van Hiele Model of Geometric Thinking

The van Hiele Levels

The van Hiele Model of Geometric Thinking describes how students learn geometric concepts. The van Hiele Model contains five levels.

Level 0	Level 1	Level 2	Level 3	Level 4
Visualization	Analysis	Informal Deduction/ Abstraction	Deduction	Rigor

Note: Different researchers use different numbering systems to describe the van Hiele levels.

In the van Hiele Model, a student must progress through each of the levels in order. Each level introduces its own concepts, symbols, and language. As students' move through the levels, what is introduced and developed at a lower level, is extended and applied at the following level. Concepts implicitly understood at one level become explicitly understood at the next level. Levels are not age dependent. Instead they are thought to be based upon instruction. Students in the same classroom may possibly be at more than two van Hiele levels in the same classroom. Therefore, instruction needs to match the individual student's level.

While most students will progress from Level 0 to Level 2 in grades K-8, that is not true for all students. An elementary school student may advance to Level 2 and a high school student may remain at Level 0. Students at different levels of understanding may find it difficult to communicate their understandings and/or misunderstandings with others that are at a higher or lower level. Teachers may need to scaffold instructional activities to address varying student levels within a classroom. As students engage in geometric activities pay careful attention to the observations students are making and the language they are using to communicate their understanding.

It is important to note that students may be at different levels for different topics. (See the appendix for a description of van Hiele levels for different topics.) Therefore, a student could be at Level 2 with respect to plane shapes and Level 1 with respect to three-dimensional figures.

Phases of Learning

To help students progress from one level to the next, instruction should be designed strategically using the phases of learning. Students need to cycle (and possibly recycle) through the phases for a given topic in order to reach the next van Hiele level.

INFORMATION/INQUIRY

The focus of this phase is to assess students' prior knowledge through inquiry and vocabulary acquisition, so the teacher can design further instruction accordingly. Other characteristics of this phase include the following:

- Observations are made; questions are raised.
- Teachers learn student prior knowledge; Students gain direction.
- Ideas are introduced for students to think about.
- Level-specific vocabulary is introduced.

Phases of Learning, continued

DIRECTED ORIENTATION

The focus of this phase is for teachers to design carefully sequenced activities to explore a topic. Other characteristics of this phase include the following:

- Activities should reveal geometric structures characterized by the level.
- Tasks and materials should make the geometric concept very evident.
- Tasks should be short and elicit specific responses.
- Students are actively engaged exploring and manipulating objects (folding, measuring, etc.)

EXPLICATION

The focus of this phase is for students to verbalize understandings. Other characteristics of this phase include the following:

- The teacher role is minimal, but uses questioning techniques to draw out student thinking.
- The teacher bridges students' informal descriptions of concepts to using more precise mathematical vocabulary during discussion, but only after they are able to articulate the concept in their own words.
- The teacher should draw distinctions between the common usage of vocabulary and its mathematical usage.

FREE ORIENTATION

The focus of this phase is for students to solve problems and make connections. The tasks are more complex, multi-stepped, and/or open-ended.

- Students find their own ways of solving problems.
- The teacher's role is to select appropriate problems with multiple-solution pathways and to encourage students to reflect and elaborate.
- The teacher may introduce terms, concepts, and relevant problem-solving processes if needed.

INTEGRATION

The focus of this phase is for students reflect on what they learned and determine how the learning fit in the overall mathematical structure. Other characteristics of this phase include the following:

- The emphasis is on students understanding mathematical structures.
- The students review and summarize what they have learned.
- Teacher can furnish global summaries to help students synthesize concepts but should not present anything new.
- Ideally, the new level of thought replaces the old, and students can proceed to the next van Hiele level. If not, the students need to recycle through all or several of the phases using new tasks.

Appendix 1: Shapes

LEVEL 0 (VISUALIZATION)
This level can be characterized by the student doing some or all of the following:

- identifying shapes as visual-wholes;
- recognizing geometric figures by their appearance alone;
- not identifying the properties of figures; and/or
- grouping figures that look "alike."

LEVEL 1 (ANALYSIS)
This level can be characterized by the student doing some or all of the following:

- recognizing geometric figures by their shape;
- seeing classes of shapes;
- listing all the properties of a shape, but not seeing relationships between properties; and/or
- likely seeing all properties as being equally important.

LEVEL 2 (INFORMAL DEDUCTION/ABSTRACTION)
This level can be characterized by the student doing some or all of the following:

- noticing relationships within and between properties and figures;
- forming meaningful definitions;
- justifying reasoning using informal arguments;
- drawing logical maps and diagrams; and/or
- not yet understanding formal deduction.

LEVEL 3 (DEDUCTION)
This level can be characterized by the student doing some or all of the following:

- constructing proofs and understanding the necessity of proofs;
- understanding the role and relationships between of axioms, theorems, definitions, and postulates;
- differentiating between necessary and sufficient conditions; and/or
- making logical conclusions.

