Ohio's Model Curriculum | Mathematics

with Instructional Supports

Grade 8

Ohio | Department of Education

Mathematics Model Curriculum
with Instructional Supports
Grade 8

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Introduction

PURPOSE OF THE MODEL CURRICULUM

Just as the standards are required by Ohio Revised Code, so is a model curriculum that supports the standards. Throughout the development of the standards (2016-17) and the model curriculum (2017-18), the Ohio Department of Education (ODE) has involved educators from around the state at all levels, Pre-K–16. The model curriculum reflects best practices and the expertise of Ohio educators, but it is not a complete a curriculum nor is it mandated for use. The purpose of Ohio's model curriculum is to provide clarity to the standards, a foundation for aligned assessments, and guidelines to assist educators in implementing the standards. The model curriculum is not a collection of lessons nor a full curriculum; it does not suggest pace, sequence, or amount of time spent on topics. It provides information about a topic related to the standards including ideas for examples, strategies for teaching, possible connections between topics, and some common misconceptions.

To be noted, the emphasis of the model curriculum is on <u>instruction</u> not on assessment; therefore, the examples in the Instructional Strategies section are written for instructional purposes not necessarily as assessment items. Also, examples presented in this document may need to be rewritten to accommodate the needs of each individual classroom.

COMPONENTS OF THE MODEL CURRICULUM

The model curriculum contains two sections: Expectations for Learning and Content Elaborations.

Expectations for Learning: This section begins with an introductory paragraph describing the cluster's position in the respective learning progression, including previous learning and future learning. Following are three subsections: Essential Understandings, Mathematical Thinking, and Instructional Focus.

- **Essential Understandings** are the important concepts students should develop. When students have internalized these conceptual understandings, application and transfer of learning results.
- **Mathematical Thinking** statements describe the mental processes and practices important to the cluster.
- **Instructional Focus** statements are key skills and procedures students should know and demonstrate.

Together these three subsections guide the choice of lessons and formative assessments and ultimately set the parameters for aligned state assessments.

Content Elaborations: This section provides further clarification of the standards, links the critical areas of focus, and connects related standards within a grade or course.

Continued on next page

Introduction, continued

COMPONENTS OF INSTRUCTIONAL SUPPORTS

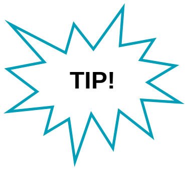
The Instructional Supports section contains the **Instructional Strategies** and **Instructional Tools/Resources** sections which are designed to be fluid and improving over time, through additional research and input from the field. The **Instructional Strategies** are descriptions of effective and promising strategies for engaging students in observation, exploration, and problem solving targeted to the concepts and skills in the cluster of standards. Descriptions of common misconceptions as well as strategies for avoiding or overcoming them and ideas for adapting instructions to meet the needs of all students are threaded throughout. The **Instruction Tools/Resources** are links to relevant research, tools, and technology. In our effort to make sure that our Instructional Supports reflect best practices, this section is under revision and will be published in 2018.

There are several icons that help identify various tips in the instructional strategies section:

 = a common misconception

 = a technology tip

 = a career connection



= a general tip which may include diverse leaner or English learner tips.

Standards for Mathematical Practice—Grade 8

The Standards for Mathematical Practice describe the skills that mathematics educators should seek to develop in their students. The descriptions of the mathematical practices in this document provide examples of how student performance will change and grow as students engage with and master new and more advanced mathematical ideas across the grade levels.

MP.1 Make sense of problems and persevere in solving them.
In Grade 8, students solve real-world problems through the application of algebraic and geometric concepts. Students seek the meaning of a problem and look for efficient ways to represent and solve it. They may check their thinking by asking themselves, "What is the most efficient way to solve the problem?", "Does this make sense?", and "Can I solve the problem in a different way?"

MP.2 Reason abstractly and quantitatively.
In Grade 8, students represent a wide variety of real-world contexts through the use of real numbers and variables in mathematical expressions, equations, and inequalities. They examine patterns in data and assess the degree of linearity of functions. Students contextualize to understand the meaning of the number(s) or variable(s) as related to the problem and decontextualize to manipulate symbolic representations by applying properties of operations.

MP.3 Construct viable arguments and critique the reasoning of others.
In Grade 8, students construct arguments using verbal or written explanations accompanied by expressions, equations, inequalities, models, and graphs, tables, and other data displays (e.g., box plots, dot plots, histograms, etc.). They further refine their mathematical communication skills through mathematical discussions in which they critically evaluate their own thinking and the thinking of other students. They pose questions like "How did you get that?", "Why is that true?" "Does that always work?" They explain their thinking to others and respond to others' thinking.

MP.4 Model with mathematics.
In Grade 8, students model problem situations symbolically, graphically, in tables, and contextually. Working with the new concept of a function, students learn that relationships between variable quantities in the real-world often satisfy a dependent relationship, in that one quantity determines the value of another. Students form expressions, equations, or inequalities from real-world contexts and connect symbolic and graphical representations. Students solve systems of linear equations and compare properties of functions provided in different forms. Students use scatterplots to represent data and describe associations between variables. Students need many opportunities to connect and explain the connections between the different representations. They should be able to use all of these representations as appropriate to a problem context. Students should be encouraged to answer questions such as "What are some ways to represent the quantities?" or "How might it help to create a table, chart, graph, or ____?"
Continued on next page

Standards for Math Practice, continued

MP.5 Use appropriate tools strategically.

Students consider available tools (including estimation and technology) when solving a mathematical problem and decide when certain tools might be helpful. For instance, students in Grade 8 may translate a set of data given in tabular form to a graphical representation to compare it to another data set. Students might draw pictures, use applets, or write equations to show the between the angles created by a transversal that intersects parallel lines. Teachers might ask, "What approach are you considering?" or "Why was it helpful to use ____?"

MP.6 Attend to precision.

In Grade 8, students continue to refine their mathematical communication skills by using clear and precise language in their discussions with others and in their own reasoning. Students use appropriate terminology when referring to the number system, functions, geometric figures, and data displays. Teachers might ask, "What mathematical language, definitions, or properties can you use to explain ____ ?"

MP.7 Look for and make use of structure.

Students routinely seek patterns or structures to model and solve problems. In Grade 8, students apply properties to generate equivalent expressions and solve equations. Students examine patterns in tables and graphs to generate equations and describe relationships. Additionally, students experimentally verify the effects of transformations and describe them in terms of congruence and similarity.

MP.8 Look for and express regularity in repeated reasoning.

In Grade 8, students use repeated reasoning to understand the slope formula and to make sense of rational and irrational numbers. Through multiple opportunities to model linear relationships, they notice that the slope of the graph of the linear relationship and the rate of change of the associated function are the same. For example, as students repeatedly check whether points are on the line with a slope of 3 that goes through the point (1, 2), they might abstract the equation of the line in the form $\frac{y-2}{x-1} = 3$. Students should be encouraged to answer questions such as "How would we prove that ____ ?" or "How is this situation like and different from other situations using these operations?"

Mathematics Model Curriculum
with Instructional Supports
Grade 8

STANDARDS	MODEL CURRICULUM (8.NS.1-2)
THE NUMBER SYSTEM **Know that there are numbers that are not rational, and approximate them by rational numbers.** **8.NS.1** Know that real numbers are either rational or irrational. Understand informally that every number has a decimal expansion which is repeating, terminating, or is non-repeating and non-terminating. **8.NS.2** Use rational approximations of irrational numbers to compare the size of irrational numbers, locate them approximately on a number line diagram, and estimate the value of expressions, e.g., π^2. *For example, by truncating the decimal expansion of $\sqrt{2}$, show that $\sqrt{2}$ is between 1 and 2, then between 1.4 and 1.5, and explain how to continue on to get better approximations.*	**Expectations for Learning** In previous grades, students learned to use long division to convert fractions to decimals. The resulting decimals either terminated or repeated. Until this point, their exploration of the real number system was limited to rational numbers. In eighth grade, students will extend their knowledge of real numbers to include irrational numbers whose decimal representation is non-repeating and non-terminating. In future grades, this leads to the understanding that there are numbers that are not real such as imaginary numbers and computations with radicals. **ESSENTIAL UNDERSTANDINGS** **Real Numbers** • Every real number can be classified as repeating, terminating, or non-repeating, non-terminating. • Real numbers are either rational or irrational. • A rational number is any number that can be written as the quotient or fraction of two integers, $\frac{p}{q}$, where *p* is the numerator and *q* is the non-zero denominator. • Rational numbers when written as a decimal expansion are repeating or terminating. • Irrational numbers when written as a decimal expansion are non-repeating and non-terminating. • A number is classified by its simplest form, e.g., $\sqrt{25}$ is rational because 5 is rational. *Continued on next page*

STANDARDS	MODEL CURRICULUM (8.NS.1-2)
8.NS.1-2, continued	**Expectations for Learning, continued** **ESSENTIAL UNDERSTANDINGS, CONTINUED** **Real Numbers, continued** • Square roots may be negative, e.g., Both 6 and –6 are square roots of 36, and $\sqrt{36}$ means only the positive square root whereas $-\sqrt{36}$ means only the negative square root. • A negative sign cannot be inside a square root number at this grade level. • The roots of perfect squares and perfect cubes of whole numbers are rational. • Square roots of whole numbers that are not perfect squares are irrational. • Cube roots of whole numbers that are not perfect cubes are irrational. • Given two distinct numbers, it is possible to find both a rational and an irrational number between them. **MATHEMATICAL THINKING** • Use precise mathematical language. • Determine reasonableness of results. • Generalize concepts based on patterns. • Compute accurately and efficiently. **INSTRUCTIONAL FOCUS** • Identify if a real number is rational or irrational. • Convert rational numbers from fractions to their decimal expansions. • Discover that a rational number in simplest fractional form has a terminating decimal only if the prime factorization of its denominator contains only 2s and 5s. • Approximate the value of irrational numbers. • Explain how to get more precise approximations of irrational numbers. • Approximate the location of irrational numbers on a number line. • Compare and order rational and irrational numbers. *Continued on next page*

STANDARDS	MODEL CURRICULUM (8.NS.1-2)
8.NS.1-2, continued	**Content Elaborations** • Ohio's K-8 Critical Areas of Focus, Grade 8, Number 4, page 55 • Ohio's K-8 Learning Progressions, The Number System, pages 16-17 **CONNECTIONS ACROSS STANDARDS** • Know that $\sqrt{2}$ is irrational (8.EE.2). • Side lengths of right triangles can be irrational (8.G.6-8).

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (8.NS.1-2)

Instructional Strategies

Note: The model curriculum is not a collection of lessons nor a full curriculum; it does not suggest pace, sequence, or amount of time spent on topics. It provides information about a topic related to the standards including ideas for examples, strategies for teaching, common misconceptions, and possible connections between topics. In addition, the emphasis of the model curriculum is on instruction not on assessment; therefore, the examples in the Instructional Strategies section are written for instructional purposes not necessarily as assessment items. Examples may need to be rewritten to accommodate the needs of each individual classroom.

Standards for Mathematical Practice
This cluster focuses on but is not limited to the following practices:
MP.6 *Attend to precision.*
MP.7 *Look for and make use of structure.*
MP.8 *Look for and express regularity in repeated reasoning.*

Every real number has a place on the number line and therefore has a decimal expansion. Placing numbers, both rational and irrational, on a number line is vital to building number sense. Discussions should include the concepts of denseness and of infiniteness such that between any two rational numbers there are an infinite number of numbers.

EXAMPLE
Pick two rational numbers that are "really close." Have a partner find 5 numbers between them.

RATIONAL NUMBERS
A **rational number** is a number that can be written in the form *a*/*b*, where *a* and *b* are both integers, and *b* is not 0; their decimal expansion either repeats or terminates.

Students may incorrectly think that all numbers written in fraction form are rational numbers. However, $\frac{\pi}{2}$ is not a rational number since π is not an integer. However, $\frac{3.2}{4.5}$ is a rational number because it is equivalent to $\frac{32}{45}$.

In previous grades, students become familiar with rational numbers called decimal fractions. In Grade 7, students carry out the long division and recognize that the remainders may repeat in a predictable pattern—a pattern creates the repetition in the decimal representation (see 7.NS.2.d). In Grade 8, they explore its occurrence.

Ask students what will happen in long division once the remainder is 0. They can reason that the long division is complete, and the decimal representation terminates. However, if the remainder never becomes 0, then the remainder will repeat in a cyclical pattern. The important understanding is that students can see that the pattern will continue to repeat.

Explore differences between terminating and repeating decimals.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (8.NS.1-2)

EXAMPLE (EXTENSION)

$\frac{1}{2}, \frac{3}{10}, \frac{1}{8}, \frac{1}{3}, \frac{3}{80}, \frac{5}{8}, \frac{1}{4}, \frac{1}{14}, \frac{39}{40}, \frac{1}{5}, \frac{4}{25}, \frac{1}{6}, \frac{7}{12}, \frac{1}{7}, \frac{15}{16}, \frac{11}{15}, \frac{1}{9}, \frac{17}{125}, \frac{5}{6}, \frac{5}{18}, \frac{3}{7}, \frac{9}{10}, \frac{2}{70}$

a. Sort the numbers above into the category of either repeating decimals or terminating decimals.
b. Work with a partner to determine what all the terminating decimals have in common.
c. Create a rule to determine if a fraction terminates without using division.

Discussion: Students should come to the conclusion that the prime factorization of the denominator of terminating decimals contains only powers of 2s and 5s because terminating decimals have to be fractions with denominators containing powers of 10s, e.g., 10^0, 10^1, 10^2, 10^3…

IRRATIONAL NUMBERS

Students should come to the understanding that— (1) every rational number has a decimal representation that either terminates or repeats and (2) every terminating or repeating decimal is a rational number. Then, they can use that information to reason that on the number line, **irrational numbers** (i.e., those that are not rational) must have decimal representations that neither terminate nor repeat. The surprising fact, now, is that there are numbers on the number line that cannot be expressed as *a*/b, with *a* and *b* being both integers, and these are called irrational numbers.

Some students are surprised that the decimal representation of π does not repeat. Some students believe that if we keep looking at digits farther and farther to the right, eventually a pattern will emerge. Instead of telling students to stop trying to find the repetition, use this idea as an opportunity to reiterate that mathematics is alive and an exciting realm to explore!

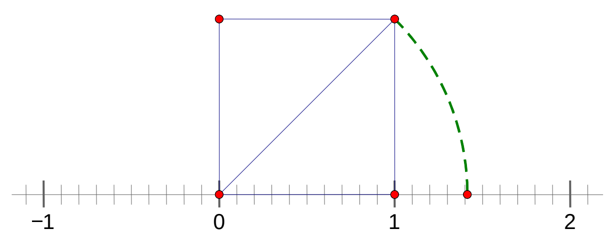
TIP!

To show students that irrational numbers lie on the number line, students can connect irrational numbers to the Pythagorean Theorem (8.G.8).

EXAMPLE

a. Construct a right isosceles triangle with legs of 1 unit.

b. Use the Pythagorean Theorem to determine the length of the hypotenuse. *(Link to 8.G.6-8)*



c. What type of number is the hypotenuse? Explain.

d. Rotate the hypotenuse back to the original number line. What conclusions can you make about irrational numbers?

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (8.NS.1-2)

Technology, such as Desmos or Geogebra, allows students to rotate lines more easily.

EXAMPLE

a. Place $\sqrt{1}, \sqrt{4}, \sqrt{9}, \sqrt{16}$ on a number line.
b. Then place $\sqrt{2}, \sqrt{3}, \sqrt{5}, \sqrt{6}, \sqrt{7}, \sqrt{8}, \sqrt{10}, \sqrt{11}, \sqrt{12}, \sqrt{13}, \sqrt{14}, \sqrt{15}$ on the same number line.
c. How many irrational numbers do you predict will be between the numbers 4 and 5 on the number line? Why?
d. What pattern do you notice?

A few irrational numbers are given special names (π and *e*), and much attention is given to $\sqrt{2}$. Because we name so few irrational numbers, students sometimes incorrectly conclude that irrational numbers are unusual and rare. In fact, irrational numbers are much more plentiful than rational numbers, in the sense that they are "denser" on the number line.

EXAMPLE

Order the following numbers on a number line: 3.5 x 10^{-4}, 3.5, $\frac{3}{1000}$, $\sqrt{0.31}$

Students often incorrectly think that a number written in scientific notation with a negative exponent is a negative number. Having students order numbers in scientific notation on a number line will help break this misconception.

EXAMPLE

Write an inequality to compare $-\frac{4}{9}$ and -0.44444.

Discussion: An interesting extension could be a discussion about whether $0.\overline{9}$ equals 1.

APPROXIMATING IRRATIONAL NUMBERS

In previous grades, students learned processes that can be used to locate any rational number on the number line: Divide the interval from 0 to 1 into *b* equal parts; then, beginning at 0, count out *a* of those parts. Now they can use similar strategies to locate irrational numbers on a number line.

Use an interactive number line to allow students to see how a number line can be infinitely divided. One resource is Zoomable Number Line by MathisFun.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (8.NS.1-2)

Although students at this grade do not need to be able to *prove* that $\sqrt{2}$ is irrational, they minimally need to *know* that $\sqrt{2}$ is irrational (see 8.EE.2), which means that its decimal representation neither terminates nor repeats. Nonetheless, they should be able to approximate irrational numbers such as $\sqrt{2}$ without using the square root key on the calculator.

The $\sqrt{2}$ caused Greek Matematicians many problems, for although they could construct it using tools, but they could not measure it. Integrating math history into the lesson may be interesting for some students.

TIP!

