<br>**BUILDING FUNCTIONS**<br>**Build new functions from existing functions.**<br>**F.BF.4** Find inverse functions.<br>**a.** Informally determine the input of a function when the output is known. (A1, M1) | **Expectations for Learning**<br>In eighth grade, students learn that functions map inputs to outputs. In this cluster, students informally reverse this to find the input of a function when the output is known. In later classes, (+) some students more fully develop the concepts, procedures, and notation for inverses of functions.<br><br>**ESSENTIAL UNDERSTANDINGS**<br>• Sometimes the input of a function can be found when the output is given.<br><br>**MATHEMATICAL THINKING**<br>• Explain mathematical reasoning.<br><br>**INSTRUCTIONAL FOCUS**<br>** Limit to situations where inverse values are unique. Exclude formal notation; exclude finding the inverse algebraically; exclude switching x and y; exclude reflecting about the line* $y = x$.<br>• Use graphs and tables to find the input value of a function when given an output, and interpret the values in context.<br><br>**Content Elaborations**<br>**OHIO'S HIGH SCHOOL CRITICAL AREAS OF FOCUS**<br>• Math 1, Number 2, pages 4-6<br><br>**CONNECTIONS ACROSS STANDARDS**<br>• Understand the concept of a function and use function notation (F.IF.1-2). |

# INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (F.BF.4)

## Instructional Strategies

*Note: The model curriculum is not a collection of lessons nor a full curriculum; it does not suggest pace, sequence, or amount of time spent on topics. It provides information about a topic related to the standards including ideas for examples, strategies for teaching, common misconceptions, and possible connections between topics. In addition, the emphasis of the model curriculum is on instruction not on assessment; therefore, the examples in the Instructional Strategies section are written for instructional purposes not necessarily as assessment items. Examples may need to be rewritten to accommodate the needs of each individual classroom.*

***Standards for Mathematical Practice***

*This cluster focuses on but is not limited to the following practices:*

***MP.1** Make sense of problems and persevere in solving them.*

***MP.8** Look for and express regularity in repeated reasoning.*

### INVERSE OF FUNCTIONS

Provide examples of inverse relations that are not purely mathematical to introduce the idea of inverse of functions. For example, given a function that names the capital of a state, $f$(Ohio) = Columbus, ask questions such as what is $x$, when $f(x)$ = Denver. Students can conclude that $x$ = Colorado. Build on this concepts by looking at numerical input and output values. Keep it simple, informal, and free of inverse function notation. Instead focus in on the idea of "going backwards." Ask question such as "What is the input when the output is known?"

The habit of immediately swapping $x$ and $y$ values may be confusing to some students. A better conceptual approach is to more clearly work backwards, keeping the letters the same. For example, suppose $f(x) = 3x + 5$. Call it $y = 3x + 5$, and solve for $x$ to get $x = \frac{y-5}{3}$. Therefore if $x = g(y)$, and $g(y) = \frac{y-5}{3}$, then $f(x)$ and $g(y)$ are inverses of each other.

**TIP!** Swapping variables in an equation will lead to students misunderstanding notation in later mathematics courses when they will be required to use inverse function notation. For example, in later mathematics they may confuse the inverse of $y = f(x)$ with $y = f^{-1}(x)$. instead of the correct $x = f^{-1}(y)$. See "Inverse Functions: What Our Teachers Didn't Tell Us" article by Wilson, Adamson, Cox, and O'Bryan for further explanation.

For example, students might determine that folding a piece of paper in half 5 times results in 32 layers of paper. Then if they are given that there are 32 layers of paper, they can solve to find how many times the paper would have been folded in half.

## INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (F.BF.4)

***EXAMPLE***

Mark earns $9 an hour working at Best Buy. Write a function to describe the situation using *a* to represent the number of hours worked, and *b* to represent the total money earned.

*Discussion:* Discuss why both $b = 9a$ and $a = \frac{b}{9}$ could both describe the situation. Discuss when it would be more useful to have the hours be the independent variable and when it would be more useful to have the total money earned be the independent variable. Connect with rearranging equations and formulas in A.CED.4. In some circumstances, it is more useful to have the hours be assigned as the independent variable, and in other situations it is more useful to have the total money earned be assigned as the independent variable.

## Instructional Tools/Resources

*These tools and resources may be helpful for instruction. Before use, they should be evaluated at the local level to determine their appropriateness for instruction.*

**Manipulatives/Technology**

- Use the book, *The Sneeches*, by Dr. Seuss to introduce students to the concept of inverse functions.
- Desmos is a free graphing calculator that is available to students as website or an app.
- Geogebra is a free graphing calculator that is available to students as website.
- Wolframalpha is dynamic computing tool.

**Inverse of Functions**

- Temperature in Degrees Fahrenheit and Celsius is an Illustrative Mathematics task that uses a real-world example of inverse functions.

**General Resources**

- Arizona High School Progressions on Functions is an informational text for teachers. This cluster is addressed on pages 12-13.
- High School Coherence Map by UnboundED is a map of the relationships among high school mathematics standards. Please note: This resource is aligned to the Common Core not Ohio's Learning Standards. Although the tool is useful, it needs to be checked with respect to Ohio's Learning Standards.

# INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (F.BF.4)

**References**


- Common Core Standards Writing Team. (2013, March 1). *Progressions for the Common Core State Standards in Mathematics (draft). Grade 8, High School, Functions.* Tucson, AZ: Institute for Mathematics and Education, University of Arizona.
- Fernández, M. (February 2001). Graphical transformations and calculator greeting cards. *Mathematics Teacher*, *94(2),* 106-110.
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- Weiss, M. (December 2016/January 2017). Solving and graphing quadratics with symmetry transformation. *Mathematics Teacher*, *110(5),* 394-397.
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| STANDARDS | MODEL CURRICULUM (F.LE.1-2) |
|---|---|
| **Functions**<br>**LINEAR, QUADRATIC, AND EXPONENTIAL MODELS**<br>**Construct and compare linear, quadratic, and exponential models, and solve problems.**<br>**F.LE.1** Distinguish between situations that can be modeled with linear functions and with exponential functions.★<br>**a.** Show that linear functions grow by equal differences over equal intervals and that exponential functions grow by equal factors over equal intervals.<br>**b.** Recognize situations in which one quantity changes at a constant rate per unit interval relative to another.<br>**c.** Recognize situations in which a quantity grows or decays by a constant percent rate per unit interval relative to another.<br>**F.LE.2** Construct linear and exponential functions, including arithmetic and geometric sequences, given a graph, a description of a relationship, or two input-output pairs (include reading these from a table).★ | **Expectations for Learning**<br>In eighth grade, students interpret the rate of change and initial value of a linear function in terms of the situation it models and in terms of its graph or a table of values. Students also see examples of non-linear functions and learn and apply the properties of integer exponents. In Math 1, students focus on comparing linear and exponential functions. In Math 2, students compare across linear, exponential, and quadratic functions.<br><br>**ESSENTIAL UNDERSTANDINGS**<br>• Linear functions have a constant additive change.<br>• Exponential functions have a constant multiplicative change.<br>• Linear and exponential functions both have initial values.<br>• To highlight the constant growth/decay rate, *r*, often expressed as a percentage, exponential functions can be written in the form, $f(n) = a(1 + r)^n$.<br>• To highlight the growth/decay factor, *b*, exponential functions can be written in the form, $f(n) = a(b)^n$.<br>• An arithmetic sequence is a linear function, and a geometric sequence is an exponential function.<br><br>**MATHEMATICAL THINKING**<br>• Represent a concept symbolically.<br>• Make and modify a model to represent mathematical thinking.<br>• Make connections between concepts, terms, and properties within the grade level and with previous grade levels.<br>*Continued on next page* |

| STANDARDS | MODEL CURRICULUM (F.LE.1-2) |
|---|---|
| | **Expectations for Learning, continued**<br>**INSTRUCTIONAL FOCUS**<br>• Aim toward a multifaceted understanding of additive versus multiplicative change across different representations.<br>• For linear functions (arithmetic sequences), focus on the constant rate of change across the tables, graphs, contexts, and the explicit and recursive representations.<br>• For exponential functions (geometric sequences), focus on the constant growth/decay rate (or factor) across the tables, graphs, contexts, and the explicit and recursive representations.<br><br>**Content Elaborations**<br>**OHIO'S HIGH SCHOOL CRITICAL AREAS OF FOCUS**<br>• Math 1, Number 2, pages 4-6<br><br>**CONNECTIONS ACROSS STANDARDS**<br>• Build a function that models a relationship between two quantities (F.BF.1a, 2).<br>• Interpret functions that arise in applications in terms of the context (F.IF.4a, 5a).<br>• Analyze functions using different representations (F.IF.7a, e).<br>• Summarize, represent, and interpret data on two categorical and quantitative variables (S.ID.6c).<br>• Interpret linear models (S.ID.7).<br>• Interpret the structure of expressions (A.SSE.1).<br>• Interpret the parameters in a linear or exponential function in terms of a context (F.LE.5). |

# INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (F.LE.1-2)

## Instructional Strategies

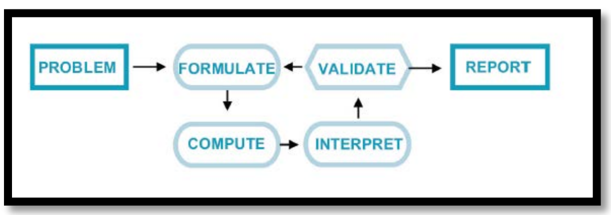
*Note: The model curriculum is not a collection of lessons nor a full curriculum; it does not suggest pace, sequence, or amount of time spent on topics. It provides information about a topic related to the standards including ideas for examples, strategies for teaching, common misconceptions, and possible connections between topics. In addition, the emphasis of the model curriculum is on instruction not on assessment; therefore, the examples in the Instructional Strategies section are written for instructional purposes not necessarily as assessment items. Examples may need to be rewritten to accommodate the needs of each individual classroom.*

This cluster is recommended to be taught in conjunction with F.BF.1-2 and F.IF.1-3. The main difference between F.BF.1-2 and F.IF.1-3 is that in F.IF.1-3 the emphasis is on sequences being functions with integer domains, whereas F.BF.1-2 focuses on writing functions recursively and explicitly. The focus of the cluster F.LE.1-3 is comparing the different types of functions including those formed by sequences and using the appropriate type to solve problems.

***Standards for Mathematical Practice***
*This cluster focuses on but is not limited to the following practices:*
***MP.1*** *Make sense of problems and persevere in solving them.*
***MP.2*** *Reason abstractly and quantitatively.*
***MP.4*** *Model with mathematics.*
***MP.7*** *Look for and make use of structure.*
***MP.8*** *Look for and express regularity in repeated reasoning.*

### MODELING

This cluster is included in the modeling standards. Geometric models that students can build are especially helpful to see patterns. See page 13 for more information about modeling.




### COMPARING LINEAR AND EXPONENTIAL GROWTH

The phrase "increasing exponentially" in everyday language means "really fast." In mathematics, increasing exponentially means increasing using the model $y = ab^x$ (where $\boldsymbol{a} > \mathbf{0}$ and $\boldsymbol{b} > \mathbf{1}$), but it may be increasing incredibly slowly ($\boldsymbol{b} = \mathbf{1.01}$). Because a graph of an exponential function eventually curves up, it will eventually have output values greater than a linear or quadratic (or polynomial) function. To understand this, students need to compare two graphs to see where the two graphs intersect. Then they will see that function behavior for values of $x$ close to 0 is different than large (positive) values of $x$. Note: If $\boldsymbol{y} = \boldsymbol{ab^x}$ with $\boldsymbol{a} > \mathbf{0}, \mathbf{0} < \boldsymbol{b} < \mathbf{1}$, then we have exponential decay.



Ohio Department of Education

# INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (F.LE.1-2)

***EXAMPLE***

**a.** Is it better to be paid a penny on the first day, and then double that amount each day thereafter for a month, or is it better to be paid $100 a day for the month?

**b.** If we add a third option to get $1 times the square of the day number, which of the three options would take? Explain.

*Discussion:* Have students make both a tabular representation and a graph of the situation before writing an equation. (An alternate introductory lesson could be on the fable "One Grain of Rice" by Demi. See Instructional Tools/Resources section for more resources on "One Grain of Rice.")

Explore simple linear and exponential functions by engaging in hands-on experiments. For example, students can measure the diameters and corresponding circumferences of several circles and discover that a function that relates the diameter to the circumference is a linear function with a first common difference. Then they can explore the value of an investment for an account that will double in value every 12 years and see that it is an exponential function with a base of 2.

Use a graphing calculator or computer program to compare tabular and graphic representations of exponential and polynomial functions to show how the *y*- (output) values of the exponential function eventually exceed those of linear functions

Students may also incorrectly believe that the end behavior of all functions depends on the situation and not on the fact that exponential function values will eventually get larger than those of any other polynomial function. Provide situations where students can discover this concept.

***EXAMPLE***

Suppose you wanted to join a gym that charges a $80 initiation fee, and then quotes you that it will cost you $230 for 6 months. Three price functions are given, all of which meet the quoted price, where *k* is the time in months and *P(k)* is the total cost. Which is the best model for you and why. Which is the best model for the gym?

**Plan A:** $P(k) = 80 + \frac{115}{3}k$

**Plan B:** $P(k) = 80(1.25327)^k$

*Discussion:* Students should discover that the *y*-intercept of each graph represents the initiation fee. If you only needed a gym membership for less than 6 months and there is no cancellation fee, then Plan B would be best for the consumer. In contrast after 6 months, Plan B is significantly better for the gym. The best plan for the consumer who wants a membership for 6 months or longer is Plan A.

# INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (F.LE.1-2)

## COMPARING LINEAR AND EXPONENTIAL MODELS

**Linear Function**

| x | f(x) | |
|---|---|---|
| -2 | -9 | |
| | | +0.5 |
| -1 | -8.5 | |
| | | +0.5 |
| 0 | -8 | |
| | | +0.5 |
| 1 | -7.5 | |
| | | +0.5 |
| 2 | -7 | |

**Exponential Function**

| x | f(x) | |
|---|---|---|
| -2 | $\frac{1}{2}$ | |
| | | ×4 |
| -1 | 2 | |
| | | ×4 |
| 0 | 8 | |
| | | ×4 |
| 1 | 32 | |
| | | ×4 |
| 2 | 128 | |

Compare tabular representations of a variety of functions to show that over equal $x$-intervals linear functions have a constant first difference (equal differences over equal $x$-intervals), while exponential functions do not (instead function values grow by equal factors over equal $x$-intervals). Also, quadratic functions have a constant second difference over equal $x$-intervals. Have students explore these concepts instead of just telling them. Require them to explain why these patterns hold true and justify their thinking.

### *EXAMPLE*

Give each student 2 tables, and tell them to create an equation for each type of function. Then complete the table.

**Linear:**___________

| x | y | Patterns in y |
|---|---|---|
| | | |
| | | |
| | | |
| | | |
| | | |

**Exponential:**___________

| x | y | Patterns in y |
|---|---|---|
| | | |
| | | |
| | | |
| | | |
| | | |

**a.** What pattern did you notice about the linear functions?
**b.** What pattern did you notice about the exponential functions?
**c.** Share your equations with 5 other people in the class. How were the observations about your patterns similar or different than your classmates? Explain.
**d.** Will the patterns that you found, always hold true? Explain and justify you thinking.

# INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (F.LE.1-2)

Apply linear and exponential functions to real-world situations. For example, a person earning $10 per hour experiences a constant rate of change in salary given the number of hours worked, while the number of bacteria on a dish that doubles every hour will have equal factors over equal intervals.

## CONSTRUCT ARITHMETIC AND GEOMETRIC SEQUENCES

Provide examples of arithmetic and geometric sequences in graphical, verbal, or tabular forms, and have students generate formulas and equations that describe the patterns. Explicit and recursive representations of functions could be constructed by analyzing the representations of linear and exponential functions.

Make connection to S.ID.6-7 with respect to exponential growth. For example have students create a simulation to model the exponential growth of cancer cells.

## Instructional Tools/Resources

*These tools and resources may be helpful for instruction. Before use, they should be evaluated at the local level to determine their appropriateness for instruction.*

**Manipulatives/Technology**

- Desmos is a free graphing calculator that is available to students as website or an app.
- Geogebra is a free graphing calculator that is available to students as website.
- Wolframalpha is dynamic computing tool.
- Visual Patterns is a website that shows pictures of linear, exponential, and quadratic patterns.
- Patterns Posters for Algebra 1 from Finding Ways by Fawn Nguyen is a blog about a class activity where students are given patterns and they have to make posters from them. She is the creator of the visual patterns link above.

# INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (F.LE.1-2)

**One Grain of Rice and The King's Chessboard**

Both are children's stories that can be used to teach exponential growth.

- *One Grain of Rice: A Mathematica Folktale* by Demi is a children's book about a rajah who takes his people's rice very year until a wise girl develops a clever plan using exponential growth.
- One Grain of Rice in an NCTM Illuminations lesson on exponential growth. *NCTM now requires a membership to view their lessons.*
- One Grain of Rice is another lesson from the Jim Wilson's University of Georgia's webpage using Demi's story and a spreadsheet.
- One Grain of Rice: Exponential Growth is a YouTube video on the story.
- *The Kings Chessboard,* by David Birch and Devis Grebu is a children's book about exponential growth as a wise man who refuses a king's reward for a favor instead takes a payment of rice.
- The Legend of a Chessboard: Teaser is a YouTube video that puts the quantity of rice in the context of different places such as a chessboard, a room, cities, and the country of Switzerland.
- The Legend of a Chessboard is a YouTube video based on the story.

**Exponential Growth**

- Exponential Models: Rhinos and M&M's is lesson from PBS that uses paperfolding, M&M's, and Rhinos to show exponential growth and decay.
- Exponential Growth & Decay (Ashby) is a lesson by Achieve the Core that uses several different representations to demonstrate exponential growth and decay.
- An Intro to Exponential Growth and Decay is a Desmos activity that also models exponential growth and decay using pennies and M&M's.
- Growth and Decay is a Desmos activity that uses March Madness brackets to illustrate exponential growth and decay.
- Overrun by Skeeters-Exponential Growth and Skeeter Populations and Exponential Growth are lessons from Annenberg Learner where students model functions that represent exponential growth about skeeters (mosquitoes).
- Predicting your Financial Future is an NCTM Illumination's lesson about compound interest. *NCTM now requires a membership to view their lessons.*
- Fry's Bank is a 3-Act Math Task by Dan Meyer that introduces exponential growth.
- Pixel Pattern is a 3-Act Math Task by Dan Meyer that explores patterns.
- Identifying Exponential Functions is a task by Illustrative Mathematics that introduces exponential functions by experimenting with the parameters of the function.
- Two Points Determine an Exponential Function I and Two Points Determine an Exponential Function II are tasks by Illustrative Mathematics where students have to find the values of *a* and *b* in an exponential function given two points.

# INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (F.LE.1-2)

**Comparing Linear and Exponential Growth**

- Piles of Paper is a CPalms activity where students fold paper to demonstrate linear and exponential growth.
- National Debt and Wars is a lesson by NCTM Illuminations where student collect information about the National Debt, plot the data by decade, and decide whether an exponential curve is a good fit. *NCTM now requires a membership to view their lessons.*
- Shrinking Candles, Running Waters, Folding Boxes is a lesson by NCTM Illuminations that has students determine which function type best fits the data. Skip the "Weather, It's a Function" section as it is above grade-level. *NCTM now requires a membership to view their lessons.*
- Birthday Gifts and Turtle Problem is a Mathematics Design Collaborative lesson from the State of Georgia Department of Education that explores the rates of changes of linear functions versus exponential functions.
- Representing Linear and Exponential Growth by Mathematics Assessment Project is a lesson that has students interpret exponential and linear functions.

**Curriculum and Lessons from Other Sources**

- EngageNY Algebra 1, Module 3, Topic A, Lesson 5: The Power of Exponential Growth, Lesson 6: Exponential Growth—U.S. Population and World Population, Lesson 7: Exponential Decay are lessons that pertain to this cluster.
- EngageNY Algebra 1, Module 3, Topic D, Lesson 21: Comparing Linear and Exponential Models Again, Lesson 22: Modeling an Invasive Species Population, Lesson 23: Newton's Law of Cooling are lessons that pertain to this cluster.
- The Georgia Standards of Excellence Curriculum Frameworks for Algebra 1, Unit 5: Comparing and Contrasting Functions compares and contrasts functions. There are many tasks in this document that align with this cluster.
- The Mathematics Vision Project Secondary Math 1, Module 2: Linear and Exponential Functions has many task that align with this cluster.
- A lesson on Exponential Modeling developed by the Virginia Department of Education that uses a graphing calculator. It has the following activities: Who Wants to be a Millionare?, Paper Folding, M&M Decay, Decaying Dice Game, Population Growth, and Baseball Players' Salaries.

**General Resources**

- Arizona High School Progression on Functions is an informational document for teachers. This cluster is addressed on pages 16-17.
- Arizona High School Progression on Modeling is an informational document for teachers. This cluster is addressed on page 5.
- High School Coherence Map by UnboundED is a map of the relationships among high school mathematics standards. Please note: This resource is aligned to the Common Core not Ohio's Learning Standards. Although the tool is useful, it needs to be checked with respect to Ohio's Learning Standards.

## INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (F.LE.1-2)

**References**


- Chan, H. (May 2015). How do they grow? *Mathematics Teaching in the Middle School*, *20(9),* 548-555.
- Common Core Standards Writing Team. (2013, March 1). *Progressions for the Common Core State Standards in Mathematics (draft). Grade 8, High School, Functions.* Tucson, AZ: Institute for Mathematics and Education, University of Arizona.
- Common Core Standards Writing Team. (2013, July 4). *Progressions for the Common Core State Standards in Mathematics (draft). High School, Modeling*. Tucson, AZ: Institute for Mathematics and Education, University of Arizona.
- Kara, M., Eames, C., Miller, A., & Chieu, A. (May 2015). Staircases, towers, and castles. *Mathematics Teacher*, *108(9),* 663-670.
- Kirwan, J. (April 2017). Using visualization to generalize on quadratic patterning tasks. *Mathematics Teacher*, *110(8),* 588-593.
- Yarma, C. & Sampson, J. (October 2001). Just say "charge it!" *Mathematics Teacher*, *94(7),* 558-564.

<table>
<tr><th>STANDARDS</th><th>MODEL CURRICULUM (F.LE.5)</th></tr>
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## Functions

**LINEAR, QUADRATIC, AND EXPONENTIAL MODELS**

**Interpret expressions for functions in terms of the situation they model.**

**F.LE.5** Interpret the parameters in a linear or exponential function in terms of a context.★

</td><td>

## Expectations for Learning

This standard does not present new expectations for student learning. Rather, it emphasizes important habits to complement F.LE.1-3. In this cluster, students connect their understanding of the defining characteristics of linear functions (initial value and rate of change) to the defining characteristics of exponential functions (initial value and growth rate/growth factor) and by interpreting them in the context of a real-world problem.

### ESSENTIAL UNDERSTANDINGS

- Linear functions have a constant additive change.
- Exponential functions have a constant multiplicative change.
- Linear and exponential functions both have initial values.
- To highlight the constant growth/decay rate, *r*, often expressed as a percentage, exponential functions can be written in the form, $f(n) = a(1 + r)^n$.
- To highlight the growth/decay factor, *b*, exponential functions can be written in the form, $f(n) = a(b)^n$.
- An arithmetic sequence is a linear function, and a geometric sequence is an exponential function.

### MATHEMATICAL THINKING

- Connect mathematical relationships to contextual scenarios.
- Use accurate mathematical vocabulary to describe mathematical reasoning.
- Attend to meaning of quantities.
- Make connections between concepts, terms, and properties within the grade level and with previous grade levels.

### INSTRUCTIONAL FOCUS

- For linear functions (arithmetic sequences), focus on the constant rate of change across the tables, graphs, contexts, and the explicit and recursive representations.
- For exponential functions (geometric sequences), focus on the constant growth/decay rate (or factor) across the tables, graphs, contexts, and the explicit and recursive representations.

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| STANDARDS | MODEL CURRICULUM (F.LE.5) |
|---|---|
| | **Content Elaborations**<br>**OHIO'S HIGH SCHOOL CRITICAL AREAS OF FOCUS**<br>• Math 1, Number 2, pages 4-6<br><br>**CONNECTIONS ACROSS STANDARDS**<br>• Build a function that models a relationship between two quantities (F.BF.1a, 2).<br>• Interpret functions that arise in applications in terms of the context (F.IF.4-5).<br>• Analyze functions using different representations (F.IF.7a, e).<br>• Summarize, represent, and interpret data on two categorical and quantitative variables (S.ID.6c).<br>• Interpret linear models (S.ID.7).<br>• Interpret the structure of expressions (A.SSE.1). |

# INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (F.LE.5)

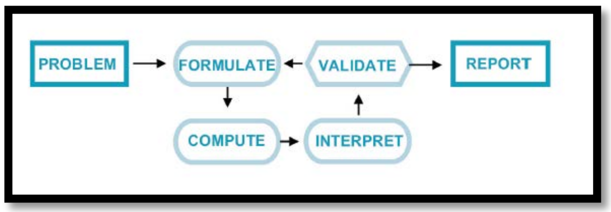
## Instructional Strategies

*Note: The model curriculum is not a collection of lessons nor a full curriculum; it does not suggest pace, sequence, or amount of time spent on topics. It provides information about a topic related to the standards including ideas for examples, strategies for teaching, common misconceptions, and possible connections between topics. In addition, the emphasis of the model curriculum is on instruction not on assessment; therefore, the examples in the Instructional Strategies section are written for instructional purposes not necessarily as assessment items. Examples may need to be rewritten to accommodate the needs of each individual classroom.*

***Standards for Mathematical Practice***
*This cluster focuses on but is not limited to the following practices:*
***MP.2*** *Reason abstractly and quantitatively.*
***MP.7*** *Look for and make use of structure.*

### MODELING

This cluster is included in the modeling standards. Geometric models that students can build are especially helpful to see patterns. See page 13 for more information about modeling.




### INTERPRETING PARAMETERS

Emphasis should be put on using units to understand problems. Students should recognize the meaning of the parameters of a function. Draw attention to the units of the parameters and connect them to the context.

Use real-world contexts to help students understand how the parameters of linear and exponential functions depend on the context. For example, a plumber who charges \$50 for a house call and \$85 per hour would be expressed as the function $f(x) = 85x + 50$, and if the rate were raised to \$90 per hour, the function would become $f(x) = 90x + 50$. On the other hand, an equation of $f(x) = 8{,}000(1.04)^x$ could model the rising population of a city with 8,000 residents when the annual growth rate is 4%. Students can examine what would happen to the population over 25 years if the rate were 6% instead of 4% or the effect on the equation and function of the city's population were 12,000 instead of 8,000.

Students may incorrectly believe that the first term in a linear equation is always the rate of change. However, in the equation $y = 10 + 2x$, 10 is the constant value (or $y$-intercept) not the rate of change.

TIP!

Students may want to multiply the initial value by the base before raising the base to its exponential value. However, $3{\cdot}2^4$ is not equivalent to $(3{\cdot}2)^4$. Review the properties of exponents using expanded form.

# INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (F.LE.5)

***EXAMPLE***

*Illegal Fish*

A fisherman illegally introduces some fish into a lake, and they quickly breed. The growth of the population of this new species (within a period of a few years) is modeled by $P(x) = 5b^x$ where $x$ is the time in weeks following the introduction and $b$ is a positive unknown base.

- **a.** Exactly how many fish did the fisherman release into the lake?
- **b.** Find $b$ if you know the lake contains 33 fish after eight weeks. Show step-by-step work. What's the percent growth rate by week?
- **c.** Instead, now suppose that $P(x) = 5b^x$ and $b = 2$. What is the weekly percent growth rate in this case? What does this mean in every-day language?

Task from Illustrative Mathematics. For solutions and discussion, see https://www.illustrativemathematics.org/content-standards/tasks/579.

Provide students with opportunities to research raw data on the internet (such as increases in gasoline consumption in China over $x$ number of years) and graph and make generalizations about trends in growth, determining whether the growth is linear or exponential. Working in pairs or small groups, students can be given different parameters of a function to manipulate and compare the results to draw conclusions about the effects of the changes.

Graphs and tables can be used to examine the behaviors of functions as parameters are changed, including the comparison of two functions. For example, what would happen to a population if it grew by 500 people per year in contrast to the population rising an average of 8% per year over the course of 10 years?

***EXAMPLE***

- **a.** John deposited $400 in the bank at 2.25% simple interest rate for 6 months. Write an equation modeling the situation. How long will it take him to make $500 in interest?
- **b.** Jasmine deposited $400 in the bank at 2.25% rate compounded every 6 months. Write an equation modeling the situation. How long will it take her to earn $500 in interest?
- **c.** Compare the parameters of the equations in both situations.

# INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (F.LE.5)

## Instructional Tools/Resources

*These tools and resources may be helpful for instruction. Before use, they should be evaluated at the local level to determine their appropriateness for instruction.*

**Manipulatives/Technology**

- Graphing calculators or computer software that generates graphs and tables of functions
- Web sites and other sources that provide raw data, such as the cost of products over time, population changes, etc.
- Desmos is a free graphing calculator that is available to students as website or an app.
- Geogebra is a free graphing calculator that is available to students as website.
- Wolframalpha is dynamic computing tool.

**Interpreting Parameters**

- Illegal Fish is a task from Illustrative Mathematics where students interpret the relevant parameters of exponential growth in a real-world context.
- Taxi is a task from Illustrative Mathematics where students interpret the parameters of a linear equation.
- U.S. Population 1982-1998 is a modeling task from Illustrative Mathematics using U.S. Census data. Students are required of make predictions using a linear model without using an equation.
- DDT-cay is a task from Illustrative Mathematics that allows students to encounter negative exponents in a contextual situation.
- Avi & Benita's Repair Shop is a Desmos activity focusing on parameters of linear and exponential functions in a real-world context.

**Curriculum and Lessons from Other Sources**

- EngageNY Algebra 1, Module 3, Topic D, Lesson 21: Comparing Linear and Exponential Models Again, Lesson 22: Modeling an Invasive Species Population, Lesson 23: Newton's Law of Cooling, Lesson 24: Piecewise and Step Functions in Context are lessons that pertain to this cluster.
- The Georgia Standards of Excellence Curriculum Frameworks for Algebra 1, Unit 5: Comparing and Contrasting Functions compares and contrasts functions. This cluster is addressed on pages 61-69.
- The Mathematics Vision Project Secondary Math 1, Module 2: Linear and Exponential Functions has many tasks that align with this cluster.

# INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (F.LE.5)

**General Resources**

- Arizona High School Progression on Functions is an informational document for teachers. This cluster is addressed on page 17.
- Arizona High School Progression on Modeling is an informational document for teachers. This cluster is addressed on page 12.
- High School Coherence Map by UnboundED is a map of the relationships among high school mathematics standards. Please note: This resource is aligned to the Common Core not Ohio's Learning Standards. Although the tool is useful, it needs to be checked with respect to Ohio's Learning Standards.

**References**


- Common Core Standards Writing Team. (2013, March 1). *Progressions for the Common Core State Standards in Mathematics (draft). Grade 8, High School, Functions.* Tucson, AZ: Institute for Mathematics and Education, University of Arizona.
- Common Core Standards Writing Team. (2013, July 4). *Progressions for the Common Core State Standards in Mathematics (draft). High School, Modeling*. Tucson, AZ: Institute for Mathematics and Education, University of Arizona.

<table>
<tr><th>STANDARDS</th><th>MODEL CURRICULUM (G.CO.1-5)</th></tr>
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## Geometry

**CONGRUENCE**

**Experiment with transformations in the plane.**

**G.CO.1** Know precise definitions of ray, angle, circle, perpendicular line, parallel line, and line segment, based on the undefined notions of point, line, distance along a line, and arc length.

**G.CO.2** Represent transformations in the plane using, e.g., transparencies and geometry software; describe transformations as functions that take points in the plane as inputs and give other points as outputs. Compare transformations that preserve distance and angle to those that do not, e.g., translation versus horizontal stretch.

**G.CO.3** Identify the symmetries of a figure, which are the rotations and reflections that carry it onto itself.

**a.** Identify figures that have line symmetry; draw and use lines of symmetry to analyze properties of shapes.

**b.** Identify figures that have rotational symmetry; determine the angle of rotation, and use rotational symmetry to analyze properties of shapes.

**G.CO.4** Develop definitions of rotations, reflections, and translations in terms of angles, circles, perpendicular lines, parallel lines, and line segments.

*Continued from previous page*

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## Expectations for Learning

In middle school, students first learn about the basic rigid motions (translations, rotations, and reflections) and verify their properties experimentally. In this cluster, students formalize the notion of a transformation as a function from the plane to itself. Building on their hands-on work, students develop mathematical definitions of the basic rigid motions. These definitions serve as a logical basis for the theorems that students prove in Geometry. An important step in high school is to perform appropriate transformations and give precise descriptions of sequences of basic rigid motions that carry one figure onto another. Transformations provide language to be precise about symmetry; this is the first time students have encountered formal symmetry.

The student understanding of this cluster begins at van Hiele Level 1 (Analysis) and moves to Level 2 (Informal Deduction/Abstraction).

### ESSENTIAL UNDERSTANDINGS

- A transformation is a function from the plane to itself; input and output values are points, not numbers.
- Rigid motions are transformations that preserve distance and angle.
- Some transformations preserve distance and angle measures, and some do not.
- In order to perform a translation, a distance and a direction is required.
- A rotation requires a center and an angle.
- A reflection requires a line.
- The symmetries of a figure are the transformations that carry the figure onto itself.

### MATHEMATICAL THINKING

- Use accurate and precise mathematical vocabulary and symbolic notations.
- Make connections between concepts, terms, and properties within the grade level and with previous grade levels.
- Recognize, apply, and justify mathematical concepts, terms, and their properties.

*Continued on next page*

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| STANDARDS | MODEL CURRICULUM (G.CO.1-5) |
|---|---|
| **G.CO.5** Given a geometric figure and a rotation, reflection, or translation, draw the transformed figure using items such as graph paper, tracing paper, or geometry software. Specify a sequence of transformations that will carry a given figure onto another. | **Expectations for Learning, continued**<br>**INSTRUCTIONAL FOCUS**<br>• Know precise definitions of basic terms: ray, angle, circle, perpendicular line, parallel line, and line segment.<br>• Develop and use appropriate geometric notation.<br>• Formalize definitions of basic rigid motions (translations, rotations, and reflections).<br>• Perform and identify transformations using a variety of tools.<br>• Identify the symmetries shown in a figure (rotational and line symmetries).<br><br>**Content Elaborations**<br>**OHIO'S HIGH SCHOOL CRITICAL AREAS OF FOCUS**<br>• Math 1, Number 5, pages 9-10<br><br>**CONNECTIONS ACROSS STANDARDS**<br>• Understand congruence in terms of rigid motion (G.CO.6-8).<br>• Prove and apply geometric theorems (G.CO.9).<br>• Make formal geometric constructions (G.CO.12).<br>• Justify the slope criteria for parallel and perpendicular lines (G.GPE.5).<br>• Reason quantitatively (N.Q.2-3). |

# INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.CO.1-5)

## Instructional Strategies

*Note: The model curriculum is not a collection of lessons nor a full curriculum; it does not suggest pace, sequence, or amount of time spent on topics. It provides information about a topic related to the standards including ideas for examples, strategies for teaching, common misconceptions, and possible connections between topics. In addition, the emphasis of the model curriculum is on instruction not on assessment; therefore, the examples in the Instructional Strategies section are written for instructional purposes not necessarily as assessment items. Examples may need to be rewritten to accommodate the needs of each individual classroom.*

***Standards for Mathematical Practice***
*This cluster focuses on but is not limited to the following practices:*
***MP.1*** *Make sense of problems and persevere in solving them.*
***MP.4*** *Model with mathematics.*
***MP.5*** *Use appropriate tools strategically.*
***MP.6*** *Attend to precision.*

There are many approaches to teaching the concepts of congruence and similarity. The standards define congruence and similarity in terms of transformations. This allows the concepts to be grounded in hands-on experiences. In Grade 8, students should have used many manipulatives to *explore* geometric concepts and definitions. Now in high school, students will use transformations to *prove* geometric concepts and experience definitions. Rigid motions will be used to lay the foundation of proving theorems.