LEVEL 4 (RIGOR)
This level can be characterized by the student doing some or all of the following:

- establishing and comparing mathematical systems;
- describing the effect of adding or deleting an axiom in a given system; and/or
- understanding non-Euclidean systems.

Appendix 2: Transformations/Location

LEVEL 0 (VISUALIZATION)
This level can be characterized by the student doing some or all of the following:

- identifying a slide, flip, and turn as maintaining the same shape;
- identifying whether a slide, flip, and/or turns creates an image from a preimage; and/or
- understanding location on a coordinate grid.

LEVEL 1 (ANALYSIS)
This level can be characterized by the student doing some or all of the following:

- extending a coordinate grid to four quadrants;
- using square grids to examine transformations; and/or
- recognizing, using, and describing properties of translations, rotations, and reflections.

LEVEL 2 (INFORMAL DEDUCTION/ABSTRACTION)
This level can be characterized by the student doing some or all of the following:

- analyzing relationships within and between transformations;
- algebraically expressing transformations: reflections about the *x*- and *y*-axes, and rotations of 45°, 90°, and 180° about the origin;
- understanding and justifying how to use perpendicular bisectors of corresponding points to locate the turn center of a rotation;
- understanding how the composition of two reflections can be a translation or a rotation; and/or
- following a rationale if shown proof.

LEVEL 3 (DEDUCTION)
This level can be characterized by the student doing some or all of the following:

- Understanding and creating formal proofs using transformations.

LEVEL 4 (RIGOR)
This level can be characterized by the student doing some or all of the following:

- establishing and comparing mathematical systems;
- describing the effect of adding or deleting an axiom in a given system; and/or
- understanding non-Euclidean systems.

Appendix 3: Visualization

LEVEL 0 (VISUALIZATION)

This level can be characterized by the student doing some or all of the following:

- thinking about shapes and solids based on the way that they look;
- looking at different orientations is a challenge;
- starting to think of solids in the terms of faces or sides;
- starting to think about positions of lines and features of solids;
- comparing solids based on features (faces, edges, vertices) without paying attention to properties such as angle sizes or edge lengths;
- unable to visualize solids they cannot see;
- moving solids by guess and check; and/or
- lacking coordination between desired movement of solids (which can be shown by a movement of the hands) and what happens on computer screen.

LEVEL 1 (ANALYSIS)

This level can be characterized by the student doing some or all of the following:

- showing a greater degree of attention to properties of shapes and solids;
- building 3D figures from 2D images and 2D drawings from 3D figures;
- viewing a figure from front, back, left, and right positions of solids;
- visualizing cross-sections when slicing solids;
- comparing solids based on properties;
- using observation as a basis for explanations; and/or
- understanding that a movement is made, then after observing the result, the next movement is selected, etc.

LEVEL 2 (INFORMAL DEDUCTION/ABSTRACTION)

(Note: The significant difference between Level 1 and Level 2 is using logical reasoning. Level 1 activities can be turned into Level 2 by logical reasoning).

This level can be characterized by the student doing some or all of the following:

- using mathematical analysis of solids before performing any movements;
- giving informal justifications based on isolated properties; and/or
- analyzing beginning and final positions and making a plan to transform figures using a sequence of movements.

LEVEL 3 (DEDUCTION)

This level can be characterized by the student doing some or all of the following:

- analyzing solids based on structure including properties from definitions;
- preplanning the movement of solids; and/or
- using movements efficiently.

LEVEL 4 (RIGOR)

This level can be characterized by the student doing some or all of the following:

- establishing and comparing mathematical systems;
- describing the effect of adding or deleting an axiom in a given system; and/or
- understanding non-Euclidean systems.

Appendix 4: Three-Dimensional Figures

LEVEL 0 (VISUALIZATION)

This level can be characterized by the student doing some or all of the following:

- judging solids visually and holistically;
- not considering properties;
- unable to visualize what they cannot see; and/or
- manipulating solids by guess and check.

LEVEL 1 (ANALYSIS)

This level can be characterized by the student doing some or all of the following:

- identifying components;
- informally describing properties;
- not logically relating properties; and/or
- visualizing simple movements from one visible position to another.

LEVEL 2 (INFORMAL DEDUCTION/ABSTRACTION)

This level can be characterized by the student doing some or all of the following:

- understanding properties
- logically classifying solids;
- understanding logic of definitions;
- comparing solids by analyzing;
- visualizing positions that are not visible by reasoning; and/or
- matching corresponding parts.

LEVEL 3 (DEDUCTION)

This level can be characterized by the student doing some or all of the following:

- proving theorems about solids; and/or
- linking visualization to properties of solids.

LEVEL 4 (RIGOR)

This level can be characterized by the student doing some or all of the following:

- establishing and comparing mathematical systems;
- describing the effect of adding or deleting an axiom in a given system; and/or
- understanding non-Euclidean systems.