EXAMPLE

Approximate $\sqrt{2}$ to one, two, and then three places to the right of the decimal point without using a calculator

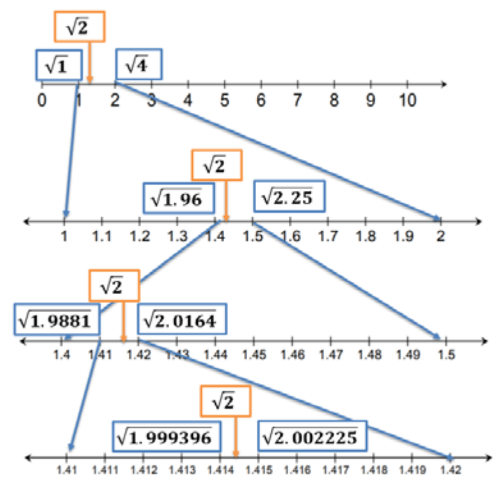
Method 1: Using Tables

x	x^2
1.0	1.00
1.1	1.21
1.2	1.44
1.3	1.69
1.4	1.96
1.5	2.25
1.6	2.56
1.7	2.89
1.8	3.24
1.9	3.61
2.0	4.00

x	x^2
1.40	1.9600
1.41	1.9881
1.42	2.0164
1.43	2.0449
1.44	2.0736
1.45	2.1025
1.46	2.1316
1.47	2.1609
1.48	2.1904
1.49	2.2201
1.50	2.2500

x	x^2
1.410	1.988100
1.411	1.990921
1.412	1.993744
1.413	1.996569
1.414	1.999396
1.415	2.002225
1.416	2.005056
1.417	2.007889
1.418	2.010724
1.419	2.013561
1.420	2.016400

Method 2: Using Number Lines



Discussion: From knowing that $1^2 = 1$ and $2^2 = 4$ (or from the Pythagorean Theorem activity), students can reason that there is a number between 1 and 2 whose square is 2. In the first table above, students can see that between 1.4 and 1.5, there is a number whose square is 2. Then in the second table, they locate that number between 1.41 and 1.42. And in the third table they can locate $\sqrt{2}$ between 1.414 and 1.415. Eventually students can develop more efficient methods for this work. For example, they might have begun the first table with 1.4. And once they see that $1.42^2 > 2$, they do not need generate the rest of the data in the second table.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (8.NS.1-2)

Technology such as the Number Line Generator by HelpingWithMath.com could be used to zoom in on the number line. There are several methods for approximating irrational numbers such as pi (π). As an extension, students can explore other irrational numbers such as the golden ratio phi (ϕ). The golden ratio can lend itself to cross-curricular extensions in classes such as art.

There are many applets that guide students in approximating pi such as Computing Pi by NCTM Illuminations and Approximating Pi by Desmos.

In the real-world, all measurements and constructions are approximate. Nonetheless, it is possible to see the distinction between rational and irrational numbers in their decimal representations. Given two distinct numbers, it is possible to find both a rational and an irrational number between them.

PRECISION

Have students do explorations where precision matters. Discuss situations where precision is vital and other situations where reasonableness is more vital than precision. Although learning about significant digits formally takes place in high school, it is appropriate to talk about how intermediate rounding affects precision. The display of irrational numbers on a calculator could also be used for a discussion point. Another discussion could take place comparing results using the pi button compared to the typical approximation of 3.14.

EXAMPLE

The teacher asked the class to multiply $\frac{5}{9} \times \frac{1}{6}$

- April multiplied $\frac{5}{9} \times \frac{1}{6}$.
- Cindy multiplied 0.5×0.2.
- Carlos multiplied 0.55×0.17.
- Jacob multiplied 0.555×0.167.
- Stephanie multiplied 0.5555×0.1667.
- Elijah multiplied 0.55555×0.166667.

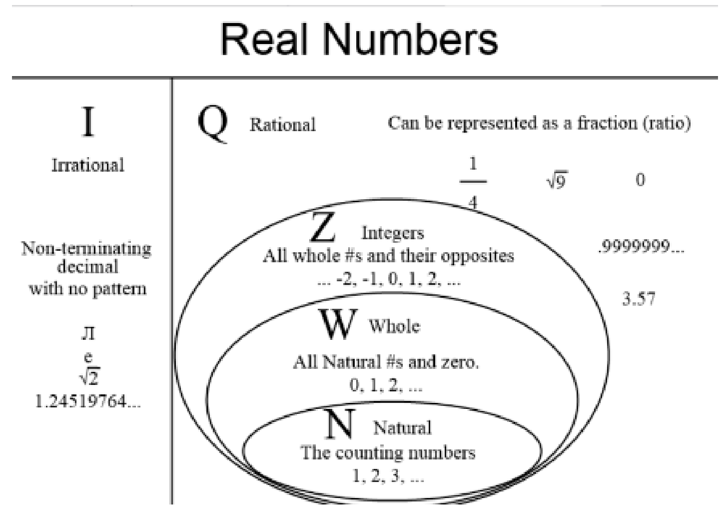
a. Whose answer was the most precise? Explain.
b. Find the error in approximation between the students' value and the actual value.
c. When is it acceptable to round before solving and when is it not?
d. When it is acceptable to round before solving, how do you know what place to round to?

The concept of precision should also be tied to real-world contexts. For example, we do not buy 3.5 apples, but we may buy 3.5 lbs of ground beef, so rounding to a whole number is a precise number in terms of buying apples. Therefore, being precise should relate to the real-world context of the problems.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (8.NS.1-2)

CLASSIFYING REAL NUMBERS

It could be appropriate to use Venn diagrams/set diagrams or flow charts to show the relationships among real, rational, irrational numbers, integers, and natural numbers. The diagram should show that the all real numbers (numbers on the number line) are either rational or irrational.



It may be helpful to incorporate the Venn diagram by revealing each category set by set. For example, first reveal the counting numbers, then whole numbers and so forth, revealing a larger and more complex "universe" as time progresses. The sets can help students understand the "why" of math. At each level of depth the new set was developed to answer new types of questions. For example any two whole numbers can be added to get an answer, but it is not possible to subtract any two whole numbers without introducing integers. Integers enlarge the number system so that subtraction IS possible for any two integers. Similar reasoning extends to other sets. The idea of zero (as a number and not just a placeholder) was not thought about in Europe until the 12th century; it was even outlawed at one point. Without the concept of zero, civilization would not have progressed as it had because calculous could not be invented, which was fundamental to the advancement in physics, engineering, computers, and financial and economic theory. Adding the human element and history to a math concept it makes the idea easier to incorporate into a student's mind.

TIP!

For kinesthetic learner have students create a Venn diagram foldable to show the relationships between the different types of numbers.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (8.NS.1-2)

Instructional Tools/Resources

These tools and resources may be helpful for instruction. Before use, they should be evaluated at the local level to determine their appropriateness for instruction.

- Card Sort: Real Number Statements by Mathycathy is a Desmos activity that encourages deeper thinking about the relationships of real numbers using the words *always*, *sometimes*, and *never*.
- Approximating Irrationals by Matthew Kim is a Desmos activity where students write and graph approximations of irrational numbers.
- Computing Pi is an applet by NCTM Illuminations that allows students to approximate pi using circumscribed and inscribed polygons.
- A Ratio that Glitters: Exploring the Golden Ratio is a lesson by NCTM Illuminations that has students explore and approximate the golden ratio, phi. *NCTM now requires a membership to view their lessons.*
- Comparing and Rational and Irrational Numbers is task by Illustrative Mathematics where students decide which number in a pair is greater.
- A Real Number System Foldable by Professor Hutchinson is a Slideshare that shows how to make a foldable about the Real Number System.

Curriculum and Lessons from Other Sources

- EngageNY Grade 8, Module 7, Topics A and B have a variety of lesson for this cluster. They are as follows: Lesson 2: Square Roots, Lesson 6: Finite and Infinite Decimals, Lesson 7: Infinite Decimals, Lesson 8: The Long Division Algorithm, Lesson 9: Decimal Expansion of Fractions, Part 1, Lesson 11: The Decimal Expansion of Some Irrational Numbers, Lesson 12: Decimal Expansion of Fractions, Part 2, Lesson 13: Comparing Irrational Numbers, and Lesson 14: Decimal Expansion of Pi are lessons that pertain to this cluster.
- Illustrative Mathematics, Grade 8, Unit 8: Pythagorean Theorem and Irrational Numbers, Lesson 1: The Area of Squares and their Side Lengths, Lesson 3: Rational and Irrational Numbers, Lesson 4: Square Roots on a Number Line, Lesson 5: Reasoning About Square Roots, and Lesson 14: Decimal Representations of Rational Numbers are lessons that pertain to this cluster. *You must create an account to view Illustrative Mathematics lessons on OpenUp Resources.*
- Decimal Approximations of Squares and Roots is a lesson from Georgia Standards of Excellence Framework Grade 8, that allows students approximate square roots using grid paper. Then students approximate square roots using inequalities. This lesson is found on pages 18-27.

General Resources

- Arizona 6-8 Progression on the Number System and High School Progression on Number
  This cluster is addressed on pages 14-15.
- Coherence Map by Achieve the Core is an interactive website that shows the progression and interconnectedness of the Common Core standards. To be noted, there are some differences between Ohio's Learning Standards and the Common Core, so although the tool is useful, it needs to be checked with respect to Ohio's Learning Standards.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (8.NS.1-2)

General Resources, continued

- High School Coherence Map by UnboundED is a map of the relationships among high school mathematics standards. Please note: This resource is aligned to the Common Core not Ohio's Learning Standards. Although the tool is useful, it needs to be checked with respect to Ohio's Learning Standards.

References

- Common Core Standards Writing Team. (2013, July 4). *Progressions for the Common Core State Standards in Mathematics (draft). Grades 6–8, The Number System; High School, Number*. Tucson, AZ: Institute for Mathematics and Education, University of Arizona.
- Griffith, L. (February 2016). Exponent rummy. *Mathematics Teaching in the Middle School*, *21(6)*, 376-377.
- Nowling, D. (2007). Precision: The neglected part of the measurement standard. *Mathematics Teacher*, *100(5)*, 356-360.

STANDARDS	MODEL CURRICULUM (8.EE.1-4)
EXPRESSIONS AND EQUATIONS **Work with radicals and integer exponents.** **8.EE.1** Understand, explain, and apply the properties of integer exponents to generate equivalent numerical expressions. *For example,* $3^2 \times 3^{-5} = 3^{-3} = 1/3^3 = 1/27$. **8.EE.2** Use square root and cube root symbols to represent solutions to equations of the form $x^2 = p$ and $x^3 = p$, where p is a positive rational number. Evaluate square roots of small perfect squares and cube roots of small perfect cubes. Know that $\sqrt{2}$ is irrational. **8.EE.3** Use numbers expressed in the form of a single digit times an integer power of 10 to estimate very large or very small quantities and to express how many times as much one is than the other. *For example, estimate the population of the United States as 3×10^8; and the population of the world as 7×10^9; and determine that the world population is more than 20 times larger.* *Continued on next page*	**Expectations for Learning** In previous grades students were exposed to writing expressions with whole number exponents. In this cluster students should use patterns to extend their knowledge of exponents to include zero and negative powers. They will develop an initial understanding of the properties of exponents in numerical operations. Students are not expected to know or memorize the algorithmic rules for exponents but rather derive the rules using patterns. In high school, students will be using the properties of exponents more formally with algebraic expressions. In grade eight, students will also develop an understanding of square and cube roots and their symbols to solve equations. In previous grades students were exposed to understandings of powers of ten. In this cluster, students will write very small and very large numbers in scientific notation. They will use this knowledge to perform calculations and comparisons using scientific notation. **ESSENTIAL UNDERSTANDINGS** **Exponents** *Note: In 8th Grade all exponents are integers, so all notes refer to cases using integer exponents.* • $x^0 = 1$, when $x \neq 0$. • $x^{-n} = \frac{1}{x^n}$, when $x \neq 0$. • A coefficient is different than an exponent. For example, n^2 is different than $2n$ because n^2 means $n \times n$ and $2n$ means $2 \times n$, and n^3 is different than $3n$ because n^3 means $n \times n \times n$ and $3n$ means $3 \times n$. • A positive exponent denotes repeated multiplication of the base. • A negative exponent denotes repeated division of the base. • A negative exponent does not change the sign of the base. *Continued on next page*

STANDARDS	MODEL CURRICULUM (8.EE.1-4)
8.EE.4 Perform operations with numbers expressed in scientific notation, including problems where both decimal notation and scientific notation are used. Use scientific notation and choose units of appropriate size for measurements of very large or very small quantities, e.g., use millimeters per year for seafloor spreading. Interpret scientific notation that has been generated by technology.	**Expectations for Learning, continued** **ESSENTIAL UNDERSTANDINGS, CONTINUED** **Roots** • The equation $x^2 = p$ has two solutions for x: $\sqrt{p}$ and $-\sqrt{p}$. For example, in describing the solutions to $x^2 = 36$, students can write $x = \pm\sqrt{36} = \pm 6$. • The $\sqrt{p}$ is defined to be the positive solution to the equation $x^2 = p$. For example, it is **not** correct to say that $\sqrt{36} = \pm 6$ instead it should be $\sqrt{36} = 6$. • $x^3 = p$ has only one solution for x: $\sqrt[3]{p}$. • The square root of 2 is irrational. **Scientific Notation** • Scientific notation is a mathematical expression written as a decimal number greater than or equal to one but less than 10 multiplied by a power of ten, e.g. 3.1×10^4. • A number expressed in scientific notation that has a negative exponent is between negative one and positive one. • A number expressed in scientific notation that has a positive exponent is greater than one or less than negative one. • Powers of ten can be used to compare numbers written in scientific notation. **MATHEMATICAL THINKING** • Compute accurately and efficiently with grade-level numbers. • Pay attention to and make sense of quantities. • Recognize and use a pattern or structure. • Use technology to deepen understanding. • Represent a concept symbolically. *Continued on next page*

STANDARDS	MODEL CURRICULUM (8.EE.1-4)
8.EE.1-4, continued	**Expectations for Learning, continued** **INSTRUCTIONAL FOCUS** **Exponents** • Use patterns and structure to develop an understanding of the properties of exponents. • Create equivalent numerical expressions using the properties of integer exponents including negative and zero exponents. **Roots** • Solve $x^2 = p$ and $x^3 = p$, where p is a positive rational number. Show solutions with square and cube root symbols. • Evaluate square roots of perfect squares and cube roots of perfect cubes. **Scientific Notation** • Identify numbers in scientific notation. • Convert numbers from scientific notation to decimal notation or from decimal notation to scientific notation. • Compare numbers in scientific notation. • Perform operations on numbers where both scientific notation and decimal notation are used. • Interpret how different technology devices show scientific notation. • Choose appropriately sized units when dealing with very large or very small quantities. **Content Elaborations** • Ohio's K-8 Critical Areas of Focus, Grade 8, Number 4, page 55 • Ohio's K-8 Learning Progressions, Expressions and Equations, pages 18-19 **CONNECTIONS ACROSS STANDARDS** • Define and approximate irrational numbers (8.NS.1). • Estimate irrational numbers (8.NS.2). • Apply the Pythagorean Theorem (8.G.6-8).

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (8.EE.1-4)

Instructional Strategies

Note: The model curriculum is not a collection of lessons nor a full curriculum; it does not suggest pace, sequence, or amount of time spent on topics. It provides information about a topic related to the standards including ideas for examples, strategies for teaching, common misconceptions, and possible connections between topics. In addition, the emphasis of the model curriculum is on instruction not on assessment; therefore, the examples in the Instructional Strategies section are written for instructional purposes not necessarily as assessment items. Examples may need to be rewritten to accommodate the needs of each individual classroom.

Standards for Mathematical Practice
This cluster focuses on but is not limited to the following practices:
***MP.2** Reason abstractly and quantitatively*
***MP.6** Attend to precision.*
***MP.7** Look for and make use of structure.*
***MP.8** Look for and express regularity in repeated reasoning.*

Although students begin using whole-number exponents in Grades 5 and 6, it is in Grade 8 when students are first expected to understand, explain, and apply the properties of exponents and to extend the meaning beyond counting-number exponents. It is no accident that these expectations are simultaneous because it is the properties of counting-number exponents that provide the rationale for the properties of integer exponents. In other words, students should not be told these properties but rather should derive them through experience and reason.

EXPONENTS

The focus in Grade 8 is on using patterns to find exponent rules. They should not be given the rules to memorize, but they should develop an understanding of why they work. Although students may use bases that are variables as an extension, the focus is on using numerical bases to create understanding. Exponents will also be limited to integers.

Students should use multiplicative reasoning and expanded notation to gain understanding of non-positive exponents.

When talking about the meaning of an exponential expression, it is easy to say (incorrectly) that "3^5 means 3 multiplied by itself 5 times." But by writing out the meaning, $3^5 = 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3$, students can see that there are only 4 multiplications but 5 factors, so a better description is "3^5 means 5 3s multiplied together." Students also need to realize that these simple descriptions work only for counting-number exponents. When extending the meaning of exponents to include 0 and negative exponents, these descriptions are limiting; it is not sensible to say "3^0 means 0 3s multiplied together" or that "3^{-2} means -2 3s multiplied together."

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (8.EE.1-4)

EXAMPLE

a. Finish filling out the table. Write your solutions as whole numbers or fractions (no decimals). Look for patterns to help you.
b. What does 5^0 = ? Why?
c. What does 3^0 = ? Why?
d. What does any number raised to the 0th power equal? Why?
e. What does 5^{-1} = ? Why?
f. What does 3^{-1} = ? Why?
g. What did you notice about negative exponents?
h. What does any number raised to the negative exponent mean?
i. If positive exponents mean repeated multiplication, what do negative exponents mean?

Exponent Form	Expanded Form	Solution
5^4		
5^3	$5 \cdot 5 \cdot 5$	125
5^2		
5^1		
5^0		
5^{-1}		
5^{-2}		
5^{-3}		

Discussion: Students can use the third row of the chart to deduce that as you move from one row to the row above it, you multiply by 5. In addition, as you move from one row to the one below it, you divide by 5. Using that logic $5^3 \div 5$ is 25 or 5^2, $5^2 \div 5$ is 5 or 5^1, $5^1 \div 5$ is 1 or 5^0, $5^0 \div 5$ is $\frac{1}{5}$ or 5^{-1}, $5^{-1} \div 5$ is $\frac{1}{5^2}$ or $\frac{1}{25}$ or 5^{-2}, etc. Another way to approach non-positive integer exponents is to connect it to powers of 10, thereby connecting to prior learning in 5.NBT.1-3.

Students oftentimes incorrectly think that a negative exponent means a negative number. Stress that the negative exponent is the multiplicative inverse of the rational number.

Properties of Exponents

Students should not simply be given the exponent rules. Many students who simply "memorize" the rules without understanding may confuse the rules when trying to apply them at later times. Instead students should be encouraged to discover the rules using tables, patterns, and expanded notation. As they have multiple experiences simplifying numerical expressions with exponents, these properties become natural and obvious.