Effective teaching requires students to use a variety of geometric tools, such as graph paper, transparencies, tracing paper, dynamic geometry software, straightedge, compass, and protractor to obtain images of a given figure under specified transformations both on and off the coordinate plane. Different tools lead to different understandings. For example, transparencies are especially useful because two copies of the plane can be seen (the paper and the transparency). The original piece of paper can be easily seen as the domain and the transparency can be seen as the range. The preservation of distance and angle can also be clearly seen using a transparency since the transparency is not torn, stretched, or distorted. Emphasize to students that a transformation acts on the entire plane mapping each point to another corresponding point.

### VAN HIELE CONNECTION

In Math 1 students are expected to move from Level 1(Analysis) to Level 2 (Informal Deduction/Abstraction). In van Hiele Level 1 students use coordinates to examine transformations and recognize, use, and describe properties of transformations. Whereas van Hiele Level 2 can be characterized by the student doing some or all of the following:

- analyzing relationships within and between transformations;
- algebraically expressing transformations: reflections about the *x*- and *y*-axes, and rotations of 45°, 90°, and 180° about the origin; *Note: In high school, students should be given an opportunity to experience rotations around any point (not just the origin) and onto any angle, not just multiples of 90. Also, reflections are not limited to the x-and y-axes but could be about any line.*
- understanding and justifying how to use perpendicular bisectors of corresponding points to locate the turn center of a rotation;
- understanding how the composition of two reflections can be a translation or a rotation; and/or
- following a rationale if shown proof.

See the van Hiele pdf on ODE's website for more information about van Hiele levels.

# INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.CO.1-5)

## DEFINING GEOMETRIC TERMS

One of the major differences between geometric concepts in middle school and geometry concepts in high school is the usage of precise language and exact definitions. Standard G.CO.1 fits nicely alongside the cluster G.CO.12-13 where students make geometric constructions. Remind students that although they may be familiar with many geometric concepts, those taught in high school require greater precision with respect to definitions.

TIP!

Many students have trouble distinguishing between undefined terms, definitions, properties, axioms, formulas etc. Helping students understand the difference between them will aid in the use of precise language.

Explain to students that multiple definitions can exist for a given object. For example, both a rectangle with all equal sides and a rhombus with four right angles are definitions that can be used to define a square. A correct definition will encompass all examples of squares and exclude all examples of non-squares.

### Precise Definitions Based Upon Undefined Notions

Some definitions in mathematics are precisely defined, while others are built upon undefinable concepts (notions) such as a point, line, plane, distance along a line, betweenness, space, and arc length etc. They are concepts (although representations can be drawn) that only exist as an idea or mental image. For example, a point is a location in space. It has no dimension (length, area, or volume). Although, to convey the idea of a point, it is typically drawn (thus giving it a perceived dimension). Undefinable concepts are the building blocks needed for creating precise definitions.

# INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.CO.1-5)

***EXAMPLE***

**Comparing Dots and Points**

- How do you draw a point?
- How big should the representation of the point be?
- What is the biggest dot you could use to represent a point?
- What is the smallest dot you could use to represent a point?
- What are the advantages and disadvantages to using different size dots to represent a point?
- What do you feel is the ideal sized dot used to represent a point? Explain.

*Discussion:* Students often confuse a point with a dot. A point is a location in space. It has no dimensions such as length, width, volume, or area. Although, it is usually represented by a dot, it is not the dot as dots have dimensions. Push students towards abstracting a point by comparing different dots that students make to represent a point. Discuss how the bigger the dot, the less precise it is as far as determining location. However, if a dot is drawn too small, it becomes hard to read.

Some geometric assumptions are—

- Through any two distinct points there is exactly one line that contains both.
- Through at least three noncollinear points, there is exactly one plane that contains all three.
- There exists a real number distance for every pair of points. The distance from $A$ to $B$ is equal to the distance from $B$ to $A$. If the distance between the two points is 0, then the points coincide. Otherwise the distance must be greater than 0.
- Every line may have a coordinate number line associated with it.

See EngageNY, Geometry, Module 1, Topic G, [Lesson 33: Review of the Assumptions] for a list of all geometric assumptions.

**TIP!**

Challenge students to find and correct imprecise definitions in resources such as websites, worksheets, textbooks, etc.

Students come to high school with less precise definitions. Push them to extend their preconceived definitions to more precise definitions. For example many students enter high school defining parallel lines as lines that do not intersect. Push them to formalize their understandings of parallel lines by using the Parallel Line Postulate: For a line and a point, not on the line, there is exactly one line parallel to the line through the point which would be coplanar.

In addition, students come to high school with the notion that perpendicular lines intersect at 90° angles and that parallel lines have the same slope. These are not incorrect; however, at the high school level, they should build toward the following definitions: Two lines are perpendicular if the four angles formed by their intersection are congruent, and two lines in the same plane are parallel if they have no points in common.

# INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.CO.1-5)

Students may have difficulty interpreting expressions that contain symbols such as ⊥ or ∥. Students often incorrectly interpret these symbols as operators instead of as an indicator of a relationship, whereas still others have difficulty expressing the meaning of the symbol. It may be helpful to use written words when introducing notation along with the symbol "line $m$ is perpendicular to line $n$ ($m \perp n$), so students associate the symbols with the words. Eventually, students can be weaned off the written expression.

A good geometric definition—

- names the term being defined;
- classifies the term and differentiates it from others in a similar class;
- uses precise language;
- stands against counterexamples;
- uses only previously defined terms;
- uses proper symbolism; and
- is reversible.

Give each student in the class a term to define. As they present their definition, encourage their classmates to challenge their definitions using counterexamples. Stress the idea that every single word in the definition matters and cannot be disregarded.

***EXAMPLE***

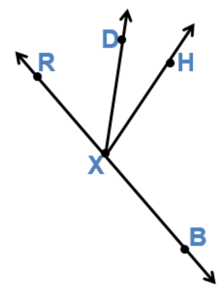
**Precise Definitions**

Carol gets the word "supplementary" to define.

*Discussion:*

- For example, Carol defines supplementary angles as "Supplementary angles are angles that add up to 180°." Mike draws the picture on the right and asks "Is this what you mean?"
- Carol then revises her definition based on Mike's challenge: "Supplementary angles are **two** angles that add up to 180°."

Since $m\angle RXD + m\angle DXH + m\angle HXB = 180°$, therefore they are supplementary angles.

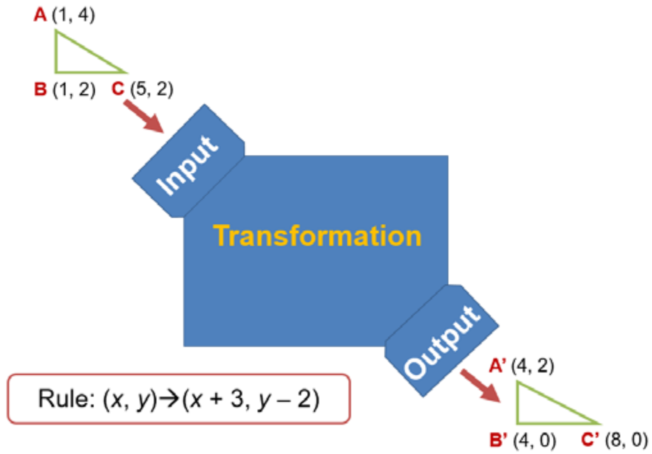



Review vocabulary associated with transformations (e.g. point, line, segment, angle, circle, polygon, parallelogram, perpendicular, rotation, reflection, translation, parallel, arc length, and ray). Be sure that students are aware of the differences between defined terms, undefined terms, properties, formulas etc.

*Betweenness* is the quality or state of being between two others in an ordered mathematical set. This allows us to state if on the line a point $X$ is between two points $A$ and $B$, the length of the line segment $AX$ plus the length of the line segment $XB$ will equal the length of the line segment $AB$, which is presented as Segment Addition Postulate. Once betweenness has been established, it can be used as a foundation for Segment Addition Postulate.

# INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.CO.1-5)

## TRANSFORMATIONS AS FUNCTIONS




A transformation, $T$, is a function that assigns to each point $P$ of the plane a unique point $P'$such that $P' = T(P)$. A rigid motion $R$ is a transformation that maps any pair of distinct points $P$ and $Q$ of the plane onto a pair of distinct points $P'$ and $Q'$ of the plane, so that $P' = R(P)$, $Q' = R(Q)$ and $\overline{PQ} = \overline{P'Q'}$. *Note: Throughout the Model Curriculum different notations are used such a function notation and prime notation etc. As each districts' resources are different, it is up to each district to determine the notation that students need to use. It may be helpful to show students a variety of notations, so they can be mathematically literate when seeing different notations in different resources/courses/schools.*

Connect geometric transformations to function transformations that students learn in Algebra standards. Make the transition from transformations as physical motions to the concept of a function that takes all points in the plane as inputs and give other points as outputs. The correspondence between the original points and their final corresponding points determines the transformation. A function machine may be useful to illustrate the similarities and differences. Point out that the rule maps all points on the figure as well as the plane, not just the vertices.

**TIP!** In an algebraic function, the *x*-coordinate is the independent variable and the *y*-coordinate is the dependent variable. However, in a geometric transformation function all the ordered pairs (that include both the *x*- and *y*-coordinates) that make up the preimage are considered the independent variable (of which there are an infinite amount) and all the ordered pairs that make up the image (of which there are an infinite amount) are considered the dependent variable. This is a shift for many students.

| Preimage ABC (x, y) | Image A'B'C' (x + 3, y – 2) |
|---|---|
| (1, 4) | (4, 2) |
| (1, 2) | (4, 0) |
| (5, 2) | (8, 0) |
| (1, 2.25) | (4, 0.25) |
| (1.00001, 2) | (4.00001, 0) |
| (4.84, 2) | (7.84, 0) |
| (1, 3.9999) | (4, 1.9999) |

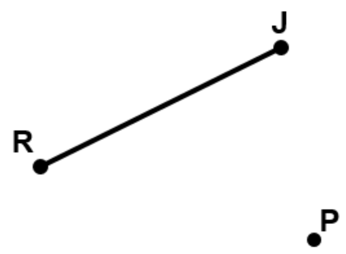
Teachers can help students connect input/preimage to domain and output/image to range. Draw attention to the fact that the image consists of all points that lie on the figure, not just the vertices.

Making constructions using a straightedge and a compass can highlight functional aspects of transformations.

# INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.CO.1-5)

***EXAMPLE***

**Highlighting Functional Aspects of a Translation**




**a.** Given line segment $\overline{RJ}$ and a point $P$ that does not lie on $\overline{RJ}$, construct a line through $P$ that is parallel to $\overline{RJ}$, using a compass and straight edge.
**b.** Mark off the distance $RJ$ along the new parallel line containing $P$ and label it $R'J'$.
**c.** Describe why $\overline{R'J'}$ is a translation of $\overline{RJ}$.
**d.** How are $\overline{RJ}$ and $\overline{R'J'}$ related to the concept of a function?

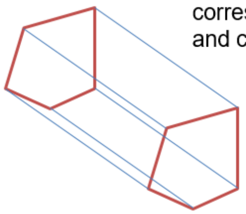
## DEFINITIONS OF RIGID MOTIONS

A rigid motion is a function that keeps distances unchanged. Using hands-on experiences in Grade 8, students should come to high school Geometry with at least three assumptions about rigid motions (translations, reflections, rotations):

1. They map lines to lines, rays to rays, and segments to segments.
2. They preserve distance.
3. They preserve angle measure.

Provide both individual and small-group activities, allowing adequate time for students to explore and verify conjectures about rigid motions and develop precise definitions of rotations, reflections, and translations both on and off the coordinate plane. Students should come to the realization that translations move points a specified distance along parallel lines; rotations move points along a circular arc with a specified center and angle, and reflections move points across a specified line of reflection so that the line of reflection is the perpendicular bisector of each line segment connecting corresponding pre-image and image points.

Students should visualize rigid motions in terms of these definitions not just coordinate rules. For example, they could use perpendicular lines and distance to identify a reflection. Direct students to pay careful attention to properties of figures that are preserved. For example, as the result of a translation, a line segment is both parallel and congruent to its corresponding line segment in its pre-image.

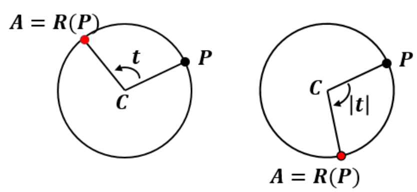


In this two-dimensional translation, all blue lines connecting the corresponding vertices are parallel and congruent.

# INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.CO.1-5)

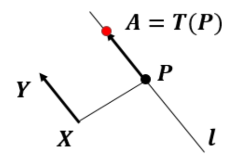
When students use coordinate notation such as ($x$, $y$)→( –$x$, $y$), they may incorrectly believe that –$x$ is always negative instead of understanding that it indicates the opposite sign of the *x-coordinate*. See Model Curriculum 7.NS.1-3 for more common errors with negative numbers.

**Definition of Rotation**



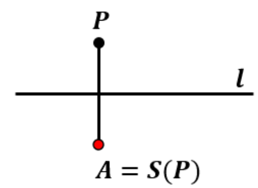

The rotation $R$ around the Point $C$ through the angle $t$ is a rigid motion takes a point $P$ to the point $A = R(P)$ as follows. If $P = C$, then $R(P) = C$. If $P \neq C$ and $t \geq 0$, then $A$ is on the circle with center $C$ and radius $|CP|$ so that $\angle PCA = t°$ and $A$ is counterclockwise from $P$. If $t < 0$, we rotate clockwise by $|t|°$.

**Definition of Translation**




The translation $T$ along the directed line segment $\overrightarrow{XY}$ is a rigid motion takes the point $P$ to the point $A = T(P)$ as follows: The line $l$ passing through $P$ and parallel to line through $X$ and $Y$. Point $A$ is the point on $l$ so that the direction from $P$ to $A$ is the same as the direction from $X$ to $Y$ and so that $|PA| = |XY|$.

**Definition of Reflection**



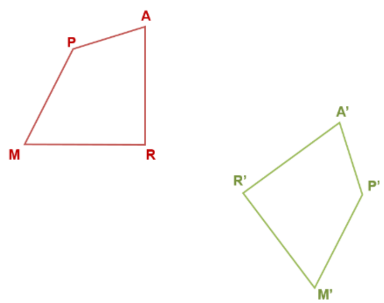

The reflection $S$ across the line $l$ is a rigid motion that takes each point on $l$ to itself, and takes any other point $P$ to the point $A = S(P)$ which is such that $l$ is the perpendicular bisector of the line segment $PA$.

# INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.CO.1-5)

***EXAMPLE***

**Lines of Reflection**

Draw a line of reflection for the figure PARM and its reflected image P'A'R'M'.




*Discussion:* A task like this reinforces that the line of reflection is equidistant from corresponding vertices of the two figures. To find the line of reflection students should connect the vertices of the original figure and its image, and then find the perpendicular bisector of the connecting lines using a compass and straight edge. Once students have practice finding the perpendicular bisector without coordinates, they can move toward using coordinates and the midpoint formula. Explain to students that coordinates are used to represent rigid motions as functions that map points in the plane to other points in the plane. See 8.G.1-4 Model Curriculum for a similar example using the coordinate plane.

In Grade 8 students should have used transparencies or tracing paper to have figures translate onto one another. Some students may need to review these concepts with transparencies. Emphasize that the plane (transparency or tracing paper) is moving and not the figure. In Grade 8, students described translations in terms of horizontal and vertical movements. Move students toward an informal understanding of a vector that is represented by a directed line segment. A vector, where one of its points designated as the starting point and the other as the ending point, shows the direction and the amount of translations (magnitude) of a plane.

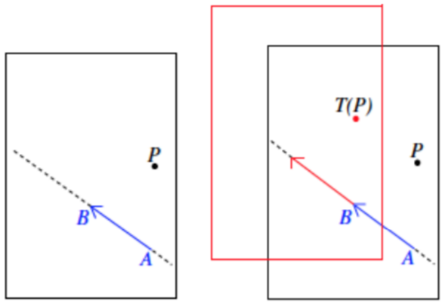
# INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.CO.1-5)

***EXAMPLE***

**Translating a Point**

Use a transparency or tracing paper to perform a translation of a point.

- **a.** Draw point $P$ and a dotted line $AB$ on a piece of paper representing a plane. (Remember a plane extends infinitely in all directions.) Draw vector $\overrightarrow{AB}$ on top of the dotted line $AB$.
- **b.** Place a transparency or tracing paper over the piece of original diagram from part **a**. Trace $P$ and the vector $\overrightarrow{AB}$ in another color such as red.
- **c.** Slide the transparency along line $AB$ moving in the direction from $A$ to $B$ noting the distance from $A$ to $B$ fixed. The translation, $T$, moves the given point $P$ to its image $T(P)$ denoted by the red dot.
- **d.** Repeat the process using geometric figures in place of $P$ and different vectors.
- **e.** Draw attention to the fact that $Translation_{\overrightarrow{AB}}$ is different than $Translation_{\overrightarrow{BA}}$ since the direction is different.




From EngageNY Grade 8, Module 2, Lesson 2.

*Discussion:* Connect translating along a vector to the Pythagorean Theorem and the Distance Formula. Previously in Grade 8 a student might translate a figure 4 units to the right and 3 units up. This is the same as moving 5 units at an approximate 36.87° angle. Once students have an understanding of how a vector is a transformation of a plane that maps each point on the plane to its image, have students discover that the segment with endpoints $P$ and $T(P)$ is parallel and congruent to vector $\overrightarrow{AB}$. Once they are able to make these connections to the particular segments, help them to realize that these properties hold true for all translations.

Provide real-world examples of rigid motions using precise definitions (e.g. a Ferris wheels for a rotation and arc length; mirrors for a reflection; moving vehicles for a translation).

In a rotation the center point is fixed, and the rest of the plane moves the same number of degrees along the arc of the circle. Students need to make a connection between a rotation, as a rigid motion, and the circular movement with the center at point $C$ that rotates any point $P$ in a way that $\overline{CP} \cong \overline{CP'}$ that are the radii of a circle. In high school, students should be given an opportunity to experience rotations around any point (not just the origin) and onto at any angle, not just a multiple of 90.

Rotations should be addressed again when looking at arc length and sector area. For example rotating a point on a circle creates an arc or rotating a radius creates a sector.

# INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.CO.1-5)

Translations and rotations preserve orientation (Orientation is a geometric notion that in two-dimensions allows one to say when a figure goes around clockwise or counterclockwise, and in three-dimensions when a figure is left-handed or right-handed. This is different than the common usage of orientation); reflections do not. Sometimes students want to incorrectly say that a rotation changes the orientation of the shape because the shape has been turned to create a shape that looks different. This is not an orientation change because the vertices can still be named in a clockwise fashion. However, in a reflection, the order of vertices is reversed, thus the orientation is not preserved.

Some students may struggle to use the correct center of rotation if they have memorized the rule $(x, y) \rightarrow (y, -x)$. However this rule only holds true for 90° clockwise rotations when the center of rotation is the origin. Provide students with problems that include a variety of centers of rotation and using angles other than multiples of 90°.

## TRANSFORMATIONS THAT PRESERVE DISTANCE AND THOSE THAT DO NOT

Focus attention on the attributes (e.g. distances or angle measures) of a geometric figure that remain constant under various transformations. Also explore transformations that do (dilations) and do not (horizontal or vertical stretch) preserve angle measure and/or length. For example, use control points on electronic pictures and work with stretching and cropping images.

It may be useful to connect the idea that a translation is a sequence (a composition) of two reflections over parallel lines and a rotation is the composite of two reflections over intersecting lines.

TIP!

Connect transformations, rigid motions, and symmetries to works of art. M.C. Escher has examples of translations, reflections, and/or rotations in some of his pieces such as "Two Birds," "Fish," "Clown," "Lizard," or "Three Birds." Examples of his work can be found at http://www.mcescher.com/gallery/switzerland-belgium/

Discuss the concept of symmetry and how it connects to the symmetry of graphs of functions and shapes of data displays.

TIP!

# INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.CO.1-5)

***EXAMPLE***

**Finding the Center or Rotation**

Find the center of rotation of the image using a compass and straightedge, transparency, folding, or another method. Explain how you found it.

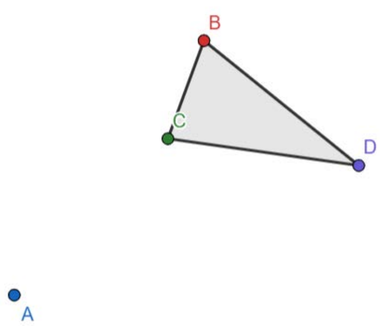

## REPRESENTING AND DRAWING TRANSFORMATIONS

Remember a basic rigid motion $F$ is a rule so that, for each point $P$ of the plane, $F$ assigns (or moves) a point $F(P)$ to $P$ where the distance between the two points are preserved. Emphasize the importance of understanding a transformation as the correspondence between initial and final points, rather than the physical motion. Be precise with respect to the descriptions of sequences of rigid motions. This differentiates high school Geometry from Grade 8. For rotations students must state the center and angle of rotation. For reflections student must state the line of reflection. For a translation, students must state the direction and the distance.

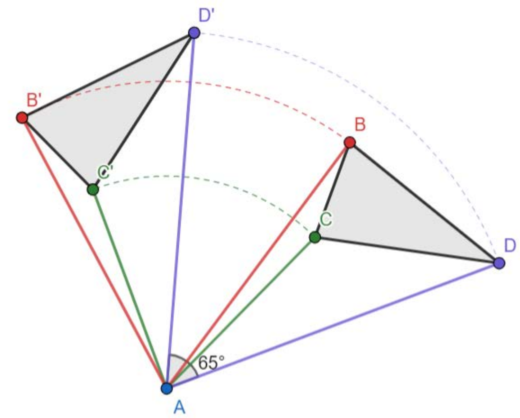
***EXAMPLE***

**Rotating a Figure**

Rotate triangle $BCD$ 65° counterclockwise around point A.




*Discussion:* Using a straight edge, create a line segment containing points $A$ and $D$. Using a protractor measure 65° counterclockwise and make a mark. Sketch a light line that goes through point $A$ and the new mark. Then use a compass to make an arc centered at $A$ and containing point $D$. The intersection point of the sketched 65° line and the arc containing $D$ is $D$'. Repeat this process for points $B$ and $C$. Finish by connecting the vertices of the triangle.

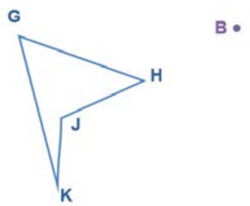


# INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.CO.1-5)

***EXAMPLE***

**Rotating a Figure**

Rotate quadrilateral $GHJK$ 125° clockwise around point $B$.



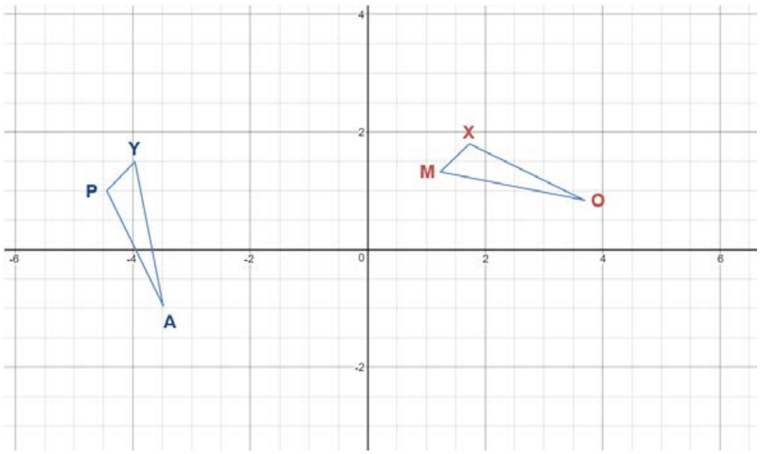

*Discussion:* Draw students' attention to the fact that 125° clockwise rotation is the same as a –235° rotation.

Provide students with a pre-image and a final image, and ask them to describe the steps required to generate the final image both on and off the coordinate plane. Show examples with more than one answer (e.g. a reflection might result in the same image as a translation).

***EXAMPLE***

**Describing Steps for a Transformation**

Describe the steps required to transform the image $\triangle YAP$ onto $\triangle MOX$. Be precise in your description.




*Discussion:* One method that could be used to transform triangle $YAP$ to triangle $MOX$ is a 90° rotation around point $A$ (–3.5, –1), and then a reflection across *x* = –1 and then a translation (*x*, *y*)→(*x* + 1, *y* + 2). Students may use a variety of methods. To indicate corresponding points, students may wish to use function notation.



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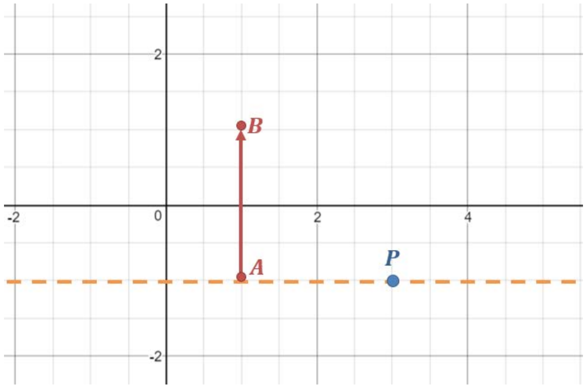
# INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.CO.1-5)

***EXAMPLE***

**Analyzing a Transformation**




**a.** Does it matter if you translate point $P$ along $\overrightarrow{AB}$ and then reflect it across $y = -1$ or if you first reflect point $P$ across $y = -1$ and then translate it along the vector AB? Explain.

**b.** Are sequences of translations commutative? Explain.

Work backwards to determine a sequence of transformations that will map one figure onto another of the same size and shape both on and off the coordinate plane.

## SYMMETRY

A symmetry of a figure is a basic rigid motion that maps the figure back onto itself. Students in Ohio no longer have any explicit learning standards on symmetry until this course, so students may or may not have an understanding of the concept.

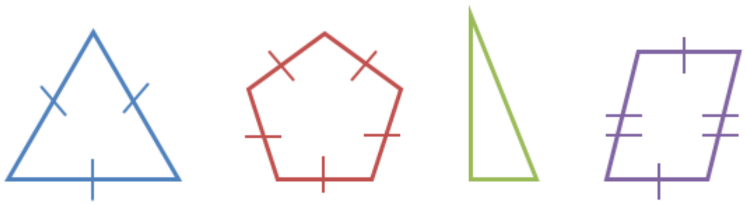
Analyze various figures (e.g. regular polygons, folk art designs, or product logos) to determine which rotations and reflections carry (map) the figure onto itself. These transformations are the "symmetries" of the figure. Use symmetry in proofs of properties of geometric figures. For example, use symmetry to prove that the diagonals of a rectangle are congruent. Include finding the line of symmetry on the coordinate plane (for a rectangle, for example). Include point symmetry as a subset of rotational symmetry. Remind students that if the figure has no symmetry other than 360° rotation, the figure has no rotational symmetry.

Transparencies work well to represent the symmetries of a figure because they keep the original and transposed figures on separate planes. Geometry software, although very useful in many ways in the classroom, requires students to imagine this concept, whereas transparencies are more concrete.

# INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.CO.1-5)

***EXAMPLE***

**Lines of Symmetry**



**a.** Use a compass and straightedge to draw a line of symmetry onto each object. Sketch in any remaining lines of symmetry.
**b.** Justify that your lines are truly lines of symmetry.
**c.** How do you know that you found all the lines of symmetry?

## Instructional Tools/Resources

*These tools and resources may be helpful for instruction. Before use, they should be evaluated at the local level to determine their appropriateness for instruction.*

**Manipulatives/Technology**

- Tracing paper (patty paper)
- Transparencies
- Graph paper
- Ruler
- Protractor
- Compass
- Mira
- Computer dynamic geometry software (such as Geometer's Sketchpad®, Desmos, Cabri®, or Geogebra®).
- Graphing calculators and other handheld technology such as TI-Nspire™.
- *Flatland: A Romance of Many Dimensions* by Edwin Abbott.

## INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.CO.1-5)

**Analyzing Definitions**

- Defining Parallel Lines by Illustrative Mathematics is a task that has students examine and analyze three different definitions for parallel lines.
- Defining Perpendicular Lines by Illustrative Mathematics is a task that has students examine and analyze three different definitions for perpendicular lines.
- Practice: Geometric Definitions by Khan Academy has students analyze three definitions and match teacher's critiques to the definitions.
- Defining Rotations by Illustrative Mathematics is a task that encourages students to be precise in their definitions.
- Defining Reflections by Illustrative Mathematics is a task that has students compare and contrast different definitions.

**Connecting Functions in Geometry to Algebra**

- Connect Geometry and Algebra Through Functions by Technologically Embodied Geometric Functions is a webpage with interactive activities connecting *Flatland* with Geometry and Algebraic Functions.
- Transformations as Functions by MathBitsNotebook is an informational page about transformations as functions.

**Comparing Functions that Preserve Distances to Those that Do Not**

- Horizontal Stretch of the Plane by Illustrative Mathematics is a task that has students compare transformations of those that preserve distance and angles and those that do not.

**Analyzing Rigid Motions**

- Seven Circles II by Illustrative Mathematics is an instructional task that has students analyze rigid motions.
- Symmetries of a Quadrilateral I and Symmetries of a Quadrilateral II by Illustrative Mathematics is a task that has students examine rigid motions in the context of a quadrilateral.

**Representing and Drawing Rigid Motions**

- Showing a Triangle Congruence: the General Case and Showing Triangle Congruence: A Particular Case by Illustrative Mathematics are tasks that have students compare different transformations to obtain an image from an pre-image.
- Transformation Golf: Rigid Motions has students use transformations to play a round of golf.

# INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.CO.1-5)

**Curriculum and Lessons from Other Sources**

- EngageNY, Grade 8, Module 2, Topic A, Lesson 2: Definition of Translation and Three Basic Properties and Lesson 3: Translating Lines are lessons that pertain to this cluster.
- EngageNY, Geometry, Module 1, Topic A, Lesson 1: Construct an Equilateral Triangle is a lesson that pertains to this cluster.
- EngageNY, Geometry, Module 1, Topic C, Lesson 12: Transformations—The Next Level, Lesson 13: Rotations, Lesson 14: Reflections, Lesson 15: Rotations, Reflections, and Symmetry, Lesson 16: Translations, Lesson 17: Characterize Points on a Perpendicular Bisector are lessons that pertain to this cluster.
- Georgia Standards of Excellence Curriculum Frameworks, Geometry, Unit 1: Transformations in the Coordinate Plane has many tasks that pertain to this cluster.
- Mathematics Vision Project, Geometry, Module 1: Transformations and Symmetry has many tasks that pertain to this cluster.

**General Resources**

- Arizona 7-12 Progression on Geometry is an informational document for teachers. This cluster is addressed on pages 13-15.
- High School Coherence Map by UnboundED is a map of the relationships among high school mathematics standards. Please note: This resource is aligned to the Common Core not Ohio's Learning Standards. Although the tool is useful, it needs to be checked with respect to Ohio's Learning Standards.
- van Hiele Model of Geometric Thinking is a pdf created by ODE that summarizes the van Hiele levels.

**References**


- Common Core Standards Writing Team. (2016, March 24). *Progressions for the Common Core State Standards in Mathematics (draft). Grades 7–8, HS, Geometry*. Tucson, AZ: Institute for Mathematics and Education, University of Arizona.
- Edwards, B. & Ward, M. (2008.) The role of mathematical definitions in mathematics and in undergraduate mathematics course. *Making the Connection: Research to Practive in Undergraduate Mathematics Education*, 73, 223-232. Retrieved from https://pdfs.semanticscholar.org/2fca/dfddcc2e87d92ce0e80079863050aaa887a4.pdf
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- Keiser, J. (2000). The role of definition. *Mathematics Teaching in the Middle School*, 5*(6),* 506-511.
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## INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.CO.1-5)

**References, continued**


- Wu, H. (2013). Teaching geometry according the common core standards. Retrieved from https://math.berkeley.edu/~wu/Progressions_Geometry.pdf

| STANDARDS | MODEL CURRICULUM (G.CO.6-8) |
|---|---|
| **Geometry**<br>**CONGRUENCE**<br>**Understand congruence in terms of rigid motions.**<br>**G.CO.6** Use geometric descriptions of rigid motions[G] to transform figures and to predict the effect of a given rigid motion on a given figure; given two figures, use the definition of congruence in terms of rigid motions to decide if they are congruent[G].<br>**G.CO.7** Use the definition of congruence in terms of rigid motions to show that two triangles are congruent if and only if corresponding pairs of sides and corresponding pairs of angles are congruent.<br>**G.CO.8** Explain how the criteria for triangle congruence (ASA, SAS, and SSS) follow from the definition of congruence in terms of rigid motions. | **Expectations for Learning**<br>In middle school, students understand congruence through a sequence of basic rigid motions (reflections, rotations, and translations). In this cluster, students will build on this knowledge to prove that two figures are congruent if there is a sequence of rigid motions carrying one onto the other. The triangle congruence criteria can then be established using the definition of congruence in terms of rigid motions. This is the time when students are first exposed to the criteria for triangle congruence; students should know and be able to use AAS, ASA, SAS, and SSS and understand that the criteria follow from rigid motions.<br><br>The student understanding of this cluster begins at van Hiele Level 1 (Analysis) and moves to Level 2 (Informal Deduction/Abstraction).<br><br>**ESSENTIAL UNDERSTANDINGS**<br>• Two figures are defined to be congruent if one can be mapped onto the other by rigid motions.<br><br>**MATHEMATICAL THINKING**<br>• Explain mathematical thinking.<br>• Recognize, apply, and justify mathematical concepts, terms, and their properties.<br>• Represent concepts symbolically.<br>• Use formal and informal reasoning.<br>• Use accurate and precise mathematical vocabulary.<br><br>**INSTRUCTIONAL FOCUS**<br>• Use rigid transformations to determine if the figures are congruent<br>• Given congruent triangles, describe the rigid transformations that map one triangle onto the other<br>• Establish the criteria for triangle congruence (AAS, ASA, SAS, and SSS) in terms of rigid motions.<br>• Know and be able to use triangle congruence (AAS, ASA, SAS, and SSS) in solving problems.<br>*Continued on next page* |

| STANDARDS | MODEL CURRICULUM (G.CO.6-8) |
|---|---|
| | **Content Elaborations**<br>**OHIO'S HIGH SCHOOL CRITICAL AREAS OF FOCUS**<br>• Math 1, Number 5, pages 9-10<br><br>**CONNECTIONS ACROSS STANDARDS**<br>• Experiment with transformations in the plane (G.CO.1-5).<br>• Prove and apply theorems about triangles (G.CO.10).<br>• Prove and apply theorems about parallelograms (G.CO.11).<br>• Reason quantitatively (N.Q.2-3) |

# INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.CO.6-8)

## Instructional Strategies

*Note: The model curriculum is not a collection of lessons nor a full curriculum; it does not suggest pace, sequence, or amount of time spent on topics. It provides information about a topic related to the standards including ideas for examples, strategies for teaching, common misconceptions, and possible connections between topics. In addition, the emphasis of the model curriculum is on instruction not on assessment; therefore, the examples in the Instructional Strategies section are written for instructional purposes not necessarily as assessment items. Examples may need to be rewritten to accommodate the needs of each individual classroom.*

***Standards for Mathematical Practice***

*This cluster focuses on but is not limited to the following practices:*

***MP.3*** *Construct viable arguments and critique the reasoning of others.*

***MP.4*** *Model with mathematics.*

***MP.5*** *Use appropriate tools strategically.*

***MP.6*** *Attend to precision.*

### VAN HIELE CONNECTION

In Math 1 students are expected to move from Level 1(Analysis) to Level 2 (Informal Deduction/Abstraction). In van Hiele Level 1 students use coordinates to examine transformations and recognize, use, and describe properties of transformations. Whereas van Hiele Level 2 can be characterized by the student doing some or all of the following:

- analyzing relationships within and between transformations;
- algebraically expressing transformations: reflections about the *x*- and *y*-axes, and rotations of 45°, 90°, and 180° about the origin; *Note: In high school, students should be given an opportunity to experience rotations around any point (not just the origin) and onto any angle, not just multiples of 90. Also, reflections are not limited to the x-and y-axes but could be about any line.*
- understanding and justifying how to use perpendicular bisectors of corresponding points to locate the turn center of a rotation;
- understanding how the composition of two reflections can be a translation or a rotation; and/or
- following a rationale if shown proof.