Appendix 5: Geometric Measurement (Length, Area, and Volume)

Note: Most of the research with Geometric Measurement is aligned to Battista's CBA Levels, which is a significant elaboration on the original van Hiele levels, because of this the concepts do not necessarily correspond to the van Hiele levels. For consistency purposes, an alignment to van Hiele is created below, but for a more accurate and precise description of level with respect to geometric measurement see Battista, M. (2012). *Cognition-Based Assessment and Teaching of Geometric Measurement*. Portsmouth, NH: Heinemann.

LEVEL 0 (VISUALIZATION)

This level can be characterized by the student doing some or all of the following:

- visually comparing lengths, area, and volume holistically;
- comparing lengths, areas, or volumes directly or indirectly;
- using numbers unconnected to the concept of measuring length, area, or volume;
- lack of understanding of a length, area, or volume unit;
- lacking proper spatial structure or organization of area or volume units;
- organizing measurement unit iteration in ways inconsistent with the properties of measurement (gaps, overlaps, different size units);
- incorrect counting or double counting;
- decomposing lengths/paths, shapes, or solids incorrectly; and/or
- applying formulas without understanding a figure's dimensions or why the formula works.

LEVEL 1 (ANALYSIS)

This level can be characterized by the student doing some or all of the following:

- comparing length, area, or volume by manipulating and matching parts;
- visually comparing shapes by composing/decomposing;
- visualizing structured iteration of length, area, or volume units;
- organizing area and volume units into (2D, 3D) array structure without gaps or overlaps.
- using a single unit, row, or layer repeatedly (iterating) to correctly measure or construct length, area, or volume respectively.
- determining measurement without having to show every unit instead of using only numbers (no visible units or repeated units); and/or
- creating composite units, columns, rows, or layers to find length, area, or volume.

LEVEL 2 (INFORMAL DEDUCTION/ABSTRACTION)

This level can be characterized by the student doing some or all of the following:

- fully understanding the procedures and formula for determining measurement of simple, familiar shapes;
- generalizing unit-measure enumeration to include fractional units;
- understanding and making unit conversions; and/or
- generalizing measurement to non-squares for area and non-cubes for volume.

Appendix 5: Geometric Measurement, continued

LEVEL 3 (DEDUCTION)

This level can be characterized by the student doing some or all of the following:

- comparing shapes by property-preserving transformations/decompositions;
- using geometric properties and variables to understand and solve problems involving formulas for non-familiar shapes; and/or
- comparing length, area, and volume by using geometric properties or transformations.

LEVEL 4 (RIGOR)

This level can be characterized by the student doing some or all of the following:

- establishing and comparing mathematical systems;
- describing the effect of adding or deleting an axiom in a given system; and/or
- understanding non-Euclidean systems.

References

Battista, M. (2012). *Cognition-Based Assessment and Teaching of Geometric Measurement*. Portsmouth, NH: Heinemann.

Battista, M. (2011). Conceptualizations and issues related to learning progressions, learning trajectories, and levels of sophistication. *The Mathematics Enthusiast*, 8(3), p. 507.

Breyfogle, M. & Lynch, C. (November 2010). van Hiele revisited. *Mathematics Teaching in the Middle School*, 16(4), 232-238.

Clements, D. (January 2003). Teaching and learning geometry. *A Research Companion to Principles and Standards for School Mathematics*. Reston, VA: National Council of Teacher of Mathematics.

Clements, D. & Battista, M., (1992) Geometry and spatial reasoning. In D.A. Grouws (ed.), *Handbook of Research on Mathematics Teaching and Learning* (pp.420-464). New York: Macmillan.

Crowley, M. (1987). The van Hiele model of development of geometric thought. *In Learning and Teaching Geometry , K-12, 1987 Yearbook of the National Council of Teachers of Mathematics.* Edited by Mary Montgomery Lindquist, 1-16. Reston, VA: National Council of Teacher of Mathematics.

George, W. (January 2017). Bringing van Hiele and Piaget together: a case for topology in early mathematics learning. *Journal of Humanistic Mathematics*, 7(1), 105-116: doi: http://scholarship.claremont.edu/cgi/viewcontent.cgi?article=1158&context=jhm

Groth, R. (August 2005). Linking theory and practice in teaching geometry. *Mathematics Teacher*, 99(1), 27-30.

Gutiérrez, A. (1992). Exploring the links between van Hiele levels and 3-dimensional geometry. *Structural Topology*, 31-47.

Howse, T. & Howse, M. (December 2014/January 2015). Linking van Hiele theory to instruction. *Teaching Children Mathematics*, 21(5), 304-312.

Malloy, C. (October 1999). Perimeter and area through the van Hiele model. *Mathematics Teaching in the Middle School*, 5(2), 87-90.

Wheatley, J. (August 2011). An Investigation of Three-Dimensional Problem Solving and Levels of Thinking Among High School Geometry Students. A Project Report Presented to the Graduate Faculty of Central Washington University. Retrieved from https://pdfs.semanticscholar.org/994a/d70f32b0dd2b38342da230b6275e08f9032e.pdf

Whittaker, H. & Johnson, I. (October 2005). Interactive geometry in the B.C. (before computers) Era. *Mathematics Teacher*, 99(3), 170-176.