Properties of Integer Exponents

For any nonzero real numbers *a* and *b* and integers *n* and *m*:

1. $a^0 = 1$
2. $a^{-n} = \frac{1}{a^n}$
3. $a^n a^m = a^{n+m}$
4. $(a^n)^m = a^{nm}$
5. $a^n b^n = (ab)^n$
6. $\frac{a^n}{a^m} = a^{n-m}$

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (8.EE.1-4)

EXAMPLE

	Expanded Form	Exponent Form
$2^3 \cdot 2^5$	$(2 \cdot 2 \cdot 2) \cdot (2 \cdot 2 \cdot 2 \cdot 2 \cdot 2)$	2^8
$4^2 \cdot 4^3$		
$3^4 \cdot 3^3$		
$5^2 \cdot 5^3$		
$6^2 \cdot 6^0$		
$4^{-1} \cdot 4^{-2}$	$\frac{1}{4^1} \cdot \frac{1}{4^2}$	$\frac{1}{4^3} = 4^{-3}$
$2^{-3} \cdot 2^{-4}$		
$6^{-2} \cdot 6^3$		

a. Fill out the table.
b. What is $3^{-4} \cdot 3^2$?
c. Create a rule for multiplying powers ($x^n \cdot x^y$).
d. Would your rule work for $3^4 \cdot 5^2$? Explain.
e. Based on your response to Part **d.**, does your rule need to be revised? If so, write the revision below.

Discussion: If students reason about these examples with a sense of generality about the numbers, they begin to articulate the properties. For example, "I see that 3 twos are being multiplied by 5 twos, and the results is 8 twos being multiplied together, where the 8 is the sum of 3 and 5, the number of twos in each of the original factors." This method would also work for a base other than two (as long as the bases are the same).

Students can use similar tables as the example above to explore the other exponent rules:

- $(5^3)^4 = (5 \cdot 5 \cdot 5) \cdot (5 \cdot 5 \cdot 5) \cdot (5 \cdot 5 \cdot 5) \cdot (5 \cdot 5 \cdot 5) = 5^{12}$
- $(3 \cdot 7)^4 = (3 \cdot 7) \cdot (3 \cdot 7) \cdot (3 \cdot 7) \cdot (3 \cdot 7) = (3 \cdot 3 \cdot 3 \cdot 3) \cdot (7 \cdot 7 \cdot 7 \cdot 7) = 3^4 \cdot 7^4$
- $\frac{3^6}{3^2} = \frac{3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3}{3 \cdot 3} = 3 \cdot 3 \cdot 3 \cdot 3 = 3^4$

TIP! Students often confuse and misapply the properties of exponents. Have students expand the expressions with exponents until they internalize the rules.

Another way to view the meaning of 0 and negative exponents is by applying the following principle: *The properties of counting-number exponents should continue to work for integer exponents.*

- Therefore, the properties for exponents can also be used to help students understand $x^0 = 1$. For example, consider the following expression and simplification: $3^0 \cdot 3^5 = 3^{0+5} = 3^5$. This computation shows that the when 3^0 is multiplied by 3^5, the result should be 3^5, which implies that 3^0 must be 1. Because this reasoning holds for any base other than 0, it can be reasoned that $a^0 = 1$ for any nonzero number *a*.
- The properties of exponents can also help students make sense of negative exponents. To make a judgment about the meaning of 3^{-4}, the approach is similar: $3^{-4} \cdot 3^4 = 3^{-4+4} = 3^0 = 1$. This computation shows that 3^{-4} should be the reciprocal of 3^4, because their product is 1. And again, this reasoning holds for any nonzero base. Thus, we can reason that $a^{-n} = \frac{1}{a^n}$.

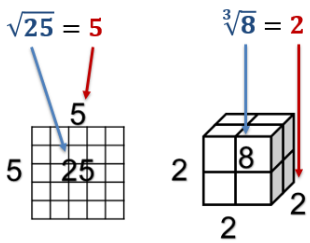
INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (8.EE.1-4)

In Grade 8 students should also have practice using negative numbers as bases. For example, it is very difficult for students to differentiate between -3^2 and $(-3)^2$. To help students differentiate between the two forms, encourage them to write it out in expanded form: $(-3)^2 = -3 \cdot -3$ and $-3^2 = -1 \cdot 3 \cdot 3$.

Have students identify the bases before solving problems as many students incorrectly only attribute the exponent to the nearest number. For example, instead of realizing that $(4 + 2)^3 = (4 + 2)\cdot(4 + 2)\cdot(4 + 2) = 216$, they incorrectly calculate $4 + 2^3 = 12$.

TIP!

Students should also have practice using non-integer bases such as $(1.5)^3$ or $\left(\frac{3}{4}\right)^{-2}$.



ROOTS

In this cluster students will learn that the inverse operation for squaring a number is to find its square roots, and the inverse operation for cubing a number is finding its cube root and vice versa. To help students build a conceptual understanding, connect roots to area and volume models where the area and volume are the radicand and the solution is the length of the side of the model. Another way to explain it is that the area and volume of the square or cube represents *n*; and the square and cube's side length is represented by $\sqrt{n}$ and $\sqrt[3]{n}$ respectively.

TIP!

Although students should realize that the index for a square root is assumed, it may be helpful for some students to put an index in front of the root sign for a square root, $\sqrt[2]{}$. Once students become familiar with square roots the index for square roots can be dropped.

Have students use geoboards, square tiles, graph paper, or unit cubes to build squares and cubes reviewing exponents in the process. Problems such as the Painted Cube Problem, (See Instruction Tools/Resources), can then be modified to extend to square and cube roots.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (8.EE.1-4)

EXAMPLE

Part 1

Every time a square has a birthday, a one-inch square grows one inch in all directions. *Note: On its first birthday, its dimensions are 1 inch by 1 inch.*

- **a.** Model the squares first five birthdays and fill out the table using graph paper or tiles.
- **b.** What would the square look like on its 12th birthday?
- **c.** What is the relationship between a square's birthday and its area?
- **d.** If you were told that a square had an area of 81 in^2, could you find its birthday?
- **e.** Create a rule to find a square's area given its birthday.
- **f.** Create a rule to find a square's birthday given its area.
- **g.** Apply your rules to find the length of squares where the area is not a whole number.

Birthday (in)	Area (in^2)
1	
2	
3	
4	
5	
12	
	81
b	
	a

Length (in)	Area (in^2)
	0.49
	0.09
	1.21
	0.0025

Part 2

Every time a cube has a birthday, it grows one inch in all directions. On each birthday, the cube is dipped in paint, so each of its outside faces are painted a new color. *Note: On its first birthday, its dimensions are 1 inch by 1 inch by 1 inch.*

- **a.** Build the cube's first five birthdays and fill out the table.
- **b.** What would the cube look like on its 11th birthday?
- **c.** What is the relationship between a cube's birthday and its volume?
- **d.** If you were told that a cube had a volume of 512 in^3, could you find its birthday?
- **e.** Create a rule to find a cube's volume given its birthday. Explain why it works.

Example for Part 2 continued on next page

Birthday (in)	Volume of large cube (in^3)	Volume of large cube (in^3) written in exponent form	Total number of small cubes	Number of small cubes with 3 painted faces	Number of small cubes with 2 painted faces	Number of small cubes with 1 painted faces	Number of small cubes with 0 painted faces
1							
2							
3	$3 \cdot 3 \cdot 3$	3^3	27	8	12	6	1
4							
5							
11							
			512				
b							
			v				

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (8.EE.1-4)

Length (in)	Volume (in^3)
	729
	343
	1,000
	1728
	0.125
	0.027
	0.216
	3.375
	0.000001

- **f.** Create a rule for finding how many cubes have 3 painted faces?
- **g.** Create a rule for finding how many cubes have 2 painted faces?
- **h.** Create a rule for finding how many cubes have 1 painted faces?
- **i.** Create a rule for finding how many cubes have 0 painted faces?
- **j.** Why do you think the chart omits 4 or 5 painted faces?
- **k.** Create a rule to find a cube's birthday given its volume.
- **l.** Find the lengths of different cubes given their volumes.

Adapted from the Painted Cube Problem.

Also, provide practical opportunities for students to flexibly move between forms of squared and cubed numbers. For example, If $3^2 = 9$ then $\sqrt{9} = 3$. This flexibility should be experienced symbolically and verbally with manipulatives and with drawings.

A common misconception is the failure to understand $\sqrt{9} = 3$ and not -3 since it indicates the principal square root, while $x^2 = 9$ has two solutions 3 and -3.

Connecting Irrational Roots to the Pythagorean Theorem

Opportunities to understand irrational numbers conceptually should be developed. One way is for students to draw a square that is one unit by one unit and find the diagonal using the Pythagorean Theorem. The diagonal drawn has an irrational length of $\sqrt{2}$. Other irrational lengths can be found using the same strategy by finding diagonal lengths of rectangles with various side lengths. See cluster 8.NS.2 for more information on irrational roots.

SCIENTIFIC NOTATION

The meanings of integer exponents, especially with respect to 0 and negatives, can be further explored in a place-value chart:

Thus, integer exponents support writing any decimal in expanded form like the following: $3247.568 = 3 \cdot 10^3 + 2 \cdot 10^2 + 4 \cdot 10^1 + 7 \cdot 10^0 + 5 \cdot 10^{-1} + 6 \cdot 10^{-2} + 8 \cdot 10^{-3}$.

thousands	*hundreds*	*tens*	*ones*		*tenths*	*hundredths*	*thousandths*
10^3	10^2	10^1	10^0		10^{-1}	10^{-2}	10^{-3}
3	2	4	7	.	5	6	8

Expanded form and the connection to place value is important for helping students make sense of scientific notation, which allows very large and very small numbers to be written concisely, enabling easy comparison. Students can make sense of scientific notation and negative exponents by using patterns.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (8.EE.1-4)

EXAMPLE

a. Fill in the table.
b. What do you notice about numbers written in scientific notation where the exponent is *positive*?
c. What do you notice about numbers written in scientific notation where the exponent is *negative*?
d. Michael said that 4.2×10^{-6} is a negative number. Is he correct? Explain.
e. Explain why $10^0 = 1$. Revise your rule if necessary in Part **b.**
f. What do you think 3.1×10^9 will be in standard notation?
g. What do you think 3.1×10^{-7} will be in standard notation?
h. Create a rule for converting scientific notation to standard notation. Will it always work? Explain.
i. What do you think 41,000,000,000 is in scientific notation?
j. What do you think 0.0000000056 is in scientific notation?
k. Create a rule for converting standard notation to scientific notation. Will it always work? Explain.
l. What is the relationship between 3.1×10^3 and 3.1×10^2?
m. What is the relationship between 3.1×10^4 and 3.1×10^3?
n. What is the relationship between 3.1×10^{-3} and 3.1×10^{-4}?
o. Generalize relationships between the rows in the table.

Scientific Notation				Standard Notation
3.1×10^4		$3.1 \times 10 \times 10 \times 10 \times 10$	$3.1 \times 10{,}000$	31,000
3.1×10^3				
3.1×10^2				
3.1×10^1				
3.1×10^0				
3.1×10^{-1}				
3.1×10^{-2}	$3.1 \times \frac{1}{10^2}$	$3.1 \times \frac{1}{10 \cdot 10}$	$3.1 \times \frac{1}{100}$	0.031
3.1×10^{-3}				
3.1×10^{-4}				

Students often think the decimal point moves, but the decimal point really stays fixed and the values of the digits change their place-value position by a factor of 10. Since the value of the digits change by a factor of 10, the digits, not the decimal point, moves.

Students often confuse which way to move the digits in relation to the decimal point. Train them to ask themselves "Is the number really small or really large?" and then move it accordingly.

Students may incorrectly think that the exponent tells the number of zeros to add. Continually ask the students if there is a relationship between the number of zeros and the exponents.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (8.EE.1-4)

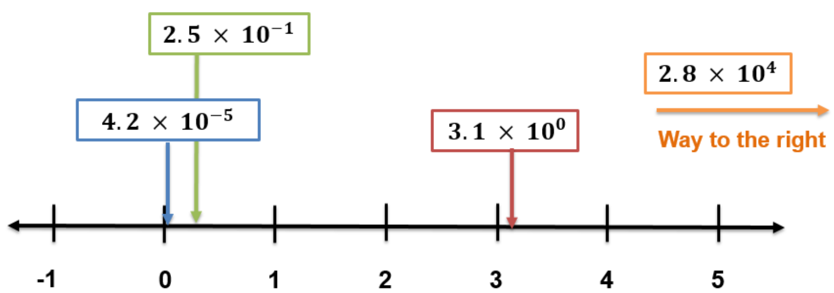
EXAMPLE

- **a.** Create a two-column table displaying scientific notation and standard form for the number 6.2 multiplied by various powers of 10 (Use both positive and negative exponents).
- **b.** Study the table, is there a rule between the number of zeros and the exponent. Explain.
- **c.** Does your rule in part **b.** hold true for 5.27×10^3or 4.621×10^5or even 2.3489×10^6. Explain.
- **d.** Does your rule in part **b.** hold true for 5.27×10^{-3}or 4.621×10^{-5}or even 2.3489×10^{-6}. Explain.
- **e.** Revise your rule if necessary in part **b.** depending on what you learned in parts **c.** and **d.**

To develop familiarity, go back and forth between standard notation and scientific notation for numbers. Compare numbers, where one is given in scientific notation and the other is given in standard notation. Have students place numbers written in scientific notation on a number line and order them without converting them to standard form. Students should come to the conclusion that when determining value, the power is more important than the coefficient.

EXAMPLE

Order 4.2×10^{-5}, 2.8×10^4, 3.1×10^0, and 2.5×10^{-1} on a number line.



Students often think that numbers written in scientific notation with negative exponents are negative numbers instead of really small numbers between 0 and 1. To break this misconception have students practice placing numbers written in scientific notation on a number line.

Real-world problems can help students compare quantities and make sense about their relationships. Conversely scientific notation can also help students make sense of really large or small numbers by modeling situations. Students should as often as possible have a real-world situation to model when using scientific notation to help their understanding of the concept.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (8.EE.1-4)

EXAMPLE

- Lake Erie has about 1.276×10^{14} cubic gallons of water. If Lake Superior is about 25 times the size of Lake Erie, how big is Lake Superior?
- How does temperature affect water levels in the great lakes? (An investigation like this could possibly tie into the statistics standards and modeling.)

At the very end of all their previous exploration with exponent rules, have students apply rules to operations using scientific notation.

EXAMPLE

$$\frac{9.3 \times 10^5}{3.1 \times 10^3}$$

a. Evaluate. Write your solution in scientific notation.
b. How many times bigger is 9.3×10^5 than 3.1×10^3?

Discussion: Students should recognize that $9.3 \div 3.1$ is 3 and that $\frac{10^5}{10^3}$ is equivalent to 10^2 so it equals 3×10^2.

Instructional Tools/Resources

These tools and resources may be helpful for instruction. Before use, they should be evaluated at the local level to determine their appropriateness for instruction.

Manipulatives/Technology

- Square tiles and cubes to develop understanding of squared and cubed numbers
- Calculators to verify and explore patterns
- Webquests using data mined from sites like the U.S. Census Bureau, scientific data (planetary distances)
- Place value charts to connect the digit value to the exponent (negative and positive)

Painted Cube or Birthday Problem

- The Painted Cube by Oregon Department of Education
- Painted Cubes from the Mathematics Centre
- Painted Sides of a Cube from The University of Georgia
- Painted Cube from NRICH Enriching Mathematics
- The Painted Cube by CollecEDNY

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (8.EE.1-4)

Powers of Ten

- Secret Worlds: The Universe Within by Molecular Expressions: Science, Optics and You allows students to move through and view the universe by powers of 10. There is an accompanying activity called Perspectives: Powers of 10 where students make drawings based on powers of ten.
- Powers of Ten by Scale of the Universe is a video that shows that moves through and views the universe by powers of ten.
- Orders of Magnitude is a task by Illustrative Mathematics that helps students understand powers of ten.

Exponents

- Extending the Variation of Exponents, Variation 1 is a task by Illustrative Mathematics that has students create meaning of negative integer exponents.
- Ponzi Pyramid Scheme by Essential Understanding of the Common Core has students develop exponent rules by using the Ponzi Pyramid Scheme, Tables, a Spinner Game, and Bingo.
- Exponents Jeopardy Game by Math-Play.Com is an online math game which allows students to review writing exponents. (Note: Do not click download; click "The game" in blue font.)

Scientific Notation

- USADebtClock.com is a clock that shows the national debt growing in real time. It can be used to demonstrate large numbers as a tie-in to scientific notation.
- Estimate and Compare Measurements in Scientific Notation by Mobius Math is a worksheet that builds the conceptual understanding of scientific notation.
- Ants versus Humans is a task by Illustrative Mathematics that requires students to work with very large and small numbers expressed in scientific notation and standard form.
- Choosing Appropriate Units is a task by Illustrative Mathematics that requires students to use scientific notation in choosing units to report quantities.
- Giantburgers is a task by Illustrative Mathematics that requires students do think logically about scientific notation to solve problems.
- Applications of Scientific Notation by NASA is a worksheet with astronomical problems that use scientific notation.
- Jeopardy Review: Exponents-Scientific Notation by jeopardylabs.com is a review game about exponent rules and scientific notation
- Scientific Notation by jeopardylabs.com is a Jeopardy Game reviewing scientific notation and comparing scientific notation.