See the van Hiele pdf on ODE's website for more information about van Hiele levels.

### CONGRUENCE AND RIGID MOTIONS

In Grade 8 students experimented with congruence. They should have noticed that when rigid motions are performed, size and shape of figures are preserved. They may have even defined congruence as "same size and shape." Although, this idea helps create an intuitive understanding of congruence, high school students need to use a more precise definition. Students should develop the relationship between transformations and congruency and understand that two figures are defined to be congruent if and only if there is a sequence of rigid motions that maps one onto the other.

Students should identify rigid motions (translations, reflections, and/or rotations) that map one figure onto another to determine if two figures are congruent. Allow adequate time and provide hands-on activities for students to visually and physically explore the relationship between rigid motions and congruence. The use of graph paper, tracing paper, and/or dynamic geometry software to obtain images of a given figure under specified rigid motions should be used to achieve this experience.

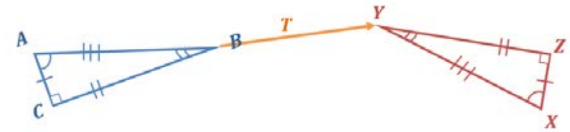
## INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.CO.6-8)

Avoid circular reasoning in establishing the definition of congruency. Define congruence in terms of rigid motions before referring to proof statements about corresponding parts.

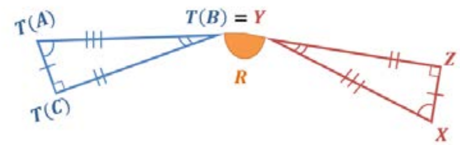
***EXAMPLE***

**Congruency of Rigid Motions**

Prove that $\triangle ABC$ and triangle $\triangle XYZ$ are congruent given that the corresponding side lengths and corresponding angle measures are equal.


*Discussion:* A student may translate $\triangle ABC$ along line segment $BY$, so that $T(B) = Y$. Then, apply a rotation, $R$, clockwise about point $Y$ through the angle $\angle T(C)YX$. Because rigid motions preserve angles and sides lengths, the line segment through points $Y$ and $Z$ coincides with the line segment through points $R(T(B))$ and $R(T(C))$ and the line segment that goes through points $Y$ and $X$ coincides with the line segment that goes through $(T(B))$ and $R(T(A))$ and the line segment that goes through points $X$ and $Z$ coincides with the line segment that goes through points $(T(A))$ and $R(T(C))$. Since all three points coincide, there is a sequence of rigid motions that maps $\triangle ABC$ onto $\triangle XYZ$, and therefore the triangles are congruent. *Note: A point $T(C)$ is the image of point $C$ after it was translated along line segment $BY$. A Point $R(T(C))$ is the image of Point $C$ after it was first translated along line segment $BY$ and then rotated about Point $Y$ on $\angle T(C)YX$. The same logic applies to other points on the figure.*





Work backwards – given two figures that are congruent, find a sequence of rigid motions that will map one onto the other and prove it.

Some students may incorrectly believe that all transformations, including dilations, are rigid motions. Provide opportunities for students to create counterexamples of this misconception, including dilations and vertical and horizontal stretch.

# INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.CO.6-8)

Some students may incorrectly believe that any two figures that have the same area are the result of rigid transformation(s). Students should recognize that the preservation of area does not guarantee the preservation of side lengths and angle measures.

A symmetry is a rigid motion that carries a figure to itself. It is nothing other than a congruence of an object with itself. When a figure has many rigid motions that map a figure onto itself, it is because the symmetries in the objects are being compared.

TIP!

Students may incorrectly believe that two angles cannot be congruent if the rays forming the angles have different lengths or different directions. Show students two congruent angles, one with longer rays than the other. Have students measure the angles to determine congruency. Then discuss that although one appears larger than the other, they are really the same angle measure since rays continue forever.

## CORRESPONDING PARTS OF CONGRUENT FIGURES ARE CONGRUENT

Although there is a relationship between correspondence and congruence, correspondences do not necessarily imply congruency. Correspondence is simply matching. Since any two parts of figures can be compared or matched, not all correspondences imply congruency. The two corresponding figures may or may not have the same measures (angles and side lengths), and they may or may not be able to be mapped using rigid motions, for example similar triangles. An example of this happens in construction when parts of old building have settled and no longer have square corners, yet windows or doors (with square corners) still need to be replaced with new objects. In this case workers need to adjust the matching parts to make the corresponding (not congruent) parts fit. However, the opposite is true: if there is congruence between two figures there is a correspondence between their congruent parts. This is because an image created by a rigid motion produces a one-to-one correspondence between its parts (points, angle measures, and side lengths) because the two figures can be mapped onto each other. Therefore, corresponding parts of congruent figures are congruent.

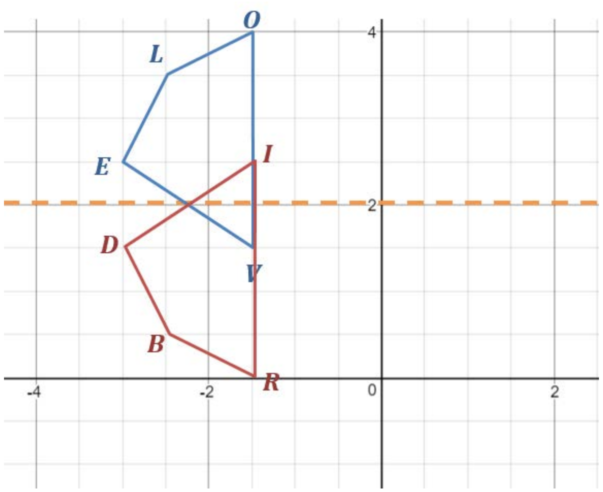
# INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.CO.6-8)

***EXAMPLE***

**Exploring Correspondence and Congruence**

The figure LOVE (shown in blue) has been mapped to the second figure (shown in red) by a reflection across $y = 2$.

- **a.** Identify the corresponding vertices, sides, and angles.
- **b.** State the congruency between the two figures.
- **c.** Explain why you can state that the two figures are congruent.




| Corresponding Vertices | Corresponding Segments | Corresponding Angles | Congruent Figures |
|---|---|---|---|
| | | | |
| | | | Explanation of Congruency: |
| | | | |
| | | | |

*Discussion:* Emphasize to students that the two figures are congruent because they can be mapped onto each other using a rigid motion—not just because their corresponding parts can be located. Other situations, such as similar figures, also have corresponding parts, but are not congruent.

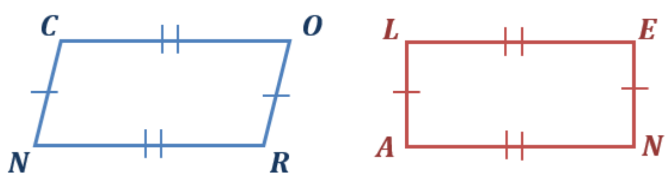
**TIP!**

Some students may believe that it is not important to list corresponding vertices of congruent figures in order; however, it is useful to stress the value of listing corresponding vertices in the same order so that corresponding sides and angles can be easily identified and that included sides or angles are apparent.

# INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.CO.6-8)

***EXAMPLE***

**Exploring Correspondence and Congruence**




**a.** List all the corresponding parts between the two figures so that sides of the first quadrilateral are congruent to the sides of the second quadrilateral.

**b.** Are the two figures congruent? Explain.

*Discussion:* The figures are not congruent. The definition of rigid motions does not apply, because not all the distances between pairs of corresponding points remain equal. Therefore, the corresponding angles are not congruent. The parts will not coincide if quadrilateral CORN is mapped onto quadrilateral LEAN as required by rigid motions.

**Corresponding Parts of Congruent Triangles are Congruent (CPCTC)**

Special attention is given to the corresponding parts of congruent triangles that are congruent usually written as CPCTC. Students should describe rigid motion(s) (translations, reflections and/or rotations) that map one triangle onto another to determine if two triangles are congruent. Compare a given triangle and its image to verify that corresponding sides and corresponding angles are congruent.

**Criteria for Triangle Congruence**

Since basic rigid motions preserve segments, lines, and angles, congruent figures must have all pairs of corresponding equal sides and all pairs of corresponding equal angles. The opposite (converse) is also true: If a figure has all corresponding congruent pairs of sides and all corresponding congruent pairs of angles, the figures are congruent. The same logic applies to triangles; however instead of needing six conditions to prove congruency, there are some shortcuts using only three conditions.

Build on previous learning of transformations and congruency to develop a formal criterion for proving the congruency of triangles. This is the time when students are first exposed to the criteria for triangle congruence; students should know and be able to use ASA, SAS, and SSS and understand that the criteria follow from rigid motions. See the EngageNY, Geometry, Module 1, Topic D lessons listed in the Instructional Resources section for guidance on establishing congruency of triangles.

# INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.CO.6-8)

Students should construct pairs of triangles that satisfy the ASA, SAS, or SSS congruence criteria, and use rigid motions to verify that they satisfy the definition of congruent figures. Investigate rigid motions and congruence both algebraically (using coordinates) and logically (using proofs).

Students should be able to use ASA, SSS, and SAS in formal and informal proofs. Traditionally, AAS and HL are included in the list of congruence theorems. These are special cases of triangle congruence, for example HL is a specific case of SAS for right triangles (via Pythagorean theorem) and AAS is a specific case of ASA (via Third Angle Theorem).

Some students may incorrectly believe that combinations such as SSA or AAA are also a congruence criterion for triangles. Provide opportunities to expose students counterexamples to confront this misconception.

Some students may incorrectly believe that SSS, ASA, and SAS apply to all figures, not just triangles. Provide counterexamples to help students.

## Instructional Tools/Resources

*These tools and resources may be helpful for instruction. Before use, they should be evaluated at the local level to determine their appropriateness for instruction.*

**Manipulatives/Technology**

- Tracing paper (patty paper)
- Transparencies
- Graph paper
- Ruler
- Protractor
- Compass
- Mira
- Computer dynamic geometry software (such as Geometer's Sketchpad®, Desmos, Cabri®, or Geogebra®).
- Graphing calculators and other handheld technology such as TI-Nspire™.

# INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.CO.6-8)

**Triangle Congruency**

- Evaluating Conditions for Congruency by Mathematics Assessment Project has students analyze the truth about conjectures of congruency.
- Why Does SSS Work? by Illustrative Mathematics is a task that has students explore SSS criteria by performing different rigid motions. There is also an attached GeoGebra file to be used with this activity.
- Why Does ASA Work? by Illustrative Mathematics is a task that has students explore ASA criteria.
- Why Does SAS Work? by Illustrative Mathematics is a task that has students explore SAS criteria.
- Side-Angle-Side Congruence by Basic Rigid Motions is a video that explains a proof H. Wu's "Teaching Geometry According to the Common Core Standards" publication.
- When Does SSA Work to Determine Triangle Congruence? by Illustrative Mathematics is a task that has students explore congruency using different criteria to determine congruence.
- SSS Congruence Criterion by Illustrative Mathematics is a task that has students establish SSS congruence using rigid motions.

**Curriculum and Lessons from Other Sources**

- EngageNY, Geometry, Module 1, Topic C, Lesson 19: Construct and Apply a Sequence of Rigid Motions, Lesson 20: Applications of Congruence in Terms of Rigid Motions, Lesson 21: Correspondence and Transformations are lessons that pertain to this cluster.
- EngageNY, Geometry, Module 1, Topic D, Lesson 22: Congruence Criteria for Triangles—SAS, Lesson 23: Base Angles of Isosceles Triangles, Lesson 24: Congruence Criteria for Triangles—ASA and SSS, Lesson 25: Congruence Criteria for Triangles—AAS and HL are lessons that pertain to this cluster.
- Georgia Standards of Excellence Curriculum Frameworks, Geometry, Unit 2: Similarity, Congruence, and Proofs. This cluster is addressed on pages 25-38.
- Mathematics Vision Project, Geometry, Module 2: Congruence, Construction, and Proof has many tasks that pertain to this cluster.

**General Resources**

- Arizona 7-12 Progression on Geometry is an informational document for teachers. This cluster is addressed on pages 13-15.
- High School Coherence Map by UnboundED is a map of the relationships among high school mathematics standards. Please note: This resource is aligned to the Common Core not Ohio's Learning Standards. Although the tool is useful, it needs to be checked with respect to Ohio's Learning Standards.
- van Hiele Model of Geometric Thinking is a pdf created by ODE that summarizes the van Hiele levels.

## INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.CO.6-8)

**References**


- Common Core Standards Writing Team. (2016, March 24). *Progressions for the Common Core State Standards in Mathematics (draft). Grades 7–8, HS, Geometry*. Tucson, AZ: Institute for Mathematics and Education, University of Arizona.
- Eggleton, P. (October 2001). Triangles a la fettuccine: A hands-on approach to triangle-congruence theorems. *Mathematics Teacher*, *94(7),* 534-537.
- Piatek-Jimenz, K. (February 2008). Building intuitive arguments for the triangle congruence conditions. *Mathematics Teacher*, *101(6),* 463-466.
- Wu, H. (2013). Teaching geometry in Grade 8 and high school according to Common Core Standards. Retrieved from https://math.berkeley.edu/~wu/CCSS-Geometry_1.pdf
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| STANDARDS | MODEL CURRICULUM (G.CO.9-11) |
|---|---|
| **Geometry**<br>**CONGRUENCE**<br>**Prove geometric theorems both formally and informally using a variety of methods.**<br>**G.CO.9** Prove and apply theorems about lines and angles. *Theorems include but are not restricted to the following: vertical angles are congruent; when a transversal crosses parallel lines, alternate interior angles are congruent and corresponding angles are congruent; points on a perpendicular bisector of a line segment are exactly those equidistant from the segment's endpoints.*<br>**G.CO.10** Prove and apply theorems about triangles. *Theorems include but are not restricted to the following: measures of interior angles of a triangle sum to 180°; base angles of isosceles triangles are congruent; the segment joining midpoints of two sides of a triangle is parallel to the third side and half the length; the medians of a triangle meet at a point.*<br>*Continued on next page* | **Expectations for Learning**<br>In middle school, students informally define and apply the relationships of lines, angles, triangles, and parallelograms. For this cluster, students now develop conjectures and construct valid proofs about lines, angles, triangles, and parallelograms. They should begin with informal proof and work toward formal proof using a variety of methods including coordinate-based methods. Also, students should apply these relationships to real-world settings and to proofs.<br><br>The student understanding of this cluster begins at van Hiele Level 2 (Informal Deduction/Abstractions) and moves to Level 3 (Deduction).<br><br>*Note: Although Math 1 students may use coordinate-based methods for proofs for all standards in this cluster, it is only required that they use coordinate based methods for G.CO.9. This is due to the movement of G.GPE.4 to Math 2.*<br><br>**ESSENTIAL UNDERSTANDINGS**<br>• The process of proof can vary from informal to formal reasoning.<br>• A proof is a deductive argument that explains why a claim must be true.<br>• Proof can rely on formal and informal language; there are many ways to justify a claim, not all of which rely on technical vocabulary.<br>• Students should demonstrate a knowledge of the content listed in the standards and be able to apply those concepts in various problem solving settings.<br>• Coordinate proof is a method that uses algebraic techniques to prove geometric theorems and properties.<br>*Continued on next page* |

<table>
<tr><th>STANDARDS</th><th>MODEL CURRICULUM (G.CO.9-11)</th></tr>
<tr><td>G.CO.11 Prove and apply theorems about parallelograms. Theorems include but are not restricted to the following: opposite sides are congruent, opposite angles are congruent, the diagonals of a parallelogram bisect each other, and conversely, rectangles are parallelograms with congruent diagonals.</td><td>

<h2>Expectations for Learning, continued</h2>
<h3>MATHEMATICAL THINKING</h3>
<ul>
<li>Explain mathematical thinking.</li>
<li>Recognize, apply, and justify mathematical concepts, terms, and their properties.</li>
<li>Represent concepts symbolically.</li>
<li>Use formal and informal reasoning.</li>
<li>Use accurate and precise mathematical vocabulary.</li>
<li>Plan a solution pathway.</li>
<li>Make and analyze mathematical conjectures.</li>
<li>Solve real-world and mathematical problems accurately.</li>
<li>Create a drawing and add components as appropriate.</li>
</ul>
<h3>INSTRUCTIONAL FOCUS</h3>
<ul>
<li>Form conjectures about geometric relationships and examine their validity, providing evidence to support or refute the claim.</li>
<li>Using previously established facts about lines, angles, triangles, and parallelograms, construct a valid argument for why a conjecture is true or not true.</li>
<li>Solve problems involving lines, angles, triangles, and parallelograms by applying theorems.</li>
</ul>
<h2>Content Elaborations</h2>
<h3>OHIO'S HIGH SCHOOL CRITICAL AREAS OF FOCUS</h3>
<ul>
<li>Math 1, Number 5, pages 9-10</li>
</ul>
<h3>CONNECTIONS ACROSS STANDARDS</h3>
<ul>
<li>Experiment with transformations in the plane (G.CO.1, 3, 4).</li>
<li>Understand congruence in terms of rigid transformations (G.CO.6-8).</li>
<li>Use coordinates to prove simple geometric theorems algebraically (G.GPE.5).</li>
<li>Reason quantitatively (N.Q.1-3).</li>
<li>Create equations that describe relationships (A.CED.1-2).</li>
<li>Reason with equations by explaining steps (A.REI.1, 3).</li>
</ul>
</td></tr>
</table>

# INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.CO.9-11)

## Instructional Strategies

*Note: The model curriculum is not a collection of lessons nor a full curriculum; it does not suggest pace, sequence, or amount of time spent on topics. It provides information about a topic related to the standards including ideas for examples, strategies for teaching, common misconceptions, and possible connections between topics. In addition, the emphasis of the model curriculum is on instruction not on assessment; therefore, the examples in the Instructional Strategies section are written for instructional purposes not necessarily as assessment items. Examples may need to be rewritten to accommodate the needs of each individual classroom.*

***Standards for Mathematical Practice***

*This cluster focuses on but is not limited to the following practices:*

***MP.1** Make sense of problems and persevere in solving them.*

***MP.2** Reason abstractly and quantitatively.*

***MP.3** Construct viable arguments and critique the reasoning of others.*

***MP.5** Use appropriate tools strategically.*

***MP.6** Attend to precision.*

***MP.7** Look for and make use of structure.*

A proof is an argument that demonstrates that a statement is always true. A collection of examples is not a proof unless the examples exhaust all possible cases.

The purposes of proof are to verify and explain results, promote discovery, and to communicate and formalize truth. Although proof does not end in the knowledge of the truth but goes further to explain the "why" something is true. The idea is to communicate truth (of mathematics or other disciplines) to skeptics. A student will continue to use proof, verification, and justification throughout not only his/her math career, but throughout life.

Proofs are beneficial forms of communication when they convince others *that* a statement must be true. Proofs are especially useful when they explain *why* the statement must be true. Proofs are usually organized into steps, each of which follows from previous steps and from accepted theorems, definitions, and assumptions. Proofs can vary in level of formality and amount of detail. Sometimes, for example, a brief sketch of a proof is more informative than a formal proof that attends precisely to every detail. At other times, it might be important to check every step very carefully.

Proofs can be presented in various formats, including paragraphs, flow charts, or two columns. In geometry, proofs can be based on transformations, triangle congruence, or coordinates. Geometry proofs can make use of algebra; algebra proofs can make use of geometry. This sort of diversity is worth encouraging, for understanding is more stable and flexible when it is informed by multiple perspectives.

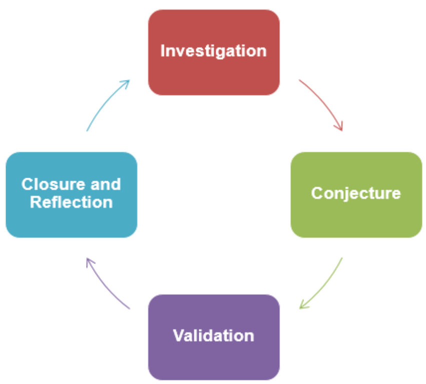
When aiming to prove a statement, it helps to reframe it in the form, "If A then B." For example,

| Statement | "If A then B" form |
|---|---|
| The sum of two odd numbers is even. | If two numbers are odd, then their sum is even. |
| Opposite sides of a parallelogram are congruent. | If a figure is a parallelogram then its opposite sides are congruent. |

# INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.CO.9-11)

In this form, A is called the hypothesis, and B is called the conclusion. The idea is, "Whenever (hypothesis) A is true, it follows that (conclusion) B must also true." Proofs may consist of one or more statements. Each justified statement follows from the previous one(s). Each conclusion may be supported either by a particular definition, a particular postulate, a particular theorem, particular property, etc.

**Process of Proving**




Uncertainty drives proof and makes it meaningful. Avoid asking students to prove a routine theorem they already know to be true for it can be demotivating. Instead provide them with tasks that need proofs, so they can participate in the process of proving as they have a reason to provide a proof. Give students an opportunity to investigate figures. Have them list things that they are most likely true, most likely false, or things that tend to be true or false. Use their observations as a launching pad into proving different theorems. Point out that since nothing has yet to be proven, students are just making conjectures based on their observations. Challenge students when they give imprecise conjectures or definitions. (This cluster should reemphasize the precise use of definitions in G.CO.1.) It may be more useful to highlight proofs that promote understanding.

A conjecture is an educated guess or opinion. To tell whether a conjecture is true or false, students should usually start by examining instances. For conjectures about geometric figures, this means that drawings are made and explored. If even one counterexample is found, the conjecture is not true. If a counterexample is not found, there is evidence that the conjecture is true. Still, for a conjecture to be accepted as true for all cases, it must be proved.

Students may incorrectly think that a conjecture is true because it worked in all examples that were explored, when there may be some types of examples that were not explored, and therefore the conjecture may be false. Teachers should challenge students to find another method (instead of using examples) that is valid.

Students may incorrectly think that they only need to provide an example or two to make a conclusion, but a proof is needed because it covers all possible examples.

## INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.CO.9-11)

### VAN HIELE CONNECTION

In Math 1 students are expected to move from Level 2 (Informal Deduction/Abstraction) to Level 3 (Deduction). Whereas in Level 2 students use informal arguments, Level 3 requires a more formal approach to thinking.

Level 3 can be characterized by the student doing some or all of the following:

- constructing proofs and understanding the necessity of proofs;
- understanding the role and relationships between of axioms, theorems, definitions, and postulates;
- differentiating between necessary and sufficient conditions; and/or
- making logical conclusions.

### PROOFS

Here is general information about proof instruction:

- The main concept that should be gleaned from these standards is that students need to be able to explain reasons for their thinking and why/how something is true, much like in the ELA writing standards where *evidence* must be included for any claim.
- Geometry is visual and should be taught in ways that leverage this aspect. Sketching, drawing and constructing figures and relationships between geometric objects should be central to any geometric study and certainly to proof. (Niven, 1987)
- Exploring the history of geometry and real-world applications may help students develop conceptual understandings before they begin to use formal proof.
- Teachers should select a level of formality that is appropriate for the content, for their students, and the reader of the proof. However, every level of formality includes students' ability to formally/informally reference the appropriate source—a definition, a property, a law, an axiom, a theorem, etc. Make sure that the level of formality does not distract from the main idea of the proof.
- Proof formats could include but are not limited to the following: deductive, inductive, two-column, paragraph, flow chart, visual, and/or counterexample.
- Using dynamic geometry software allows students to make conjectures that can, in turn, be formally proven. For example, students might notice that the base angles of an isosceles triangle always appear to be congruent when manipulating triangles on the computer screen. This could lead to a more formal discussion of why this occurs every single time.
- The emphasis should be on a progression toward proof and not an emphasis on formal proof. Students need to be able to come up with their own conjectures and then provide mathematically sound justification for the conjectures' validity. Ultimately students should be moving toward constructing a complete argument, in a variety of formats, to move from given information to a conclusion.
- Direct instruction is not the best way to introduce formal proof. Instead the focus should "be on justifying ideas while helping students build the visual and empirical foundation for higher levels of geometric thought, we can lead students to appreciate the need for formal proof." (Battista and Clements, 1995)

## INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.CO.9-11)

- A complete argument should include a marked diagram, when applicable. For example, without a marked diagram a student may mistake congruent triangles by AAS instead of ASA.
- Students continue to use precise language and relevant vocabulary to justify steps in their work to construct viable arguments that defend their method of solution.
- All statements need to be examined on their own merit—conditionals and their converse statements are not always both true. For the times when both conditionals and their converses are true, biconditional statements can be written.
- The concepts of inverse and contrapositive are not emphasized but can be explored.
- A valuable activity is to have students critique each other's work. A good resource with potential problems on page 7 (labeled p. 5): Introduction to Proof.
- Coordinate proof is not that main focus of this cluster, but it can be used. Notice that coordinate proof is addressed in G.GPE.4 and 5.

TIP!

Students may think that justifications are not needed for statements they view as "obvious." Emphasis should be placed on providing students with examples contradicting the "obvious" or giving reasoning for every claim to create a complete argument.

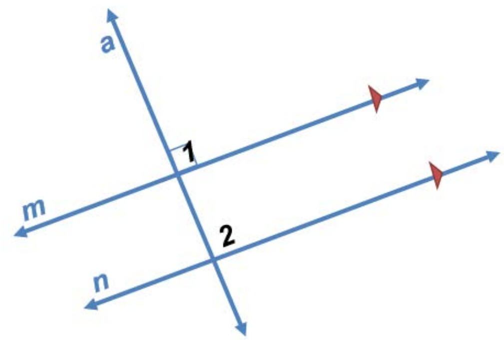
**Proofs about Lines and Angles and their Applications**

- Problem situations should involve algebraic relationships with and without labeled diagrams.
- Some of the theorems listed in this cluster (e.g. the ones about alternate interior angles and the angle sum of a triangle) are logically equivalent to the Euclidean Parallel Postulate and should be acknowledged.

# INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.CO.9-11)

***EXAMPLE***

If you are given that $a \perp m$ and $m \parallel n$, what kind of conjectures about lines and angles can you draw from the given information?




*Proof:* A student may want to prove that if a transversal is perpendicular to one of the lines included in the set of parallel lines, then it is perpendicular to the rest of the parallel lines. Angle 1 is 90° because it is formed by two perpendicular lines. Since $\angle 1$ and $\angle 2$ are formed by parallel lines cut by a transversal, they are corresponding angles, and by the Corresponding Angles Criteria they are congruent. Therefore $\angle 2$ is also 90° and line $a \perp n$. Therefore, in a plane, if a transversal is perpendicular to one of the lines included in the set of parallel lines, then it is perpendicular to the rest of the parallel lines.

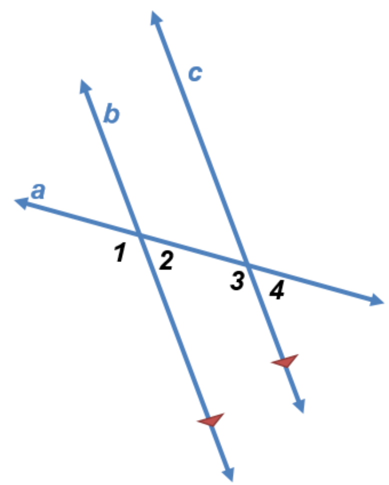
Students may incorrectly try to apply the Transitive Property by writing statements with mixed symbols such as if $a \perp m$ and $m \perp n$, then $a \parallel n$. Explain to students that the symbols must be the same in all three statements in order to apply the Transitive Property.

***EXAMPLE***

Prove that if two parallel lines are cut by a transversal, then the consecutive exterior angles are supplementary.
*Given:* Lines b and c are parallel.
*Prove:* Angle 1 and Angle 4 are supplementary angles.

| Statement | Reason |
|---|---|
| Lines *b* and *c* are parallel and cut by a transversal. | Given |
| $\angle 1$ and $\angle 2$ are supplementary. | Two angles that form a linear pair are supplementary. |
| $\angle 3$ and $\angle 4$ are supplementary. | Two angles that form a linear pair are supplementary. |
| $\angle 1 \cong \angle 3$ | If two ∥ lines are cut by a transversal, then corresponding angles are ≅. |
| $\angle 1$ and $\angle 4$ are supplementary. | Substitution |

| Statement | Reason |
|---|---|
| Lines *b* and *c* are parallel and cut by a transversal. | Given |
| $\angle 1$ and $\angle 2$ are supplementary. | Two angles that form a linear pair are supplementary. |
| $\angle 2 \cong \angle 4$ | If two ∥ lines are cut by a transversal, then corresponding angles are ≅. |
| $\angle 1$ and $\angle 4$ are supplementary. | Substitution |

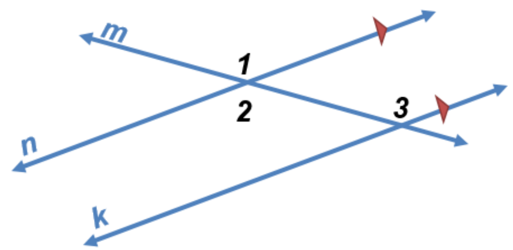



*Discussion:* There are generally several acceptable ways to prove a statement. As students become more advanced in their mathematical thinking, push them toward more efficient proofs.

# INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.CO.9-11)

***EXAMPLE***

Assuming that the Vertical Angles Conjecture (If two angles are vertical, they are equal in measure.) is true and that the Corresponding Angles Conjecture (If two parallel lines are cut by a transversal, then their corresponding angles are congruent.) is true. Then write a proof showing the Alternate Interior Angles conjectures (If two parallel lines are cut by a transversal, then their alternate interior angles are congruent.) is also true.




*Proof:* We can draw a pair of vertical angles and label them $\angle 1$ and $\angle 2$. Then we can draw line *k* so that it is parallel to line *n*. Angle $\angle 3$ would be a corresponding angle to $\angle 1$ and an alternate interior angle to $\angle 2$. Since $\angle 2 \cong \angle 1$ (vertical angles) and $\angle 1 \cong \angle 3$ (corresponding angles), then $\angle 2 \cong \angle 3$ (alternate interior angles) is true because of the Transitive Property.

*Discussion:* This is an example of a paragraph proof. After doing several proofs about pairs of angles formed by parallel lines and a transversal, it is important for students to understand that any pair of congruent angles such as alternate interior angles (not only corresponding angles are congruent) can be used as the given to prove that the other pair of angles (e.g., alternate exterior angles) is also congruent.

Students may incorrectly think that the converse of a statement is always true. For example, if the statement is "Parallel lines do not intersect.", then the converse would be "Lines that do not intersect are parallel.", which is not always true.

### Proofs about Triangles and their Applications

When being introduced to proofs, students can begin with labeled diagrams to draw conclusions. Then require students to label/mark their own diagrams to draw conclusions. Students should move towards writing a complete argument, in a variety of formats, to move from given information to a conclusion, including drawing and marking a diagram.

After students are familiar with the transformational definition of congruence in terms of rigid motions, have them prove that the base angles of an isosceles triangle are congruent.

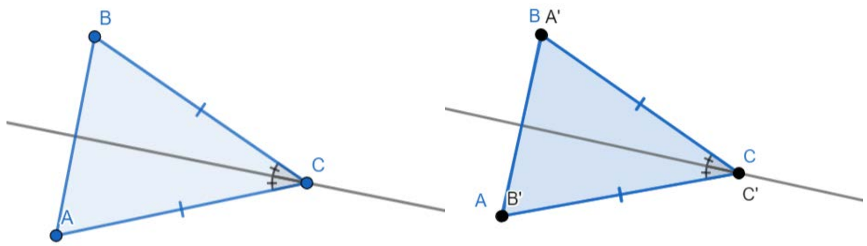
# INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.CO.9-11)

***EXAMPLE***

**Base Angles of an Isosceles Triangle**

Prove that the base angles of an isosceles triangle are congruent.

*Discussion:* Students may also use transformations to prove theorems. This can be done using transparencies, tracing paper, or dynamic geometric software. *Note: In order to use congruence in an explanation or a proof, the rigid motion(s) need to be specified.* By the definition of an isosceles triangle, the two sides of the triangle are congruent. A student can create an angle bisector through the vertex of the isosceles triangle creating two congruent angles. A student may choose to use the angle bisector as a line of reflection, reflecting the entire triangle over the angle bisector (so A' maps to B and B' maps to A). Because reflections preserve angle measure and side length, the two congruent sides are taken to each other and maps the triangle on to itself. Since the triangle is mapped onto itself, it means that the base angles also map onto each other, so they must be congruent.




After proving a generalization, students may still incorrectly believe that exceptions to the generalization might exist. Ask students to find an example that counters the generalization and see if this example exists.

# INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.CO.9-11)

***EXAMPLE***

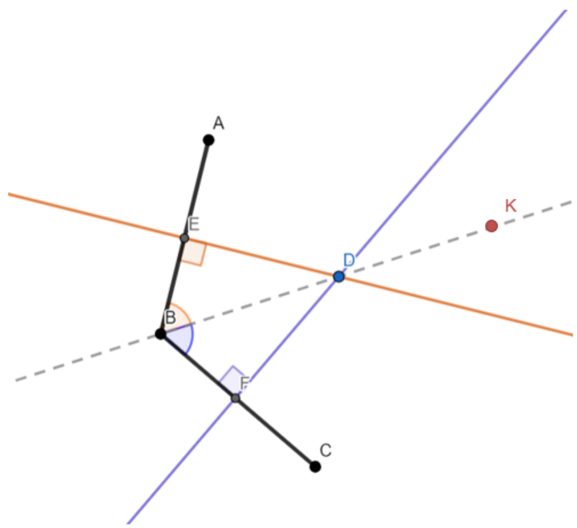
**Angle Bisectors**

Prove that if a point is on the angle bisector, then it is equidistant from the sides of the angle.

*Discussion:*

**Method 1**

Using geometric software such as GeoGebra, students can construct an angle bisector. Then they can construct a perpendicular line between the point on the angle bisector to each side of the angle. The advantage of using geometric software is that students can manipulate the angle and point on the angle bisector to show how the angle bisector and the perpendicular lines remain even when the angle size is adjusted or the point on the angle bisector moves. Then students could use triangle congruence to prove that the point on the angle bisector is equidistant from the sides.




- $\overline{BD} \cong \overline{BD}$ because it's the same line segment
- $\triangle EBD \cong \triangle FBD$ because of AAS
- $\overline{ED} \cong \overline{DF}$ because of CPCTC

Therefore, a point on an angle bisector is equidistant from the sides of its angle.

**Method 2**

To prove that a point on the angle bisector is equidistant from the sides of angle, the definition of the distance between the point and a line should be recalled: a distance between a point and a line is the length of a perpendicular line segment that goes from that point toward the line. Select any point $D$ on the angle bisector $\overline{BK}$. Draw two perpendicular lines from point $D$ toward the sides of the angle $BAC$. The lengths of the line segment $\overline{DE}$ and $\overline{DF}$ respectively represent the distances from a point $D$ to the sides $\overline{BA}$ and $\overline{BC}$. Two right triangles $EBD$ and $FBD$ are congruent by AAS (Angle $BED$ is congruent to angle $BFD$ since they are both right angles. Angle $EBD$ is congruent to angle $FBD$ by the definition of the angles bisector. $\overline{BD} \cong \overline{BD}$ by the Reflexive Property). Since sides $\overline{ED}$ and $\overline{DF}$ are congruent by CPCTC, their lengths are equal. Since point $D$ was selected randomly, it is proved that any point on the angle bisector is equidistant from the sides of the angle.

# INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.CO.9-11)

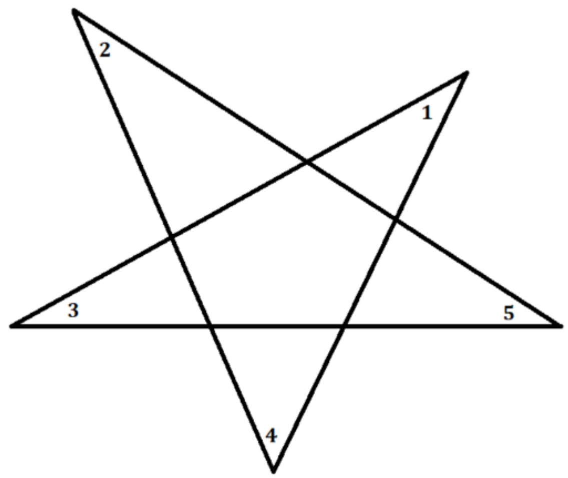
***EXAMPLE***

**Exploring a Proof Without Words**

**Part 1**

What is the sum of the 5 vertex angles?

**Part 1 Diagram**




**Part 2**

**a.** How was it determined that one of the angles equals $m\angle 2 + m\angle 4$ and the other equals $m\angle 1 + m\angle 3$?

**b.** What does the dotted line going upward from angle 5 represent (paying attention to the labeled angles)?

**c.** Would the ideas represented by the image be applicable to 5-pointed stars of any shape?

**Part 2 Diagram**

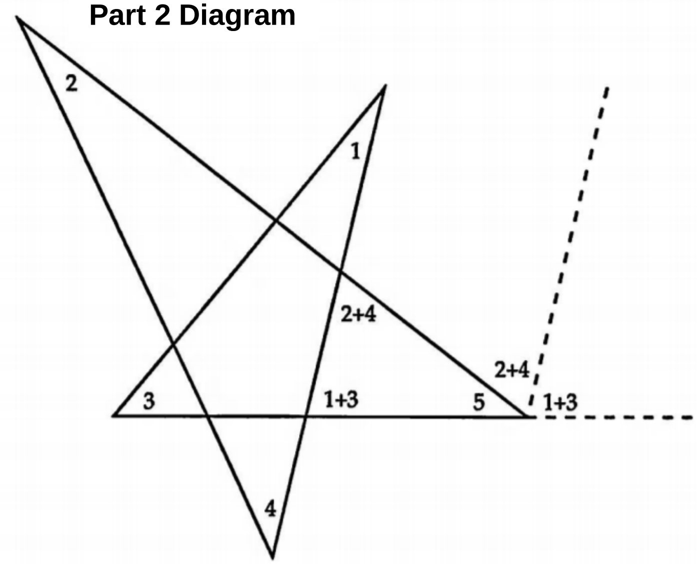



Image taken from Fouad Nakhli's Proof Without Words published by The Mathematical Association

*Discussion:* Students do not always need to use words for proofs. Proofs without words are a viable method for proving theorems and conjectures at this level. Although a proof without words can be useful as a context for discussion, a proof without words can also stand alone. Students should explore how to explain what is represented and possibly add their own written descriptions. (Some conjectures/theorems lend themselves more to this strategy than others.) For part **a.** students can utilize the fact the sum of the angles in a triangle = 180° with various triangles within the star. For part **b.** students realized that the dotted line is drawn in and is parallel to the line segment adjacent to angle 1 and angle 4 so that students make the connection to the properties of parallel lines intersected by a transversal. Part **c.** shows how this image can be a "proof" instead of just an example.

# INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.CO.9-11)

**TIP!** Explicitly modeling mathematical thinking in classroom discussion helps students reason through proofs on their own.

## Proofs about Parallelograms and their Applications

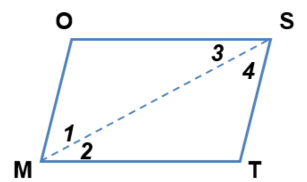
Students will combine the definition of a parallelogram with triangle congruence criteria and previously proved theorems to yield properties of parallelograms assumed in earlier grades. A variety of proof formats can be used here, including those based in symmetrical and transformational arguments.

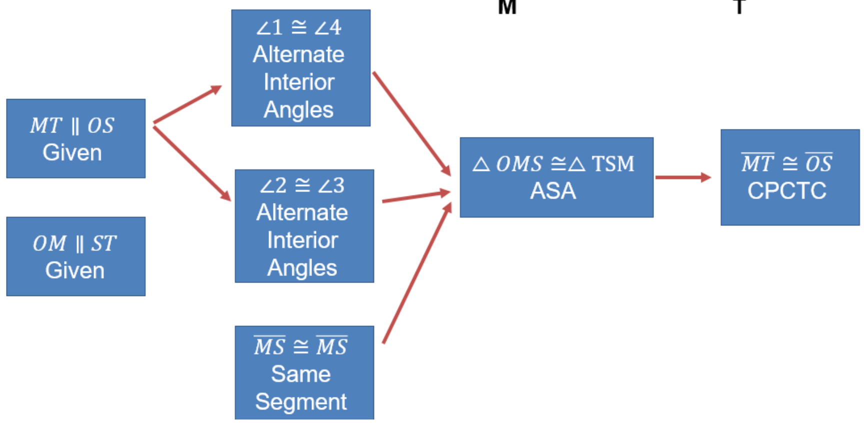
***EXAMPLE***

**Proof about a Parallelogram**

**Given:** $MT \parallel OS$
$OM \parallel ST$

**Prove:** $\overline{MT} \cong \overline{OS}$




*Discussion:* This proof could be given after students are asked what they know about parallelograms. Some students may say that opposite sides are congruent. Then students could be challenged about how they know that its true (besides the teacher told them). This example shows a flow chart proof. A flow chart could be scaffolded for struggling learners in many ways. One way is by giving them the boxes and having them put them in order. Another way could be to give students the boxes with the statements and have them come up with the rationales or give students the rationales and have them come up with the statements. Another way could be by only giving students some of the boxes. The goal should be for students to prove the statement independently without any scaffolding, but it may take time and experience.

# INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.CO.9-11)

TIP!

Have some students present their proofs and other students critique the proofs. This promotes communication, critical thinking, and mathematical reasoning.

**Indirect Proof**

Explain to students that sometimes it is easier to disprove a conjecture than to prove it. This is called an indirect proof. Practice in developing statements that lead to contradiction helps bridge students to indirect proof.

Students may incorrectly think that more than one counterexample is necessary for disproving a statement. It would be valuable to explore multiple counterexamples, but teachers should emphasize that only one is necessary.

TIP!

Although not required in the standards, some students may benefit from exploring rules of inference. Truth tables may help students develop logical reasoning and see how conditional statements apply under different circumstances.

## Instructional Tools/Resources

*These tools and resources may be helpful for instruction. Before use, they should be evaluated at the local level to determine their appropriateness for instruction.*

**Manipulatives/Technology**

- Computer dynamic geometry software (such as Geometer's Sketchpad®, Desmos, Cabri®, or Geogebra®).
- Graphing calculators and other handheld technology such as TI-Nspire™.
- Tracing paper (patty paper)
- Transparencies
- Graph paper
- Ruler
- Protractor
- Compass
- Proofs Without Words: Exercises in Visual Thinking

# INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.CO.9-11)

**Proofs about Lines and Angles**

- Points Equidistant from Two Points in the Plane by Illustrative Mathematics is a task where students prove that the perpendicular is exactly the set of points equidistant to the endpoints of the segment.
- Congruent Angles Made by Parallel Lines and a Transverse by Illustrative Mathematics is a task where students prove congruence of vertical angles.

**Proofs about Triangles**

- Circles in Triangles by Mathematics Assessment Project has students apply theorems for circles inscribed in triangles.
- Classifying Triangles by Illustrative Mathematics is a task where students synthesize their knowledge of triangles.
- Midpoints of Triangle Sides by Illustrative Mathematics is a task where students use similarity transformations to relate two triangles.
- Congruent Angles in Isosceles Triangles by Illustrative Mathematics is a task where students establish that the base angles in an isosceles triangle are congruent.
- Sum of Angles in a Triangle by Illustrative Mathematics is a task where students provide an argument for the sum of the angles in a triangle at the high school level.

**Proofs about Quadrilaterals**

- Quadrilaterals is a Performance Assessment Task by Inside Mathematics where studetns use geometric properties to solve a problem.
- Midpoints of the Sides of a Parallelogram by Illustrative Mathematics is a task where students use previously known facts about parallelograms to prove new facts.
- Is This a Parallelogram? by Illustrative Mathematics is a task where students develop an alternative characterization of a parallelogram in terms of congruence of opposite sides.

**Curriculum and Lessons from Other Sources**

- EngageNY, Geometry, Module 1, Topic C, Lesson 18: Looking More Carefully at Parallel Lines is a lesson that pertains to this cluster.
- EngageNY, Geometry, Module 1, Topic D, Lesson 26: Triangle Congruency Proofs, Lesson 27: Triangle Congruency Proofs are lessons that pertain to this cluster.
- Georgia Standards of Excellence Curriculum Frameworks, Geometry, Unit 2: Similarity, Congruence and Proof has many tasks that pertain to this cluster.
- EngageNY, Geometry, Module 1, Topic E, Lesson 28: Properties of Parallelograms, Lesson 29: Special Lines in Triangles, Lesson 30: Special Lines in Triangles are lessons that pertain to this cluster.
- EngageNY, Geometry, Module 1, Topic G, Lesson 33: Review of the Assumptions, Lesson 34: Review of the Assumptions, are lessons that pertain to this cluster.
- Mathematics Vision Project, Geometry, Module 2: Congruence, Construction, and Proof has many tasks that pertain to this cluster.

# INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.CO.9-11)

**General Resources**

- Arizona 7-12 Progression on Geometry is an informational document for teachers. This cluster is addressed on pages 15-16.
- High School Coherence Map by UnboundED is a map of the relationships among high school mathematics standards. Please note: This resource is aligned to the Common Core not Ohio's Learning Standards. Although the tool is useful, it needs to be checked with respect to Ohio's Learning Standards.
- van Hiele Model of Geometric Thinking is a pdf created by ODE that summarizes the van Hiele levels.

**References**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| STANDARDS | MODEL CURRICULUM (G.CO.12-13) |
|---|---|
| **Geometry**<br>**CONGRUENCE**<br>**Make geometric constructions.**<br>**G.CO.12** Make formal geometric constructions with a variety of tools and methods (compass and straightedge, string, reflective devices, paper folding, dynamic geometric software, etc.).<br>*Copying a segment; copying an angle; bisecting a segment; bisecting an angle; constructing perpendicular lines, including the perpendicular bisector of a line segment; and constructing a line parallel to a given line through a point not on the line.*<br>**G.CO.13** Construct an equilateral triangle, a square, and a regular hexagon inscribed in a circle. | **Expectations for Learning**<br>In elementary and middle school, students learn to use measurement tools to informally draw geometric shapes with given conditions. In this cluster, students make formal and precise constructions using a variety of tools, and they understand the geometric relationships upon which the constructions are based.<br><br>The student understanding of this cluster begins at van Hiele Level 1 (Analysis) and moves to Level 2 (Informal Deduction/Abstraction).<br><br>**ESSENTIAL UNDERSTANDINGS**<br>• Construction is a process of reasoning that does not use a scale and does not use measurement.<br>• Simple constructions can be used to develop an understanding of mathematical relationships.<br><br>**MATHEMATICAL THINKING**<br>• Make sound decisions about using tools.<br>• Strategically use technology to deepen understanding.<br>• Plan a pathway to complete constructions.<br>• Determine accuracy of results.<br>• Create a drawing and add components as appropriate.<br><br>**INSTRUCTIONAL FOCUS**<br>• Distinguish between a rough sketch, a careful drawing with measurements, and a construction with compass and straightedge.<br>• Use a variety of geometric tools to make precise constructions.<br>*Continued on next page* |

| STANDARDS | MODEL CURRICULUM (G.CO.12-13) |
|---|---|
| | **Content Elaborations**<br>**OHIO'S HIGH SCHOOL CRITICAL AREAS OF FOCUS**<br>• Math 1, Number 5, pages 9-10<br><br>**CONNECTIONS ACROSS STANDARDS**<br>• Experiment with transformations in the plane (G.CO.1, 5).<br>• Understand and apply theorems about circles (G.C.3, (+) 4).<br>• Prove and apply geometric theorems (G.CO.9-11).<br>• Reason quantitatively (N.Q.1-3). |

# INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.CO.12-13)

## Instructional Strategies

*Note: The model curriculum is not a collection of lessons nor a full curriculum; it does not suggest pace, sequence, or amount of time spent on topics. It provides information about a topic related to the standards including ideas for examples, strategies for teaching, common misconceptions, and possible connections between topics. In addition, the emphasis of the model curriculum is on instruction not on assessment; therefore, the examples in the Instructional Strategies section are written for instructional purposes not necessarily as assessment items. Examples may need to be rewritten to accommodate the needs of each individual classroom.*

***Standards for Mathematical Practice***

*This cluster focuses on but is not limited to the following practices:*

***MP.1** Make sense of problems and persevere in solving them.*
***MP.2** Reason abstractly and quantitatively.*
***MP.4** Model with mathematics.*
***MP.5** Use appropriate tools strategically.*
***MP.6** Attend to precision.*

### VAN HIELE CONNECTION

In Math 1 students are expected to move from Level 1(Analysis) to Level 2 (Informal Deduction/Abstraction). In van Hiele Level 1 students may list all the properties of a shape, but they may not see the relationships between properties. Whereas van Hiele Level 2 can be characterized by the student doing some or all of the following:

- noticing relationships within and between properties and figures;
- forming meaningful definitions;
- justifying reasoning using informal arguments;
- drawing logical maps and diagrams; and/or
- not yet understanding formal deduction.

This cluster when combined with G.CO.9-11 will also move into Level 3 (Deduction) where students start to construct proofs and understand the necessity of proofs.

See the van Hiele pdf on ODE's website for more information about van Hiele levels.

### MAKE FORMAL CONSTRUCTIONS

Mathematicians have been making constructions using a compass and straight edges without markings since ancient Greece. This is because handheld measurement tools, such as protractor or ruler, result in approximations while constructions, supported by formal Geometry, are theoretically perfect. Constructions also allow students to have a kinesthetic learning experience and make proofs relevant as they have to justify the correctness of their construction. This cluster should be taught in conjunction with other clusters such as G.CO.1-5 and G.CO.9-11.

It is more important that students understand why the construction works instead of manually performing the construction by memorizing steps. For students who are struggling because of dexterity problems, geometric software may be useful.

TIP!

Some students may believe that a construction is the same as a sketch or drawing. Emphasize the need for precision and accuracy when doing constructions.

# INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.CO.12-13)

Provide opportunities for students to practice executing basic constructions. Here is a list of some important constructions. Classroom instruction should include but not be limited to these constructions:

- Reproduce a line segment on a ray with a specified endpoint.
- Construct an equilateral triangle on a given side.
- Reproduce and angle with one side specified.
- Construct a line perpendicular to a given line from a given point.
- Construct the perpendicular bisector of a line segment.
- Construct an angle bisector.
- Construct a line parallel to a given line through a given point.
- Divide a given line segment into any number of equal segments.
- Construct an equilateral triangle inscribed in a square.
- Construct a regular hexagon inscribed in a circle.
- Draw tangents to a circle from a point outside the circle.
- Construct the sum, difference, product, and quotient of two given positive numbers.
- Construct the square root of a positive number.

(List taken from Wu, 2013, Teaching Geometry in Grade 8 and High School According to the Common Core Standards, page 145)

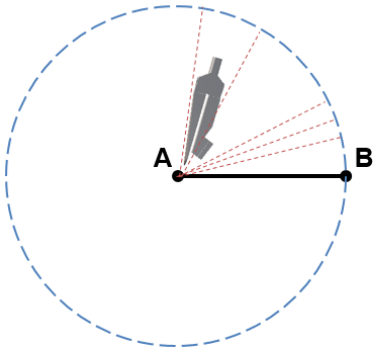
Students should not only construct figures, but they should be able to justify or prove why they are correct. Oftentimes the students will utilize the congruence theorems.

## INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.CO.12-13)

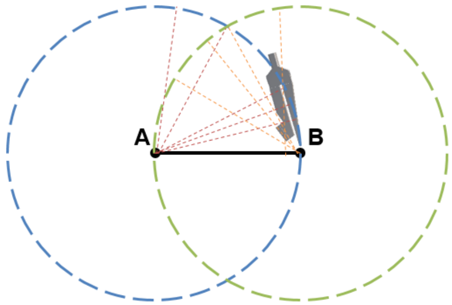
***EXAMPLE***

**Constructing and Equilateral Triangle**

Construct an equilateral triangle using $\overline{AB}$ as one of its sides.

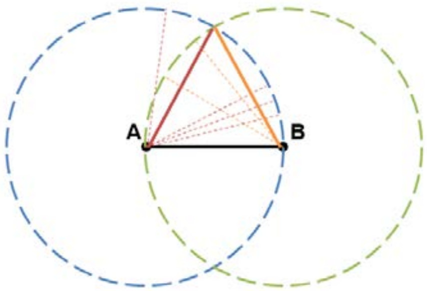




*Discussion:* Ask students what they know about equilateral triangles. They should recall that an equilateral triangle has all equal sides and angles. They may also know that all angles have a measure of 60°. Prepare students to connect their future constructions to the circles. Ask students what they know about circles. They should say that all radii in a circle have the same length. Constructing a circle may be helpful for constructing an equilateral triangle using two distinct radii of the circle as two out of the three congruent sides of an equilateral triangles is the initial thought. A student may start with their compass point on point $A$ and extend it to point $B$ to make the circle with radius $\overline{AB}$, but now the question must be asked "Which radius will be part of the equilateral triangle?" Although, some students may try to eyeball one that is 60°, emphasize that it must be exact. This may also be good place to give students the opportunity to recall the formal definition of an equilateral triangle and a circle.




Ask students how they could get another side of the triangle. Hopefully a student will come up with the idea of drawing another circle using the other point on the line segment (in the diagram point $B$.) Again have students explain that all radii in a circle are congruent. Have students decide which radii should be used to make an equilateral triangle. They should come to the conclusion that the radii needed are those that intersect each other and also the intersection point of the circle. This is because the point of intersection of the two circles along with points $A$ and $B$ will become the vertices of the equilateral triangle.

# INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.CO.12-13)



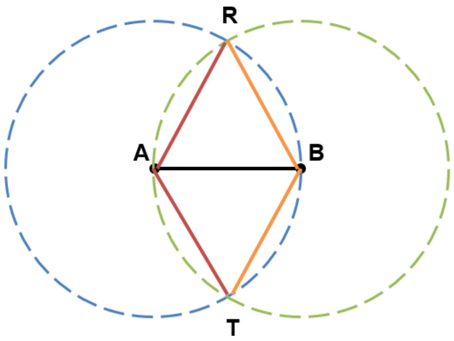

To reinforce the idea, ask students how they can prove that the triangle they constructed is in fact an equilateral triangle. They should state that it is an equilateral triangle because all the sides of the triangle are radii of congruent circles. Therefore, all three side lengths are congruent, and the triangle is an equilateral triangle.

***EXAMPLE***

**Making Observations and Justifying Thinking in Relation to Constructions**

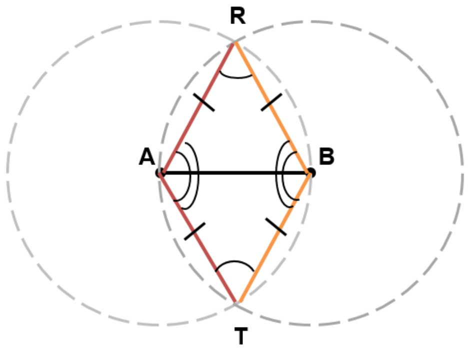
Construct the perpendicular bisector of a line segment.

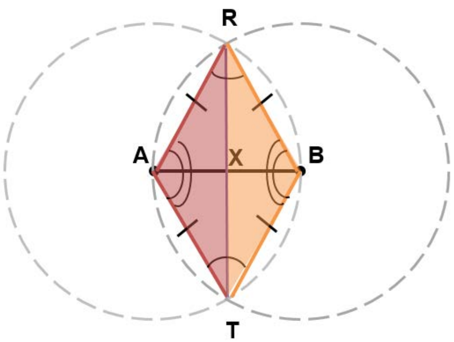

*Discussion:* Before having students construct, use what students learned from the previous example about equilateral triangles. Label where the circles intersect $R$ and $T$, and draw line segments connecting $R$ and $T$ to $A$ and $B$ forming equilateral triangles. Instead of telling students what to prove, ask students what they are able to prove. Tell students to share everything they know to be true about the picture, justifying their thinking. (*Note: Letting students direct the instruction may result in many other theorems being proved before arriving at constructing the perpendicular bisector.)* Although your goal may be to have students construct the perpendicular bisector of a line segment, let them form their own goals. The ultimate goal is to let students make observations and justify their thinking to think mathematically about constructions. *Note: Students may skip all the proof steps in this example if they have already proved previously in class that the diagonals of a rhombus are perpendicular bisectors.*

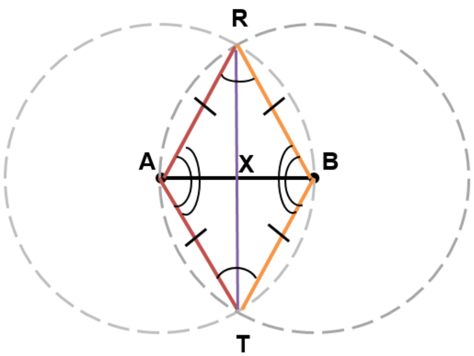


# INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.CO.12-13)

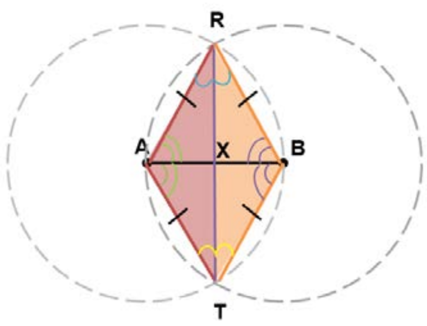
- Students should realize that $\triangle ARB \cong \triangle ATB$ because they are both equilateral triangles whose side lengths are radii of two congruent circles.
- They should eventually come to the conclusion that, by definition, quadrilateral $ARBT$ is a rhombus because sides $\overline{AR}, \overline{RB}, \overline{BT},$ and $\overline{AT}$ are congruent since the side lengths are radii of two congruent circles.
- Since $\triangle ARB \cong \triangle ATB$, students should realize that $\angle ARB \cong \angle ATB$ since they are both angles in an equilateral triangle.
- Students know that $\angle RAB \cong \angle BAT \cong \angle RBA \cong \angle ABT$ because they are angles in two congruent equilateral triangles. Then they can see that $\angle RAT \cong \angle RBT$ because $\angle RAB + \angle BAT \cong \angle RBA + \angle ABT$ due to the Substitution Property and the Angle Addition Postulate. **Therefore, they have just proved that opposite angle in a rhombus are congruent.**



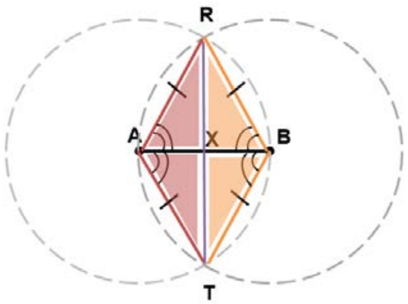

- Now draw a line segment connecting point $R$ and point $T$, and ask students to share everything they now know about the picture to be true and justify their thinking.
- They may say that $\triangle ART \cong \triangle BRT$ by SAS.

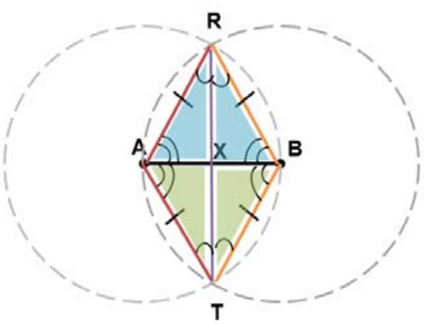


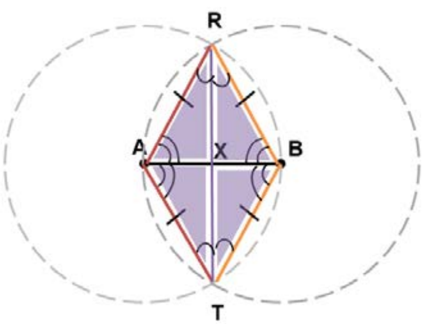

- Since $\triangle ART \cong \triangle BRT$, then $\angle BRX \cong \angle ARX$ and $\angle ATX \cong \angle BTX$ by CPCTC, which means $\overline{RT}$ bisects $\angle ARB$ and $\angle BTA$.
- Since $\angle RAX \cong \angle TAX$ and $\angle RBX \cong \angle TBX$ because they are all angles in congruent equilateral triangle, they should come to the conclusion that $\overline{AB}$ bisects $\angle RAT$ and $\angle RBT$ which means the students just proved **that the diagonals of a rhombus bisect its angles.**

# INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.CO.12-13)

- Since $\overline{AX} \cong \overline{AX}$ and $\overline{XB} \cong \overline{XB}$ due to the Reflexive Property $\angle RAX \cong \angle TAX$ and $\angle RBX \cong \angle TBX$ because they are all angles in congruent equilateral triangle, they should also see that $\triangle RAX \cong \triangle TAX$ and $\triangle RBX \cong \triangle TBX$ by SAS.

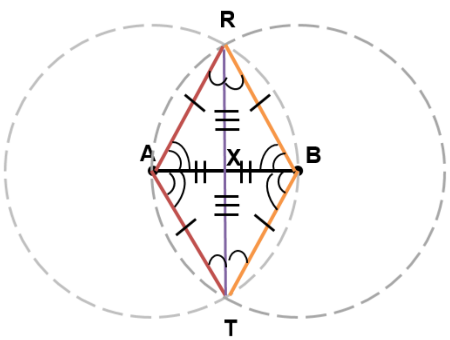


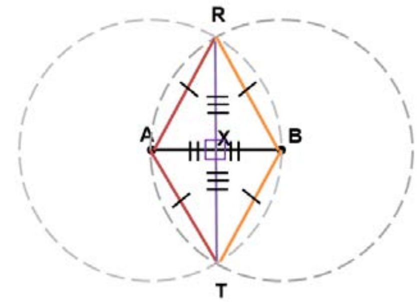

- Students can also say $\triangle RAX \cong \triangle RBX$ and $\triangle TAX \cong \triangle TBX$ by ASA, so by the Substitution Property $\triangle RAX \cong \triangle RBX \cong \triangle TAX \cong \triangle TBX$.

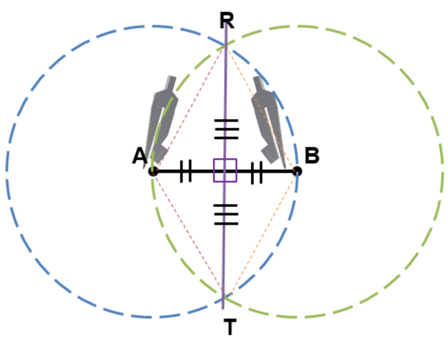
- This means that $\overline{AX} \cong \overline{XB}$ and $\overline{RX} \cong \overline{XT}$ by CPCTC. Therefore $X$ is the midpoint of line segments $\overline{AB}$ and $\overline{RT}$, and $\overline{RT}$ bisects $\overline{AB}$ at point $X$ and $\overline{AB}$ bisects $\overline{RT}$ at point $X$, so students just proved that the **diagonals of a rhombus bisect each other.**



# INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.CO.12-13)

Students should be able to reason that the measure of each angle in an equilateral triangle equals 60° so $\angle RAX, \angle TAX, \angle TBX, \angle ATB,$ and $\angle ARB = 60°$, and since $\overline{RT}$ bisects $\angle ATB$ and $\angle ARB$ angles $\angle ARX, \angle BRX, \angle ATX,$ and $\angle BTX = 30°$. Since all angles in a right triangle must equal 180°, $\angle AXR, \angle BXR, \angle AXT,$ and $\angle BXT = 90°$ because $180° - 60° - 30° = 90°$, therefore students just proved **that diagonals of a rhombus are perpendicular bisectors of each other**.




Now bring students back to the original question: Construct the perpendicular bisector of a line segment. Ask students if they can see that $\overline{RT}$ is the perpendicular bisector of $\overline{AB}$ and vice versa. Point out that the endpoints of the two line segments of $\overline{AB}$ and $\overline{RT}$ become vertices of the rhombus. So, by constructing two equilateral triangles with the common side $\overline{AB}$, they constructed a rhombus whose diagonals are perpendicular segments and angle bisectors. Therefore, they can conclude that if they construct a rhombus, they should also get a perpendicular bisector. Now, have them construct a rhombus using a compass and straight edge. If they get stuck, prompt them to start with constructing an equilateral triangle.

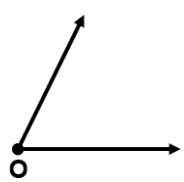
TIP!

Draw attention to the fact that in many constructions there are congruent triangles that often make the construction work. For example, applying the Perpendicular Bisector Theorem (If a point is on the perpendicular bisector of a segments, then it is equidistant from the segment's endpoints.) helps create congruent triangles.

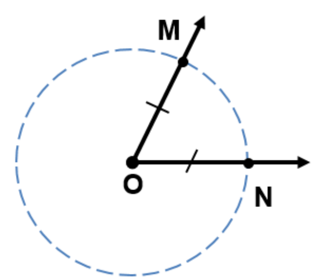
***EXAMPLE***

**Making Observations and Justifying Thinking in Relation to Constructions**

Construct the bisector of an angle.

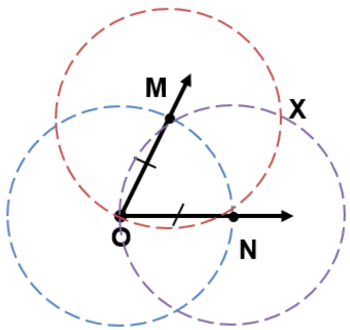



*Discussion:* Have students start with what they know. If they construct a circle with the center at the vertex $O$, then $\overline{OM} \cong \overline{ON}$ since they are both radii of the same circle. Ask students questions such as "What shapes have equal angles?" and then discuss which shapes with equal angles could be useful.

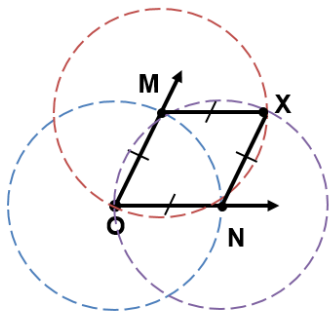


## INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.CO.12-13)

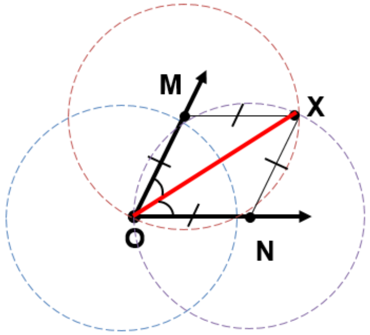
If students keep their compass the same length as $\overline{OM}$ and create two new circles with the center at $M$ and $N$, students will see the new circles intersect at Point $X$.




Students should also observe that $\overline{OM} \cong \overline{ON} \cong \overline{MX} \cong \overline{XN}$ since they are all radii of congruent circles, and quadrilateral $OMXN$ is a rhombus.




In the previous example students discovered that the diagonal of the rhombus was proved to be also an angle bisector. Now they can conclude that drawing the segment $\overline{XO}$ will bisect $\angle MON$.




Have students use different tools to make constructions. For instance, challenge students to perform the same construction using a variety of tools, for example, a compass or string. Have students compare dynamic geometry software commands to sequences of compass-and-straightedge steps.

# INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.CO.12-13)

## Instructional Tools/Resources

*These tools and resources may be helpful for instruction. Before use, they should be evaluated at the local level to determine their appropriateness for instruction.*

**Manipulatives/Technology**

- Compass
- Straightedge/Ruler
- String
- Origami paper
- Tracing paper
- Protractor
- Mira
- Computer dynamic geometry software (such as Geometer's Sketchpad®, Desmos, Cabri®, or Geogebra®).
- Graphing calculators and other handheld technology such as TI-Nspire™.

**Constructions**

- Students will construct perpendicular bisectors while designing a delivery route for a local pizza shop, as described in Dividing a Town into Pizza Delivery Regions by NCTM Illuminations. Lead a class discussion where students will identify application of these skills across various career fields (e.g., landscaping, agriculture, construction, architecture, logistics). Students will apply the information to their plan for education and training through high school and beyond. *NCTM now requires a membership to view their lessons.*
- Constructions and Concurrencies by MathBitsNotebook.com is a webpage that contains many links to pages that explain constructions.
- Construction by Math Open Reference is a webpage that has links for tutorials on constructions. The students can follow through step-by-step or watch the entire way through independently.
- Geometric Constructions by Geogebra is an interactive course on geometric constructions.
- Desmos Geometry allows students to make geometric constructions on a computer.
- Create My Constructions by Andrew Stadel is a Desmos activity that contains 6 challenges that help students and teachers become more familiar with the Desmos tool.

# INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.CO.12-13)

**Curriculum and Lessons from Other Sources**

- EngageNY, Geometry, Module 1, Topic A, Lesson 1: Construct an Equilateral Triangle, Lesson 2: Construct an Equilateral Triangle, Lesson 3: Copy and Bisect and Angle, Lesson 4: Construct a Perpendicular Bisector, Lesson 5: Points of Concurrencies are lessons that pertain to this cluster.
- Georgia Standards of Excellence Curriculum Frameworks, Geometry, Unit 2: Similarity, Congruence, and Proof has lessons that pertain to this cluster. Lessons on construction start on page 77.
- Mathematics Vision Project, Geometry, Module 2: Congruence, Construction, and Proof has several lessons that pertain to this cluster.

**General Resources**

- Arizona 7-12 Progression on Geometry is an informational document for teachers. This cluster is addressed on page 16.
- High School Coherence Map by UnboundED is a map of the relationships among high school mathematics standards. Please note: This resource is aligned to the Common Core not Ohio's Learning Standards. Although the tool is useful, it needs to be checked with respect to Ohio's Learning Standards.
- van Hiele Model of Geometric Thinking is a pdf created by ODE that summarizes the van Hiele levels.

**References**


- Common Core Standards Writing Team. (2016, March 24). *Progressions for the Common Core State Standards in Mathematics (draft). Grades 7–8, HS*, *Geometry*. Tucson, AZ: Institute for Mathematics and Education, University of Arizona.
- Prasad, P. (November 2016). Leveraging interactive software to prompt discussion. *Mathematics Teaching in Middle School*, 22*(4),* 226-232.
- Wu, H. (2013). Teaching geometry according the Common Cores Standards. Retrieved from https://math.berkeley.edu/~wu/Progressions_Geometry.pdf
- Wu, H. (2013). Teaching geometry in Grade 8 and high school according the Common Cores Standards. Retrieved from https://math.berkeley.edu/~wu/CCSS-Geometry_1.pdf

| STANDARDS | MODEL CURRICULUM (G.CO.14) |
|---|---|
| **Geometry**<br>**CONGRUENCE**<br>**Classify and analyze geometric figures.**<br>**G.CO.14** Classify two-dimensional figures in a hierarchy based on properties. | **Expectations for Learning**<br>In elementary school, students learn to classify two-dimensional figures based on their properties. In middle school, students focus on drawing quadrilaterals and triangles with given conditions. Now in high school, they learn to analyze and relate categories of two-dimensional shapes explicitly based on their properties. Based on analysis of properties, students create hierarchies for two-dimensional figures.<br><br>The student understanding of this cluster begins at van Hiele Level 1 (Analysis) and moves to Level 2 (Informal Deduction/Abstraction).<br><br>**ESSENTIAL UNDERSTANDINGS**<br>• There is a distinction between the definition of a figure and its properties, e.g., side lengths, angles, parallel/perpendicular sides, diagonals, symmetry.<br>• Figures may be categorized in different ways based on their properties.<br><br>**MATHEMATICAL THINKING**<br>• Use accurate mathematical vocabulary to describe geometric relationships.<br>• Make connections between terms and properties.<br>• Recognize, apply, and justify mathematical concepts, terms, and their properties.<br>• Generalize concepts based on patterns.<br>• Use formal reasoning.<br>*Continued on next page* |

| STANDARDS | MODEL CURRICULUM (G.CO.14) |
|---|---|
| | **Expectations for Learning, continued**<br>**INSTRUCTIONAL FOCUS**<br>• Explain the difference between the definition of a figure and its properties.<br>• Know precise definitions of special polygons, e.g., rhombus, parallelogram, rectangle, square, kite, trapezoid, isosceles trapezoid, equilateral triangle, isosceles triangle, and regular polygon.<br>• Compare and contrast definitions of quadrilaterals, including both definitions of trapezoids.<br>• Know and apply properties of special polygons and use them to classify figures.<br>• Explain the relationships among special quadrilaterals.<br>• Explain the relationships among special triangles.<br>• Create hierarchies in order to represent the relationship between pairs of figures and among several figures.<br><br>**Content Elaborations**<br>**OHIO'S HIGH SCHOOL CRITICAL AREAS OF FOCUS**<br>• Math 1, Number 5, pages 9-10<br><br>**CONNECTIONS ACROSS STANDARDS**<br>• Prove and apply theorems about quadrilaterals and triangles (G.CO.10-11).<br>• Justify the slope criteria for parallel and perpendicular lines (G.GPE.5).<br>• Know precise definitions (G.CO.1). |

# INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.CO.14)

## Instructional Strategies

*Note: The model curriculum is not a collection of lessons nor a full curriculum; it does not suggest pace, sequence, or amount of time spent on topics. It provides information about a topic related to the standards including ideas for examples, strategies for teaching, common misconceptions, and possible connections between topics. In addition, the emphasis of the model curriculum is on instruction not on assessment; therefore, the examples in the Instructional Strategies section are written for instructional purposes not necessarily as assessment items. Examples may need to be rewritten to accommodate the needs of each individual classroom.*

### VAN HIELE CONNECTION

In Math 1 students are expected to move from Level 1(Analysis) to Level 2 (Informal Deduction/Abstraction). In van Hiele Level 1 students may list all the properties of a shape, but may not see the relationships between properties. Whereas van Hiele Level 2 can be characterized by the student doing some or all of the following:

- noticing relationships within and between properties and figures;
- forming meaningful definitions;
- justifying reasoning using informal arguments;
- drawing logical maps and diagrams; and/or
- not yet understanding formal deduction.

***Standards for Mathematical Practice***

*This cluster focuses on but is not limited to the following practices:*

***MP.1*** *Make sense of problems and persevere in solving.*

***MP.3*** *Construct a viable argument and critique the reasoning of others.*

***MP.4*** *Model with mathematics.*

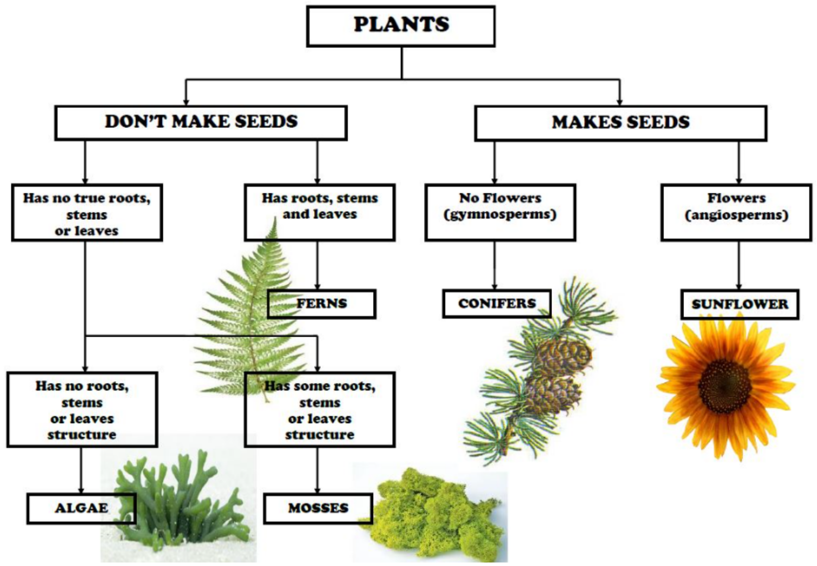
***MP.6*** *Attend to precision.*

***MP.7*** *Look for and make use of structure.*

***MP.8*** *Look for and express regularity in repeated reasoning.*

# INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.CO.14)

See the van Hiele pdf on ODE's website for more information about van Hiele levels.




## HIERARCHIES

This standard has moved from Grade 5 to high school, for research has shown that students are not ready to classify shapes into a hierarchy until high school. Hierarchies could include tree diagrams, Venn diagrams, or other conceptual maps (whatever that could be) and could be organized based on different properties of figures (diagonals, sides, angles, symmetry) and/or using different figures (quadrilaterals, triangles, polygons). Before students make a hierarchy of geometric figures, it may be useful to introduce hierarchies using non-mathematical ideas or by making cross curricular connections.

Image taken from Plant Classification Chart posted on Iman's Home-School page found at https://imanshomeschool.wordpress.com/2013/05/01/plant-classification-chart/

***EXAMPLE***

**Classifying Objects**

Think of a real-life situation that you could classify. Create a flow chart and a Venn Diagram to illustrate your classification.

*Discussion:* Students can use cross-curricular situations in science or social studies to create classifications or even find creative ways to classify animals or students in their classroom or school.

Before showing students a quadrilateral hierarchy, challenge students to create their own quadrilateral hierarchy using a flow chart or Venn diagram. It may be helpful to give groups of students cards with pictures of shapes on each card and have students sort them by using their own established criteria. See Polygon Capture in the Instructional Tools and Resources for a worksheet containing quadrilaterals that can be cut out and sorted.

Students may incorrectly think that orientation of a figure effects the classification. Show several examples of figures with different orientations.

In order to classify shapes, students need to understand the conditions that define a regular, equilateral, or equiangular figures, especially triangles and quadrilaterals. Although, the quadrilateral hierarchy is the focus of this cluster, a hierarchy can be used to organize other polygons.

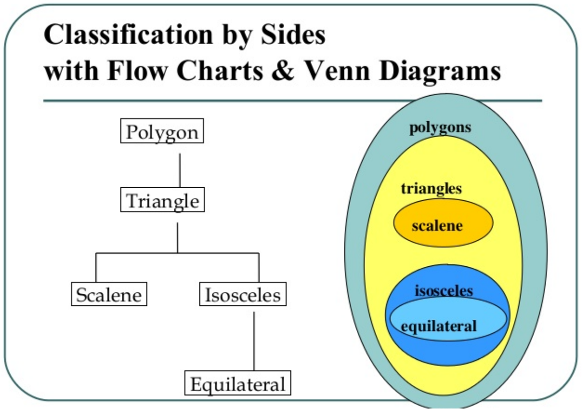
# INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.CO.14)

Extend classification of shapes to more than 4 sides, when it is appropriate for the students. Special polygons include but are not limited to rhombus, parallelogram, rectangle, square, kite, trapezoid, isosceles trapezoid, equilateral triangle, isosceles triangle, and regular polygons.

TIP! Compare and contrast different definitions of quadrilaterals. This is an opportunity to include both definitions of a trapezoid or a kite. Students can explore how these different definitions affect a potential hierarchy.

Have students become fluid in converting a hierarchy presented in a flow chart to a Venn diagram and vice versa.

While students come with experience about properties such as side lengths and angles in polygons, especially triangles and quadrilaterals, this may be their first formal experience with properties of diagonals. Students should investigate the relationships of diagonals in terms of being perpendicular, bisecting each other, and being congruent. See cluster G.CO.12-13 for an example of students proving properties of the diagonals of a rhombus.




Taken from a SlideShare titled Classify Polygons from Dreams4school published on Dec 25, 2014.

Students may incorrectly think that diagonals must be slanted. Show several examples of polygons with vertical and/or horizontal diagonals.

Students need to be able to, given specific properties, determine what figure is being described. For example, questions like "I have perpendicular diagonals and four congruent sides. What shape am I?" could be useful. Also, one could cover up part of a figure and ask "What shape could I be?"

When learning about geometric concepts in high school, it is important that students are able to reason logically about properties of figures. Questions involving replies such as *Sometimes, Always, Never* are appropriate.

***EXAMPLE***

**Reason Logically About Properties of Figures**

Use *Sometimes*, *Always*, or *Never* to answer the question. Then explain why.

- Is a square a rectangle?
- Is a rectangle a square?
- Is a rhombus a rectangle?
- Is a rectangle a rhombus?

# INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.CO.14)

## Instructional Tools/Resources

*These tools and resources may be helpful for instruction. Before use, they should be evaluated at the local level to determine their appropriateness for instruction.*

**Materials/Manipulatives**

- Tracing paper (patty paper)
- Transparencies
- Graph paper
- Ruler
- Protractor
- Compass
- Mira
- Computer dynamic geometry software (such as Geometer's Sketchpad®, Desmos, Cabri®, or Geogebra®).
- Graphing calculators and other handheld technology such as TI-Nspire™.

**Classifying Quadrilaterals**

- Polygon Capture by NCTM Illuminations is a game that could be used to motivate students to examine relationships among geometric properties.
- What is a Trapezoid? (Part 2) is a task by Illustrative Mathematics that has students compare different definitions of trapezoids and relate them to a Venn diagram.
- What Do These Shapes Have in Common? is a task by Illustrative Mathematics that has students classify shapes based on their properties.
- Bridges in Mathematics, Grade 3 Supplement, Set C4 Geometry: Quadrilaterals is a compilation of activities that relate to the sorting of quadrilaterals.
- Quadrilateral Properties and Relationships Lesson Plan includes an "Who Am I?" activity and a quadrilateral sorting activity using geometric software.
- Don Steward's Complete the Quadrilateral by Fawn Nguyen published on February 8, 2013 is an activity where students use quadrilateral definitions to create quadrilaterals on dot paper.
- Parallelograms and Translations by Illustrative Mathematics is a task where students apply the definition of a parallelogram in the context of a geometric construction.

# INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.CO.14)

**Diagonals**

- Geometry: Exploring Diagonals of Quadrilaterals by Texas Instruments is a lesson where students understand and identify quadrilaterals based solely on their diagonals.
- Diagonals of Quadrilaterals by gschettini is a GeoGebra activity where students explore the diagonals of quadrilaterals.
- Investigate Quadrilateral Diagonals by David Petro is a Desmos investigation where students explore diagonals and use them to identify a quadrilateral.
- EngageNY, Grade 5, Module 5, Topic D, Lesson 16: Draw Trapezoids to Clarify Their Attributes, and Define Trapezoids Based on Those Attributes, Lesson 17: Draw Parallelograms to Clarify their Attributes, and Define Parallelograms Based on those Attributes, Lesson 18: Draw Rectangles and Rhombuses to Clarify their Attributes, and Define Rectangles and Rhombuses Based on Those Attributes, Lesson 19: Draw Kites and Squares to Clarify their Attributes, and Define Kites and Squares Based on those Attributes, Lesson 20: Classify Two-Dimensional Figures in a Hierarchy Based on Properties, Lesson 21: Draw and Identify Varied Two-Dimensional Figures from Given Attributes are lessons that pertain to this cluster. Some of the earlier lessons in this module need to be adjusted to meet the rigor of high school Geometry, but they include some useful resources.
- Georgia Standards of Excellence Curriculum Frameworks, Fifth Grade, Unit 5: 2D Figures has many lessons that pertain to this cluster.

**General Resources**

- Arizona K-6 Progression on Geometry is an informational document for teachers. This cluster is addressed on pages 17-18.
- van Hiele Model of Geometric Thinking is a pdf created by ODE that summarizes the van Hiele levels.

**References**


- Common Core Standards Writing Team. (2013, September 19). *Progressions for the Common Core State Standards in Mathematics (draft). Grades K–5*, Geometry. Tucson, AZ: Institute for Mathematics and Education, University of Arizona.
- McGraw, R. (March 2017). Quadrilaterals: Diagonals and area. *Mathematics Teacher*, *110(7),* 498-504.
- Szydlik, J., Parrott, A., & Belnap, J. (March 2016). Conversations to transform Geometry class. *Mathematics Teacher*, *109(7),* 507-513.

| STANDARDS | MODEL CURRICULUM (G.C.2-4) |
|---|---|
| **Geometry**<br>**CIRCLES**<br>**Understand and apply theorems about circles.**<br>**G.C.2** Identify and describe relationships among angles, radii, chords, tangents, and arcs and use them to solve problems. *Include the relationship between central, inscribed, and circumscribed angles and their intercepted arcs; inscribed angles on a diameter are right angles; the radius of a circle is perpendicular to the tangent where the radius intersects the circle.*<br>**G.C.3** Construct the inscribed and circumscribed circles of a triangle; prove and apply the property that opposite angles are supplementary for a quadrilateral inscribed in a circle.<br>(+) **G.C.4** Construct a tangent line from a point outside a given circle to the circle. | **Expectations for Learning**<br>In middle school, students have worked with measurements of circles such as circumference and area. In this cluster, students solve problems using the relationships among the arcs and angles created by radii, chords, secants, and tangents. They will also construct inscribed and circumscribed circles of a triangle. In Math 2, students extend their understanding of similarity to circles.<br><br>The student understanding of this cluster begins at van Hiele Level 1 (Analysis) and moves to Level 2 (Informal Deduction/Abstraction).<br><br>**ESSENTIAL UNDERSTANDINGS**<br>• The measure of an arc is equal to the measure of its corresponding central angle.<br>• The measure of an inscribed angle is half the measure of its corresponding central angle.<br>• Inscribed angles on a diameter of a circle are right angles (special case of inscribed angles).<br>• A tangent is perpendicular to the radius at the point of tangency.<br>• A secant is a line that intersects a circle at exactly two points.<br>• A circumscribed angle is created by two tangents to the same circle from the same point outside the circle.<br>• The center of the circumscribed circle is the point of concurrency of the perpendicular bisectors because it is equidistant from the vertices of the triangle.<br>• The center of the inscribed circle is the point of concurrency of the angle bisectors because it is equidistant from the sides of the triangle.<br>• While all triangles can be inscribed in a circle, a quadrilateral can be inscribed in a circle if and only if the opposite angles in the quadrilateral are supplementary.<br>*Continued on next page* |

| STANDARDS | MODEL CURRICULUM (G.C.2-4) |
|---|---|
| | **Expectations for Learning, continued**<br>**MATHEMATICAL THINKING**<br>• Use accurate mathematical vocabulary.<br>• Make connections between concepts, terms, and properties.<br>• Recognize, apply, and justify mathematical concepts, terms, and their properties.<br>• Solve mathematical and real-world problems accurately.<br>• Determine reasonableness of results.<br>• Consider mathematical units involved in a problem.<br>• Make sound decisions about using tools.<br><br>**INSTRUCTIONAL FOCUS**<br>• Given a diagram, identify radii, chords, secants, and tangents, and the arcs and angles formed by them.<br>• Solve mathematical and real-world problems involving angles and arcs formed by radii, chords, secants, and tangents.<br>• Construct the angle bisectors of a triangle to locate the incenter, and then construct the inscribed circle.<br>• Construct the perpendicular bisectors of a triangle to locate the circumcenter, and then construct the circumscribed circle.<br>• Provide an informal argument for why the opposite angles of an inscribed quadrilateral are supplementary based on the arcs the angles intercept and their corresponding central angles.<br>• Solve problems using the property that opposite angles are supplementary for a quadrilateral inscribed in a circle.<br>• (+) Construct a tangent line from a point outside a given circle to the circle.<br>*Continued on next page* |

| STANDARDS | MODEL CURRICULUM (G.C.2-4) |
| --- | --- |
| | **Content Elaborations**<br>**OHIO'S HIGH SCHOOL CRITICAL AREAS OF FOCUS**<br>• Math 1, Number 7, page 12<br><br>**CONNECTIONS ACROSS STANDARDS**<br>• Experiment with transformations in the plane (G.CO.1).<br>• Make geometric constructions (G.CO.12).<br>• Reason quantitatively, and use units to solve problems (N.Q.1-3). |

# INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.C.2-4)

## Instructional Strategies

*Note: The model curriculum is not a collection of lessons nor a full curriculum; it does not suggest pace, sequence, or amount of time spent on topics. It provides information about a topic related to the standards including ideas for examples, strategies for teaching, common misconceptions, and possible connections between topics. In addition, the emphasis of the model curriculum is on instruction not on assessment; therefore, the examples in the Instructional Strategies section are written for instructional purposes not necessarily as assessment items. Examples may need to be rewritten to accommodate the needs of each individual classroom.*

It is important to reinforce the precise definition of a circle from G.CO.1 as it is the foundation for the rest of the learning in this cluster.

***Standards for Mathematical Practice***

*This cluster focuses on but is not limited to the following practices:*

***MP.1** Make sense of problems and persevere in problem solving.*

***MP.3** Construct viable arguments and critique the reasoning of others.*

***MP.5** Use appropriate tools strategically.*

### VAN HIELE CONNECTION

In Math 1 students are expected to move from Level 1(Analysis) to Level 2 (Informal Deduction/Abstraction). In van Hiele Level 1 students may list all the properties of a shape, but not see the relationships between properties. Whereas van Hiele Level 2 can be characterized by the student doing some or all of the following:

- noticing relationships within and between properties and figures;
- forming meaningful definitions;
- justifying reasoning using informal arguments;
- drawing logical maps and diagrams; and/or
- not yet understanding formal deduction.

This cluster when combined with G.CO.9-11 will also move into Level 3 (Deduction) where students start to construct proofs and understand the necessity of proofs.

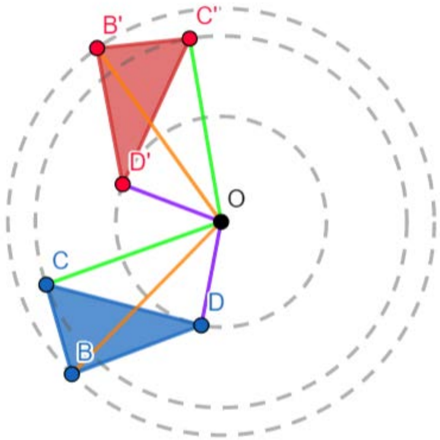
See the van Hiele pdf on ODE's website for more information about van Hiele levels.

# INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.C.2-4)

## RELATIONSHIPS AMONG PARTS OF A CIRCLE

Students should be able to identify and describe relationships among angles, radii, chords, secants, tangents, and arcs to solve problems. For example, students should be able to find measures of inscribed angles given the measure of the intercepted arc. For angles with a vertex outside of the circle, only the case of two tangents needs to be considered. A case formed by two secants or a secant and a tangent could be explored as an extension. Also, use properties of congruent triangles and perpendicular lines to prove theorems about diameters, radii, chords, and tangent lines when appropriate.

Although students should have some exposure to proving theorems such as Chord – Center, Arc – Chord Congruence, Secant Length, Angle – Chord, Angle – Secant, Tangent - Chord, Tangent - Secant, and Tangent Square, spend more time on the applicational aspect of the theorems; the emphasis on G.C.2 is on using the relationships of the parts of a circle to solve problems. *Note: Secant-Length and Tangent Square Theorem use similarity in their proofs and may not be appropriate for Math 1.*



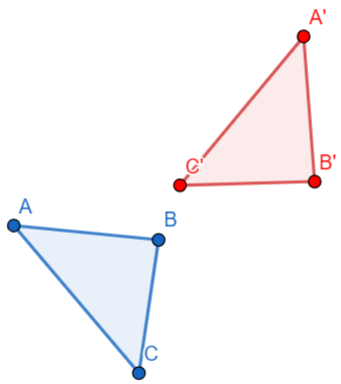

Note how the study of circles can be used when dealing with rotations of figures. For example, students should see that every point on a rotated figure moves along a different circle centered around the same center. They should also notice that the each vertex and its image are equidistant from the center of rotation since they are radii of the same rotated circle.

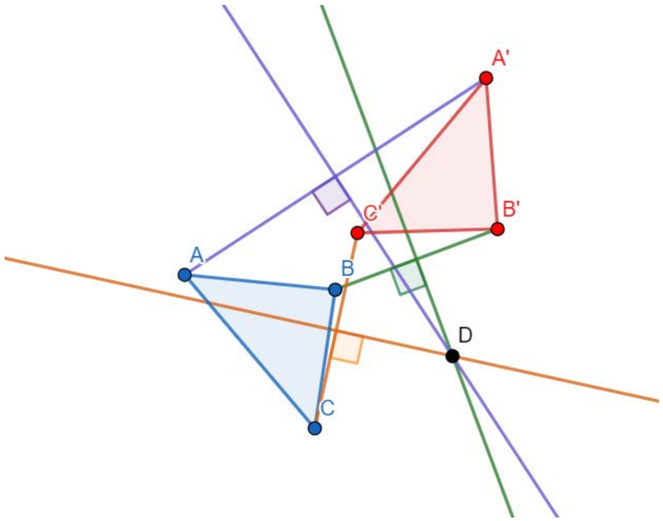
# INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.C.2-4)

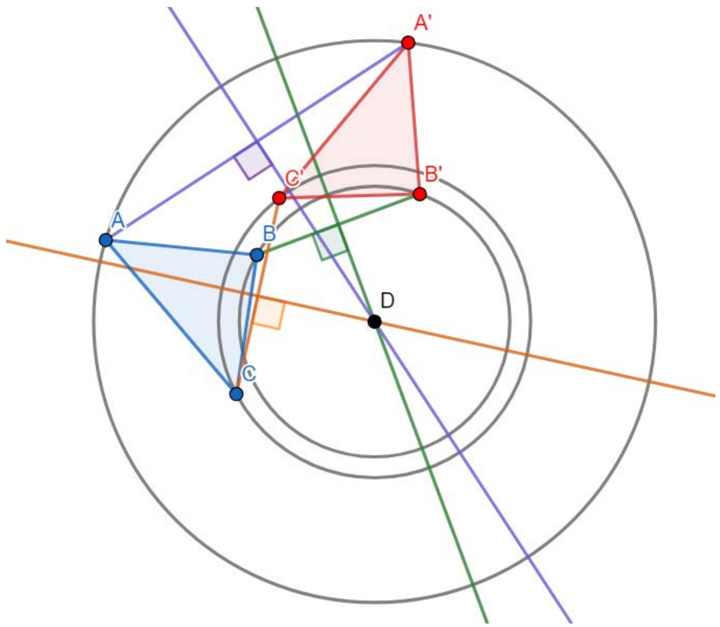
***EXAMPLE***

**Finding the Center of Rotation**

Find the center of rotation that maps $\triangle ABC$ onto $\triangle A'B'C'$.




*Discussion:* To find the center of rotation students must find the perpendicular bisector of two or more segments that connects parts of corresponding vertices. Students should justify why perpendicular bisectors are concurrent at the center of rotation. They should remember that each vertex and its corresponding vertex lies on the same circle. One way to justify the concurrency of perpendicular bisectors is to use the chords that connect the corresponding vertices and the idea that a perpendicular bisector includes all points equidistant from the endpoints. Since the endpoints of a chord are equidistant from the center of the circle, the perpendicular bisector must include the center of the circle. Other methods could be used such as using the perpendicular bisectors of each base of the isosceles triangles which are also angle bisectors. Use that to make connection between the angle bisectors and the point of concurrency.

GeoGebra can help show how to rotate a figure around a point.

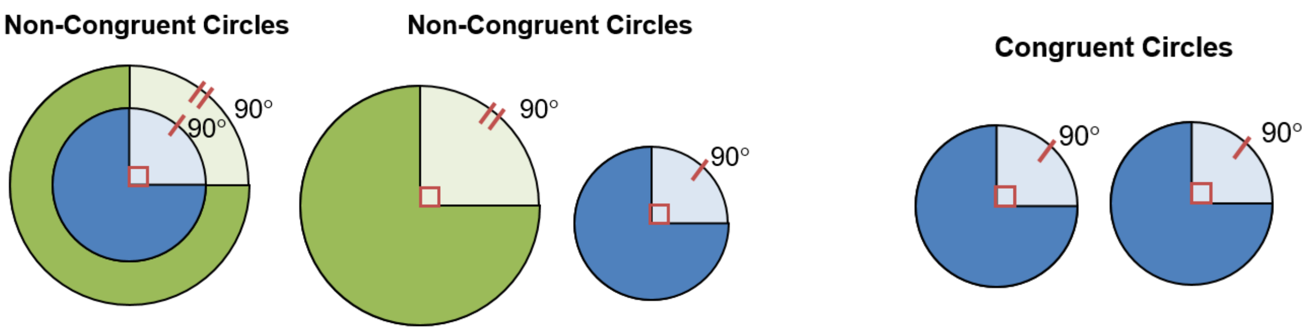
- Create a figure and a point of rotations.
- Create a slider.
- Then use the transform rotation button.
- When placing the rotation substitute $\alpha$ in for the number of degrees. That connects the slider to the object and center of rotation.

# INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.C.2-4)

## ARC LENGTH

Students need to understand that angle measure and angular arc measure are each an amount of rotation and are measured in the same units. Both the angular arc measure and the angle measure of the whole circle equal 360°. One way to show this is to relate a central angle of 90° to a quarter of a circle.

Discuss what it means for arcs to be congruent. Students need to understand that congruent central angles in non-congruent circles have non-congruent corresponding intercepted arcs. Therefore, congruent arcs will have equal angular measures and equal arc lengths and either be on the same or congruent circles.

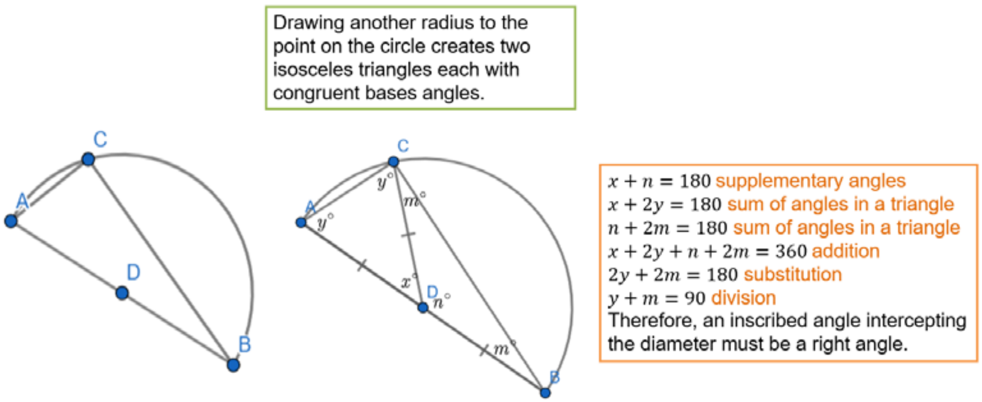



Many students incorrectly think that arc measure (the amount of rotation about the center) and arc length (a distance along the arc) are the same. Show students two concentric circles that have the same angular measure but different lengths.

Some students may incorrectly think that the arc of a larger size circle will have the same size and shape of a smaller circle. Have students map one circle onto the other (without using dilation), so they can see the overlap is not perfect because the curvatures are different.

# INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.C.2-4)

## Angle on the Diameter Inscribed on a Semi-Circle




Starting with the special case of an angle inscribed in a semicircle, use the fact that the angle sum of a triangle is 180° to show that this angle is a right angle. Dynamic geometry software could be a great tool for developing this concept. Students can grab a point on a circle and move it to see that the measure of the inscribed angle passing through the endpoints of a diameter is always 90°. It is important to note, that having students construct the situations themselves can help solidify understanding of the relationships. Once students gain an intuitive sense of this concept using software, it may be useful to have students prove it using Thales' Theorem: If $A$, $B$, and $C$ are distinct points on a circle where the line $AB$ is a diameter, then the angle $\angle ACB$ is a right angle. Then extend the result to any inscribed angle.

## CENTRAL AND INSCRIBED ANGLES

Have students explore the relationship between central angles and inscribed angles using dynamic geometric software. Once students observe that the inscribed angle is always half of the central angle, have them prove why its true.



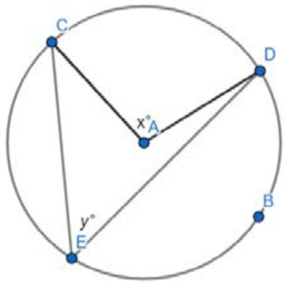
Ohio Department of Education

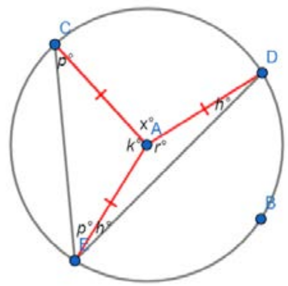
# INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.C.2-4)

***EXAMPLE***

**Proving the Relationship between Inscribed and Central Angles**

Prove that the measure of the inscribed angle $\angle CED$ is half of the central angle $\angle CAD$.




*Discussion:* One way to prove this theorem is to draw a radius of the circle through points $A$ and $E$. This makes two isosceles triangles: $\triangle CAE$ and $\triangle DAE$. Since the base angles of isosceles triangles are congruent, the base angles can be labeled with the same letter. By the Angle Addition Postulate $m\angle CEA + m\angle DEA = m\angle CED$ or $p° + h° = y°$. Some applications of central angles could include an analog clock, a ferris wheel, a baseball park etc

Students should also realize that $x + k + r = 360°$ because the sum of the angles of a circle equals 360° and $k + 2p = 180°$ and $r + 2h = 180°$ because the sum of the angle of a triangle equals 180°. Then students can use algebraic manipulation.

$k + 2p = 180\ and\ r + 2h = 180, so$

$By\ the\ Substitution\ Property: k + 2p + r + 2h = 360$

$By\ the\ Associative\ Property: k + r + 2p + 2h = 360$

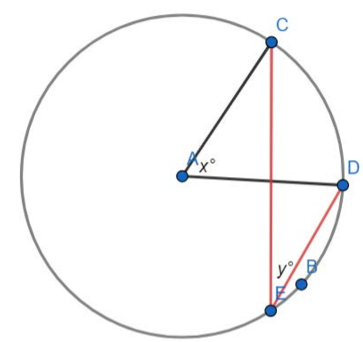
$Since: x + k + r = 360$

$By\ the\ Substitution\ Property: x + k + r = k + r + 2p + 2h$

$And: x = 2p + 2h$

$by\ the\ Distributive\ Property\ x = 2(p + h)$

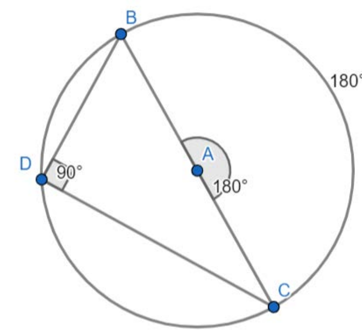
$By\ the\ Substituion\ Property\ since\ p + h = y, then\ x = 2y$




Using the diagram at the right, students can prove that $x° = 2y°$ or $y° = \frac{1}{2}x°$ because $x$ and $y$ intercept the same arc.

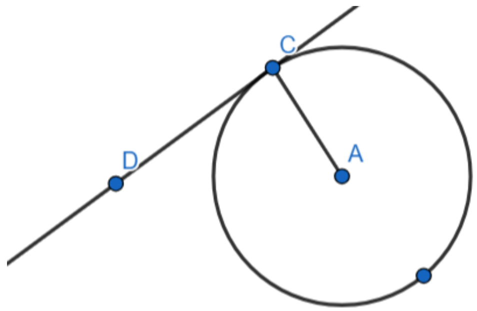
## INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.C.2-4)

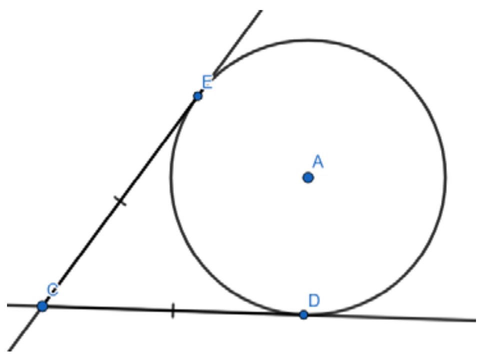
Students sometimes confuse the relationships of the measures of the intercepted arc and the inscribed angles and central angles. Use the simple case of a diameter to help students remember which angle is equal to the arc measure and which angle is half.




**Tangent Lines to a Circle**

Have students explore the relationship between a tangent line and the radius at the point of tangency using dynamic geometric software. They should discover that a tangent line to a circle is perpendicular to the radius. After students discover this, challenge them to prove it. Students may use rigid motions and the line of reflection or they may use an indirect proof. In addition, students can discover that from any external point, there are exactly two lines tangent to the circle. The distances from the exterior point to each point of tangency are equal.



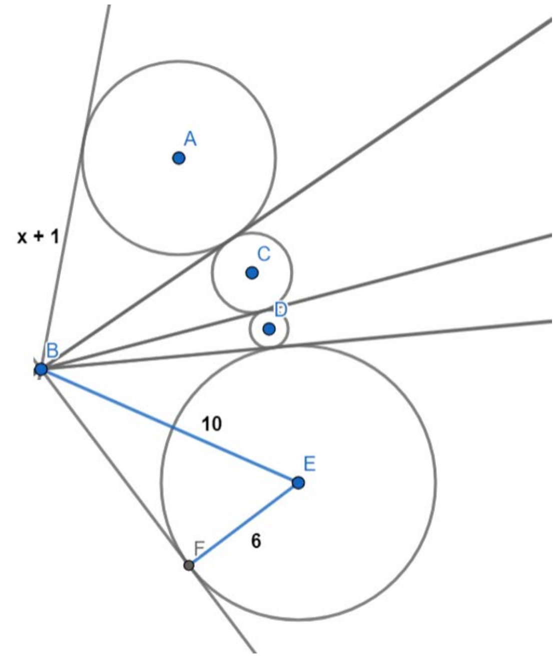




Students may incorrectly think they can tell by inspection whether a line intersects a circle at exactly one point. However, it may be beneficial to point out that a tangent line is the line perpendicular to a radius at the point where the radius intersects the circle.

# INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.C.2-4)

***EXAMPLE***

Find the value of *x*. Explain how you found your answer.




*Discussion:* Students should recognize that $\triangle BEF$ is a right triangle since a tangent line makes a perpendicular line with the radius at the point of tangency, so by the Pythagorean Theorem $\overline{BF}$ is 8 units. Since the distance from the exterior point to each point of tangency is the same, $\overline{BF}$ is congruent to the other segment that is tangent to Circle $E$, which is also 8 units. The other line segment that is tangent to Circle $E$ through Point $B$ is also tangent to Circle $D$ though Point $B$. Therefore, the other line segment that is tangent to Circle $D$ through Point $B$ is also congruent to $\overline{BF}$ by the Transitive Property. Since Circle $C$ and Circle $A$ share tangent lines through Point $B$, the tangent of Circle $A$ through Point $B$ is also 8, so $x + 1 = 8$, and $x = 7$.

**Circumscribed Angles**

Explore that a circumscribed angle and the associated radii at the points of tangency create a kite with two right angles. Once students have a firm grasp of the fact that a tangent line creates a right angle to the radius at the point of tangency, they should be able to prove that the circumscribed angle and the central angle intercepting the same arc are supplementary.

## CONSTRUCTIONS WITH CIRCLES

Standards G.C.3 and G.C4 connect with G.CO.12 and G.CO.13.