Roots

- Visualizing Squares and Square Roots contains two lessons by NCTM Illuminations where students use geoboards to visualize the geometric meanings of square and square root. *NCTM now requires a membership to view their lessons.*
- Squares, diagonals, and Square Roots is a lesson by NCTM Illuminations where students explore the relationship between the length of the sides and diagonals of a square. *NCTM now requires a membership to view their lessons.*

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (8.EE.1-4)

Roots, continued

- Square Roots Go Rational is a lesson by NCTM Illuminations where students explore no-so-perfect squares. *NCTM now requires a membership to view their lessons.*
- Stacking Squares is a lesson by NCTM Illuminations where students explore ways of arranging square to represent square and cube roots. *NCTM now requires a membership to view their lessons.*

Curriculum and Lessons from Other Sources

- EngageNY Grade 8, Module 1, Topic A, Lesson 1: Exponential Notation, Lesson 2: Multiplication and Division of Numbers in Exponential Form, Lesson 3: Numbers in Exponential Form Raised to a Power, Lesson 4: Numbers Raised to the Zeroth Power, Lesson 5: Negative Exponents and the Laws of Exponents, Lesson 6: Proof of the Laws of Exponents are lessons about exponential notation and the properties of integer exponents.
- EngageNY Grade 8, Module 1, Topic B, Lesson 7: Magnitude, Lesson 8: Estimating Quantities, Lesson 9: Scientific Notation, Lesson 10: Operations with Numbers in Scientific Notation, Lesson 11: Efficacy of Scientific Notation, Lesson 12: Choice of Unit, Lesson 13: Comparison of Numbers Written in Scientific Notation and Interpreting Scientific Notation Using Technology are lessons about scientific notation.
- Illustrative Mathematics, Grade 8, Unit 7: Exponents and Scientific Notation has many lessons that pertain to this cluster. *You must create an account to view Illustrative Mathematics lessons on OpenUp Resources.*
- Illustrative Mathematics, Grade 8, Unit 8: The Pythagorean Theorem, Lesson 2: Side Lengths and Areas, Lesson 3: Rational and Irrational Numbers, Lesson 4: Square Roots and the Number Line, Lesson 5: Reasoning About Square Roots, Lesson 12: Edge Lengths and Volumes, Lesson 13: Cube Roots are lessons that pertain to this clusters. *You must create an account to view Illustrative Mathematics lessons on OpenUp Resources.*

General Resources

- Coherence Map by Achieve the Core is an interactive website that shows the progression and interconnectedness of the Common Core standards. To be noted, there are some differences between Ohio's Learning Standards and the Common Core, so although the tool is useful, it needs to be checked with respect to Ohio's Learning Standards.
- Arizona 6-8 Progression on Expressions and Equations
  This cluster is addressed on page 11.
- High School Coherence Map by UnboundED is a map of the relationships among high school mathematics standards. Please note: This resource is aligned to the Common Core not Ohio's Learning Standards. Although the tool is useful, it needs to be checked with respect to Ohio's Learning Standards.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (8.EE.1-4)

References

- Common Core Standards Writing Team. (2013, July 4). *Progressions for the Common Core State Standards in Mathematics (draft). 6-8, Expressions and Equations*. Tucson, AZ: Institute for Mathematics and Education, University of Arizona.
- Davis, B. (August 2015). Exponentiation: a new basic? *Mathematics Teaching in the Middle School*, *21(1)*, 34-41.
- Schultz, K., & Bismarck S. (November 2013). Radical thoughts on simplifying square roots. *Mathematics Teaching in the Middle School*, *19(4)*, 222-228.

<table>
<tr><th>STANDARDS</th><th>MODEL CURRICULUM (8.EE.5-6)</th></tr>
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EXPRESSIONS AND EQUATIONS

Understand the connections between proportional relationships, lines, and linear equations.

8.EE.5 Graph proportional relationships, interpreting the unit rate as the slope of the graph. Compare two different proportional relationships represented in different ways. *For example, compare a distance-time graph to a distance-time equation to determine which of two moving objects has greater speed.*

8.EE.6 Use similar triangles to explain why the slope m is the same between any two distinct points on a non-vertical line in the coordinate plane; derive the equation $y = mx$ for a line through the origin and the equation $y = mx + b$ for a line intercepting the vertical axis at b.

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Expectations for Learning

In previous grades students were exposed to unit rates and the constant of proportionality. In this cluster students will extend their vocabulary to include rate of change, rise over run, and slope. Students will understand that these are equivalent to the unit rate and constant of proportionality and is represented by the variable m. Students will use their knowledge of similar triangles to derive the slope of a line. Students will be introduced to the concept of the y-intercept. They will construct equations in the form of $y = mx$ or $y = mx + b$ to describe the relationship between two variables. These linear relationships provide the initial foundation for an understanding of functions.

ESSENTIAL UNDERSTANDINGS

- The slope is a constant ratio between the rise and the run for any two points on a line.
- A graph of a proportional relationship is a line that passes through the origin.
- Only the slope, m, of the equation $y = mx$ represents a proportional relationship.
- Slope is represented by m in the equation $y = mx$ or $y = mx + b.$
- Corresponding angles in similar right triangles are equal.
- Corresponding sides of similar triangles are proportional.
- A line in the form $y = mx$ and intersects the origin.
- A line in the form $y = mx + b$ intersects the y-axis at (0, b) with b being the y-intercept. *Note: A linear function has neither a slope nor a y-intercept. But the graph of a linear function has both.*
- A relationship between two variables can be represented as a graph, table, equation, or verbal description.

MATHEMATICAL THINKING

- Use accurate mathematical vocabulary to describe mathematical reasoning.
- Represent a concept symbolically.
- Solve real-word problems accurately.
- Compute using strategies or models.
- Make connections between concepts, terms, and properties within the grade level and with previous grade levels.

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STANDARDS	MODEL CURRICULUM (8.EE.5-6)
8.EE.5-6, continued	**Expectations for Learning, continued** **INSTRUCTIONAL FOCUS** • Graph proportional relationships, identifying the constant of proportionality (unit rate) as the slope of the line. • Compare different representations of a proportional relationship (graph, table, equation, verbal description). • Use similar right triangles to derive the slope of the line. • Determine the relationship between the two variables by writing an equation in the form $y = mx$ or $y = mx + b$. • Compare the equations $y = mx$ and $y = mx + b$. **Content Elaborations** • Ohio's K-8 Critical Areas of Focus, Grade 8, Number 1, page 50-51 • Ohio's K-8 Critical Areas of Focus, Grade 8, Number 3, pages 53-54 • Ohio's K-8 Learning Progressions, Expressions and Equations, pages 18-19 **CONNECTIONS ACROSS STANDARDS** • Describe the effect of dilations (8.G.3). • Use informal arguments to establish facts about the angle sum and exterior angles of triangles and angles created when parallel lines are cut by a transversal (8.G.5). • Compare properties of two functions each represented in a different way (8.F.2). • Interpret the equation $y = mx + b$ as defining a linear function, whose graph is a straight line (8.F.3). • Construct a function to model a linear relationship between two quantities (8.F.4) • Determine the rate of change and initial value from a table or graph (8.F.4). • Use the equation of a linear model to solve problems in the context of bivariate measurement data, interpreting slope and y-intercept (8.SP.4).

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (8.EE.5-6)

Instructional Strategies

Note: The model curriculum is not a collection of lessons nor a full curriculum; it does not suggest pace, sequence, or amount of time spent on topics. It provides information about a topic related to the standards including ideas for examples, strategies for teaching, common misconceptions, and possible connections between topics. In addition, the emphasis of the model curriculum is on instruction not on assessment; therefore, the examples in the Instructional Strategies section are written for instructional purposes not necessarily as assessment items. Examples may need to be rewritten to accommodate the needs of each individual classroom.

Standards for Mathematical Practice
This cluster focuses on but is not limited to the following practices:
MP.2 *Reason abstractly and quantitatively.*
MP.6 *Attend to precision.*
MP.7 *Look for and make use of structure.*
MP.8 *Look for and express regularity in repeated reasoning.*

This cluster focuses on extending the understanding of ratios and proportions. Unit rates have been explored in Grade 6 as the comparison of two different quantities where the second unit is a unit of one, (unit rate). In Grade 7, unit rates were expanded to complex fractions and percents through solving multi-step problems such as discounts, interest, taxes, tips, and percent of increase or decrease. Proportional relationships were applied in scale drawings, and students should have developed an informal understanding that the steepness of the graph is the unit rate. Now in Grade 8 unit rates are addressed formally, and slope is introduced in graphical representations, in algebraic equations, and in geometry through similar triangles.

UNIT RATE AS THE SLOPE

Students in Grade 7 represented proportional relationships as equations such as $y = kx$ or $t = pn$. They also graphed proportional relationships discovering that a graph of a proportion must go through the origin, and that in the point $(1, r)$, r is the unit rate. Now in Grade 8, the unit rate of a proportion is used to introduce "the slope" of the line.

One can see the constant of proportionality as the relationship between two quantities in equations, tables, graphs, and function machines. Now students need to make connections between the different representations in order to come to a unified understanding that the different representations are in essence different ways of modeling the same information. Explicit connections need to be made between the multiplicative factor, the slope, scale factor, and an increment in a table. They need to analyze information given in multiple representations to answer questions when comparing. Information can be given in graphs, tables, equations, and/or words.

TIP!

Use the Frayer Model© to help students describe the connections between a table, a graph, a proportional relationship, and a relationship described in words and with an equation.

To reinforce the relationships between the *x* and the *y*, students should continually name quantities for the real-world problem they represent. They should also identify the independent and dependent variables. (See 6.EE.9 for more information.) Also, use graphs where the scale is not always one and present problems that require graphing rational numbers that are not limited to integers.

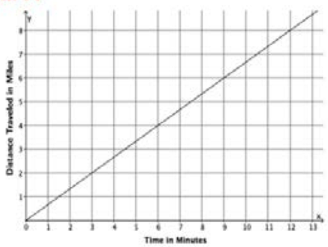
Distance-time problems

Distance-time problems are common in mathematics. In this cluster, they serve the purpose of illustrating how the rates of two objects can be represented, analyzed, and described in different ways: verbally, graphically, tabularly, and algebraically. Emphasize the creation of representative graphs and the meaning of various points. Then compare the same information when represented in an equation.

EXAMPLE

Which car is going faster, Car A, Car B, or Car C?

Car A



Car B

Time in minutes (x)	Distance in miles (y)
15	12.5
30	25
45	37.5

Car C

Car C is traveling 1/6 miles per minute faster than car B.

DISCOVERING SLOPE THROUGH SIMILAR TRIANGLES

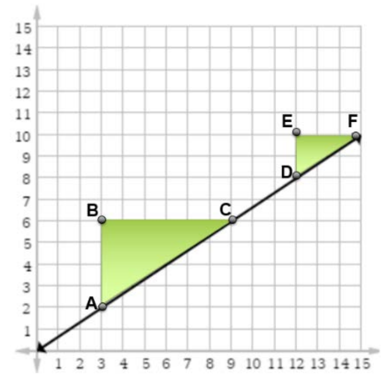
By using coordinate grids and various sets of similar triangles, students can prove that the slopes of the corresponding sides are equal, thus making the unit rate or rate of change equal. Two triangles are similar if the second can be obtained by the first from a sequence of rotations, reflections, translations, and *dilations.* (8.G.4).

Therefore if one triangle can be mapped on top of the other by a sequence of transformations, the triangles are similar, which means the sides are proportional.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (8.EE.5-6)

EXAMPLE

Prove that a line has a constant rate of change.



Discussion: Have students draw a line, and then draw two right triangles where the hypotenuse is on the line. Then have students prove that they are similar by a series of transformations, so one triangle is mapped onto the other. For example, if ΔABC is translated 9 units to the right and 6 units up and dilated by ½, with center of dilation at vertex A, they can discover that the triangles are similar. They can also discover that both hypotenuses have the same slope.

Also, it may be useful to also have students prove the converse.

EXAMPLE

Prove that if a collection of ordered pairs has constant rate of change, then it forms a line.

Affording students many experiences with multiple sets of triangles allow students to discover and to generalize the slope to *y* = *mx* for a line through the origin and *y* = *mx* + *b* for a line through the vertical axis at *b*. It is also very important to connect these concepts with real-world examples.

EXAMPLE

Jonna runs 100 meters in 14 seconds.

- If she maintains a constant speed, use slope triangles to show some other combinations of meters and seconds that she could run.
- What do you notice about all your triangles?
- Could you write an equation that demonstrates Jonna's speed?

Students should discover that *y* = *mx* is a special case of *y* = *mx* + *b* where *y* = *mx* is a proportion and *y* = *mx* + *b* is not. They may want to connect it to 8.G.2 with the idea that *y* = *mx* + *b* is in essence just a vertical translation of *y* = *mx*.

Use graphing utilities such as Desmos to show the lines in the form of *y* = *mx* + *b* as vertical translations of the equation *y* = *mx*.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (8.EE.5-6)

Instructional Tools/Resources

These tools and resources may be helpful for instruction. Before use, they should be evaluated at the local level to determine their appropriateness for instruction.

- Slopes Between Points on a Line is a task by Illustrative Mathematics that helps students understand why the calculated slope will be the same for any two points on a line.
- Peaches and Plums, Comparing Speeds in Graphs and Equations, and Who Has the Best Job? are tasks by Illustrative Mathematics that helps students reason about unit rates using a real-world problem.
- Sore Throats, Variation 2, Stuffing Envelopes, and Coffee by the Pound are tasks by Illustrative Mathematics that connects the 8th grade standard 8.EE.5 to its predecessors 7.RP.2 and 6.EE.9.
- Walk Out is a 3-Act Task by TapIntoTeenMinds that helps students match a distance-time graph to a situation.
- Tech Weight is a 3-Act Task by TapIntoTeenMinds that works with direct variation situations.
- How Fast Can Bamboo Grow? by BBC's Wild China has a Youtube video about the linear growth of bamboo. Some bamboo species grow nearly 1.5 inches per hour.

Curriculum and Lessons from Other Sources

- EngageNY Grade 8, Module 4, Topic B, Lesson 10: A Critical Look at Proportional Relationships and Lesson 11: Constant Rate discuss proportions, linear equations in two variables, and their graphs.
- EngageNY Grade 8, Module 4, Topic C, Lesson 15: The Slope of a Non-Vertical Line and Lesson 16: The Computation of the Slope and a Non-Vertical Line use similar triangles to explore slope.
- Illustrative Mathematics, Grade 8, Unit 2: Dilations, Similarity, and Introducing Slope, Lesson 10: Meet Slope, Lesson 11: Writing Equations for Lines, Lesson 12: Using Equations for Lines are lessons that pertain to this cluster. *You must create an account to view Illustrative Mathematics lessons on OpenUp Resources.*
- Illustrative Mathematics, Grade 8, Unit 3: Linear Relationships, Lesson 1: Understanding Proportional Relationships, Lesson 2: Graphs of Proportional Relationships, Lesson 3: Representing Proportional Relationships, Lesson 4: Comparing Proportional Relationships, Lesson 5: Introduction to Linear Relationships, Lesson 7: Representations of Linear Relationships, Lesson 8: Translating to $y = mx + b$, are lessons that pertain to this cluster. *You must create an account to view Illustrative Mathematics lessons on OpenUp Resources.*

General Resources

- Arizona 6-8 Progression on Expressions and Equations
  This cluster is addressed on pages 11-12
- Coherence Map by Achieve the Core is an interactive website that shows the progression and interconnectedness of the Common Core standards. To be noted, there are some differences between Ohio's Learning Standards and the Common Core, so although the tool is useful, it needs to be checked with respect to Ohio's Learning Standards.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (8.EE.5-6)

General Resources, continued

- High School Coherence Map by UnboundED is a map of the relationships among high school mathematics standards. Please note: This resource is aligned to the Common Core not Ohio's Learning Standards. Although the tool is useful, it needs to be checked with respect to Ohio's Learning Standards.

References

- Baltus, C. (April 2010). Connected representations from proportion to linear functions. *Mathematics Teacher*, *103(8)*, 590-596.
- Common Core Standards Writing Team. (2013, July 4). *Progressions for the Common Core State Standards in Mathematics (draft). 6-8, Expressions and Equations*. Tucson, AZ: Institute for Mathematics and Education, University of Arizona.
- Langrall, C., & Swafford, J. (December 2000). Two balloons for two dollars: developing proportional reasoning. *Mathematics Teaching in the Middle School*, *6(4)*, 254-261.
- Williams-Candek, C. (October 2016). All talk and more action. *Mathematic Teaching in the Middle School*, *22(3)*, 162-170.

STANDARDS	MODEL CURRICULUM (8.EE.7-8)
EXPRESSIONS AND EQUATIONS **Analyze and solve linear equations and pairs of simultaneous linear equations.** **8.EE.7** Solve linear equations in one variable. **a.** Give examples of linear equations in one variable with one solution, infinitely many solutions, or no solutions. Show which of these possibilities is the case by successively transforming the given equation into simpler forms, until an equivalent equation of the form $x = a$, $a = a$, or $a = b$ results (where a and b are different numbers). **b.** Solve linear equations with rational number coefficients, including equations whose solutions require expanding expressions using the distributive property and collecting like terms. **8.EE.8** Analyze and solve pairs of simultaneous linear equations graphically. **a.** Understand that the solution to a pair of linear equations in two variables corresponds to the point(s) of intersection of their graphs, because the point(s) of intersection satisfy both equations simultaneously. *Continued on next page*	**Expectations for Learning** In prior grades, students learn to solve and graph linear equations algebraically and through real-world examples. In this cluster, students will continue to solve and graph linear equations that include rational coefficients, variables on both sides, and distributive property. Students will solve pairs of linear equations through graphing and simple inspection. They will determine if a pair of linear equations in two variables has no solutions, one solution, or infinitely many solutions. This cluster sets the foundation to solving systems of equations in high school Algebra. **ESSENTIAL UNDERSTANDINGS** **Linear Equations** • Linear equations can have no solutions, one solution, or infinitely many solutions. • Linear equations are solved by using inverse operations. • Linear equations that are equivalent to $x = a$ have one solution. • Linear equations that are equivalent to $a = a$ have infinitely many solutions. • Linear equations that are equivalent to $a = b$ have no solutions. **Pairs of Linear Equations** • Pairs of linear equations can have no solutions, one solution, or infinitely many solutions. • Pairs of linear equations in two variables that intersect at one point have one solution. • Pairs of linear equations in two variables that are parallel have no solutions. • Pairs of linear equations in two variables that have all points in common have infinitely many solutions. • The solution(s) to a pair of linear equations in two variables make both equations true. • A solution to a pair of linear equations in two variables is often written as an ordered pair. *Continued on next page*

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<tr><th>STANDARDS</th><th>MODEL CURRICULUM (8.EE.7-8)</th></tr>
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b. Use graphs to find or estimate the solution to a pair of two simultaneous linear equations in two variables. Equations should include all three solution types: one solution, no solution, and infinitely many solutions. Solve simple cases by inspection. *For example, $3x + 2y = 5$ and $3x + 2y = 6$ have no solution because $3x + 2y$ cannot simultaneously be 5 and 6.*

c. Solve real-world and mathematical problems leading to pairs of linear equations in two variables. *For example, given coordinates for two pairs of points, determine whether the line through the first pair of points intersects the line through the second pair.* (Limit solutions to those that can be addressed by graphing.)

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Expectations for Learning, continued

MATHEMATICAL THINKING

- Create a model to make sense of a problem.
- Determine reasonableness of results.
- Apply grade-level concepts, terms, and properties.
- Recognize and use a pattern or structure.
- Recognize, apply, and justify mathematical concepts, terms, and their properties.