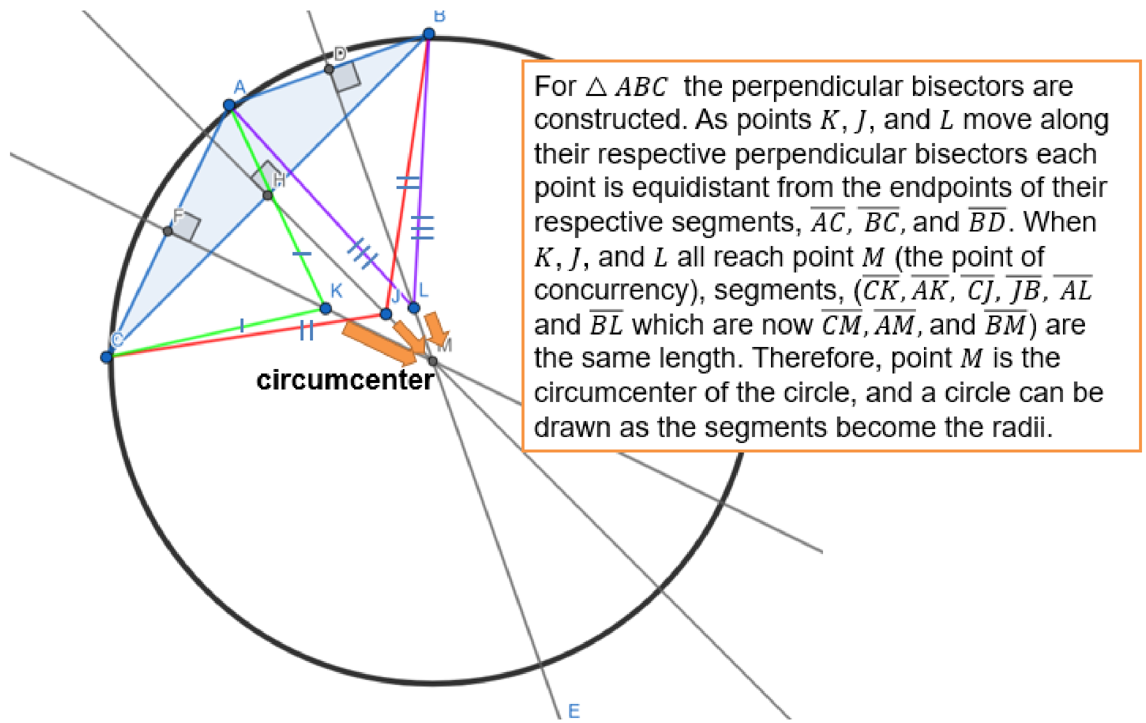
**Construct the Inscribed and Circumscribed Circles of a Triangle**

Use formal geometric constructions to construct perpendicular bisectors and angle bisectors of a given triangle. Their intersections are the centers of the circumscribed and inscribed circles. The center of the circumscribed circle is the circumcenter, and the center of the inscribed circle is incentre. Have students explain why the perpendicular bisectors create the circumcenter of the circumscribed circle and the angle bisectors create incenter of the inscribed circle.

# INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.C.2-4)

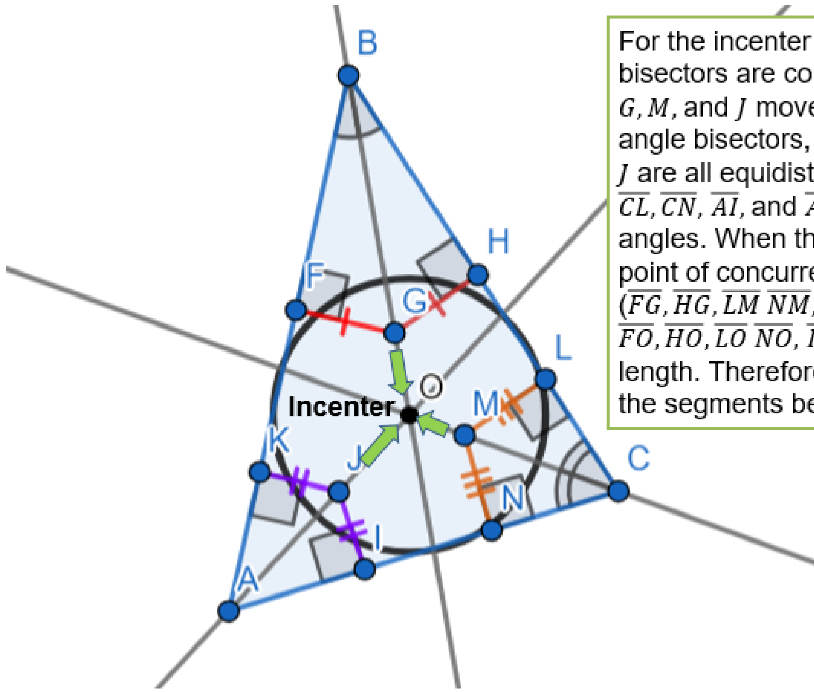
Relate the perpendicular bisectors of the sides of a triangle to the theorem that states any point on the perpendicular bisector of a line segment is the same distance from the two endpoints. Students should then reason through this idea to connect it to the circumcenter of the circle.

Students can use geometric software such as GeoGebra to explore the circumcenter and incenter of a circle by dragging points along perpendicular and angle bisectors.



# INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.C.2-4)

To understand the location of the incenter, students can relate the angle bisectors of a triangle to the theorem that states any point that lies on the bisector of an angle is equidistant from the sides of the angle.

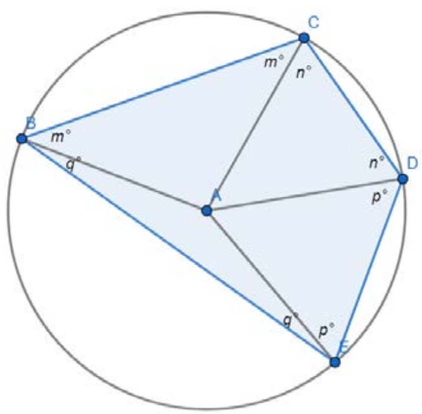


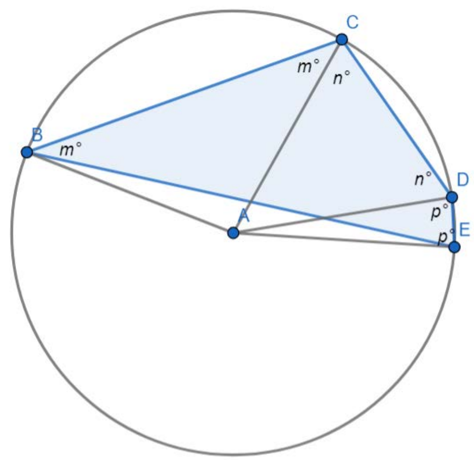

For the incenter of $\triangle ABC$, the angle bisectors are constructed. As points $G, M,$ and $J$ move along their respective angle bisectors, $\overline{BO}$, $\overline{OC}$, and $\overline{OA}$, $G, M,$ and $J$ are all equidistant from the rays, $\overline{BF}$, $\overline{BH}$, $\overline{CL}$, $\overline{CN}$, $\overline{AI}$, and $\overline{AK}$ of their respective angles. When they all reach point $O$ (the point of concurrency), all the segments, ($\overline{FG}$, $\overline{HG}$, $\overline{LM}$ $\overline{NM}$, $\overline{IJ}$, and $\overline{KJ}$ which are now $\overline{FO}$, $\overline{HO}$, $\overline{LO}$ $\overline{NO}$, $\overline{IO}$, and $\overline{KO}$ ) are the same length. Therefore, a circle can be drawn as the segments become the radii of the circle.

# INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.C.2-4)

**Opposite Angles are Supplementary for Quadrilaterals Inscribed in a Circle**
There are three possible methods of proving opposite angles of an inscribed quadrilateral are supplementary.

**Method 1:** Dissect an inscribed quadrilateral into four isosceles triangles formed by radii and the sides of the quadrilateral.



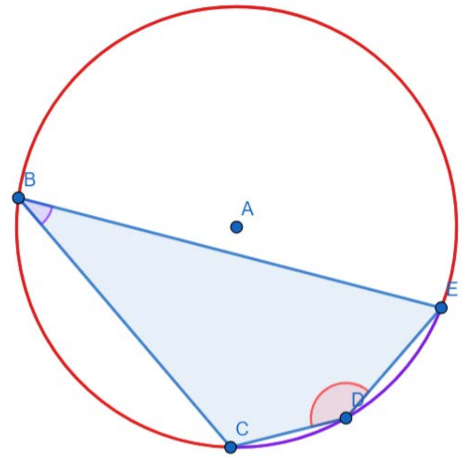




*Discussion:* Make sure to show different examples of quadrilaterals inscribed in a circle, so students see that the proof works despite the shape of the quadrilateral. Focus on the diagram on the left with the angles' measures indicated to prove that opposite angles of the inscribed quadrilateral are supplementary.

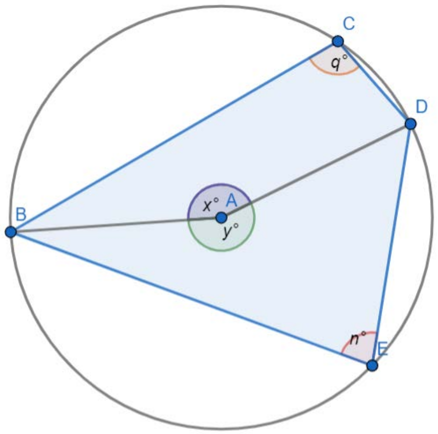
$\angle CDE$ and $\angle CBE$ are opposite angles.
$\angle CDE = p + n$ and $\angle CBE = m + q$ (Angle Addition Postulate)
$2m + 2n + 2p + 2q = 360$ (The sum of all angles in the quadrilateral equals 360°.)
$m + n + p + q = 180$ (Multiplication/Division Property of Equality)
$(m + q) + (n + p) = 180$ (Associative Property)
$\angle CDE + \angle CBE = 180$, (Substitution)
Therefore, opposite angles are supplementary.

## INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.C.2-4)

**Method 2:**
Use a of pair opposite angles of a quadrilateral inscribed in a circle.
A quadrilateral $BEDC$ is inscribed in a circle $A$. Angle $CDE$ and angle $EBC$ are opposite angles.
$m\widehat{CDE} + m\widehat{EBC} = 360°$ (The sum of all arc angular measures in a circle equal 360°.)
$m\widehat{EBC} = 2m\angle CDE$ (The measure of the inscribed angle is ½ the measure of the intercept arc.)
$m\widehat{CDE} = 2m\angle EBC$ (The measure of the inscribed angle is ½ the measure of the intercept arc.)
$2m\angle CDE + 2m\angle EBC = 360°$ (Substitution Property)
$m\angle CDE + m\angle EBC = 180°$ (Multiplication Property of Equality)
Therefore, the sum of measures of opposite angles in a quadrilateral is 180°, or the angles are supplementary.




**Method 3:**
Deconstruct a quadrilateral inscribed in a circle into two quadrilaterals that have one angle lying on the center and the angle measures as indicated.

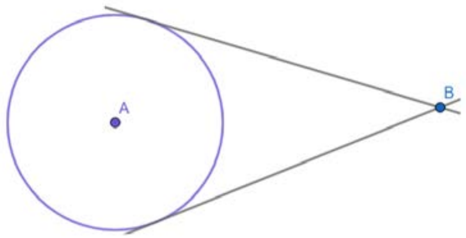



$\angle x + \angle y = 360°$ (2 angles that make up a circle equal 360°)
$\angle q = \frac{1}{2}\angle y$ (An inscribed angle is ½ of its corresponding central angle.)
$\angle y = 2\angle q$ (Multiplication Property of Equality)
$\angle n = \frac{1}{2}\angle x$ (An inscribed angle is ½ of its corresponding central angle.)
$\angle x = 2\angle n$ (Multiplication Property of Equality)
$2\angle q + 2\angle n = 360°$ (Substitituion Property)
$\angle q + \angle n = 180°$ (Multiplication/Division Property of Equality)
Therefore, two opposite angles of a quadrilateral inscribed in a circle are supplementary.

# INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.C.2-4)

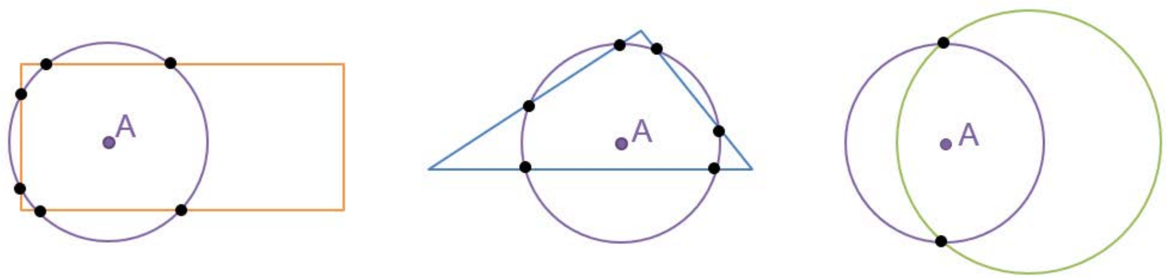
**Construct a Tangent Line from a Point Outside the Given Circle to the Circle (+)**
One way to help students reason through this construction is by using questioning techniques forcing the students to think through the logic of the construction.

***EXAMPLE***
Construct two distinct lines tangent to the circle $A$ from a point $B$ that is outside of the given circle.



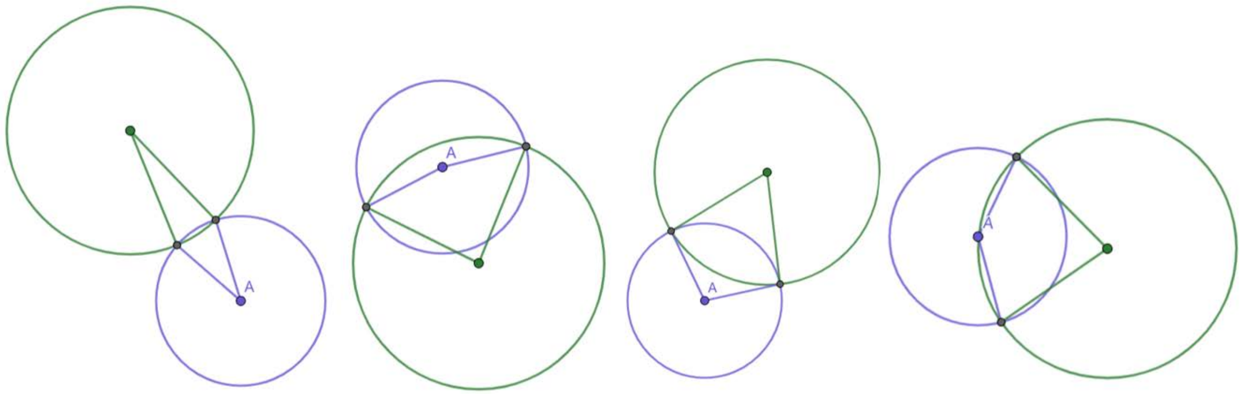

*Discussion:* [This activity can be demonstrated by using two transparencies, by geometric software or even by having students drag shapes on PowerPoint, Word, or on a Smartboard.] Start by having the students sketch the end product, so they keep the goal in mind. A sample conversation is recorded below.

*Teacher:* This picture shows two tangent lines passing through a Point B that is outside the circle. Tell me what you know about tangent lines.
*Student:* A tangent intercepts a circle at only one point.
*Teacher:* What is that point called?
*Student:* The point of tangency.
*Teacher:* So, if you have two tangent lines, how many points of tangency should you have?
*Student:* Two.
*Teacher:* What are some shapes that we can construct that will intersect the circle at a maximum of two points? [Have students use geometric software to move some different shapes around to show that only circles can intersect another circle at a maximum of two points.]
*Student:* Circles.

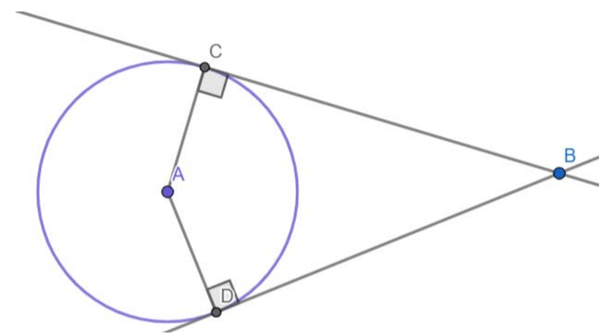


## INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.C.2-4)

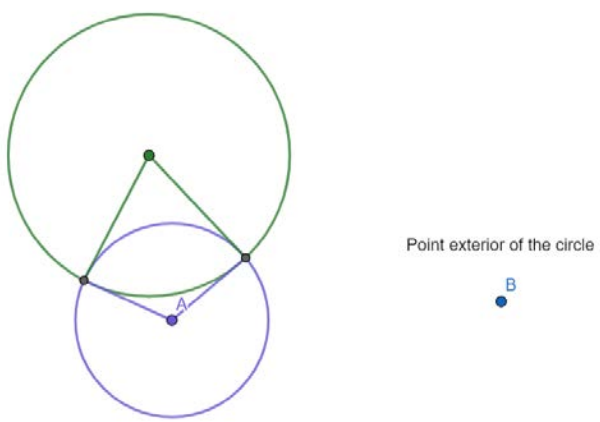
*Teacher:* As you move the circle around what do you notice about the intersection points?




*Student:* They create radii using the center of both circles.
*Teacher:* Now recall the goal of our task to construct two tangent lines from a point outside the circle. How do the tangent lines and radii relate to one another?
*Student:* A tangent line should be perpendicular to the radius of a circle at the point of tangency.

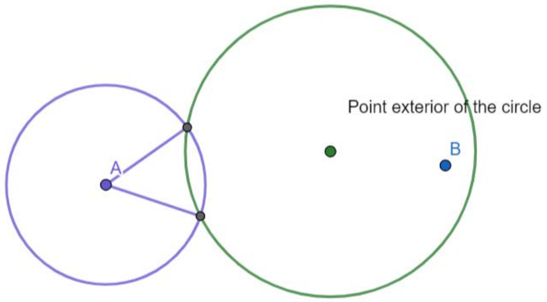


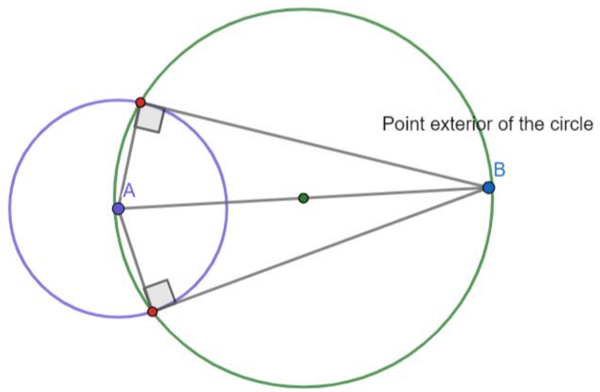

*Teacher:* So, you need your final construction to look like this (pictured on the right):
*Teacher:* So which circle's radius should be the focus?
*Student:* The original circle.
*Teacher:* How do you think you will get there with your intersecting circles? Will any intersection points of two intersecting circles create points of tangency?



# INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.C.2-4)

*Student:* No, they will have to line up.
*Teacher:* What do you mean? Will this work? Where do you have to place the circle and how big does the circle have to be in order to ensure that at the point of tangency the radii and the tangent lines will form right angles? Can you explain?




*Student:* What about if we create right angles in the new (green) circle and then line them up? We already learned the theorem that an inscribed triangle that is built on the diameter is a right angle. So we can create right triangles using that theorem. Therefore, the new circle (green) would have to pass through the center, $A$, of the original circle (purple) and the point $B$ exterior to the original circle.

*Teacher:* So if you didn't have geometric software, how would you construct two tangent lines using a compass and straight edge?

*Student:* You would have to create the diameter of a new circle connecting the exterior point (Point $B$) and the center of the original circle. Then find the midpoint of the diameter, and use half the diameter as the radius to create the new intersecting (green) circle. Next you would have to find the intersection points of the two circles and connect those points to the given exterior point. Since the constructed triangles are right triangles, their legs are perpendicular. Then you will have correctly constructed a tangent line going the point exterior to the circle that is perpendicular to the circle at the point of tangency.

Constructing a tangent to a circle from a given point could also be addressed when inscribing triangles, quadrilaterals, and hexagons.

# INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.C.2-4)

## Instructional Tools/Resources

*These tools and resources may be helpful for instruction. Before use, they should be evaluated at the local level to determine their appropriateness for instruction.*

**Manipulatives/Technology**

- Tracing paper (patty paper)
- Transparencies
- Graph paper
- Ruler
- Protractor
- Compass
- Mira
- Computer dynamic geometry software (such as Geometer's Sketchpad®, Desmos, Cabri®, or Geogebra®).
- Graphing calculators and other handheld technology such as TI-Nspire™.
- Geogebra Geometry is the Geometry application of Geogebra.

**Similar Circles**

- Similar Circles by Illustrative Mathematics is a task that has students prove that circles are similar using a coordinate plane. There is an attached Geogebra file.

**Tangents**

- Tangent Lines and the Radius of a Circle is a task by Illustrative Mathematics that has students explain why the tangent line is perpendicular to the radius at the tangent line. It explains two possible approaches: reason directly from past results of plane geometry including the Pythagorean Theorem and a proof by contradiction or use the idea of rigid motions of the plane and the line of reflection.
- Neglecting the Curvature of the Earth is an Illustrative Mathematics task where students apply geometric concepts such as properties of tangents to modeling situations.

**Angles Inscribed in Circles**

- Right Triangles Inscribed in Circles and Right Triangles Inscribed in Circles II are Illustrative Mathematics tasks where students use properties of triangles to show that a triangle inscribed in a circle where one side is the diameter is a right triangle.

## INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.C.2-4)

**Constructing the Inscribed and Circumscribed Circles of a Triangle**

- Locating Warehouse by Illustrative Mathematics is a task that has students inscribe a circle given a context.
- Placing a Fire Hydrant by Illustrative Mathematics is a task that has students apply properties of circumscribing a circle given a context.
- Inscribing and Circumscribing Right Triangles by Mathematics Assessment Project has students find the relationship between the radii of inscribed and circumscribed circles of right triangles.

**Curriculum and Lessons from Other Sources**

- EngageNY, Geometry, Module 5, Topic A, Lesson 1: Thales' Theorem, Lesson 2: Circles, Chords, Diameters, and Their Relationships, Lesson 3: Rectangles Inscribed in Circles, Lesson 4: Experiments with Inscribed Angles, Lesson 5: Inscribed Angle Theorem and Its Applications, Lesson 6: Unknown Angle Problems with Inscribed Angles in Circles are lessons that pertain to this cluster.
- EngageNY, Geometry, Module 5, Topic B, Lesson 7: The Measure of the Arc, Lesson 8: Arcs and Chords are lessons that pertain to this cluster.
- EngageNY, Geometry, Module 5, Topic C, Lesson 11: Properties of Tangents, Lesson 12: Tangent Segments, Lesson 13: The Inscribed Angle Alternate—a Tangent Angle, Lesson 14: Secant Lines; Secant Line that Meet Inside a Circle, Lesson 15: Secant Angle Theorem, Exterior Case, Lesson 16: Similar Triangle in Circle-Secant (or Circle-Secant-Tangent) Diagrams are lessons that pertain to this cluster.
- Mathematics Vision Project, Geometry, Module 5: A Geometric Perspective has many lessons that pertain to this cluster.
- Georgia Standards of Excellence Curriculum Frameworks, Geometry, Unit 4: Circles and Volume has many lessons that pertain to this cluster.
- The Geometry of a Circle by Dennis Kapatos from Buffalo State is a 5-day unit plan using Geometers Sketchpad and other manipulatives. Lesson 2 applies tangents to a real-life situation of a satellite. Lesson 3 uses properties of chords and arcs to solve problems. Lesson 4 explores properties of inscribed angles and quadrilaterals. Lesson 5 uses graphs to explore geometric relationships.

**General Resources**

- Arizona 7-12 Progression on Geometry is an informational document for teachers.This cluster is addressed on page 17.
- High School Coherence Map by UnboundED is a map of the relationships among high school mathematics standards. Please note: This resource is aligned to the Common Core not Ohio's Learning Standards. Although the tool is useful, it needs to be checked with respect to Ohio's Learning Standards.
- van Hiele Model of Geometric Thinking is a pdf created by ODE that summarizes the van Hiele levels.

**References**


- Common Core Standards Writing Team. (2016, March 24). *Progressions for the Common Core State Standards in Mathematics (draft). Grades 7–8, HS, Geometry*. Tucson, AZ: Institute for Mathematics and Education, University of Arizona.

| STANDARDS | MODEL CURRICULUM (G.GPE.5, 7) |
| --- | --- |
| **Geometry**<br>**EXPRESSING GEOMETRIC PROPERTIES WITH EQUATIONS**<br>**Use coordinates to prove simple geometric theorems algebraically and to verify specific geometric statements.**<br>**G.GPE.5** Justify the slope criteria for parallel and perpendicular lines, and use them to solve geometric problems, e.g., find the equation of a line parallel or perpendicular to a given line that passes through a given point.<br>**G.GPE.7** Use coordinates to compute perimeters of polygons and areas of triangles and rectangles, e.g., using the distance formula.★ | **Expectations for Learning**<br>In middle school, students find the distance between two points in a coordinate system; work with linear functions; solve linear equations; and apply the Pythagorean Theorem in the coordinate system. In addition, they use square root symbols to represent solutions to equations, and they evaluate square roots of rational numbers. In this cluster, students use the coordinate system to justify slope criteria for parallel and perpendicular lines and compute perimeters and areas of geometric figures. In Math 2, these strategies are used for proof of geometric relationships and properties and partitioning line segments.<br><br>The student understanding of this cluster begins at van Hiele Level 2 (Informal Deduction/Abstractions) and moves to Level 3 (Deduction).<br><br>**ESSENTIAL UNDERSTANDINGS**<br>• Coordinate proof is a method that uses algebraic techniques to prove geometric theorems and properties.<br>• The slopes of parallel lines are equal, and the product of the slopes of perpendicular lines is $-1$, except for horizontal and vertical lines.<br><br>**MATHEMATICAL THINKING**<br>• Use accurate mathematical vocabulary to represent geometric relationships.<br>• Make connections between terms and formulas.<br>• Recognize, apply, and justify mathematical concepts, terms, and their properties.<br>• Compute using strategies or models.<br>• Determine reasonableness of results.<br>• Solve multi-step problems accurately.<br>• Discern and use a pattern or structure.<br>*Continued on next page* |

| STANDARDS | MODEL CURRICULUM (G.GPE.5, 7) |
|---|---|
| | **Expectations for Learning, continued**<br>**INSTRUCTIONAL FOCUS**<br>• Know and use the distance formula.<br>• Justify the slope criteria for parallel and perpendicular lines.<br>• Write equations of parallel and perpendicular lines.<br>• Solve geometric problems using the slopes of parallel and perpendicular lines.<br>• Use coordinates to compute perimeters and areas.<br><br>**Content Elaborations**<br>**OHIO'S HIGH SCHOOL CRITICAL AREAS OF FOCUS**<br>• Math 1 Number 6, page 11<br><br>**CONNECTIONS ACROSS STANDARDS**<br>• Know precise definitions (G.CO.1).<br>• Prove geometric theorems (G.CO.9-10).<br>• Understand and apply theorems about circles (G.C.2-4).<br>• Represent and solve equations graphically (A.REI.10).<br>• Create equations that describe numbers or relationships (A.CED.2, 4).<br>• Relate parallel and perpendicular lines as a system of equations (A.REI.5). |

# INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (GPE.5, 7)

## Instructional Strategies

*Note: The model curriculum is not a collection of lessons nor a full curriculum; it does not suggest pace, sequence, or amount of time spent on topics. It provides information about a topic related to the standards including ideas for examples, strategies for teaching, common misconceptions, and possible connections between topics. In addition, the emphasis of the model curriculum is on instruction not on assessment; therefore, the examples in the Instructional Strategies section are written for instructional purposes not necessarily as assessment items. Examples may need to be rewritten to accommodate the needs of each individual classroom.*

***Standards for Mathematical Practice***

*This cluster focuses on but is not limited to the following practices:*

***MP.2** Reason abstractly and quantitatively.*

***MP.4** Model with mathematics.*

***MP.5** Use appropriate tools strategically.*

***MP.6** Attend to precision.*

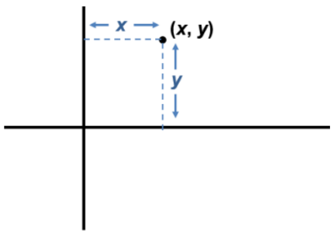
***MP.7** Look for and make use of structure.*

This cluster connects geometric to algebraic concepts by using the coordinate plane. The emphasis is on justification. The main skill that should be gleaned from these standards is for students to be able to explain reasons for their thinking and why/how something is true, much like in the ELA writing standards where *evidence* must be included for any claim. Students continue to use precise language and relevant vocabulary to justify steps in their work and construct viable arguments that defends their method of solution.

Formerly, students found the distance between two points using the Pythagorean Theorem. Now, using the Pythagorean Theorem, students should understand how to derive the Distance Formula. They should be able to explain how the Pythagorean Theorem, the slope formula, and the Distance Formula are connected.

TIP!

Show the dispalcements of points within a quadrant rather than on the *x*- and *y*- axes, so students can view the the points' distance *from* the axes in addition to its distances *along* the axis. Viewing an ordered pair as displacement of a point from the axes is more useful in proving theorems than viewing the ordered pair as horizontal and vertical distances along the axes.



# INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (GPE.5, 7)

## VAN HIELE CONNECTION

In Math 1 students are expected to move from Level 2 (Informal Deduction/Abstraction) toward Level 3 (Deduction). Van Hiele Level 2 can be characterized by the student doing some or all of the following:

- analyzing relationships within and between transformations;
- algebraically expressing transformations: reflections about the *x*- and *y*-axes, and rotations of 45°, 90°, and 180° about the origin; *Note: In high school, students should be given an opportunity to experience rotations around any point (not just the origin) and onto any angle, not just multiples of 90. Also, reflections are not limited to the x-and y-axes but could be about any line.*
- understanding and justifying how to use perpendicular bisectors of corresponding points to locate the turn center of a rotation;
- understanding how the composition of two reflections can be a translation or a rotation; and/or
- following a rationale if shown proof.

Van Hiele Level 3 can be characterized by the student doing some or all of the following:

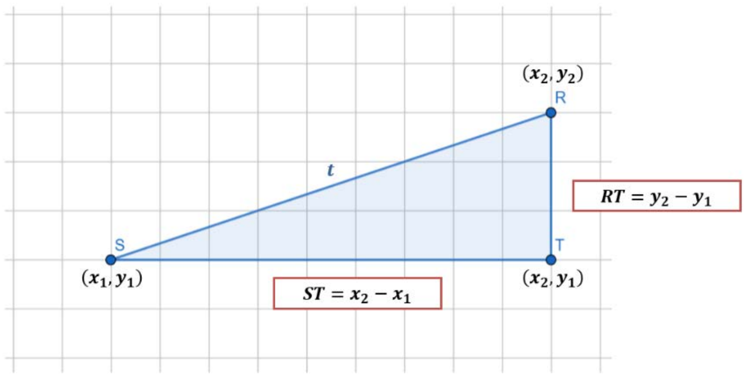
- understanding and creating formal proofs using transformations.

See the van Hiele pdf on ODE's website for more information about van Hiele levels.

## THE DISTANCE FORMULA

In Grade 8, students used the Pythagorean Theorem to find the distance between two points. Now students need to generalize those ideas to derive the Distance Formula.

Students should be able to see that if point $S$ is located at $(x_1,\ y_1)$ and point $R$ is located at $(x_2,\ y_2)$, then point $T$ must be located at $(x_2,\ y_1)$ because $\overline{ST}$ is a horizontal line segment and $\overline{RT}$ is a vertical line segment. Therefore $ST$ must be $x_2 - x_1$ and $RT$ must be $y_2 - y_1$ .




Points $S$, $R$ and $T$ form a right triangle, to which the Pythagorean Theorem can be applied $(x_2 - x_1)^2 + (y_2 - y_1)^2 = t^2$ which can be rewritten as $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = t$ or the Distance Formula. Student in Math 1 should be able to not only use the distance formula but explain how its derived from and its relationship to the Pythagorean Theorem.

# INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (GPE.5, 7)

## SLOPE

### Slope Formula

Using what students learned in Grade 8 about slope (8.EE.g) and connect it to the Slope Formula.

Students may claim that a vertical line has an infinite slope. This suggests that infinity is a number which is incorrect. The correct claim is that the slope of a vertical line is undefined. Applying the slope formula to a vertical line leads to division by zero which is makes the quotient undefined. Also, the slope of a horizontal line is 0. Students often say that the slope of vertical and/or horizontal lines is "no slope," which is also incorrect.

### Parallel and Perpendicular Lines

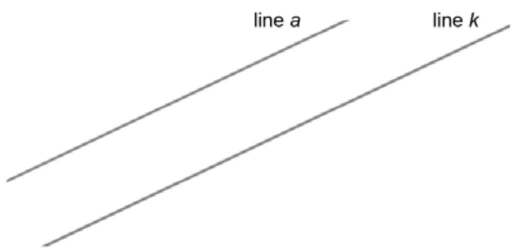
Discover pairs of parallel lines have equal slopes and pairs of perpendicular lines have slopes that are negative reciprocals of each other and have a product of –1, except for a pair of vertical and horizontal lines.

Students need to provide a convincing argument for the slope criteria for parallel and perpendicular lines. This can be done via algebraic, geometric, transformational, or indirect proof.

# INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (GPE.5, 7)

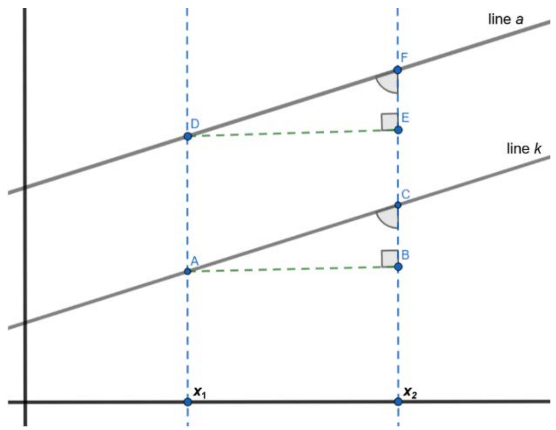
***EXAMPLE***

Prove that non-vertical parallel lines have the same slope.




*Discussion:*

1. Select two x-coordinates such that $x_1 \neq x_2$.
2. 
   - On line $k$ and line $a$ for all points with $x_1$ as their $x$-coordinate, label them as $A$ and $D$.
   - On line $k$ and line $a$ for all points with 2 as their $x$-coordinate, label them as $C$ and $F$.
   - Label the third vertex of each triangle as $B$ and $E$.
3. 
   - If $k \parallel a$ and $\overleftrightarrow{FB}$ is a transversal, the $\angle DFE \cong \angle ACB$ as corresponding angles
   - $\angle DEF \cong \angle ABC$ since both are right angles.
   - Both horizontal legs, $\overline{DE}$ and $\overline{AB}$ are $\cong$ because points are on the same vertical lines were chosen and therefore have the same distance, $x_2 - x_1$.
   - Therefore, $\triangle ABC \cong \triangle DEF$ by AAS.
   - $\overline{BC} \cong \overline{EF}$ by CPCTC.
4. Using the slope formula, the slope of line $k$ is $m_1 = \frac{BC}{AB}$ and the slope of line $a$ is $m_2 = \frac{FE}{DE}$.
5. Since $BC = EF$ and $BD = DE$, the slopes are equal or $m_1 = m_2$.




Use slopes and the Euclidean Distance Formula to solve problems about figures in the coordinate plane.

# INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (GPE.5, 7)

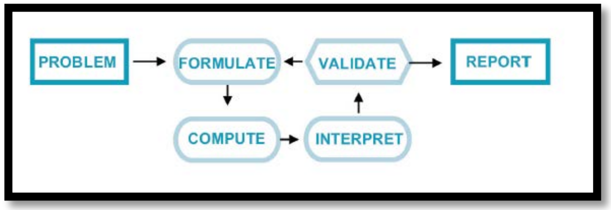
***EXAMPLE***

**Analyzing Parallel and Perpendicular Lines**

**a.** Given a line and a point not on the line, find an equation of the line through the point that is parallel to the given line.
**b.** Given a line and a point not on the line, find an equation of the line through the point that is perpendicular to the given line.
**c.** Compare finding equations of lines that are parallel to a given line to those perpendicular to the same line.

## MODELING

G.GPE.7 is a modeling standard. See page 13 for more information about modeling.




## COMPUTING PERIMETERS AND AREAS OF POLYGONS USING COORDINATES

Students should apply their knowledge of perimeter and area of polygons to the coordinate plane. This includes the application of the Distance Formula.

Use the Distance Formula to find the length of each side of a polygon whose vertices are known, and compute the perimeter of that figure.

***EXAMPLE***

An airplane flies from Cleveland Hopkins International Airport located at approximately 41.41°N, 81.85°W to Chicago O'Hare International Airport located at approximately 41.97°N and 87.90°W and then to Dallas/Fort Worth International Airport located at approximately 32.90°N and 97.04°W and then back to Cleveland Hopkins International Airport. Find the total distance the airplane flew.

*Discussion:* Point out to the students that in this situation, the use of Distance Formula can help to find the approximate distance that the airplane flew. The reason it is just an approximation is because the Distance Formula only applies to two-dimensional planes (not the flying kind); however the earth is a three-dimensional sphere. Explain to students that in navigation the Haversine Formula is used which takes into account the distance along the circular arc between two points on a sphere using their longitudes and latitudes. Also explain that flight paths are rarely the shortest distance between two points since they have to accommodate different flight patterns for safety reasons.

## INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (GPE.5, 7)

***EXAMPLE***

Given the vertices of a triangle or a parallelogram, find the equation of a line containing the altitude to a specified base and the point of intersection of the altitude and the base. Use the Distance Formula to find the length of that altitude and base, and then compute the area of the figure.

As students become proficient in using slopes and the Distance Formula to solve the kinds of problems suggested in this cluster, allow them to solve more complex problems with the aid of dynamic geometric software.

## Instructional Tools/Resources

*These tools and resources may be helpful for instruction. Before use, they should be evaluated at the local level to determine their appropriateness for instruction.*

**Manipulatives/Technology**

- Graph paper
- Patty paper
- Scientific or graphing calculators
- Dynamic geometry software (Geometer's Sketchpad®, Cabri®, Desmos®, or Geogebra®)

**Polygons on the Coordinate Grid**

- Squares on a Coordinate Grid by Illustrative Mathematics is a task where students use the Pythagorean Theorem to construct squares of different sizes on a coordinate grid.
- Triangle Perimeters by Illustrative Mathematics is a task where students apply the Pythagorean Theorem to calculate the distances and areas.
- Equal Area Triangles on the Same Base I by Illustrative Mathematics is a task where students create different triangles with the same areas.

**Parallel and Perpendicular Lines**

- Equations of Polygon Sides by Greta is a Desmos activity where students apply their knowledge of parallel and perpendicular equations to form quadrilaterals.
- Distance Between Parallel Lines by Patti is a Desmos activity to help students explore that parallel lines are equidistant at any point.
- Parallel and Perpendicular Lines by Bob Lochel is a Desmos activity where students explore parallel and perpendicular lines.
- Slope is a Mathematics Instructional Plan for Geometry created by Virginia's Department of Education.

# INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (GPE.5, 7)

**Parallel and Perpendicular Lines, continued**

- Equal Area Triangles on the Same Base II by Illustrative Mathematics is a task where students create different triangles with the same areas by applying properties of parallel lines.
- Parallel Lines in the Coordinate Plane by Illustrative Mathematics is a task where students prove the slope criterion for parallel lines.
- Triangles Inscribed in a Circle by Illustrative Mathematics is a task where students apply properties of perpendicular lines.
- Unit Squares and Triangles by Illustrative Mathematics is a task where students may apply properties of perpendicular lines as a solution strategy.
- When are Two Lines Perpendicular? by Illustrative Mathematics is a task where students explore the slope of perpendicular lines.

**Distance Formula**

- Introduction to Distance Formula by Lee-Anne Patterson is a Desmos activity that connects the Pythagorean Theorem to the Distance Formula.

**Curriculum and Lessons from Other Sources**

- EngageNY, Geometry, Module 4, Topic A, Lesson 1: Searching a Region in the Plane, Lesson 2: Finding Systems of Inequalities That Describe Triangular and Rectangular Regions, Lesson 3: Lines that Pass Through Regions, Lesson 4: Designing a Search Robot to Find a Beacon are lessons that pertain to this cluster.
- EngageNY, Geometry, Module 4, Topic B, Lesson 5: Criterion for Perpendicularity, Lesson 6: Segments that Meet at Right Angles, Lesson 7: Equations for Lines Using Normal Segments, Lesson 8: Parallel and Perpendicular Lines are lessons that pertain to this cluster.
- EngageNY, Geometry, Module 4, Topic C, Lesson 9: Perimeter and Area of Triangles in the Cartesian Plane, Lesson 10: Perimeter and Area of Polygonal Regions in the Cartesian Plane, Lesson 11: Perimeters and Areas of Polygonal Regions Defined by Systems of Inequalities are lessons that pertain to this cluster.
- Mathematics Vision Project, Geometry, Module 6: Connecting Algebra and Geometry has a task that pertains to this cluster.
- Georgia Standards of Excellence Curriculum Frameworks, Geometry, Unit 5: Geometric and Algebraic Connections has many tasks that align to this cluster.

**General Resources**

- Arizona 7-12 Progression on Geometry is an informational document for teachers. This cluster is addressed on pages 17-18.
- Arizona High School Progression on Modeling is an informational document for teachers.
- High School Coherence Map by UnboundED is a map of the relationships among high school mathematics standards. Please note: This resource is aligned to the Common Core not Ohio's Learning Standards. Although the tool is useful, it needs to be checked with respect to Ohio's Learning Standards.
- van Hiele Model of Geometric Thinking is a pdf created by ODE that summarizes the van Hiele levels.

# INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (GPE.5, 7)

**References**


- Common Core Standards Writing Team. (2016, March 24). *Progressions for the Common Core State Standards in Mathematics (draft). Grades 7–8, HS, Geometry*. Tucson, AZ: Institute for Mathematics and Education, University of Arizona.
- Common Core Standards Writing Team. (2013, July 4). P*rogressions for the Common Core State Standards in Mathematics (draft). High School, Modeling.* Tucson, AZ: Institute for Mathematics and Education, University of Arizona.

| STANDARDS | MODEL CURRICULUM (S.ID.1-3) |
|---|---|
| **Statistics and Probability**<br>**INTERPRETING CATEGORICAL AND QUANTITATIVE DATA**<br>**Summarize, represent, and interpret data on a single count or measurement variable.**<br>**S.ID.1** Represent data with plots on the real number line (dot plots[G], histograms, and box plots) in the context of real-world applications using the GAISE model.★<br>**S.ID.2** In the context of real-world applications by using the GAISE model, use statistics appropriate to the shape of the data distribution to compare center (median and mean) and spread (mean absolute deviation[G], interquartile range[G], and standard deviation) of two or more different data sets. ★<br>**S.ID.3** In the context of real-world applications by using the GAISE model, interpret differences in shape, center, and spread in the context of the data sets, accounting for possible effects of extreme data points (outliers). ★ | **Expectations for Learning**<br>In middle school students learn about the framework of the GAISE model of statistical problem solving. It consists of four steps: Formulating Questions; Collecting Data; Analyzing Data; and Interpreting Results. Students integrate this model whenever they use statistical reasoning. This process will continue throughout high school as students deepen their statistical reasoning skills. Middle school students create dot plots, histograms, and box plots and draw informal comparisons between two populations using graphs. They also summarize data sets using mean absolute deviation. In Math 1 students should use and expand their learning to more sophisticated problems and by comparing single or multiple data sets through graphical representations. Standard deviation is a new concept for students, and it builds upon their previous understanding of mean absolute deviation (MAD). In Math 3, students then extend their knowledge of mean and standard deviation from Math 1 to normal distributions.<br><br>**The GAISE Model**<br>Students will use the GAISE Model framework for statistical problem solving in all courses. The GAISE Model should not be taught in isolation. Students are building on the framework that was developed in middle school. As students progress through the courses, the learning will move towards a greater level of precision and complexity. Students in middle school start at Level A and move towards Level B. As students progress from Level A to Levels B and C, the learning becomes less teacher-driven and more student-driven.<br><br>In this cluster students are at Level B moving towards Level C, and Steps 1 and 2 continue to be emphasized with added depth on Steps 3 and 4. "Understanding the statistical concepts of GAISE model Level B enables a student to grow in appreciation that data analysis is an investigative process consisting of formulating their own questions; collecting appropriate data through various sources; analyzing data through graphs and simple summary measures; and interpreting results with an eye toward inference to a population based on a sample" (Guideline for Assessment and Instruction in Statistics Education (GAISE) Report, 2007, page 58).<br>*Continued on next page* |

| STANDARDS | MODEL CURRICULUM (S.ID.1-3) |
|---|---|
| | **Expectations for Learning, continued**<br>**<u>The GAISE Model, continued</u>**<br><br>**Step 1: Formulate the Question**<br>• Students should pose their own statistical question of interest (Level C).<br>• Students are starting to form questions that allow for generalizations of a population (Level B-C).<br><br>**Step 2: Collect Data**<br>• Students should begin to use random selection or random assignment (Level B).<br><br>**Step 3: Analyze Data**<br>• Students measure variability within a single group using MAD, IQR, and/or standard deviation (Level B).<br>• Students compare measures of center and spread between groups using displays and values (Level B).<br>• Students describe potential sources of error (Level B).<br>• Students understand and use particular properties of distributions as tools of analysis moving toward using global characteristics of distributions (Level B-C).<br><br>**Step 4: Interpret Results**<br>• Students acknowledge that looking beyond the data is feasible by interpreting differences in shape, center, and spread (Level B).<br>• Students determine if a sample is representative of a population and start to move towards generalization (Level B-C).<br>• Students note the difference between two groups with different conditions (Level B).<br>*Continued on next page* |

| STANDARDS | MODEL CURRICULUM (S.ID.1-3) |
|---|---|
| | **Expectations for Learning, continued**<br>**ESSENTIAL UNDERSTANDINGS**<br>• Univariate quantitative data can be represented using dot plots, box plots, and histograms.<br>• Mean and median are approximately equal for symmetric distributions, but tend to be different for nonsymmetric distributions.<br>• Standard deviation is a measure of variation from the mean (spread).<br>• Extreme values (outliers) have an effect on the shape, center, and spread of a distribution.<br>o The median and interquartile range are appropriate measures of center and spread if the distribution is extremely skewed or has outliers.<br>o The mean and standard deviation are appropriate measures of center and spread if the distribution is not skewed and has no extreme outliers.<br><br>**MATHEMATICAL THINKING**<br>• Use accurate and precise mathematical vocabulary.<br>• Construct formal and informal arguments to verify claims and justify conclusions.<br>• Solve real-world and statistical problems.<br>• Use appropriate tools to display and analyze data.<br>*Continued on next page* |

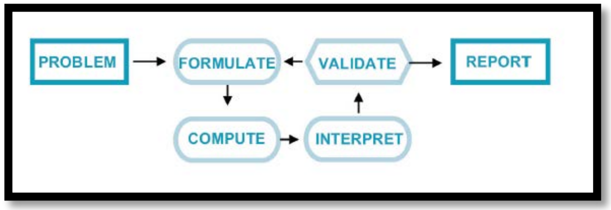
| STANDARDS | MODEL CURRICULUM (S.ID.1-3) |
|---|---|
| | **Expectations for Learning, continued**<br>**INSTRUCTIONAL FOCUS**<br>• Compare the mean to the median of the same data set and relate them to the shape of the distribution (symmetric, skewed).<br>• Develop the formula for and a conceptual understanding of standard deviation by building on the conceptual understanding and formula of mean absolute deviation.<br>• Compare two or more distributions based upon their means and standard deviations.<br>• Explain how outliers affect the mean, the median, and standard deviation.<br>• Given two or more data sets or graphs, do the following:<br>o Compare the shape (symmetric, skewed, uniform).<br>o Compare the spread (greater than, less than, equal).<br>o Compare the centers (mean, median).<br>• Interpret the mean, standard deviation, outliers, as well as differences and similarities between two or more sets of data within a context.<br><br>**Content Elaborations**<br>**OHIO'S HIGH SCHOOL CRITICAL AREAS OF FOCUS**<br>• Math 1, Number 4, page 8<br><br>**THE GAISE MODEL**<br>• GAISE Model, pages 14 – 15<br>o Focus of the cluster for Math 1 is Level B, pages 37-60<br><br>**CONNECTIONS ACROSS STANDARDS**<br>• Summarize, represent, and interpret data on two categorical and quantitative variables (S.ID.5-6). |

# INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (S.ID.1-3)

## Instructional Strategies

*Note: The model curriculum is not a collection of lessons nor a full curriculum; it does not suggest pace, sequence, or amount of time spent on topics. It provides information about a topic related to the standards including ideas for examples, strategies for teaching, common misconceptions, and possible connections between topics. In addition, the emphasis of the model curriculum is on instruction not on assessment; therefore, the examples in the Instructional Strategies section are written for instructional purposes not necessarily as assessment items. Examples may need to be rewritten to accommodate the needs of each individual classroom.*

S.ID.1-3 are modeling standards. See page 13 for more information about modeling.




***Standards for Mathematical Practice***
*This cluster focuses on but is not limited to the following practices:*
***MP.1** Make sense of problems and persevere in solving them.*
***MP.2** Reason abstractly and quantitatively.*
***MP.3** Construct viable arguments and critique the reasoning of others.*
***MP.4** Model with mathematics.*
***MP.5** Use appropriate tools strategically.*
***MP.6** Attend to precision.*
***MP.8** Look for and express the regularity in repeated reasoning.*

### THE GAISE MODEL

Opportunities should be provided for students to work through the statistical process outlined in the GAISE model that students learned in middle school. The process consists of four steps: formulating a question that can be answered by data; designing and implementing a plan that collects appropriate data; analyzing the data by graphical and/or numerical methods; and interpreting the analysis in the context of the original question.

### Step 1: Formulating Questions

By high school students should be posing their own statistical questions, designing their own data collection procedure, and conducting their own analysis that includes generalizations from a sample to a population. Teachers should provide access to real-world datasets which are readily available online. Use problems that are of interest to the students. The richer the question forumulated, the more interesting the process.

***EXAMPLE***

- Are males and females more involved in after-school activities?
- To what extent has the opiod crisis affected my community?
- How often do teens text?
- How many minutes do high schoolers spend on social media?

## INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (S.ID.1-3)

**Step 2: Collecting Data**

Although this domain addresses both categorical and quantitative data, standards S.ID.1-4 and S.ID.6 are focused on only quantitative data. Note that S.ID.5 addresses analysis for two categorical variables on the same individual. It may be helpful to contrast categorical (using bar graphs, two-way frequency tables, pie graphs, and/or the central tendency of mode) and quantitative data.

In 7th grade (7.SP.1) students developed an informal idea of sampling. By high school students should be able to differentiate between a population, a census, and a sample. When collecting data, students should begin to informally discuss the importance of random selection in terms of "fairness" especially if they have not done so in middle school. In Algebra 2 students will gain a more in-depth understanding of random sampling (S.IC.1). In this cluster even though students will not be able to calculate true randomness, they can develop the basic idea that generalizations about a population are only valid if the sample truly represents the population. This can be done by discussing biases such as only sampling one's friends. This should allow them to take the first step toward a generalization about a population. Using the probability concepts that students learned in Grade 7, students should be able to have a discussion about the basic understanding of the role of probability in random selection.

**Step 3: Analyzing Data**

Students should focus on using technology to create graphs for S.ID.1 since creating graphs by hand was one of the focuses in middle school. (See 6.SP. 4 and 7.SP3 Model Curriculum for scaffolding ideas.) In addition, students should not only become fluent in creating graphical displays but also on knowing how to choose the most appropriate graph given the data set. They should use initial graphical displays (for single or more than one group) as a first step during the exploratory stage of their analysis. Then the graphical displays could be used to prompt deeper and more meaningful questions.

Students may think that a bar graph and a histogram are the same. A bar graph is appropriate when the horizontal axis has categories and the vertical axis is labeled by either frequency (e.g., book titles on the horizontal axis and the number of students who like the respective books on the vertical axis) or a measurement of some numerical variable (e.g., days of the week on the horizontal axis and the median length of root growth of radish seeds on the vertical axis). A histogram has units of measurement of a numerical variable on the horizontal axis such as ages with intervals of equal length.

Provide students with multiple data sets that will generate different types of distributions. Give students guiding questions to discuss in order to help them determine why the distribution looks the way that it does and consider the possible implications for the distribution. They should be discussing shape, center, and spread in context. Although in middle school students had practice in describing shape, center, and spread, students should now become more fluent in these descriptions so that it becomes a habit of mind.

## INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (S.ID.1-3)

This exploratory analysis should also include the comparisons of mean and median and how they are affected by the data and the distribution. The best measures of center and spread to describe data sets without outliers are the mean and standard deviation, whereas median and interquartile range are better measures for data sets with outliers. Note on outliers: Outliers should always be included in the analysis. They can be discussed to verify the validity of the data point such as "Why was it different?" or "Could it have been a collection error?"

Students may believe that the lengths of the intervals of a boxplot (min, Q1), (Q1, Q2), (Q2, Q3), (Q3, max) are related to the number of subjects in each interval. Students should understand that each interval theoretically contains (approximately) one-fourth (25%) of the total number of subjects. Sketching an accompanying dot plot or histogram or constructing a live boxplot may help in alleviating this misconception.

### Misleading Graphs

Students should be given misleading or distorted graphs such as dot plots, histograms, and box plots to analyze. They can be used to stress the importance of labeling and proper scaling. Some examples of things that can be misleading are as follows:

- non-evenly spaced scales;
- evenly spaced scales for uneven intervals;
- non-proportional area or volume changes for a picture;
- loaded or bias labels;
- more dimensions than variables in the plotted data are used;
- histograms or bar graphs not starting at 0 (truncated graph);
- the size of the intervals in histograms;
- the bars on a histogram may not be uniform;
- a superfluous third dimension is used; and/or
- area of the box in a box and whiskers plot.

One site that has misleading graphs is 31 Misleading Graphs and Statistics by Dr. Marcel B. Finan from Arkansas Tech University.

# INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (S.ID.1-3)

**Standard Deviation and Mean Absolute Deviation**

Introduce the formula of standard deviation by reviewing the previously learned MAD (mean absolute deviation). The MAD is very intuitive and gives a solid foundation for developing the more complicated standard deviation measure.

In addition to a discussion about the shape of the distribution, students should be exploring standard deviation:

- What it means in the context of the situation?
- How the standard deviation changes as the data changes?

To introduce standard deviation, it may be helpful to calculate the standard deviation by hand for small sets of data with a focus on understanding the concept and its relationship to MAD but then transition to using technology for larger sets of data. The focus should be on the conceptual understanding in lieu of the calculations.

**Step 4: Interpreting Data**

Informally observing the extent to which two or more appropriate graphs overlap begins the discussion of drawing inferential conclusions.

TIP!

When making two or more graphs, students should be sure to use the same scale for all graphs so that comparisons can be made.

Students should explore the differences between variations and errors. Draw attention to naturally occurring errors that happen in the classroom and discuss the impact of the errors on the results. These notions can be used to explain outliers, clusters, and gaps. Understanding variability is vital for developing data sense.

## VARIABILITY

Variability is everywhere in statistics. Data vary; samples vary; distributions vary; and variation occurs both within and across samples and distributions.

Initially when given a data set, students calculate the mean or median to analyze the data set. However, relying exclusively on using a measure of center masks important features. However students who attend to variability are much more likely to predict an interval of outcomes rather than a single number.

# INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (S.ID.1-3)

Students may reason additively, proportionally, or distributively about variability. Additive reasoners focus on frequencies rather than relative frequencies. Proportional reasoners are able to make connections between sample proportions and population proportions. Distributional reasoners are able to use both center and spread to reason about a problem. Distributional reasoners like proportional reasoners, are able to make connections between the sample and population proportions, but they are also able to explicitly mention variation about the expected value. It takes time and experience for students to gain distributive reasoning. It is difficult for students to think about center, shape, and spread simultaneously (Shaughessy, 2007).

Students in high school should start to realize there is not only variability within a group but also variability between groups. Measures of spread such as mean absolute deviations (MAD) and standard deviation are ways students in Math 1 can express measures of spread. Students should be able to compare data values to a measure of center and quantify how different the data are from the measure of center. Informally discuss the benefits of imposing randomness into sampling procedures.

## Instructional Tools/Resources

*These tools and resources may be helpful for instruction. Before use, they should be evaluated at the local level to determine their appropriateness for instruction.*

**Manipulatives/Technology**

- Census at School allows students to complete a brief online survey, analyze their class census results, and compare their class with random samples for students in the United States or other countries.
- A Little Stats: Adventures in Teaching Statistics created by Amy Hogan is a list of free internet sources for real data and datasets for public use.
- Rossmanchance.com is a website by Allan Rossman and Beth Chance that has a link to many applets that could be useful for statistics and probability.
- Statistics Calculator: Box Plot is an applet that generates a box plot.
- Visual Understanding Educational Apps for Statistics
- StatKey by Lock, Lock, Lock, Lock, and Lock is an applet for representing statistics
- Box Plot by Shodor is an applet that creates a box plot.
- Graphing calculators
- Printed media (e.g., almanacs, newspapers, professional reports)
- Desmos is a free graphing calculator that is available to students as website or an app.
- Geogebra is a free graphing calculator that is available to students as website.
- Wolframalpha is dynamic computing tool.

## INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (S.ID.1-3)

**Box Plots**

- Are Female Hurricanes Deadlier than Male Hurricanes? is a lesson by Mary Richardson from the Statistics Education Web (STEW) published in June 2014. Students will apply the GAISE model and use box plots to compare two data sets.
- Colors Challenge by Reischman is a lesson from the Statistics Education Web (STEW) published June 2013. In this lesson students collect data using the GAISE model to investigate whether or not the ability to name a color of ink is more difficult when the word written in the same ink is the name of a color; students will design an experiment and use box plots.
- Representing Data with Box Plots is a lesson from Mathematics Assessment Project where students interpret box plots and create box plots from frequency graphs.
- Saga of Survival by Richardson and Rogness is a lesson from the Statistics Education Web (STEW) published July 2013. In this lesson students use demographic data from the Donner Party tragedy using box plots and two-way frequency tables.
- Did I Trap the Median? is a lesson by Parks, Steinwachs, Diaz, and Molinaro from the Statistics Education Web (STEW) published in June 2014. In this lesson students use the GAISE model to collect data about the median foot size of the class. They use box plots to evaluate the data.
- Commuting to Work is a unit by the United States Census Bureau where students use measures of central tendency and box plots to represent the number of people who bike to work.
- Speed Trap is a task by Illustrative Mathematics where students construct boxplots and use them to compare distributions.
- Haircut Costs is an introductory task by Illustrative Mathematics that introduces group comparisons using boxplots.
- Now You See It, Now You Don't: Using See It to Compare Stacked Dotplots to Boxplots by Guzman-Alvarez, Smith, Molinaro, and Diaz is a lesson from the Statistics Education Web (STEW) published May 2014. In this lesson students use the GAISE model collect data by measuring the height of their right-hand reach using dot plots and box plots.
- BMI Calculations is a CTE lesson from Achieve the Core where students evaluate BMI measurements using statistics such as a 5-Number summary and box plots.
- Differences in Earning Across Sex and Educational Attainment: Comparing Box Plots is a unit by the United States Census Bureau where students use data and boxplots to compare the earnings of men and women.

# INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (S.ID.1-3)

**Histograms**

- Census in Counties-Describing and Comparing Histograms to Understand American Life is a unit by the United States Census Bureau that uses data on employment, technology, and transportation to create histograms.
- Math Class is a lesson from Georgia Standards of Excellence Curriculum Framework's Algebra 1 course could be used to formatively assess students' ability to summarize, represent, and interpret data on a single count or measurement variable using histograms. This lesson can be found on pages 20-24.
- Georgia Standards of Excellence Curriculum Framework, Algebra 1, Unit 6: Describing Data, The Basketball Star is a lesson where students use dot plots, histograms, and box plots to compare the shape of two data sets. This lesson can be found on pages 25-32.
- Describing and Comparing Data Distributions is a unit by the United States Census Bureau where students use data on the government organization, spending, and populations at different levels (city, county, state) to compare and contrast the distributions using histograms and boxplots.
- Census in Counties—Describing and Comparing Histograms to Understand American Life is a lesson by the United States Census Bureau where students use histograms to analyze life in America.

**Frequency Graphs**

- Representing Data with Frequency Graphs by Mathematics Assessment Project has students use frequency graphs to indeify a range of measures and make sense of the data in a real-world context.

**Mean Absolute Deviation (MAD)**

- Measuring Variability in a Data Set is a task by Illustrative Mathematics that compares MAD to standard deviation.
- Georgia Standards of Excellence Curriculum Framework, Algebra 1, Unit 6: Describing Data, If the Shoe Fits? is a lesson where students explore data from shoe prints to figure out who came on school grounds without permission. They will use measures of center, IQR, Box Plots, and MAD. It also incorporates scatterplots. This lesson can be found on pages 48-61.

**Misleading Graphs**

- The Most Misleading Charts of 2015, Fixed is a website by Quartz Media that has examples of misleading graphs.
- 31 Misleading Graphs and Statistics by Dr. Marcel B. Finan of Arkansas Tech University has many misleading graphs.

# INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (S.ID.1-3)

**Curriculum and Lessons from Other Sources**

- EngageNY, Algebra 1, Module 2, Topic A, Lesson 1: Distributions and their Shapes, Lesson 2: Describing the Center of a Distribution, Lesson 3: Estimating Centers and Interpreting the Mean as a Balance Point are lessons that pertain to this cluster.
- EngageNY, Algebra 1, Module 2, Topic B, Lesson 4: Summarizing Deviations from the Mean, Lesson 5: Measuring Variability for Symmetrical Distributions, Lesson 6: Interpreting the Standard Deviation, Lesson 7: Measuring Variability for Skewed Distributions (Interquartile Range), Lesson 8: Comparing Distributions are lessons that pertain to this cluster.
- Mathematics Vision Project, Secondary Math 1, Module 9: Modeling Data is a unit on statistics. Tasks 9.1 and 9.2 are applicable to this cluster.
- Math Statistics in Schools are lessons provided by the United States Census Bureau that use real-life data.

**General Resources**

- Arizona's High School Progression on Statistics and Probability is an informational document for teachers. This cluster is addressed on pages 2-4.
- Arizona's High School Progression on Modeling is an informational document for teachers. This cluster is addressed on pages 2 and 10.
- High School Coherence Map by UnboundED is a map of the relationships among high school mathematics standards. Please note: This resource is aligned to the Common Core not Ohio's Learning Standards. Although the tool is useful, it needs to be checked with respect to Ohio's Learning Standards.
- LOCUS is an NSF Funded project focused on developing assessments of statistical understanding. Teachers must create an account to access the assessments.
- K-12 Statistics Education Resources is a collection of websites put together by the American Statistical Association for teachers.
- A Sequence of Activities for Developing Statistical Concepts by Christine Franklin & Gary Kader is an article published in *The Statistics Teacher Network*, Number 68 ,Winter 2006. It has an overview of the GAISE model and its levels, and it includes activities at each level.
- Statistics Teacher is an online journal published by the American Statistical Association (ASA) and NCTM. It includes articles, lesson plans, and professional development opportunities.
- Significance is a magazine that demonstrates the practical use of statistics and shows how statistics benefits society.
- Chance is a magazine about statistics for anyone who is interested in using data to advance science, education, and society.

## INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (S.ID.1-3)

**Research**


- Burrill, G., Franklin, C., Godbold, L., & Young. (2003). *Navigating through Data Analysis in Grades 9-12*. Reston, VA: National Council of Teachers of Mathematics.
- Common Core Standards Writing Team. (2016, March 24). *Progressions for the Common Core State Standards in Mathematics (draft). High School Statistics and Probability*. Tucson, AZ: Institute for Mathematics and Education, University of Arizona.
- Common Core Standards Writing Team. (2013, July 4). *Progressions for the Common Core State Standards in Mathematics (draft). High School, Modeling*. Tucson, AZ: Institute for Mathematics and Education, University of Arizona.
- Franklin, C., Kader, G., Mewborn, D, Moreno, J., Peck, R., Perry, M., & Scheaffer, R. (2007). *Guidelines for Assessment and Instruction in Statistics Education (GAISE) Report*. Alexandria, VA: American Statistical Association.
- Shaughnessy, J. (2007). Research on statistics learning and reasoning. In F. K. Lester (Ed.), *Second Handbook of Research on Mathematics Teaching and Learning* (pp. 957-1010). Charlotte, NC: Information Age Publishing.

| STANDARDS | MODEL CURRICULUM (S.ID.5-6) |
| --- | --- |
| **Statistics and Probability**<br>**INTERPRETING CATEGORICAL AND QUANTITATIVE DATA**<br>**Summarize, represent, and interpret data on two categorical and quantitative variables.**<br>**S.ID.5** Summarize categorical data for two categories in two-way frequency tables. Interpret relative frequencies in the context of the data (including joint, marginal, and conditional relative frequencies). Recognize possible associations and trends in the data.★<br>**S.ID.6** Represent data on two quantitative variables on a scatter plot, and describe how the variables are related.★<br>**c.** Fit a linear function for a scatterplot that suggests a linear association. (A1, M1) | **Expectations for Learning**<br>For this cluster, the GAISE Model framework continues to be used: Formulating Questions; Collecting Data; Analyzing Data; and Interpreting Results. In the middle grades, students visually approximate a linear model and informally judge its goodness of fit. In Math 1, students extend this knowledge to find the equation of a linear model, with and without technology. They will also use more precise language to describe the relationship between variables. In Math 3, concepts extend to quadratic and exponential functions as well as working with residuals.<br><br>The learning at this level is at the developmental Level B. See pages 77-78 for more information on Level B.<br><br>**ESSENTIAL UNDERSTANDINGS**<br>*Note: Students should be able to talk sensibly about the meanings of joint, marginal, and conditional frequencies within a context but should not be held responsible for precise usage of this vocabulary.*<br>• Row totals and column totals constitute the marginal frequencies.<br>• Individual table entries represent joint frequencies.<br>• A relative frequency is found by dividing the frequency count by the total number of observations for a whole set or subset.<br>    o A marginal relative frequency is calculated by dividing the row (or column) total by the table total.<br>    o A joint relative frequency is calculated by dividing the table entry by the table total.<br>    o A conditional relative frequency is calculated by restricting to one row or one column of the table.<br>• Relative frequencies are useful in considering association between two categorical variables.<br>• A linear function can be used as a model for a linear association of two quantitative variables.<br>*Continued on next page* |

## Expectations for Learning, continued

### MATHEMATICAL THINKING

- Use accurate and precise mathematical vocabulary.
- Construct formal and informal arguments to verify claims and justify conclusions.
- Solve real-world and statistical problems.
- Use appropriate tools to display and analyze data.
- Accurately make computations using data.
- Determine reasonableness of predictions.

### INSTRUCTIONAL FOCUS

**Categorical Data**

- Calculate and interpret, within a context, joint, marginal, and conditional relative frequencies.
- Recognize possible relationships (trends) in the context of the data by using percentages from two-way frequency tables.

**Quantitative Data**

- Describe, within a context, how variables are related in a linear relationship using scatter plots.
- Calculate and interpret, within a context, the slope and $y$-intercept of a linear model, given a set of data or graph, with or without technology.

*Continued on next page*

| STANDARDS | MODEL CURRICULUM (S.ID.5-6) |
|---|---|
| | **Content Elaborations**<br>**OHIO'S HIGH SCHOOL CRITICAL AREAS OF FOCUS**<br>• Math 1, Number 4, page 8<br><br>**THE GAISE MODEL**<br>• GAISE Model, pages 14 – 15<br>o Focus of this cluster for Math 1 is Level B moving toward Level C, pages 37-60<br><br>**CONNECTIONS ACROSS STANDARDS**<br>• Interpret linear models (S.ID.7-8).<br>• Build a function that models a relationship between two quantities (F.BF.1).<br>• Distinguish between situations that can be modeled with linear and exponential functions (F.LE.1).<br>• Create equations that describe numbers or relationships (A.CED.2). |

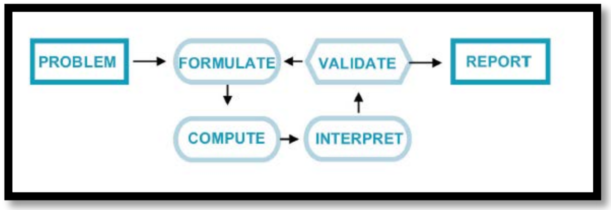
# INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (S.ID.5-6)

## Instructional Strategies

*Note: The model curriculum is not a collection of lessons nor a full curriculum; it does not suggest pace, sequence, or amount of time spent on topics. It provides information about a topic related to the standards including ideas for examples, strategies for teaching, common misconceptions, and possible connections between topics. In addition, the emphasis of the model curriculum is on instruction not on assessment; therefore, the examples in the Instructional Strategies section are written for instructional purposes not necessarily as assessment items. Examples may need to be rewritten to accommodate the needs of each individual classroom.*

***Standards for Mathematical Practice***
*This cluster focuses on but is not limited to the following practices:*
***MP.4** Model with mathematics.*
***MP.6** Attend to precision.*
***MP.7** Look for and make use of structure.*

S.ID.5-6 are modeling standards. See page 13 for more information about modeling.




This cluster builds upon what students learned in Grade 8 regarding scatter plots and two-way tables. It is also a continuation of statistical thinking using the GAISE model that was introduced in S.ID.1 – 4. Students should continue to formulate their own statistical question, collect data, analyze data (with two-way tables and scatterplots), and then interpret data (which continues in the next cluster for scatterplots).

In this cluster, the focus is on two categorical or two quantitative variables that are being measured on the same subject. Help students clearly distinguish between categorical and quantitative variables by providing multiple examples of each type.

### CATEGORICAL BIVARIATE DATA

The focus is on statistical thinking within a given data set using relative frequencies. The standard S.ID.5 is almost identical to 8.SP.4 except now in high school students delve deeper into concepts as they transition from Level A to Level B. See 8$^{th}$ Grade Model Curriculum 8.SP.1-4 for scaffolding ideas. In the categorical case, begin with two categories for each variable and represent them in a two-way table.

# INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (S.ID.5-6)

***EXAMPLE***

**Step 1: Formulate Questions**

Students came up with the question: Is there a relationship between gender and whether a student has a job in high school?

TIP! Use more open-ended types of question to promote a higher level of statistical reasoning.

**Step 2: Collect Data**

Based on the question in Step 1, students may decide that a survey can be given to Mr. Johnson's two Math 1 classes and that they can use a table to organize their data. A discussion should occur about whether the two Math 1 classes can really represent the population of the school and if not discuss how the survey should be adjusted. Some students may point out that many Math 1 students are only Freshman, so it may be better to do a survey of a $9^{th}$, a $10^{th}$, an $11^{th}$, and a $12^{th}$ grade English class. Since most honors students took Math 1 in middle school, students may wonder whether it is more likely that honors students have jobs compared to non-honors students. This could lead to a discussion about how variability not only occurs within a group, but can also occur between groups. Students should be encouraged to explore these different types of possibilities in tandem with the class discussion time permitting.

A discussion also needs to take place about clearly defining the event: What constitutes a job? Does babysitting count as a job?

Discuss how using discrete questions rather than open-ended questions is preferable when creating a survey in order to make it easier to analyze data. Discuss how loaded words or biased language could change the outcome of survey. A fun exploration could be to have the class design two similar surveys using different word choices and explore the differences in the results.

| | Job | No Job |
|---|---|---|
| Male | 30 | 35 |
| Female | 45 | 25 |

TIP! When creating surveys, students may be stuck on the idea of "fairness" instead of the idea of random selection. For example they may want to pick two boys and two girls from every class, so every student has a chance of being surveyed. Another example could be that they may think it would be better to have an optional survey so that any student can choose to participate not realizing that self-selection may lead to bias. Instead students should realize that an accurate sample does not mean that every student needs to be surveyed.

# INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (S.ID.5-6)

**Two-way Frequency Tables vs Two-way Relative Frequency Tables**

Information can be organized into Two-Way Frequency Tables or Two-Way Relative Frequencies. Students need to be exposed to both types of tables. Whereas frequency tables display counts, relative frequency tables display the data in ratios, decimals, or percents. Notice that the total in relative frequency table (bottom right-hand corner) is 1 or 100%. Students should have been exposed to both types of tables in 8th grade, but since High School students should be at GAISE Level B, the emphasis is on proportional reasoning and the two-way relative frequency table in order to interpret data in terms of fractions or percents.