INSTRUCTIONAL FOCUS

Linear Equations

- Solve multi-step equations with rational coefficients that may include variables on both sides, distributive property, and collecting like terms.
- Determine whether a linear equation has no solutions, one solution, or infinitely many solutions.
- Graph the solution to a linear equation on a number line.

Pairs of Linear Equations

- Determine whether a pair of linear equations in two variables has no solutions, one solution, or infinitely many solutions.
- Find or estimate which points are solutions to a pair of linear equations in two variables.
- Solve pairs of linear equations in two variables graphically and by simple inspection.
- Solve real-world problems using pairs of linear equations in two variables that can be addressed graphically.
- Utilize different scales when graphing pairs of linear equations in two variables.

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STANDARDS	MODEL CURRICULUM
8.EE.7-8, continued	**Content Elaborations** • Ohio's K-8 Critical Areas of Focus, Grade 8, Number 1, pages 50-51 • Ohio's K-8 Learning Progressions, Expressions and Equations, pages 18-19 **CONNECTIONS ACROSS STANDARDS** • Use the equation of a linear model to solve problems in the context of bivariate measurement data (8.SP.3). • Graph proportional relationships, and compare proportional relationships represented in different ways (8.EE.5). • Use similar triangles to explain slope (8.EE.6). • Understand that a function is a rule that assigns to each input exactly one output (8.F.1). • Compare the properties of two functions (8.F.2).

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (8.EE.7-8)

Instructional Strategies

Note: The model curriculum is not a collection of lessons nor a full curriculum; it does not suggest pace, sequence, or amount of time spent on topics. It provides information about a topic related to the standards including ideas for examples, strategies for teaching, common misconceptions, and possible connections between topics. In addition, the emphasis of the model curriculum is on instruction not on assessment; therefore, the examples in the Instructional Strategies section are written for instructional purposes not necessarily as assessment items. Examples may need to be rewritten to accommodate the needs of each individual classroom.

Standards for Mathematical Practice

This cluster focuses on but is not limited to the following practices:

***MP.2** Reason abstractly and quantitatively.*

***MP.6** Attend to precision.*

***MP.7** Look for and make use of structure.*

***MP.8** Look for and express regularity in repeated reasoning.*

SOLVING EQUATIONS IN ONE VARIABLE

In Grade 7, students learned integer operations for the first time. They also applied the properties of operations when solving two-step equations and inequalities. Now students build on the fact that solutions maintain equality and that equations may have only one solution, many solutions, or no solutions at all. Equations with infinitely many solutions may be as simple as $5 = 5$, $3x = 3x$, $3x + 5 = x + 2 + x + x + 3$, or $6x + 4x = x(6 + 4)$, where both sides of the equation are equivalent.

Properties of Operations

Table 3 on page 97 of Ohio's Learning Standards in Mathematics states the Properties of Operations and Table 4 states the Properties of Equality. Teachers should be using the correct terminology to justify steps when performing operations and solving equations. Although, eighth grade students should not be required to know the formal names of the properties, they should be encouraged to recognize them and use them to justify their steps when solving equations. For example, students may say "change order" for commutative property or "rearranging groups" for associative property, and that is acceptable at this level, but teachers should reiterate the correct vocabulary during discussions. *Note: The Addition Property of Equality and the Subtraction Property of Equality can be used interchangeable since subtracting a number is the same as adding its opposite. The same is true for the Multiplication Property of Equality and the Division Property of Equality.*

Students incorrectly think that the variable is always on the left side of the equation. Give students situations where the variable is on the right side of the equation. Emphasize using the Symmetric Property of Equality if students wish to flip the variable to the other side of the equal sign.

Solving One-Variable Linear Equations

Equation-solving in Grade 8 should involve multi-step problems that require the use of the distributive property, collecting like terms, rational coefficients, and variables on both sides of the equation.

TIP!

In Grade 7, students may have used a pan balance, number lines, or algebra tiles to solve two-step equations. Eighth grade students could review these models and build upon them. For example, algebra tiles may help prevent student errors such as incorrectly combining like terms on opposite sides of the equations.

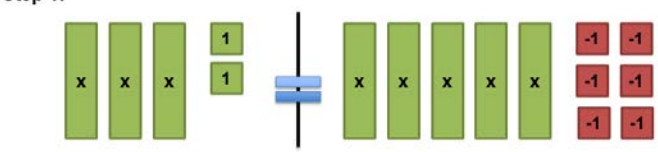
INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (8.EE.7-8)

Students incorrectly think that only the letters *x* and *y* can be used for variables. Use various letters in the classroom and on assignments to break this misconception. However, also try to avoid using variables that start with the same letter as the quantity such as *b* for bananas or *a* for apples, as this causes other misconceptions. See Model Curriculum 6.EE.5-8 and 7.EE.3-4 for more misconceptions related to variables.

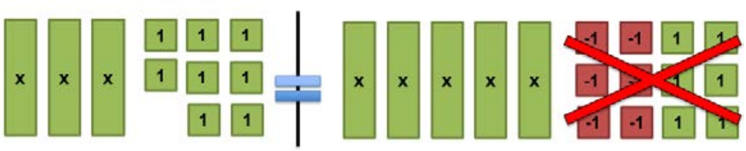
EXAMPLE

Solve 3x + 2 = 5x – 6 using Algebra tiles

Step 1:

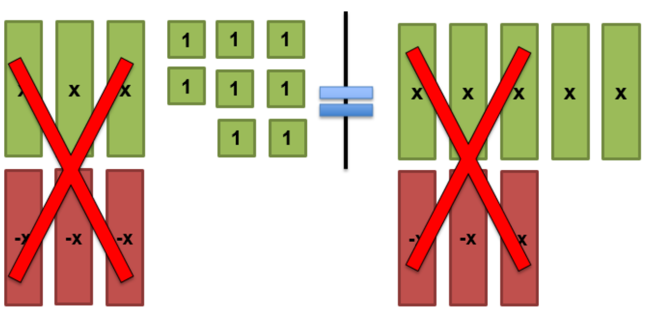


Step 2:



Step 2 Note: As the teacher initially models these steps he/she should use the properties to justify the movement of the Algebra tiles. For example, the teacher could say, "Using the Addition Property of Equality, we can add 6 positive ones to both sides of the equation. This will allow us to combine the -6 with the positive 6 on the right side on the equation to make 6 zeros using the Additive Inverse Property."

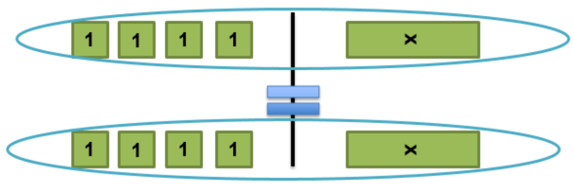
Step 3:



Step 3 Note: For example, the teacher could say, "By the Addition Property of Equality, we can add 3 negative x-terms to both sides of the equation. Then this will allow us to combine 3 positive x-terms with 3 negative x-terms to make 3 zeros on both sides of the equation using the Additive Inverse Property. However, we could also use the Subtraction Property of Equality to subtract 3 x-terms from each side of the equation, since subtraction is the same thing as adding the opposite."

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (8.EE.7-8)

Step 4:



Step 4 Note: For example, the teacher could say, "Using the Division Property of Equality, we could divide each side of the equation by 2. Although notice that dividing by 2 is the same thing as multiplying by $^1/_2$, so we could also use the Multiplicative Property of Equality to multiply each side by $^1/_2$. Then the Multiplicative Inverse Property allows us to multiply 2x by $^1/_2$ to get 1x. Then we multiply 8 by ½ to get 4, so 4 = 1x. Because of the Multiplicative Identity Property, we can just say that 4 = x. I know that some of you prefer the x on the left side of the equation, so we can flip the equation so that x = 4 by the Symmetric Property of Equality."

Students especially have difficulty with equations involving fractions. Discuss alternative methods to solving equations with fractions such as clearing the fractions by using the LCD. A similar strategy of multiplying each term by 100 can also help students clear decimals from an equation. See Model Curriculum 6.EE.5-8 and 7.EE.3-4 for scaffolding ideas for teaching fractions using models.

EXAMPLE

Solve for *h*.

$$\frac{3}{4}h + \frac{5}{12} = \frac{2}{3}$$

Method 1

$$\begin{aligned}\frac{3}{4}h + \frac{5}{12} - \frac{5}{12} &= \frac{2}{3} - \frac{5}{12}\\ \frac{3}{4}h &= \frac{8}{12} - \frac{5}{12}\\ \frac{3}{4}h &= \frac{3}{12}\\ \left(\frac{4}{3}\right)\frac{3}{4}h &= \frac{3}{12}\left(\frac{4}{3}\right)\\ h &= \frac{1}{3}\end{aligned}$$

Method 2

Clear the fractions by using the Multiplication Property of Equality, and multiply each side by the common denominator of 12.

$$\begin{aligned}\frac{12}{1}\left(\frac{3}{4}h + \frac{5}{12}\right) &= \frac{2}{3}\left(\frac{12}{1}\right)\\ 9h + 5 &= 8\\ 9h &= 3\\ h &= \frac{1}{3}\end{aligned}$$

Make sure students have a clear understanding of the multiplication of equality. Many students incorrectly try to clear fractions before distributing.

TIP!

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (8.EE.7-8)

Real-World Contexts

Connect mathematical analysis with real-life events by using contextual situations when solving equations. Students should experience—

- analyzing and representing contextual situations with equations;
- identifying whether there is one solution, no solutions, or infinitely many solutions; and then
- solving the equations to prove conjectures about the solutions.

Through multiple opportunities to analyze and solve equations, students should be able to make use of structure (S.MP.7). One way to do this is to estimate the number of solutions and possible values(s) of solutions prior to solving. Real-world tasks help ground the abstract symbolism to life, e.g., computing the number of tiles needed to put a border around a rectangular space or solving proportional problems as in doubling recipes.

Students may incorrectly think that you always need a "variable = a constant" as a solution. Draw attention to solutions that are expressions.

EXAMPLE

A photo is three times as long as its width. Bethany wants to make a frame that is one inch longer in all directions. How long is the perimeter of the frame?

- Write an equation to represent the situation. Do not forget to define the variables.
- Compare your equation to your classmates. Can there be more than one way to represent the situation? Explain.
- Solve your equation to find the perimeter of the frame. Will your classmates who had different equations get different solutions? Explain.

Discussion: Although students could solve this situation using pairs of linear equations, make sure students have the opportunity to solve these types of situations using one-variable equations. Therefore, they need to define length in terms of width or vice versa.

Students should have practice solving equations that are proportions.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (8.EE.7-8)

EXAMPLE

Mark earned $33.20 for four hours of work. If he works for 7.5 hours, how much will he make? Write a proportion equation to model the situation and solve.

$x = money\ earned$

$$\frac{33.20}{4} = \frac{x}{7.5}$$

$$\frac{7.5}{1} \cdot \frac{33.20}{4} = \frac{x}{7.5} \cdot \frac{7.5}{1}$$

$$\frac{249}{4} = x$$

$$\$62.25 = x$$

TIP! Some students may benefit from using error analysis when solving equations. Strategies may include [Pass the Pen](), *Find an Expert*, or *Journal Writing*.

SOLVING PAIRS OF LINEAR EQUATIONS GRAPHICALLY

This cluster builds on the informal understanding of slope, students gained from graphing unit rates and proportional relationships in grades 6 and 7. It also builds upon the stronger, more formal understanding of slope and the relationship between two variables from 8.EE.5-6 and 8.F.4-5. Most student experiences with pairs of simultaneous linear equations should be with graphical representations of solutions. Beginning work should involve pairs of equations with solutions that are ordered pairs of integers, making it easier to locate the point of intersection. Although students should also be able to approximate solutions that do not fall evenly onto the intersection of grid squares. In addition, students should be able to solve simple cases by inspection. For example, $x + y = 3$ and $x + y = 5$ has no solution because $x + y$ cannot equal both 3 and 5. Students should have practice working with graphs that have a variety of scales including fractions and decimals.

Real-World Contexts

Graphing pairs of linear equations should be introduced through contextual situations relevant to eighth graders, so students can create meaning. They should explore many tasks for which they must write and graph pairs of equations with different slopes and y-intercepts. This should lead to the generalization that finding one point of intersection is the single solution to the pair of linear equations. Finally, students should relate the solution to the context of the problem, commenting on the reasonableness of their solution. Emphasize that the solution must satisfy both equations.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (8.EE.7-8)

Students often confuse the solution of one-variable equations, e.g., x = 5, with the solution of two-variable equations, which have an ordered pair as the solution, e.g., (4,3).

EXAMPLE

Determine the number of movies downloaded in a month that would make the costs for two sites the same, when Site A charges $6 per month and $1.25 for each movie and Site B charges $2 for each movie and no monthly fee.

- Define your variables.
- Write equations to represent the cost at each site.
- Graph both equations on the same graph.
- What does the point of intersection represent in the situation?
- Give support for when and why you would download movies from Site A.
- Give support for when and why you would download movies from Site B.

Students often graph lines inaccurately by not graphing enough points or the ruler slipping (human error). Emphasize precision when graphing. Also emphasize reasonableness of answers and checking the solutions by substitution.

TIP!

Tasks should be structured so that students also experience equations that are parallel lines and the same line when graphed. This will help them begin to understand the relationship between different pairs of equations. For example, when the slope and the *y*-intercept of the two lines is the same, the equations represent the same line (thus resulting in infinitely many solutions), or when the slope of the two lines is the same but the *y*-intercept is different, the lines are parallel and do not have any common solutions (thus resulting in no solutions).

Students may mistake a system with infinite solutions for a system with no solutions. Infinite solutions occur when two equations have the same graph. All points that lie on the line are solutions to the system. If the graphs of the equations are parallel, then the system has no solutions because parallel lines will never intersect.

Students can use the table tool in a graphing utility to find common solutions to a pair of simultaneous linear equations. Draw attention to the fact that the common solution of the two tables is also the intersection point of the graph.

Approximation of Solutions

Solving pairs of simultaneous linear equations in Grade 8 should include estimating solutions graphically. Provide opportunities for students to see and compare simultaneous linear equations in forms other than slope-intercept form ($y = mx + b$). Students may solve pairs of simultaneous linear equations by inspection, by graphing using slope-intercept form, or by graphing using tables of values.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (8.EE.7-8)

Students have the tendency to see the intersection point of a graphed system of equations and round it to the nearest grid line cross-section. Emphasize to students, that they can sometimes find intersections in the middle of the grid squares that allow for approximations to be more precise.

Inspection

Provide ample opportunities for students to connect the graph of a pair of linear equations to the solution to a pair of simultaneous linear equations. Highlight the structure of expressions that results in the different types of equations, so students can find solutions by inspection instead of having to graph every pair of simultaneous equations.

Students could also investigate pairs of simultaneous equations using graphing calculators or online graphing resources. They could be asked to explain verbally and in writing what, in the equation and situation, makes lines shift to different locations on the graph. For example, students should recognize that when two equations (written in slope-intercept form) have the same slope but different *y*-intercepts are the same line only shifted vertically. Therefore, the lines are parallel and have no solution. They should be able to see this by inspection without having to graph both equations. Similarly, in many instances students can perceive when two equations are equivalent. Once they realize that the equations are equivalent, they can determine that there are infinitely many solutions bypassing the need for graphing the equation.

Instructional Tools/Resources

These tools and resources may be helpful for instruction. Before use, they should be evaluated at the local level to determine their appropriateness for instruction.

Manipulatives/Technology

- SMART Board's new tools for solving equations
- Graphing calculators
- Index cards with equations/graphs for matching and sorting
- Geogebra is a graphing utility that allows one to explore the effects of changes in parameters of equations on their graphs.
- Desmos is a graphing utility that allows one to explore the effects of changes in parameters of equations on their graphs. It also includes some lesson.

Pan Balance Activities

- Pan Balance-Expressions is an applet by NCTM Illuminations using a pan balance.
- Balancing to Solve Equations is a lesson from PBS Learning that uses a balance to solve equations.
- Balance Beam Equations by Passy's World of Mathematics has a lot of examples of using balance beams to solve equations.

Algebra Tiles Activities

- Solve Multi-Step Equations with Justification is a website from Henrico County Public Schools with materials on solving multi-step equations. It has a lot of material that uses Algebra tiles.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (8.EE.7-8)

Algebra Tiles Activities, continued

- Virtual Algebra Tiles is an applet from Michigan Virtual University that allows students to use Algebra tiles. This applet allows for positive and negative representation of the tiles.
- CPM Tiles is an applet from the CPM Educational Program that allows students to use Algebra tiles. The benefit of this applet is that students can change the dimensionality of *x* and *y*. However, it is limited by not allowing for a negative representation of the tiles.
- Algebra Tiles is an applet from NCTM Illuminations that has students solve equations using Algebra tiles.
- Algebra Tiles and Equation Solving is a lesson from the Virginia Department of Education on solving equations using Algebra Tiles.
- SMART Exchange has several notebook lessons with Algebra tiles.

Solving One-Variable Linear Equations

- Picture Perfect is a Desmos lesson where students use algebraic thinking to hang pictures on a wall.
- Quick Points by John Orr is a 3-Act task where students use their knowledge of writing and solving equations to come up with equation and graph solution.
- How Much Can My DVR Record? is an activity by Yummy Math that has students analyze the storage capacity of HD and non-HD programs.
- Chocolate Milk and Mixture Problems is an activity by Yummy Math that has students analyze mixture equations using chocolate milk.
- Performance Task Bank 8.EE.7 by Research and Evaluation has many performance tasks organized around 8.EE.7.
- Pentomino Puzzle is a Desmos activity that has students work through pentomino sum puzzles to help them develop an algebraic approach.