*Note: In high school, rarely will students be given the column and row for total (marginal frequency) but rather they will be expected to calculate this information independently. Students should be able to talk sensibly about the meanings of joint, marginal, and conditional frequencies within a context but should not be held responsible for precise usage of this vocabulary.*

***EXAMPLE, CONTINUED***

**Two-way Frequency Table**

| | Job | No Job | TOTAL |
|---|---|---|---|
| Male | 30 | 35 | |
| Female | 45 | 25 | |
| TOTAL | | | |

**Two-way Relative Frequency Table**

| | Job | No Job | TOTAL |
|---|---|---|---|
| Male | $^{30}/_{135} \approx$ 22% | $^{35}/_{135} \approx$ 26% | |
| Female | $^{45}/_{135} \approx$ 33% | $^{25}/_{135} \approx$ 19% | |
| TOTAL | | | $^{135}/_{135} =$ 100% |

TIP!

Some students will have more success with two-way frequency tables as opposed to two-way relative frequency tables. Depending on the data, some students may have more success changing a two-way relative frequency table into a two-way frequency table where the total is 100 or 1,000. Although this method is not always as precise, it is acceptable for Math 1 students. See article: https://opinionator.blogs.nytimes.com/2010/04/25/chances-are/ for more information.

# INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (S.ID.5-6)

**Step 3: Analyze Data**

**Joint and Marginal Relative Frequency**

Joint frequency is where the two variables "join" such as job and male. These can be found in the body of the table.

Row totals and column totals constitute the marginal frequencies. These are found in the margins of the table. Marginal frequencies can also be found by adding across columns or rows.

The ratio of the joint or marginal frequencies to the total number of subjects define relative frequencies (and percentages), respectively.

*Note: Whereas in Grade 8 students were often given the marginal frequencies, in high school students are expected calculate the marginal frequencies by themselves.*

***EXAMPLE, CONTINUED***

Discuss with students what kind of information could be found using two-tables. Discuss the pros and cons of a two-way frequency table vs a two-way relative frequency table. Strive for fluency in both, but push students to use the two-way relative frequency table.

| | Job | No Job | TOTAL |
|---|---|---|---|
| Male | $\frac{30}{135}$ | $\frac{35}{135}$ | $\frac{65}{135}$ |
| Female | $\frac{45}{135}$ | $\frac{25}{135}$ | $\frac{70}{135}$ |
| TOTAL | $\frac{75}{135}$ | $\frac{60}{135}$ | $\frac{135}{135}$ |

Joint Frequencies

Marginal Frequencies

Now that students have the data, discuss what types of questions can be asked to help them answer the initial question. Have students discuss the benefit of calculating relative frequency over frequency.

- What is the frequency that a male will have a job?
- What is the relative frequency that a male will have a job?
- If a student is selected at random, what is the probability that he or she would not have a job?

Point out that students can calculate relative frequency from a two-way frequency table, or they can work directly from a two-way *relative* frequency table.

# INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (S.ID.5-6)

## Conditional Frequency

Conditional relative frequencies are determined by focusing on a specific row or column of the table. They are particularly useful in determining any associations between the two variables. Discuss the different types of conditional frequency questions students could ask.

- Using a frequency table, what is the likelihood that a student is male assuming the he has a job? $\frac{30}{75} = \frac{2}{5}$
- Using a relative frequency table, what is the likelihood that a student is male assuming has a job? $\left(\frac{30}{135}\right) \div \left(\frac{75}{135}\right) = \frac{2}{5}$.

| | Job | No Job | TOTAL |
|---|---|---|---|
| Male | 30 | 35 | 65 |
| Female | 45 | 25 | 70 |
| TOTAL | 75 | 60 | 135 |

| | Job | No Job | TOTAL |
|---|---|---|---|
| Male | $^{30}/_{135} \approx$ 22% | $^{35}/_{135} \approx$ 26% | $^{65}/_{135} \approx$ 48% |
| Female | $^{45}/_{135} \approx$ 33% | $^{25}/_{135} \approx$ 19% | $^{70}/_{135} \approx$ 52% |
| TOTAL | $^{75}/_{135} \approx$ 55% | $^{60}/_{135} \approx$ 45% | $^{135}/_{135} =$ 100% |

Again discuss the benefits and drawbacks of calculating conditional relative frequencies using the two different tables.

Discuss how these two statements are different:

- What is the relative frequency that a female student has a job?
- What is the relative frequency that a student who has a job is female?

*Note: Although the above example is a simple example for a 2 by 2 frequency table, in high school students should gain experience and become fluent with larger tables (2 by 3, 3 by 4, etc.)*

Direct students to use percentages when making claims about data presented in two-way tables, because frequencies can be misleading.

**TIP!**

Students need practice calculating conditional frequencies with and without the word "given" being used. Have students practice writing conditional frequencies in different ways.

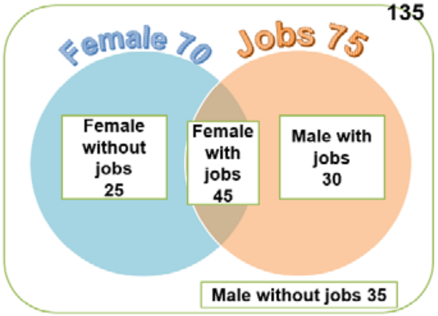
Students should also be able to convert between different data representations such as converting from a two-way table to a Venn diagram.

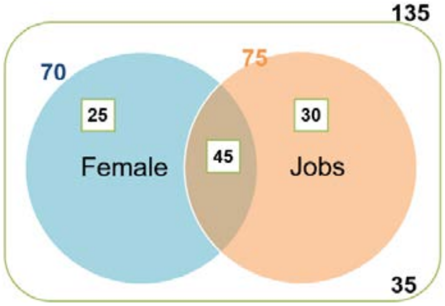
# INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (S.ID.5-6)

***EXAMPLE***

Michelle prefers using Venn diagrams over two-way frequency tables. Convert the data from the two-way frequency table to a Venn diagram to help Michelle visualize whether a female has a job. Student can move from fully labeled Venn diagrams to those where less labels are given.

| | Job | No Job | TOTAL |
|---|---|---|---|
| Male | 30 | 35 | 65 |
| Female | 45 | 25 | 70 |
| TOTAL | 75 | 60 | 135 |




**Step 4: Interpret Data**

Students should gain additional practice in writing about statistics by validating conclusions and justifying arguments (see GAISE Steps 3 and 4 Level C). Questioning could be focused around associations and trends in the data. From our previous example a question could be "Is there an association between gender and whether or not an individual has a job? Explain."

**Association**

Association is a relationship between two types of quantities so that one event is dependent on the other. Have students redefine their question to address associations such as—

- Is there an association between gender and job status?
- Does gender predict whether a high school student will have a job?

In two-way frequency tables, two categorical variables are associated if the row conditional relative frequencies (or column relative frequencies are different from the rows (or columns) of the table. Discuss with students the pairs of conditional probability questions that could be asked to determine this.

- What is the conditional relative frequency that a male has a job? (30/65 = 0.46)
- What is the conditional relative frequency that a female has a job? (45/70 = 0.64)

Since 0.46 and 0.64 are not close, gender and job status are related.

Or

- What is the conditional relative frequency that someone who has a job is male? (30/75 = 0.40)
- What is the conditional relative frequency that someone who does not have a job is male? (35/60 = 0.58)

## INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (S.ID.5-6)

Since 0.40 and 0.58 are not close, gender and job statues are related.

The greater the difference between the conditional relative frequencies, the greater the association. Although students will discuss correlation vs causation more formally in Algebra 2, a discussion about how an association does not necessarily mean a cause-and-effect relationship should occur. *Note: Association and independence will be formalized in the conditional probability standard S.CP.4 in the Geometry course.*

### Agreement-Disagreement Ratio

Students in Grade 8 learn how to find association between two quantitative variables using the Quadrant Count Ratio (QCR). A comparable method is called the Agreement-Disagreement-Ratio (ADR) which can be employed for categorical data in a 2 by 2 table for two Yes-No variables. It is calculated by taking the sum of the agreements minus the sum of the disagreements divided by the total or

ADR $= \frac{(a+d)-(b+c)}{T}$

Question 2 (Agreement: $a$, $d$; Disagreement: $b$, $c$)

| Question 1 | Yes | No | Row Total |
|---|---|---|---|
| **Yes** | $a$ | $b$ | $r_1 = a + b$ |
| **No** | $c$ | $d$ | $r_2 = c + d$ |
| **Column Total** | $c_1 = a + c$ | $c_2 = b + d$ | $T = a + b + c + d$ |

### *EXAMPLE*

A yes or no survey was given to 144 students. They were asked whether they like soccer and whether they like basketball. The purpose was to see if there is an association between liking the two sports. Find and interpret the ADR based on the results shown in the table.

Do you like soccer? (Agreement: 50, 18; Disagreement: 35, 41)

| Do you like basketball? | Yes | No | Row Total |
|---|---|---|---|
| Yes | 50 | 35 | |
| No | 41 | 18 | |
| Column Total | | | |

*Discussion:* Since the ADR $= \frac{(50+18)-(35+41)}{144} = \frac{68-76}{144} \approx -0.056$ there is an extremely weak negative association between those who like basketball and those who like soccer.

A connection could be made between frequency tables and scatterplots to connect QCR and ADR. See GAISE Report page 96 for more information.

# INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (S.ID.5-6)

## ASSOCIATION

Students should investigate problems with more emphasis placed on associations among two or more variables. They should begin to quantify their conclusions by questions such as "How strong or weak is the association?" or "Does the strength of the data association allow us to make any useful predictions?" Students should also start to be able to distinguish between "association" and "cause and effect" when discussing the relationships between variables.

### Phi Coefficient (Extension)

Another association that can be used in a 2 by 2 frequency table is the phi coefficient which is comparable to Pearson's correlation coefficient for quantitative data. Whereas the QCR and the ADR are additive in nature calculating "how many?" data values are in each quadrant or cell, Pearson's and Phi's coefficient are multiplicative in nature as they calculate "how far?" the point in each quadrant are from the center point. The Phi coefficient can be found as follows:

Phi $(\phi) = \frac{ad-bc}{\sqrt{r_1 r_2 c_1 c_2}}$

## QUANTITATIVE BIVARIATE DATA

Continue to use the GAISE model in exploring quantitative bivariate data. Emphasize using mathematical models to capture key elements of the relationship between two variables. Use real-world data and make connections to science. This cluster emphasizes the first two steps of the GAISE model: formulating questions and collecting data. The next two steps will be emphasized in the next cluster (S.ID.7-8).

### *EXAMPLE*

**Step 1: Formulate Questions**

How strong is the association between the hours spent doing homework and the hours spent playing video games?

**Step 2: Collect Data**

Use the Census at School Random Sampler, plot the data of the hours spent doing homework and the hours spent playing videogames. Have an informal discussion about sample size and a random sample.

## INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (S.ID.5-6)

In standard S.ID.7, the focus is on two quantitative variables being measured on the same subject. The paired data should be listed and then displayed in a scatterplot. What is being predicted (the dependent variable) is plotted on the vertical axis; the predictor variable (independent variable) is on the horizontal axis. If *time* is one of the variables, it usually is defined as the independent variable, so it typically is plotted on the horizontal axis.

In the numerical or quantitative case, display the paired data in a scatterplot. Quickly review creating the scatterplot and informally fitting the function by hand, but move on to more complex examples that require the use of technology.

**Mean—Mean Line Method**

To informally fit a linear function to data by hand, students can use the *Mean—Mean Line Method* to gain a conceptual understanding of the regression line.

- Order data from least to greatest based on the *x*-coordinates.
- Divide the data into a "lower half" and an "upper half." (Discard the median point if there is an odd number of data points)
- Determine the mean of the *x*-coordinates of both the "upper half" and the "lower half" of the data set.
- Determine the mean of the *y*-coordinates of both the "upper half" and the "lower half" of the data set.
- Have students write the equation of the line using the two points.
- Have students compare their regression line to one generated by technology such as a graphing calculator.

## INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (S.ID.5-6)

***EXAMPLE***

An ice cream store collected data to compare the amount of sales to the daily temperature to see if there is an association.

**a.** Graph the data on a scatterplot and fit a function to the data set.

**b.** Order the data from least to greatest based on *x*-coordinates. (See table to the right.)

**c.** Calculate the means of the upper and lower half of the data set.

| Temp (F°) | Sales ($) | |
|---|---|---|
| 32 | 250 | |
| 36 | 325 | |
| 38 | 300 | |
| 38 | 330 | Upper Half |
| 40 | 350 | |
| 42 | 345 | |
| 45 | 400 | |
| 49 | 415 | Discarded |
| 52 | 500 | |
| 56 | 525 | |
| 63 | 550 | |
| 65 | 525 | Lower Half |
| 66 | 560 | |
| 69 | 600 | |
| 72 | 650 | |

| Lower Half (7pts) | Upper Half (7pts) |
|---|---|
| Mean Temp = 38.7 | Mean Temp = 63.3 |
| Mean Sales = 328.6 | Mean Sales = 558.6 |

**d.** Write an equation of a line using the two points (38.7, 328.6) and (63.3, 558.6) and then fit that line to the data.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{558.6 - 328.6}{63.3 - 38.7} = \frac{230}{24.6} \approx 9.35$$

$$y - y_1 = m(x - x_1)$$
$$y - 328.6 = 9.35(x - 38.7)$$
$$y - 328.6 = 9.35x - 361.845$$
$$y = 9.35x - 33.245$$

**e.** Use technology to graph and compare the mean-mean line of fit you calculated with the actual line of best fit: $y = 9.05x – 18.41$**.** When you graph both lines on the scatterplot, they should be very similar.

**f.** Using both equations, predict the sales when the temperature is 80°F. How do your values differ?

**g.** Using both equations predict what temperature it would have to be in order to generate $540 in sales. How do your values differ?

The focus of this cluster is not necessarily on writing equations of lines, which is covered in F.LE.1-2. (Although it could be an application of the concept.) Rather, students should be able to pick the best model by approximating the slope and *y*-intercept and be able to informally justify this choice in context. Students should progress to more complex data sets requiring the use of technology to generate the best linear model.

## INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (S.ID.5-6)

Fitting functions to such data will help students avoid difficulties such as the interpretation of a slope when the scales are different, for example total SAT score versus grade-point average. Once students are comfortable with the same scale cases, introducing different scales in situations becomes less problematic.

Students may incorrectly believe that a 45° line in the scatterplot of two quantitative variables always indicates a slope of 1. However this is only the case when the two variables have the same scaling.

Although examples should be limited to linear models, include situations where data are not best represented by a linear model. Then have an informal discussion about how a linear model is not always the best fit for a data set. This will set the foundation for learning in Algebra 2 where students use quadratic and exponential function models in addition to linear models to make sense of scatterplots and data displays.

**Misleading Graphs**
Students should analyze misleading scatterplots. See cluster S.ID.1-3 for more information about misleading graphs.

## Instructional Tools/Resources

*These tools and resources may be helpful for instruction. Before use, they should be evaluated at the local level to determine their appropriateness for instruction.*

**Manipulatives/Technology**

- Excel/Google Sheets
- Graphing calculators.
- Desmos is a free graphing calculator that is available to students as website or an app.
- Geogebra is a free graphing calculator that is available to students as website.
- Wolframalpha is dynamic computing tool.

# INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (S.ID.5-6)

**Two-Way Frequency Tables**

- First Day Statistics Activity—Grouping Qualitative Data is a lesson by R.B. Campbell from Statistics Education Web (STEW) published in 2014 that explicitly aligns to the GAISE model.
- You Will Soon Analyze Categorical Data (Classifying Fortune Cookies Fortunes) is a lesson by Mary Richardson from Statistics Education Web (STEW) published in 2014 that explicitly aligns to the GAISE model. Students will construct two-way frequency tables and investigate their results using joint relative frequencies and marginal and conditional distributions.
- A Sweet Task by Fiedler, Huey, Jenkins, and Flinspach from Statistics Education Web (STEW) published in 2014 that explicitly aligns to the GAISE model. Students will create two-way frequency tables and interpret relative frequencies using M&M's.
- Saga of Survival (Using Data about the Donner Party to Illustrate Descriptive Statistics) by Richardson and Rogness from Statistics Education Web (STEW) published in 2013 that explicitly aligns to the GAISE model. Students explore survival rates of emigrants trapped in the Sierra Nevada winter of 1846-1847 using two-way frequency tables and box plots.
- Support for a Longer School Day? is a task by Illustrative Mathematics that provides students with an opportunity to calculate and interpret joint, marginal, and conditional relative frequencies.
- Musical Preferences is a task by Illustrative Mathematics that explores association using two-way frequency tables.
- The Case of the Careless Zookeeper by Malloure, Richardson, and Rogness from Statistics Education Web (STEW) published in 2013 that explicitly aligns to the GAISE model. This is an extension activity on two-way frequency tables that uses a chi-square test of independence.

**Scatterplots**

- Text Messaging Is Time Consuming! What Gives? by Gibson, McNelis, Bargagliotti, and Project-SET from Statistics Education Web (STEW) published in 2013 that explicitly aligns to the GAISE model. Students use the Census at School Data to create and interpret scatterplots.
- NFL Quarterback Salaries by Gibson, Bargagliotti, and Project-SET from Statistics Education Web (STEW) published in 2013 that explicitly aligns to the GAISE model. Students use scatterplots to determine which variables are the bet predictor of an NFL player's salary.
- How High Can You Jump? by Diann Reischmann from Statistics Education Web (STEW) published in 2012 that explicitly aligns to the GAISE model. Students use scatterplots and box plots to summarize the data.
- Devising a Measure: Correlation is a task by Mathematics Assessment Project involving students working with correlation involving a drive-in movie theater.
- EllipSeeIT: Visualizing Strength and Direction of Correlation is a lesson by Olvera, Dias, and Colleagues from the Statistics Education Web (STEW) published in February 2017 that explicitly aligns to the GAISE model. This lesson focuses on correlation as a way of measuring association. Students will then collect data about themselves to create a scatterplot.
- Correlation Guessing Game is an applet by Rossman/Chance where students can guess the correlation of various scatterplots.

# INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (S.ID.5-6)

**Scatterplots, continued**

- Analyzing Two Quantitative Variables in an applet by Rossman/Chance that allows students to analyze two quantitative variables on a scatterplot.
- Applying Correlation Coefficients—Educational Attainment and Unemployment by the United States Census Bureau is a lesson that uses scatterplots to explore the relationship between educational level and unemployment.
- Educational Attainment and Marital Age by the United States Census Bureau is a lesson that explores the relationship between educational attainment and marriage age.

**Curriculum and Lessons from Other Sources**

- EngageNy's Algebra 1, Module 2: Topic C, Lesson 9: Summarizing Bivariate Categorical Data, Lesson 10: Summarizing Bivariate Categorical Data with Relative Frequencies, and Lesson 11: Conditional Relative Frequencies and Association are lessons involving categorical data and two-way frequency tables.
- EngageNy's Algebra 1, Module 2: Topic D, Lesson 12: Relationships Between Two Numerical Variables and Lesson 14: Modeling Relationships with a Line are lessons involving scatterplots and the least-squares regression line.
- Mathematics Vision Project, Algebra 1, Module 9: Modeling Data has several tasks that pertain to this cluster.
- Georgia Standards of Excellence Curriculum Framework, Algebra 1, Unit 6: Describing Data has many lessons that pertain to this cluster. This cluster is addressed on pages 39-106.
- Exploring Linear Relations by Burrill and Hopfensperger is a pdf version of a textbook from the Data-Driven Mathematics series published by Dale Seymour Publications. It has some lessons that pertain to this cluster. The student edition is found here.

**General Resources**

- Arizona's High School Progression on Statistics and Probability is an informational document for teachers. This cluster is addressed on pages 4-6.
- Arizona's High School Progression on Modeling is an informational document for teachers. Statistics and Probability is discussed on page 10.
- High School Coherence Map by UnboundED is a map of the relationships among high school mathematics standards. Please note: This resource is aligned to the Common Core not Ohio's Learning Standards. Although the tool is useful, it needs to be checked with respect to Ohio's Learning Standards.
- Statistics Teacher is an online journal published by the American Statistical Association (ASA) and NCTM. It includes articles, lesson plans, and professional development opportunities.
- Significance is a magazine that demonstrate the practical use of statistics and shows how statistics benefits society.
- Chance is a magazine about statistics for anyone who is interested in using data to advance science, education, and society.
- Levels of Conceptual Understanding in Statistics (LOCUS) is an NSF funded project that has assessment questions around statistical understanding.

# INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (S.ID.5-6)

**References**


- Common Core Standards Writing Team. (2016, March 24). *Progressions for the Common Core State Standards in Mathematics (draft). Grades 6-8, Statistics and Probability*. Tucson, AZ: Institute for Mathematics and Education, University of Arizona.
- Common Core Standards Writing Team. (2013, July 4). *Progressions for the Common Core State Standards in Mathematics (draft). High School, Modeling*. Tucson, AZ: Institute for Mathematics and Education, University of Arizona.
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| STANDARDS | MODEL CURRICULUM (S.ID.7-8) |
|---|---|
| **Statistics and Probability**<br>**INTERPRETING CATEGORICAL AND QUANTITATIVE DATA**<br>**Interpret linear models.**<br>**S.ID.7** Interpret the slope (rate of change) and the intercept (constant term) of a linear model in the context of the data.★<br>**S.ID.8** Compute (using technology) and interpret the correlation coefficient of a linear fit.★ | **Expectations for Learning**<br>In middle school, students interpret the slope and *y*-intercept of a linear model. In Math 1, students build on this knowledge with more sophisticated problems. Since scales may vary, students require a deeper conceptual understanding of slope. They also need to recognize when the *y*-intercept is not always meaningful in the context of the data. This leads to the computation and interpretation of the correlation coefficient and its interpretation. In Math 3, students are introduced to and explore the distinction between correlation and causation.<br><br>The learning of standard S.ID.7 is at the developmental Level B. The learning of standard S.ID.8 is at developmental Level C. See pages 77-78 for more information on Level B, and see the Algebra 2/Math 3 Model Curriculum for more information on Level C.<br><br>**ESSENTIAL UNDERSTANDINGS**<br>• In a linear model, the slope represents the change in the predicted value for every one unit of increase in the independent (*x*) variable.<br>• When appropriate, the *y*-intercept represents the predicted value of the dependent variable when $x = 0$.<br>• In a linear model, the *y*-intercept may not always be appropriate for the context.<br>• The correlation coefficient (*r*) is a measure of the strength of a linear association in the data. Correlation coefficients are between –1 and 1, inclusive.<br>  o If *r* is close to 0, then there is a weak correlation.<br>  o If *r* is close to 1, then there is a strong correlation with a positive slope.<br>  o If r is close to –1, then there is a strong correlation with a negative slope.<br>*Continued on next page* |

| STANDARDS | MODEL CURRICULUM (S.ID.7-8) |
|---|---|
| | **Expectations for Learning, continued**<br>**MATHEMATICAL THINKING**<br>• Use accurate and precise mathematical vocabulary.<br>• Construct formal and informal arguments to verify claims and justify conclusions.<br>• Solve real-world and statistical problems.<br>• Use appropriate tools to display and analyze data.<br>• Determine reasonableness of predictions.<br><br>**INSTRUCTIONAL FOCUS**<br>• Given a linear model, interpret the slope and the *y*-intercept within a context.<br>• Compute, with technology, and interpret correlation coefficient (*r*).<br><br>**Content Elaborations**<br>**OHIO'S HIGH SCHOOL CRITICAL AREAS OF FOCUS**<br>• Math 1, Number 4, page 8<br><br>**THE GAISE MODEL**<br>• GAISE Model, pages 14 – 15<br>    o The focus of S.ID.7 is at Level B for Math 1, pages 37-60<br>    o The focus of S.ID.8 is at Level C for Math 1, pages 61-85<br><br>**CONNECTIONS ACROSS STANDARDS**<br>• Interpret the structure of functions (F.IF.4).<br>• Construct linear models, and solve problems (F.LE.1-2).<br>• Interpret expressions for functions in terms of context (F.LE.5).<br>• Build a function that models a relationship between two quantities (F.BF.1).<br>• Summarize, represent, and interpret data in two categories and quantitative variables (S.ID.6). |

# INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (S.ID.7-8)

## Instructional Strategies

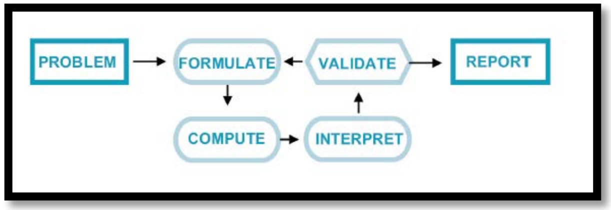
*Note: The model curriculum is not a collection of lessons nor a full curriculum; it does not suggest pace, sequence, or amount of time spent on topics. It provides information about a topic related to the standards including ideas for examples, strategies for teaching, common misconceptions, and possible connections between topics. In addition, the emphasis of the model curriculum is on instruction not on assessment; therefore, the examples in the Instructional Strategies section are written for instructional purposes not necessarily as assessment items. Examples may need to be rewritten to accommodate the needs of each individual classroom.*

***Standards for Mathematical Practice***
*This cluster focuses on but is not limited to the following practices:*
***MP.4*** *Model with mathematics.*
***MP.6*** *Attend to precision.*
***MP.7*** *Look for and make use of structure.*

As this cluster is a continuation of statistical thinking in both 8.SP.3 and S.ID.6, continue to use the GAISE model in exploring quantitative bivariate data. Emphasize using mathematical models to capture key elements of the relationship between two variables. Use real-world data and make connections to science. This cluster emphasizes the last two steps of the GAISE model: analyze data and interpret results. The first two steps were emphasized in the previous cluster (S.ID.6) with respect to quantitative data.

### MODELING

S.ID.7-8 is a modeling standard. See page 13 for more information about modeling.




### INTERPRETING THE SLOPE AND THE INTERCEPT

It will be helpful for students to write models in the context of the data. For example, when simply writing $y = 20x + 10$ to represent the height of a hot air balloon over time, students may struggle to make the connections between the variables and the model. An alternative strategy would be to write the model as follows:

$$\text{height (feet) of balloon} = 20\,\frac{feet}{minute} \cdot (x \text{ minutes}) + 10 \text{ feet}$$

The interpretation of the slope is an increase of 20 feet for every additional minute. The interpretation of the $y$-intercept would be that the balloon is at a height of 10 feet above the ground when time is equal to 0.

# INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (S.ID.7-8)

In some cases, the interpretation of the *y*-intercept (constant term) is not meaningful in the context of the data. This happens when the zero point on the horizontal axis is of considerable distance from the initial values of the horizontal variable, or in some case has no meaning at all. For example, if the horizontal axis measures student heights, it is impractical to reason that a person would have a height of zero. In Grade 8, students interpret the slope and *y*-intercept of a linear model, including situations when the *y*-intercept does not have an appropriate interpretation in context. Now since students have the vocabulary of "domain", they can explicitly state that when a *y*-intercept does not have an appropriate interpretation, 0 is excluded from the domain of the model, whereas in 8$^{th}$ grade they could not yet do so.

A visual rendering of slope makes no sense in most scatterplots. For example, students may believe that a 45° line in scatterplot of two numerical variables always indicates a slope of 1 which is the case only when the two variables have the same scaling. Because the scaling for many real-world situations varies greatly, students need to be given opportunities to compare graphs of differing scales. Asking students questions like "What would this graph look like with a different scale or using this scale?" is essential in addressing this misconception.

## CORRELATION COEFFICIENT

The correlation coefficient, *r*, measures "tightness" of the data points to a line, and the strength of the linear relationship between the two variables. Computing the formula for *r* is long and cumbersome, so students need to find the correlation coefficient using technology. The focus is on interpreting the value, not on calculating the value.

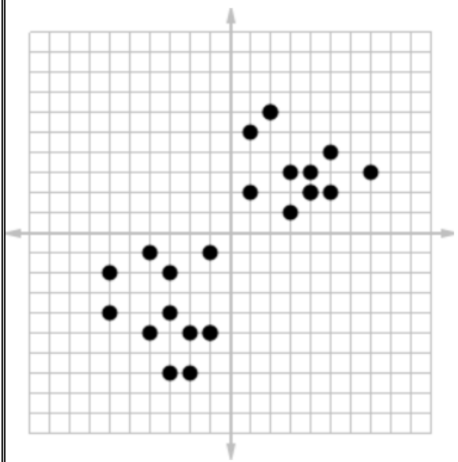
To make some sense of Pearson's r, correlation coefficient, students should recall their middle school experience with the Quadrant Count Ratio (QCR) as a measure of the relationship between two quantitative variables (See Grade 8 Model Curriculum 8.SP.1-4). The difference between QCR and Pearson's correlation coefficient is that the QCR is only based upon how many points are in each quadrant whereas Pearson's looks at how far each point is from the dividing line of each quadrant. To make the connection for students, the QCR should be reviewed and its limitations should be discussed. The primary weakness of the QCR is that all points have the same weight whether or not they are far or close to the line of best fit. Another drawback is that just because the QCR = $\pm\mathbf{1}$, the relationship between the two variables is not necessarily perfectly linear. In addition, the QCR can even be $\pm\mathbf{1}$ when it is not perfectly positive. Even though students are not required to calculate Pearson's coefficient, there should be some discussion in the classroom on how it developed. Resources such as The Evolution of Pearson's Correlation Coefficient/Exploring the Relationships Between Two Quantitative Variables by Gary Kader from the Science Education Resource Center and Carleton College and The Evolution of Pearson's Correlation Coefficient can help guide the discussion.

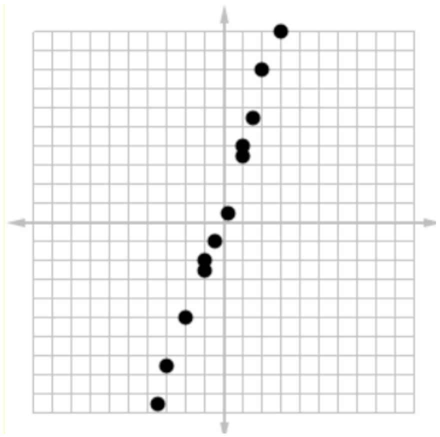
# INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (S.ID.7-8)

***EXAMPLE***

Have students discuss something similar to the following to graphs.

**Graph A** **Graph B**





**a.** Calculate the QCR for both graphs. What do you notice?

**b.** However, is one graph more positively correlated than the other? Explain.

**c.** How could we adjust the QCR formula to give more weight to the points that are closer to the regression line?

*Discussion:* Although, it is unlikely that student's will come up with the formula for $r$, this activity will help them see the need for such a formula and develop an intuitive understanding of what exactly $r$ is calculating. To extend the discussion, the development of Pearson's formula could be introduced as a discussion point, but students should be expected to do the actual calculations via technology.

Explain that in order for the correlation coefficient to be exactly 1 or $-1$, the data points must fall exactly on the line of best fit. The closer $r$ is to $\pm 1$, the stronger the correlation. The closer $r$ is to 0, the weaker the correlation. However, to be noted getting a correlation coefficient of exactly 0 is extremely unlikely.

A huge benefit for using $r$ squared (instead of $r$) is that the values of $-1$ and 1 separately do not need to be discussed separately. Instead, an $r$ squared value close to 1 is a strong correlation, while an $r$ squared value close to 0 is a weak correlation.

Students may incorrectly think that $r$ must have units, and that the units relate to the context of the problem. Instead, the only way to relate $r$ to the data is to use it to measure the strength and the direction of the relationship.

# INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (S.ID.7-8)

## UNDERSTANDING VARIABILITY

Students should explore the differences between variations and errors. Teachers can draw attention to naturally occurring errors that happen in the classroom and discuss the impact of the errors on the results. These notions can be used to explain outliers, clusters, and gaps. Understanding variability is vital for developing data sense (also in S.ID.1-3 and 8.SP.1-4).

At Level B students start to develop the ideas of covariability, induced variability, and sampling variability. See GAISE model pages 6 and 20.

## ASSOCIATION

The correlation coefficient is a quantity that measure the strength and direction of an association. Students in 8th grade learn about the Quadrant Count Ratio (QCR) which allows students to have a conceptual understanding of correlation coefficient. Students need to have an informal understanding that an association present between two variables does not necessarily mean that there is a cause and effect relationship.

## Instructional Tools/Resources

*These tools and resources may be helpful for instruction. Before use, they should be evaluated at the local level to determine their appropriateness for instruction.*

**Manipulatives/Technology**

- Excel/Google Sheets
- Graphing calculators
- Desmos is a free graphing calculator that is available to students as website or an app.
- Geogebra is a free graphing calculator that is available to students as website.
- Wolframalpha is dynamic computing tool.
- Correlation and Regression is a statistical applet that allows students to explores why the correlation and least-squares regression line changes as points are added or deleted from a scatterplot.

## INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (S.ID.7-8)

**Interpreting Linear Models**

- Coffee and Crime is a task by Illustrative Mathematics where students analyze bivariate quantitative data.
- Used Suburban Foresters II is a task by Illustrative Mathematics that emphasizes that regression lines always have a context by interpreting the slope and the intercept.
- Statistics: Does a Correlation Exist? is an activity by Texas Instruments to determine if a data set has a positive or negative correlation coefficients.
- Olympic Men's 100-meter Dash is a task by Illustrative Mathematics that has students describe the strength and direction of an association.
- Text Messaging is Time Consuming! What Gives? is a lesson by Gibson, McNelis, and Bargagliotti from the Statistics Education Web (STEW) published in August 2013. It allows students to explore relationships between two variables using data from the Census at School project. They explore variability in slope, intercepts, and correlations.
- NFL Quarterback Salaries is a lesson by Gibson and Bargagliotti from the Statistics Education Web (STEW) published in July 3013. Students will set up a statistical question and interpret a linear regression equation and the correlation coefficient.
- Guessing Correlations in an applet by Stat.istics.net where students have to match correlations with the scatterplots.
- Spurious Correlations by Tyler Vigen is a website that shows unrelated correlations. Although truly understanding the difference between correlation and causation is not required until Algebra 2, an informal discussion about association and cause-and-effect could take place.
- From the NCTM Illuminations, Impact of a Superstar is a lesson that uses technology tools to plot data, identify lines of best fit, and detect outliers. Then, students compare the lines of best fit when one element is removed from a data set, and interpret the results. *NCTM now requires a membership to view their lessons.*
- iPhone 6s Opening Weekend Sales by Clifford Pate is a Desmos lesson paired with Will the New iPhone Sales Be Huge? from Yummy Math has students make predictions about the number of iPhone units sold.
- Line of Best Fit by Desmos has students visualize a line to fit a data set, then graph that line with sliders and make predictions.

**Curriculum and Lessons from Other Sources**

- Algebra 1, Module 9: Modeling Data is a unit by the Mathematics Vision Project. Tasks 9.5, 9.6, and 9.7 align to this cluster.
- EngageNY Algebra 1 Module 2, Topic D, Lesson 19 is a lesson about interpreting the correlation coefficient as a measure of strength.

# INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (S.ID.7-8)

**General Resources**

- Arizona's High School Progression on Statistics and Probability is an informational text for teachers. This cluster is addressed on pages 6-7.
- Arizona's High School Progression on Modeling is an informational text for teachers. This cluster is addressed on pages 2, 5, and 14.
- High School Coherence Map by UnboundED is a map of the relationships among high school mathematics standards. Please note: This resource is aligned to the Common Core not Ohio's Learning Standards. Although the tool is useful, it needs to be checked with respect to Ohio's Learning Standards.
- Statistics Teacher is an online journal published by the American Statistical Association (ASA) and NCTM. It includes articles, lesson plans, and professional development opportunities.
- Significance is a magazine that demonstrate the practical use of statistics and shows how statistics benefits society.
- Chance is a magazine about statistics for anyone who is interested in using data to advance science, education, and society.
- K-12 Statistics Education Resources is a webpage put together by the American Statistics Association to provide resources for K-12 educators.

**References**


- Common Core Standards Writing Team. (2016, March 24). *Progressions for the Common Core State Standards in Mathematics (draft). Grades 6-8, Statistics and Probability*. Tucson, AZ: Institute for Mathematics and Education, University of Arizona.
- Common Core Standards Writing Team. (2013, July 4). *Progressions for the Common Core State Standards in Mathematics (draft). High School, Modeling*. Tucson, AZ: Institute for Mathematics and Education, University of Arizona.
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