Pairs of Simultaneous Linear Equations

- A Mixture of Problems by Laurie Riggs, et. al has a variety of problems including mixture problems to solve equations using different conceptual methods.
- Linear Systems Bundle by Desmos has many activities exploring linear systems.
- Supply and Demand is an activity by NCTM Illuminations that focuses on having students create and solve a system of linear equations in a real-world setting. By solving the system, students will find the equilibrium point for supply and demand. Students should be familiar with finding linear equations from two points or by using slope and *y*-intercept. This lesson was adapted from the October 1991 edition of *Mathematics Teacher. NCTM now requires a membership to view their lessons.*
- Piling Up Systems of Linear Equations is a 3-Act task by TapIntoTeenMinds where students have to write and solve systems of equations using weights of different objects.
- Counting Candy Sequel is a 3-Act task by TapIntoTeenMinds where students have to write and solve systems of equations using different colored candy.
- The Detention Buy Out is a 3-Act task by TapIntoTeenMinds where students use a system of equations to analyze the benefit of buying themselves out of detention.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (8.EE.7-8)

Pairs of Simultaneous Linear Equations, continued

- Coin Counting by Dan Meyer is a 3-Act task where students can use a system of equations to find the number of types of coins (higher level).
- How Long Until They Collide? is an activity by Yummy Math based on a Chevy car commercial where they discuss cars colliding.
- Performance Task Bank 8.EE.8 by Research and Evaluation has many performance tasks organized around 8.EE.8.
- The Intersection of Two Lines is a task by Illustrative Mathematics that introduces students to systems of equations.
- How Many Solutions? is a task by Illustrative Mathematics that has students write examples of equations with one solution, two solutions, no solutions, and infinitely many solutions.
- The Intersection of Two Lines is a task by Illustrative Mathematics that has students write and graph a pair of simultaneous linear equations.
- The Sign of Solutions is a task by Illustrative Mathematics that has students solve equations by inspection.

Curriculum and Lessons from Other Sources

- EngageNY Grade 8, Module 4, Topic A, Lesson 1: Writing Equations Using Symbols, Lesson 3: Linear Equations in *x*, Lesson 4: Solving a Linear Equation, Lesson 5: Writing and Solving Linear Equations, Lesson 6: Solutions of Linear Equations, and Lesson 7: Classifications of Solutions are lessons about one-variable equations.
- EngageNY Grade 8, Module 4, Topic D, Lesson 24: Introduction to Simultaneous Equations, Lesson 25: Geometric Interpretation of the Solutions of a Liner System, Lesson 26: Characterization of Parallel Lines are lessons about graphing pairs of simultaneous linear equations.
- Illustrative Mathematics, Grade 8, Unit 4: Linear Equations and Linear Systems has many lessons that pertain to this cluster. Lessons 1-9 are about one-variable linear equations. Lessons 10-12 have an emphasis on graphing systems of linear equations. *You must create an account to view Illustrative Mathematics lessons on OpenUp Resources.*

General Resources

- Arizona 6-8 Progression on Expressions and Equations
  This cluster is addressed on pages 12-13.
- Coherence Map by Achieve the Core is an interactive website that shows the progression and interconnectedness of the Common Core standards. To be noted, there are some differences between Ohio's Learning Standards and the Common Core, so although the tool is useful, it needs to be checked with respect to Ohio's Learning Standards.
- High School Coherence Map by UnboundED is a map of the relationships among high school mathematics standards. Please note: This resource is aligned to the Common Core not Ohio's Learning Standards. Although the tool is useful, it needs to be checked with respect to Ohio's Learning Standards.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (8.EE.7-8)

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<table>
<tr><th>STANDARDS</th><th>MODEL CURRICULUM (8.F.1-3)</th></tr>
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FUNCTIONS
Define, evaluate, and compare functions.
8.F.1 Understand that a function is a rule that assigns to each input exactly one output. The graph of a function is the set of ordered pairs consisting of an input and the corresponding output. Function notation is not required in Grade 8.
8.F.2 Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions). For example, given a linear function represented by a table of values and a linear function represented by an algebraic expression, determine which function has the greater rate of change.
8.F.3 Interpret the equation $y = mx + b$ as defining a linear function, whose graph is a straight line; give examples of functions that are not linear. For example, the function $A = s^2$ giving the area of a square as a function of its side length is not linear because its graph contains the points (1, 1), (2, 4) and (3, 9), which are not on a straight line.

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<h2>Expectations for Learning</h2>
In prior grades students have interpreted and analyzed proportions using graphs, tables, and equations. The focus for this cluster is linear equations, those in the form $y = mx + b$. Students will relate their understanding of the constant of proportionality to slope/rate of change. Students should be able to distinguish between linear and non-linear functions in different representations such as equations, graphs, and tables. These initial understandings provide the foundations for concepts that will be developed in high school math such as domain, range, function notation, as well as non-linear functions such as quadratic and exponential functions.

<h3>ESSENTIAL UNDERSTANDINGS</h3>

A function is a rule that assigns each input exactly one output.
The graph of a function is a set of ordered pairs consisting of an input and a corresponding output.
Functions can be represented as an equation, graph, table, and verbal description.
Properties of graphs of linear functions include slope/rate of change, y-intercept/initial value, x-intercept, where the slope is increasing, constant, or decreasing.
A vertical line has an undefined slope, where y is not a function of x.
A graph of a linear function is a non-vertical straight line.
A non-linear function is a function whose graph is not a straight line.
A table represents a linear function when constant differences between input values produce constant difference between output values.
Linear functions have a constant rate of change.
Some functions are not continuous.

<h3>MATHEMATICAL THINKING</h3>

Generalize concepts based on patterns.
Recognize and use a pattern or structure.
Use precise mathematical language.
Represent a concept symbolically.

Continued on next page

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STANDARDS	MODEL CURRICULUM (8.F.1-3)
8.F.1-3, continued	**Expectations for Learning, continued** **INSTRUCTIONAL FOCUS** • Determine if a table, graph, equation, or a verbal description represents a linear or nonlinear function. • Identify a set of input and output values for a function. • Compare properties of two functions in different representations. • Identify functions that are linear and nonlinear. • Give examples of functions that are nonlinear. • Interpret the slope/rate of change and *y*-intercept/initial value of a linear function. • Complete tables to show a relationship that is a function. • Determine if it is reasonable to "connect the points" on a graph based on the context. **Content Elaborations** • Ohio's K-8 Critical Areas of Focus, Grade 8, Number 2, page 52 • Ohio's K-8 Learning Progressions, Functions, page 20 **CONNECTIONS ACROSS STANDARDS** • Graph proportional relationships (8.EE.5). • Derive the equation $y = mx + b$ (8.EE.6). • Sketch a graph from a verbal description (8.F.5). • Use an equation of linear model to solve problems (8.SP.3).

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (8.F.1-3)

Instructional Strategies

Note: The model curriculum is not a collection of lessons nor a full curriculum; it does not suggest pace, sequence, or amount of time spent on topics. It provides information about a topic related to the standards including ideas for examples, strategies for teaching, common misconceptions, and possible connections between topics. In addition, the emphasis of the model curriculum is on instruction not on assessment; therefore, the examples in the Instructional Strategies section are written for instructional purposes not necessarily as assessment items. Examples may need to be rewritten to accommodate the needs of each individual classroom.

Standards for Mathematical Practice

This cluster focuses on but is not limited to the following practices:

***MP.1** Make sense of problems and persevere in solving them.*

***MP.2** Reason abstractly and quantitatively.*

***MP.5** Use appropriate tools strategically.*

***MP.8** Look for and express regularity in repeated reasoning.*

In Grade 6, students plotted points in all four quadrants of the coordinate plane. They also represented and analyzed quantitative relationships between dependent and independent variables. In Grade 7, students decided whether two quantities are in a proportional relationship. In Grade 8, students begin to call relationships functions when each input is assigned to exactly one output. Also, in Grade 8, students learn that proportional relationships are part of a broader group of linear functions, and they are able to identify whether a relationship is linear— meaning its equation can be written in $y = mx + b$ form. Nonlinear functions are included in this cluster for comparison purposes. Later, in high school, students will use function notation and will be able to identify certain types of nonlinear functions.

This cluster connects to the following Grade 8 concepts:

- Expressions and Equations–Linear equations in two variables can be used to define linear functions, and students can use graphs of functions to reason toward solutions to linear equations.
- Geometry–Similar triangles are used to show that the slope of a line is constant.
- Statistics and Probability–Bivariate data can often be modeled by a linear function.

DEFINING FUNCTIONS

To determine whether a relationship is a function, students should be expected to reason from a context, a graph, or a table, after first being clear which set represents the input (e.g., independent variable) and which set is the output (e.g., dependent variable). When a relationship is not a function, students should produce a counterexample: an "input value" with at least two "output values." If the relationship is a function, the students should explain how they verified that for each input there was exactly one output.

A function can be thought of as—

1. having two sets;
2. having a correspondence between the two sets;
3. meeting a special requirement where every input must be matched or assigned to one and only one output.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (8.F.1-3)

INDEPENDENT AND DEPENDENT VARIABLES

The dependent variable depends on the independent variable, so the independent variable can be used to predict the dependent variable. In Grade 6 students explored independent and dependent variables, and how the dependent variable changes in relation to the independent variable. In Grade 8 students need to continue identifying the independent and dependent variables in functions. Students need practice justifying the relationship between the independent and dependent variable.

Typically, the dependent variable is labeled *y* and graphed on the vertical axis, and the independent variable is labeled *x* and graphed on the horizontal axis.

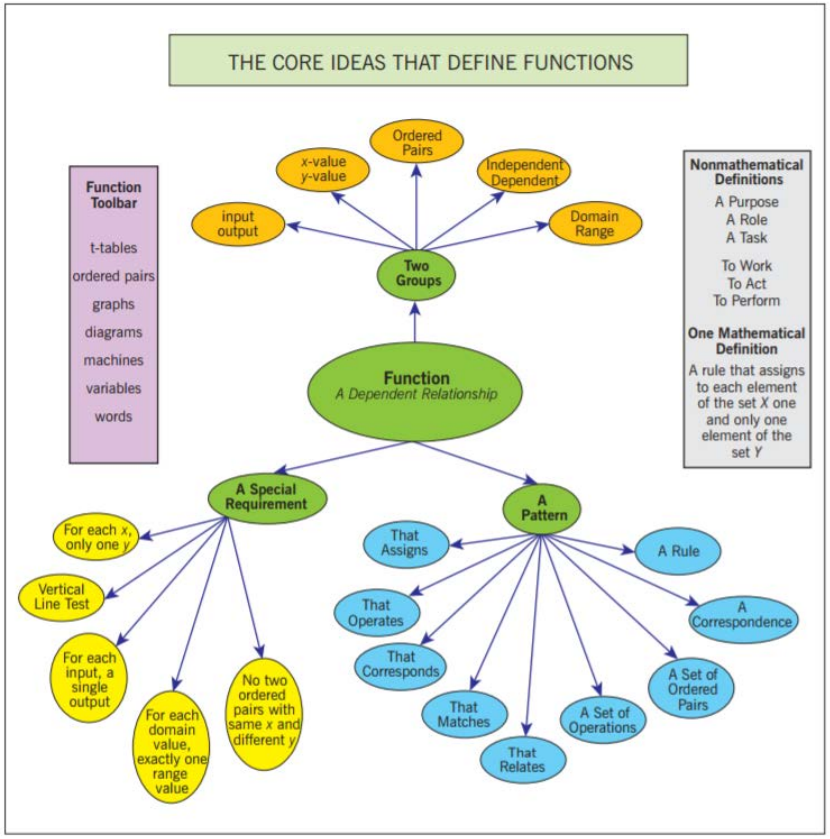
TIP! Some students confuse the meanings of *x* (independent variable) and *y* (dependent variable), particularly when graphing the line of an equation. Have students identify which one is which within a context. To help students identify which variable is which, they can ask themselves—

- ____________ changes because of ________________
- _______________ depends on __________________

Some students may infer a cause and effect between independent and dependent variables, but this is often not the case.

Usefulness of Functions

Functions allow us to make predictions, see patterns, and understand data. In order to make functions relevant to students, discuss functions are useful where in real-life. Highlight the usefulness of finding a real-life relationship between two sets. Discuss situations where having two outputs for the same input would create issues. Emphasize the *relationship* between the two sets—the input and output and that the output *depends* on the input.



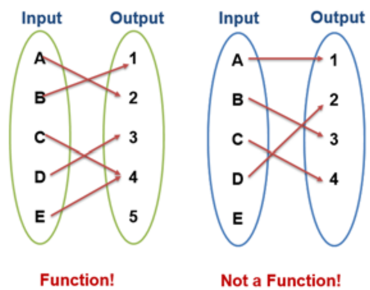
From Gregorio A. Ponce, *Mathematics Teacher*, 2007, "Critical Juncture Ahead"

Note: This graphic at this grade level is for teacher use not student use. Also, note that even though the vertical line test is shown in the graphic, using it is discouraged.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (8.F.1-3)

Example and Non-examples of Functions
Compare examples and non-examples of functions to illustrate to students the definition of a function. See High School Algebra 1/Math 1 F.IF.1-3 for similar activities.

In some situations, the independent and dependent variables are clearly defined by the situation, but in other cases defining the input and output is purely preferential. However, the assignment of the variables to the input or output may affect whether or not the correspondence between the two sets is a function. For example, when comparing a class of 25 students to their corresponding birthday month to determine whether it is a function or not, the assignment matters. If the students are defined as the input set and the birthday month is defined as the output set, the correspondence between two sets is a function because each student has exactly one birthday month. However, if the birthday month is defined as the input set and the class of 25 students is defined as the output set, the correspondence between them is not a function because a birthday month will be assigned to more than one student.



In a function, the elements from the input set must be assigned or matched to one and exactly one element from the output set. In other words, there can be no inputs left unassigned. However, the reverse is not necessarily true; there can be outputs that exist without any assigned inputs.

EXAMPLE
Determine whether the following situations are functions or not:

a. the ages of students in an 8th grade class (defined as the input) and their heights (defined as the output).
b. the names of students in an 8th grade class (defined as the input) and their eye color (defined as the output).
c. the names of students in and 8th grade class (defined as the input) and students' addresses (defined as the output).
d. the name of students in an 8th grade class (defined as the input) and zip codes (defined as the output).
e. zip codes (defined as the input) and the names of the students in an 8th grade class (defined as the output).

Discussion: Determining whether some of these situations are a function or not may depend on the inputs. For example, if a student in the class has two different color eyes, then part **b**. would not be a function. Also, some students have two or more addresses depending on their living situation. Use these opportunities to informally discuss how restricting the input (domain) affects functionality. See High School Algebra 1/Math 1Model Curriculum F.IF.3 for more information about functions.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (8.F.1-3)

In some situations, the elements used as inputs affect whether a correspondence between sets is a function or not. For example when comparing students (input) with their hair color (output) to determine if the correspondence represents a function depends on the input. If no one in the class has multi-colored hair, then the relationship represents a function. However, if even one person has highlighted or multi-colored hair, the situation does not represent a function. A discussion should take place about how restricting the input impacts determining whether a relationship is a function or not. For example, when comparing students (input) with the color clothing they are wearing (output) to determine functionality depends on the input. Since people who are wearing multi-colored clothing would be assigned to more than one color, the situation would not be representative of a function; however, if the input is restricted to only those students who are wearing monochromatic outfits, than a function would occur between the two sets.

EXAMPLE

Discuss whether each scenario is a function or not. Explain your reasoning.

- The time it takes to walk home from school
- The color of each person's shirt in the class
- The eye color of each person in the class
- The temperature converted from Fahrenheit to Celsius
- The surface area of a balloon as you blow air into it
- The month each person in the class is born
- A list of the names of the students who earned A's in the class and earned C's in the class
- The length of perimeter and area
- The hair color of each person in the class
- Students' heights in the class
- The password for a bank account

TIP!

It may be helpful to use a Function Wall Activity where categories such as clothing color(outputs) are affixed to the wall and students (inputs) who are wearing one of the assigned colors have to actually go and stand under the corresponding sign on the wall. Students who are wearing more than one color will quickly realize that they cannot go to two categories simultaneously and that if they do not have one and exactly one place to go, then the situation is not a function. See High School model curriculum F.IF.3 for more information.

Students often incorrectly think that only one-to-one correspondences are functions. For example, they may not acknowledge $y = 5$ as being a function. Provide them with situations that counter this misconception.

Some students incorrectly think that every function must have a mathematical formula. Other students may incorrectly think that every formula is a function; however, $x = 4$ or $y = \pm\sqrt{x}$ are not functions.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (8.F.1-3)

Avoid the Vertical Line Test

The "vertical line test" should be avoided because—

1. it is too easy to apply without thinking;
2. students do not need an efficient strategy at this point; and
3. it creates misconceptions for later mathematics, when it is useful to think of functions more broadly, such as whether *x* might be a function of *y* or in the context of polar coordinates.

Instead teachers should stress the definition of a function: for every input value there is one and only one output value. For more information on the uses of the vertical line test and its misapplication, see High School Algebra 1/Math 1 Model Curriculum F.IF.1-3.

Function Machine

"Function machine" pictures or models are useful for helping students imagine input and output values. A function machine contains a "hidden rule" inside the machine by which the output value is determined from the input.

Explore functions through concrete examples such as patterns or contextual problems using quantities. This should help students develop a deeper understanding of the concept of functions. In addition, these connections will help make the symbolic notation more meaningful. Making comparisons between different problem types is also helpful in making connections.

Functions as Quantities

One study found that reasoning quantitatively about functions allowed students to better understand functions as they searched for more than just patterns (Ellis, 2009). Reasoning quantitatively allowed students to extend their thinking past prototypical examples and nonuniform tables to make sense of the problem. Working with quantitative relationships allowed students to extend and justify their generalizations. Therefore, it may be wise to draw students' attention back to the quantities in the original problem.

EXAMPLE

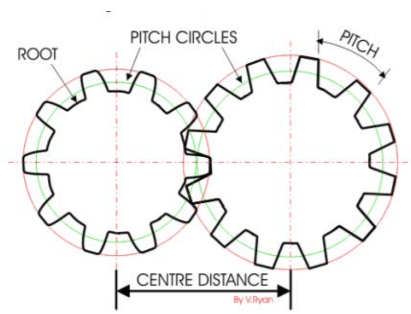
The Connected Gears Problem

You have two gears on your table. Gear A has 10 teeth, and Gear B has 12 teeth.

a. If you turn gear A a certain number of times, does gear B turn more revolutions, fewer revolutions, or the same number? How can you tell?

b. Devise a way to keep track of how many revolutions Gear A and Gear B makes. How can you keep track of both gears at the same time?

Adapted from Ellis, 2009

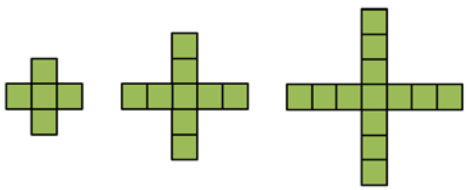


INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (8.F.1-3)

Functions as Patterns

In the elementary grades, students explored number and shape patterns (sequences), and they use rules for finding the next term in the sequence. At this point, students describe sequences both by rules relating one term to the next and also by rules for finding the n^{th} term directly. (In high school these are called recursive and explicit formulas.) Students express rules in both words and in symbols. Instruction should focus on additive and multiplicative sequences as well as sequences of square and cubic numbers.

EXAMPLE



Drawing	# of Tiles
1	
2	
3	
4	
5	
6	

a. Draw the next three images in the pattern.
b. Fill out the table.
c. Without drawing, how many tiles would be in the 10^{th} image? How do you know?
d. Write a rule to describe the pattern.
e. Would you rule change if your first image was just one square? Explain.

CONTINUOUS AND DISCRETE FUNCTIONS

When plotting points and drawing graphs, students should develop the habit of determining, based upon the context, whether it is reasonable to "connect the dots" on the graph. In some contexts, the inputs are discrete, and connecting the dots is incorrect. For example, if a function is used to model the height of a stack of n paper cups, it does not make sense to have 2.3 cups. Thus there will be no ordered pairs between $n = 2$ and $n = 3$. *Note: Students are not required to use the terminology continuous and discrete.* Emphasize these concepts throughout the year.

COMPARING FUNCTIONS

Representing the Same Functions Using Different Representations

Notice that the standards explicitly call for exploring functions numerically, graphically, verbally, and algebraically. For fluency and flexibility in thinking, students need experiences translating among these different representations.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (8.F.1-3)

Representing Different Functions Using Different Representations
Students need experience translating among the different representations using different functions. For example, they should be able to determine which function has a greater slope by comparing a table and a graph.

Representing Different Functions Using the Same Representation
Students need to compare functions using the same representation. For example, within a real-world context, students compare two graphs of linear functions and relate the graphs back to its meaning within the context and its quantities. Students should work with graphs that have a variety of scales including rational numbers.

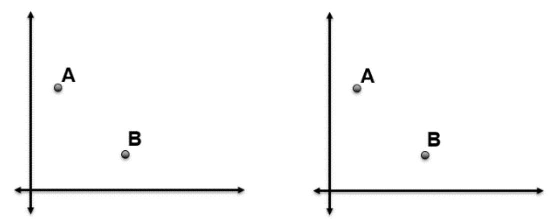
Provide multiple opportunities to examine the graphs of linear functions and use graphing calculators or computer software to analyze and compare at least two functions at the same time. Illustrate with a slope triangle where the *x*-value is "1" that the slope is the "rate of change." Students can use this technique to compare two different situations and identify which is increasing/decreasing at a faster rate.

IDENTIFYING LINEAR AND NON-LINEAR FUNCTIONS

In Grade 8, the focus is on linear functions, and students begin to recognize a linear function from its form $y = mx + b$ knowing that $y = mx$ as a special case of a linear function. Students also need experiences with nonlinear functions. This includes functions given by graphs, tables, or verbal descriptions but for which there is no formula for the rule. An example could be using a child's height as a function of his or her age because his or her growth tapers off and the rate of change is not constant.

Some students may incorrectly think that function can only be linear. Without exposure to non-linear functions, they believe all functions are linear; they do not realize that functions can have curves or bends. Give them examples to break this misconception.

EXAMPLE
Create two different graphs of functions that pass through points A and B.



Discussion: Use this exploration to highlight that functions are not necessarily linear. In the first graph most students will create a line, but the second graph should push them to create a function other than line. For example, students may use a V-shaped graph or a squiggly line to connect the dots. When drawing their representations, emphasize that the input values cannot repeat.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (8.F.1-3)

Compare linear to non-linear functions such as quadratic and exponential functions. Students can compute perimeter and area of different-size squares and identify that the first relationship is linear while the other is not by either looking at a table of values or a graph in which the side length is the independent variable (input) and the area or perimeter is the dependent variable (output). This can be tied in with 8.EE.2.

Students should also realize that an increase (or decrease) by a constant amount is a characteristic of all linear functions.

EXAMPLE

Give students a variety of linear and non-linear equations.

$y = 2x$	$y = 2^x$	$y = x^2$	$y = (x - 4)^2$
$y = 3x - 1$	$y = x^3$	$2x + 4y = -5$	$y = -\frac{3}{4}x + 7$

a. Using technology, graph the equations and determine which are linear functions.
b. Make a table of values for each of the functions (or view them using technology). Analyze any patterns of the inputs and outputs of the linear functions compared to the nonlinear functions.
c. What do all the linear functions have in common?

Many students incorrectly think that $x = 4$ is a linear function because it makes a line. Bring them back to the definition of a function to confront this misconception.

Some students will mistakenly think that a straight line must be a horizontal or vertical line only. Show students a variety of functions to break these misconceptions.

Some students think all linear functions are proportional relationships. Compare proportional linear functions to nonproportional linear functions. Emphasize that the graph of a proportional relationship must pass through the origin.

Instructional Tools/Resources

These tools and resources may be helpful for instruction. Before use, they should be evaluated at the local level to determine their appropriateness for instruction.

Manipulatives/Technology

- Graphing calculators
- Graphing software (including dynamic geometry software)
- Desmos is a graphing tool that allows one to explore the effects of changes in parameters of equations on their graphs. It also includes some lessons.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (8.F.1-3)

Patterns

- Visual Patterns is a website that shows pictures of linear, exponential, and quadratic patterns.
- Growing Linear Patterns by Mata is a slide player that has students discover linear patterns.
- Patterns Posters for Algebra 1 from Finding Ways by Fawn Nguyen is a blog about a class activity where students are given patterns, and they have to make posters from them. She is also the creator of the visual patterns link above.

Function Machines

- Function Machine is an applet by Math Playground of a function machine.
- Function Machine is an applet by Shodor of a function machine.
- Function Rules is a task by Illustrative Mathematics where students use a function machine to create a rule.

Functions

- The Customers is a task by Illustrative Mathematics where the concept of a function is introduced without an explicit algebraic representation. *You must create an account to view Illustrative Mathematics lessons on OpenUp Resources.*
- Foxes and Rabbits is a task by Illustrative Mathematics where students have to determine whether something is a function or not.
- US Garbage, Version 1 is a task by Illustrative Mathematics where the concept of a function is introduced without an explicit algebraic rule.

Graphing Functions

- How Did I Move? by NCTM Illuminations is a kinesthetic lesson to form a physical interpretation of the slope and *y*-intercept by running across the football field.
- Getting a new iPhone—Which Plan Should I Get? is a task by Yummy Math that has students analyze cell phone plans by graphing functions.
- Thank You, Mother and Father, for All Those Diapers is a task by Yummy Math that has students analyze the cost of diapers by writing equations and graphing functions.

Different Representations of Functions

- Filling the Tank is a task from Grade 8, Georgia Standards of Excellence, Unit 5: Linear Functions that compares functions using different representations. This task can be found on pages 61-72.
- Drinks at the Fair is a task by Yummy Math that has students analyze which mug is the best buy using tables and graphs.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (8.F.1-3)

Independent and Dependent Variables

- Independent and Dependent Variables is a lesson by Virginia Department of Education involving independent and dependent variables.
- Functions: Independent and Dependent Variables by Andy Schwen is a Desmos lesson where students sort independent and dependent variables and match them to the corresponding graphs.

Curriculum and Lessons from Other Sources

- Illustrative Mathematics, Grade 8, Unit 5: Functions and Volume, Lesson 1: Inputs and Outputs, Lesson 2: Introduction to Functions, Lesson 3: Equations for Functions, Lesson 4: Tables, Equations, and Graphs of Functions, Lesson 7: Connecting Representations of Functions, Lesson 8: Linear Functions, Lesson 9: Linear Models, are lessons that pertain to this cluster. *You must create an account to view Illustrative Mathematics lessons on OpenUp Resources.*
- EngageNY, Module 4, Topic A, Lesson 2: Linear and Nonlinear Expressions is a lesson about identifying linear and nonlinear functions.
- EngageNY, Grade 8, Module 5, Topic A, Lesson 1: The Concept of a Function, Lesson 2, Formal Definition of a Function, Lesson 3: Linear Functions and Proportionality, Lesson 4: More Examples of Functions, Lesson 5: Graphs of Functions and Equations, Lesson 6: Graphs of Linear Functions and Rate of Change, Lesson 7: Comparing Linear Functions and Graphs, Lesson 8: Graphs of Simple Nonlinear Functions is a series of lessons for this cluster.
- Unit 4: Functions is a unit from Grade 8 Georgia Standards of Excellence. The entire unit focuses on standards 8.F.1 and 8.F.2.

General Resources

- Arizona 8-High School Progression on Functions
  This cluster is addressed on pages 2-5.
- Coherence Map by Achieve the Core is an interactive website that shows the progression and interconnectedness of the Common Core standards. To be noted, there are some differences between Ohio's Learning Standards and the Common Core, so although the tool is useful, it needs to be checked with respect to Ohio's Learning Standards.
- High School Coherence Map by UnboundED is a map of the relationships among high school mathematics standards. Please note: This resource is aligned to the Common Core not Ohio's Learning Standards. Although the tool is useful, it needs to be checked with respect to Ohio's Learning Standards.

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INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (8.F.1-3)

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STANDARDS	MODEL CURRICULUM (8.F.4-5)
FUNCTIONS **Use functions to model relationships between quantities.** **8.F.4** Construct a function to model a linear relationship between two quantities. Determine the rate of change and initial value of the function from a description of a relationship or from two (x, y) values, including reading these from a table or from a graph. Interpret the rate of change and initial value of a linear function in terms of the situation it models, and in terms of its graph or a table of values. **8.F.5** Describe qualitatively the functional relationship between two quantities by analyzing a graph, e.g., where the function is increasing or decreasing, linear or nonlinear. Sketch a graph that exhibits the qualitative features of a function that has been described verbally.	**Expectations for Learning** Students have experienced interpreting and translating linear functions in different representations. The focus for this cluster is modeling, constructing, and analyzing linear relationships and qualitative graphs. Students should be able to describe properties of linear relationships. These initial understandings provide the foundations for key features that are developed in high school math such as domain, range, function notation, and other types of functions. **ESSENTIAL UNDERSTANDINGS** **Rate of Change** • The rate of change of a function is the slope of its graph. • The slope is a ratio of the change in y-values compared to the change in x-values between two points. • A vertical line has an undefined slope, where y is not a function of x. • The graph of a horizontal line has a slope of zero, i.e., $m = 0$, which is a constant rate of change. • The graph of a line is increasing if it rises from left to right. The slope is positive, i.e., $m > 0$. • The graph of a line is decreasing if it falls from left to right. The slope is negative, i.e., $m < 0$. • The absolute value of slope (m) is related to the steepness of a graph of a line. **Describing and Graphing Functions** • Properties of graphs of linear functions include the following: slope/rate of change; y-intercept/initial value, where the slope is increasing, constant, and decreasing. • Qualitative graphs represent essential elements of a situation in graph form. • Qualitative graphs do not necessarily need to have numerical values. • Linear functions can show proportional relationships $y = mx$ or non-proportional relationships $y = mx + b$ where $b \neq 0$. • The scale of an axis must have equal intervals. • The x-axis and y-axis do not have to have the same scale. *Continued on next page*

STANDARDS	MODEL CURRICULUM (8.F.4-5)
8.F.4-5, continued	**Expectations for Learning, continued** **MATHEMATICAL THINKING** • Generalize concepts based on patterns. • Recognize and use a pattern or structure • Use precise mathematical language. • Represent a concept symbolically. • Explain mathematical reasoning. **INSTRUCTIONAL FOCUS** **Describing and Graphing Functions** • Construct functions in different representations such as equations, graphs, tables, and verbal descriptions. • Identify and analyze whether a function is linear or non-linear. • Sketch a qualitative graph of a function from a verbal description, e.g., increasing, constant, decreasing, linear, not linear. • Analyze a qualitative graph with and without numerical context. • Given a real-world situation, sketch a graph to model the relationship that may include vertical and horizontal lines. • Given a graph of a situation, write a description of the situation that may include vertical and horizontal lines. • Describe the relationship between the inputs and outputs of a function in a qualitative way. • Choose an appropriate scale for each axis; scales should not be limited to 1. • Investigate and recognize that a line that increases from left to right has a positive slope and a line that decreases from left to right has a negative slope. • Investigate and recognize that as the absolute value of the slope increases the line of the graph gets steeper. • Investigate and recognize that the absolute value of a slope greater than one is a steeper line and less than one is a flatter line. • Recognize that in a table the y-intercept is the y-value when x is equal to 0. • Recognize that in a table the x-intercept is the x-value when y is equal to 0. • Explain why the slope of a horizontal line is zero and a vertical line is undefined. *Continued on next page*

STANDARDS	MODEL CURRICULUM (8.F.4-5)
8.F.4-5, continued	**Expectations for Learning, continued** **INSTRUCTIONAL FOCUS, CONTINUED** **Rate of Change** • Determine and interpret the rate of change and initial value of a linear function in context (equation, verbal description, and/or table). • Determine and interpret, and when appropriate approximate, the rate of change and initial value from a graph. • Find the slope of a line given two ordered pairs, a table, or a graph. **Content Elaborations** • Ohio's K-8 Critical Areas of Focus, Grade 8, Number 2, page 52 • Ohio's K-8 Learning Progressions, Functions, page 20 **CONNECTIONS ACROSS STANDARDS** • Graph proportional relationships (8.EE.5). • Derive the equation $y = mx + b$ (8.EE.6). • Interpret the equations of $y = mx + b$ (8.F.3). • Use the equation of a linear model to solve problems (8.SP.3). • Describe the effect of dilations on two-dimensional figures using coordinates (8.G.3).

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (8.F.4-5)

Instructional Strategies

Note: The model curriculum is not a collection of lessons nor a full curriculum; it does not suggest pace, sequence, or amount of time spent on topics. It provides information about a topic related to the standards including ideas for examples, strategies for teaching, common misconceptions, and possible connections between topics. In addition, the emphasis of the model curriculum is on instruction not on assessment; therefore, the examples in the Instructional Strategies section are written for instructional purposes not necessarily as assessment items. Examples may need to be rewritten to accommodate the needs of each individual classroom.

Standards for Mathematical Practice

This cluster focuses on but is not limited to the following practices:

MP.1 *Make sense of problems and persevere in solving them.*

MP.2 *Reason abstractly and quantitatively.*

MP.5 *Use appropriate tools strategically.*

MP.8 *Look for and express regularity in repeated reasoning.*

In Grade 8, students focus on linear equations and functions. Nonlinear functions are used for comparison.

Emphasize key vocabulary. Students should be able to explain what key words mean, e.g., model, interpret, initial value, functional relationship, qualitative, quantitative, linear, non-linear, increasing, decreasing, constant, slope, independent variable, dependent variable etc.

TIP! Use a "word wall" to help reinforce vocabulary.

It is important to note that a graph is a representation of a function and not the function itself. A linear function does not have a slope, but the graph of the function does.

GRAPHING AND WRITING FUNCTIONS

Students should have many opportunities to write linear functions in the form of $y = mx + b$ from a variety of contexts. However, they should also be given equations in different forms and be expected to convert them into slope-intercept form. *Note: Point-slope form is not required.* Students should be able to determine the rate of change and y-intercept from tables, graphs, and verbal descriptions in order to write the equation. Also, they should be able to explain the initial value in context of the problem.

Give students opportunities to gather their own data or graphs in contexts they understand. Students need to measure, collect data, graph data, and look for patterns, then generalize and symbolically represent the patterns. They also need opportunities to draw graphs (with qualitative features based upon experience) representing real-life situations with which they are familiar. Probe student thinking by asking them to determine which input values make sense in the problem situations.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (8.F.4-5)

Use graphing calculators or websites such as Desmos to explore linear and non-linear functions. Provide context as much as possible to build understanding of slope and *y*-intercept in a graph, especially for those patterns that do not start with an initial value of 0. It may also be worthwhile to use technology to explore various families of functions.

TIP!

Students often confuse a recursive rule with an explicit formula for a function. For example, after identifying, in a linear function, that for every increase of 1 in the input there is an increase of 2 in the output, some students will represent the equation as $y = x + 2$ instead of realizing that this means $y = 2x + b$. When tables are constructed with increasing consecutive integers for input values, then the distinction between the recursive and explicit formulas is about whether you are reasoning vertically or horizontally in the table. Both types of reasoning—and both types of formulas—are important for developing proficiency with functions. *Note: Students do not need to understand the terms recursive and explicit formulas.*

Some students may not pay attention to the scale on a graph, assuming that the scale units are always "one." Also, when making axes for a graph, some students may not use equal intervals to create the scale. Provide experiences for students to create and graph data using different scales. In addition, they also need to experience when the scales are different on the two axes, e.g., by 5s on the *x*-axis and perhaps by 100s on the *y*-axis.

TIP!

Some students do not understand that, on a graph, the *x*-axis usually represents the independent variable and the *y*-axis represents the dependent variable; therefore their graphs are reversed. Although this is still mathematically correct, emphasize that this is a convention that makes it easier to communicate.

Writing Comparison Functions

Many students have difficulties with comparison problems. They need to be exposed to these types of problems and have time and collaboration with others to make sense of them.

EXAMPLE

- There are five times as many dogs as cats at the pound. Write an equation to model the situation.
- For every 2 people who bought Apple iPhones, 8 people have bought Android phones. Write an equation to model the situation.

Discussion: This situation is commonly incorrectly written as $5d = c$, which means that there are five times as many cats as there are dogs instead of $d = 5c$. Common mistakes that students make are as follows:

- They incorrectly match the order of the words in the situation to the equation.
- They think that the larger number is placed next to the variable defining the larger group.
- They treat variables as labels.
- They treat variables as a fixed unknown rather than as a variable quantity.
- They do not treat the equal sign as representing equivalence but more like an association.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (8.F.4-5)

Students who are able to correctly write the equation understand the equation as a function. They view the equation as an operation on a variable quantity to produce another equal quantity. To help students reason through such problems have them draw pictures or ask them guiding questions such as "Are there more dogs or cats?" "Does your equation match the situation?" or "What do you have to do to the equation to make both sides equal?"

Students often misinterpret variables as labels. They confuse the notation of 3*m* (three times *m*) with 3m (3 meters).

Understanding Fractions and Writing Two-Variable Equations

Thinking about fractions as multipliers on quantities can be difficult for students. For example, many students have a difficult time explaining the meaning of $\frac{3}{4}x$. (See Model Curriculum 6.EE). Many students also have difficulty with reciprocal reasoning such that if $y = \frac{3}{4}x$, then $x = \frac{4}{3}y$. Those who lack reciprocal reasoning often lack reversibility in equation writing. Use fraction bars, graphs, and tables to help students understand that the equations are equal. See Model Curriculum 6.EE.9 for more information on reciprocal reasoning.

EXAMPLE

Stephen has a cord for his iPod that is some number of feet long. His cord is five times the length of Rebecca's cord. Draw a picture to represent this situation. Write several equivalent equations, if possible, to represent this situation (e.g., equations both in the form of $y =$ and $x =$). Do not forget to define your variables.

Taken from Hackenberg and Lee, 2015

EXAMPLE

Theo has a stack of DVD's some number of cm tall. Sam's stack is $^2/_5$ of that height. Draw a picture to represent the situation. Write an expression to define the height of Sam's stack. Write several equivalent equations, if possible, to represent the situation (e.g., equations both in the form of $y =$ and $x =$). Do not forget to define the variables.

Taken from Hackenberg and Lee, 2015

Graphing Stories

Introduce functions by graphing stories. Have students act out their own stories such as walking forward and backward and stopping slowly and quickly. Before they start, have them make a prediction about what their graph would look like on a distance/time graph. Once they collect their data, have them graph their story and discuss the meaning of a graph. Compare their graphs to their original predictions. For example, students often have difficulty realizing that a horizontal line on a distance/time graph means that an object has stopped. They may also have a difficult time realizing that walking backwards has a negative slope; they may initially think incorrectly that the line goes backwards. See Instructional Tools/Resources for resources on graphing stories.

Use a science probe to help students graph stories.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (8.F.4-5)

EXAMPLE

Represent the following situation by a graph:
Monica has a business contract with Paperclips, her local office supply store. If she buys less than 20 packages of red pens, she pays \$2.79 per package. If she buys between 20 and 40 packages of red pens, they cost \$2.59 per package. If she buys more than 40 packages, they cost \$1.99 per package.

EXAMPLE

Michael wants to take a Taxi from John Glenn International Airport to Franklin Park Conservatory and Botanical Gardens which is about $5\frac{4}{9}$ miles away.

- The initial charge (which includes loading and the first $\frac{1}{9}$ of a mile) is \$3.00.
- Each 60 seconds of wait time is \$0.45.
- Each additional $\frac{2}{9}$ of a mile is \$0.45.
- The surcharge for each trip originating at the airport is \$3.00.
- Sales Tax is 7.5%

a. Draw a distance time graph of the situation. Include possible stops along the way (such as street lights or other unexpected stops). To extend this task, feel free to use a map and even change to route slightly if you wish. This is a good opportunity to use technology such as google maps.
b. Write a story describing your graph.
c. Based on your graph, calculate the cost of Michael's taxi ride.

Discussion: This is an example of a rich task. They could take one day or several days depending on how deep the students want to go with it. They could just simply use a standardized 5.4 mile route where the stops are given or they could "make up" random stops and compare their graph and cost to the other members of their group, or they could use Google maps and map their own route even taking into account real stop lights and stop signs.

Students often incorrectly think that a function only has one type of line, but a function can be piecewise—meaning it can have two or more functions combined. Eighth graders should be exposed to some examples of graphs that are piecewise functions such as a person walking and then stopping and then walking again. *Note: The word piecewise is not required.*

Slope-Intercept Form

Students will need many opportunities and examples to figure out the meaning of $y = mx + b$. What does m mean? What does b mean? They should be able to "see" m and b in graphs, tables, and formulas or equations, and they need to be able to interpret those values in contexts.

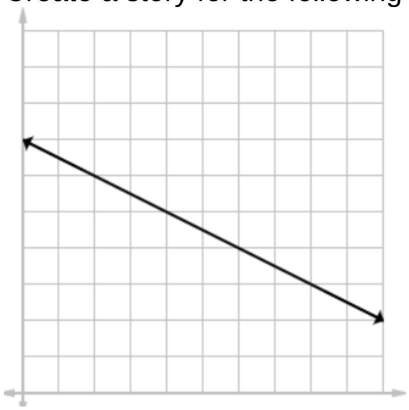
INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (8.F.4-5)

EXAMPLE
Use a function to model the height of a stack of *n* paper cups.

Discussion: The rate of change, *m*, which is the slope of the graph, is the height of the "lip" of the cup: the amount each cup sticks above the lower cup in the stack. The "initial value" in this case is not valid in the context because 0 cups would not have a height, and yet a height of 0 would not fit the equation. Nonetheless, the value of *b* can be interpreted in the context as the height of the "base" of the cup: the height of the whole cup minus its lip.

Provide students with a function in symbolic form $y = mx + b$ and ask them to create a problem situation in words to match the function. Given a graph, have students create a scenario that would fit the graph. Ask students to sort a set of "cards" to match a graphs, tables, equations, and problem situations. Have students explain their reasoning to each other. *Note: Students are not required to use function notation.*

EXAMPLE
Create a story for the following graph.



In the equation $y = mx + b$ many students assume "*b*" must be a whole number. Use examples where *b* is a rational number. Also, in some cases, the initial value is not necessarily the *y*-intercept, so make sure to give students situations where the initial value is not the *y*-intercept.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (8.F.4-5)

EXAMPLE

Monica has 3 cups of lemonade mix which serves 6 people. How many cups of lemonade mix does she need to serve 30 people? Use a graph to find the solution.

Discussion: Draw attention to the fact that the initial value is not the y-intercept.

Students also think that in order to graph an equation, it must be presented in the slope-intercept form: $y = mx + b$. Give students example of equations in other forms to prevent this misconception. In Grade 8, they can graph other forms of equations by creating a table first.

EXAMPLE

A babysitter charges $5 for driving to the house and then an additional $7 an hour.

- **a.** Write an equation to represent the situation in slope-intercept form.
- **b.** Graph the function.
- **c.** Where on the graph does it show how much she gets paid if she works 0 hours? How much does she get paid?
- **d.** Where on the graph does it show how much she gets paid if she works 1 hour? How much does she get paid?
- **e.** Where on the graph does it show how much she gets paid if she works 5 hours? How much does she get paid?
- **f.** How much does she owe if she works –2 hours based on the graph?
- **g.** Does it make sense to work –2 hours? Explain.
- **h.** How should you adjust your graph (if necessary) to represent the actual situation?

Students see the formula $y = mx + b$ as a storage place for m and b (a template for seeing the starting point and how much the next point goes up and over) instead of seeing the relationship between the x and y variables. They think x means "every time" or as a label for the slope value. To prevent this misconception, use real-world situations and focus on the dependency of x and y and the covariant relationship. Here are some examples of relational questions instead of calculation questions:

- What do the x and y represent in this situation?
- What does the slope value of 3 mean in this situation?
- What quantity is represented by $3x$?

Describing Functions

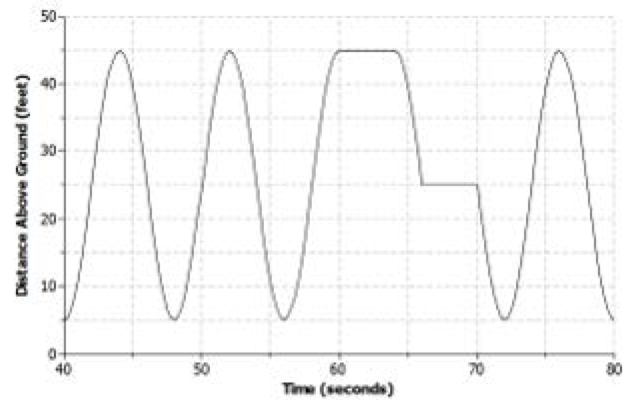
Using a variety of representations of functions, students should be able to classify and describe the function as linear or non-linear, increasing or decreasing. Provide opportunities for students to share their ideas with other students, explain their thinking, and create their own examples for their classmates.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (8.F.4-5)

EXAMPLE

Lamar and his sister continue to ride the Ferris wheel. The graph represents Lamar and his sister's distance above the ground with respect to time during the next 40 seconds of their ride.

- Name one interval where the function is increasing.
- Name one interval where the function is decreasing.
- Is this function linear or non-linear? Explain.
- What could be happening between the interval of time from 60-64 seconds?



EXAMPLE

Create a story for the following graph:

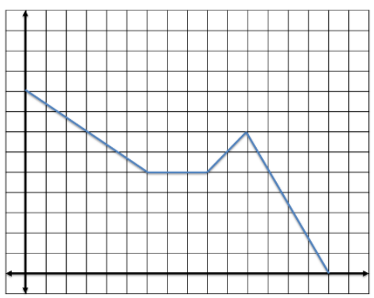


Image from EngageNY Grade 8, Module 6,

RATE OF CHANGE

Explore the rate of change in a function by analyzing its graph. Explain where it is increasing, decreasing, or constant in a contextual situation.

Slope in a linear function is the rate of change between the dependent variable (*y*-value) and the independent variable (*x*-value). Use the slope of the graph and similar triangle arguments to call attention to not just the change in *x* or *y*, but also to the rate of change, which is a ratio of the two. Students should ask themselves "When *x* increases by one, what change is there in *y*?" They should also state directionality (left/right or positive/negative) on the *x*-axis when describing Δx (change in x) instead of simply describing the direction as "over," which is too vague.

Students often confuse $\frac{\Delta y}{\Delta x}$ with the ordered pair notation (*x*, *y*).

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (8.F.4-5)

Students often confuse a horizontal line (slope of zero) with a vertical line (undefined slope) and think that a vertical line has no slope, when the slope is **undefined**.

Students mistakenly believe that a slope of zero is the same as "no slope."

TIP!

When input values are not increasing consecutive integers (e.g., when the input values are decreasing; when some integers are skipped; or when some input values are not integers), some students have more difficulty identifying the pattern and calculating the slope. It is important that all students have experience with such tables, so as to be sure that they do not overgeneralize from the easier examples.

Finding Slope

As a result of the movement of F.IF.6 from Algebra 1/Math 1 to Algebra 2/Math 3, students are expected to be knowledgeable in calculating and interpreting the slope of linear functions by the end of Grade 8. They are also expected to use the slope formula or another preferred strategy to find the slope between two points. However, before the slope formula is introduced students should spend a lot of time exploring slope conceptually.

EXAMPLE

On Friday Elijah drove 63 miles using 3 gallons of gas. On Saturday he drove 164.3 miles and used 5.3 gallons of gas. Use the slope formula to determine his average gas mileage for the two days.

Instructional Tools/Resources

These tools and resources may be helpful for instruction. Before use, they should be evaluated at the local level to determine their appropriateness for instruction.

Manipulatives/Technology

- Graphing calculators
- Graphing software for computers, including dynamic geometry software
- Data-collecting technology, such as motion sensors, thermometers, CBL's, etc.
- Desmos is a graphing utility that allows one to explore the effects of changes in parameters of equations on their graphs. It also includes some lessons.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (8.F.4-5)

Graphing Stories

- Graphing Stories is an Illustrative Math task that investigates the graphs of relationships between quantities using video clips.
- Stories from Graphs a lesson from Cobb Learning where students match distance/time graphs to stories and then create stories to go with other graphs.
- Graphing Stories is a collaboration between Dan Meyer and Buzz Math that has students graph stories based on a real-life video clip.
- Graphing Stories is activity by Desmos where students will graph stories using functions in the coordinate plane.
- Interpreting Distance-Time Graphs is a lesson by the Mathematics Assessment Project where students interpret graphs.
- Party by Inside Mathematics is a Performance Assessment Task that challenges students to demonstrate understanding of the concepts of relations and functions.

Functions

- Chicken and Steak, Variation 1 and Chicken and Steak, Variation 2 are tasks by Illustrative Mathematics that present real-world situations that can be modeled by linear functions.
- Riding by the Library is a task by Illustrative Mathematics that has students draw the graphs of two functions from verbal descriptions.
- Tides, Distance, and Bike Race are tasks by Illustrative Mathematics that has students interpret graphs.
- Vincent's Graph is a task by Inside Mathematics that challenges students to use understanding of functions to interpret and draw graphs.
- Comparing Value for Money: Baseball Jerseys is a task by Mathematics Assessment Project where students compare functions to find the best deal.
- Domino Effect is a task by Mathalicious relating equations and graphs to a real-world situation such as the price of a pizza.
- High School Graduation is a task by Illustrative Mathematics that incorporates some aspects of modeling.
- Distance Across the Channel is a task by Illustrative Mathematics that models a real-world situation.

Slope

- Comparing lines and Linear Equations is a lesson by Mathematics Assessment Project intended to help teachers assess how well students are able to interpret speed as the slope of a linear graph and translate between the equation of a line and its graphical representation.
- Steepness and Fall Hiking is a lesson by Yummy Math where students explore slope using percent grade.
- Steps and Slopes: Measuring the Rise and Run of Steps to Calculate Slope is a lesson by NCTM Illuminations. *NCTM now requires a membership to view their lessons.*
- Ramp It Up by Lynn Miller-Jones is a unit where students explore what is necessary to build a ramp that will satisfy the American Disabilities Act for wheelchair accessibility.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (8.F.4-5)

Slope-Intercept Form

- Slope-Intercept Form is a Youtube video that parodies of Thrift Shop that teaches how to graph lines using slope and y-intercept.
- Y = MX + B is a Youtube video by Glen Hamlen that connects slope-intercept form to the tune of YMCA.
- Delivering the Mail, Assessment Variation is a task by Illustrative Mathematics and Student Achievement Partners that shows an example of a multi-select problem.

Slope-Intercept Form, continued

- Compare Proportional and Nonproportional Relationships by Mobius Math is a worksheet involving slope. It is found on page 2 of the pdf.

Curriculum and Lessons from Other Sources

- EngageNY, Grade 8, Module 6, Topic A, Lesson 1: Modeling Linear Relationships, Lesson 2: Interpreting Rate of Change and Initial Value, Lesson 3: Representations of a Line, Lesson 4: Increasing and Decreasing Functions, Lesson 5: Increasing and Decreasing Functions are various lessons that address this cluster.
- Illustrative Mathematics, Grade 8, Unit 5: Functions and Volume, Lesson 5: More Graphs of Functions, Lesson 6: Even More Graphs of Functions, Lesson 10: Piecewise Linear Functions, Lesson 11: Filling Containers are lessons addressing this cluster. *You must create an account to view Illustrative Mathematics lessons on OpenUp Resources.*
- Exploring Linear Relations is a textbook from the Data-Driven Mathematics series published by the American Statistical Association that connects statistics to functions.

General Resources

- Arizona Grade 8-High School Progression on Functions
  This cluster is addressed on page 6.
- Coherence Map by Achieve the Core is an interactive website that shows the progression and interconnectedness of the Common Core standards. To be noted, there are some differences between Ohio's Learning Standards and the Common Core, so although the tool is useful, it needs to be checked with respect to Ohio's Learning Standards.
- High School Coherence Map by UnboundED is a map of the relationships among high school mathematics standards. Please note: This resource is aligned to the Common Core not Ohio's Learning Standards. Although the tool is useful, it needs to be checked with respect to Ohio's Learning Standards.

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INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (8.F.4-5)

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STANDARDS	MODEL CURRICULUM (8.G.1-5)
GEOMETRY **Understand congruence and similarity using physical models, transparencies, or geometry software.** **8.G.1** Verify experimentally the properties of rotations, reflections, and translations (include examples both with and without coordinates). **a.** Lines are taken to lines, and line segments are taken to line segments of the same length. **b.** Angles are taken to angles of the same measure. **c.** Parallel lines are taken to parallel lines. **8.G.2** Understand that a two-dimensional figure is congruent[G] to another if the second can be obtained from the first by a sequence of rotations, reflections, and translations; given two congruent figures, describe a sequence that exhibits the congruence between them. (Include examples both with and without coordinates.) **8.G.3** Describe the effect of dilations[G], translations, rotations, and reflections on two-dimensional figures using coordinates. *Continued on next page*	**Expectations for Learning** This is the first experience students have with transformations with and without coordinates. Students need many opportunities to explore transformations (rotations, reflections, translations, and dilations) of figures to verify their properties. They should discover that rotations, reflections, and translations preserve angle measures and side lengths. In contrast, they should discover that dilations only preserve angle measure whereas corresponding side lengths are proportional. Students should have many opportunities to explore and discover angle relationships in this cluster using transformations. In particular, students should explore interior and exterior angles of a triangle and the angles formed by parallel lines cut by a transversal. Students justify their findings through informal arguments involving transformations. Proofs of these informal arguments will be a component of high school mathematics. The student understanding of this cluster aligns with a van Hiele Level 2 (Informal Deduction/Abstraction). **ESSENTIAL UNDERSTANDINGS** **Angle Relationships** • Parallel lines cut by a transversal create relationships, either congruent or supplementary, between pairs of angles. • The sum of the measure of the interior angles of a triangle is 180 degrees. • Any exterior angle of a triangle is congruent to the sum of the measures of the two remote interior angles of the triangle. • If two angles in one triangle are congruent to two angles in another, then the triangles are similar. *Continued on next page*

STANDARDS	MODEL CURRICULUM (8.G.1-5)
8.G.4 Understand that a two-dimensional figure is similar to another if the second can be obtained from the first by a sequence of rotations, reflections, translations, and dilations; given two similar two-dimensional figures, describe a sequence that exhibits the similarity between them. (Include examples both with and without coordinates.) **8.G.5** Use informal arguments to establish facts about the angle sum and exterior angle of triangles, about the angles created when parallel lines are cut by a transversal, and the angle-angle criterion for similarity of triangles. *For example, arrange three copies of the same triangle so that the sum of the three angles appears to form a line, and give an argument in terms of transversals why this is so.*	**Expectations for Learning, continued** **ESSENTIAL UNDERSTANDINGS, CONTINUED** **Transformations** • Identify corresponding sides and angles of transformed figures. • Reflections, rotations, and translations preserve angle measures and side lengths. • Two figures are congruent if there is a sequence of reflections, rotations, and translations that maps one figure precisely to the other. • Reflections and rotations change location and orientation. • Translations change only location. • Reflections require a line of reflection. • Rotations require a point of rotation, a degree of rotation, and a direction of rotation. • Translations require distance and direction. • Dilations preserve angle measures while corresponding side lengths are proportional. • Dilations require a center of dilation and a scale factor. • A sequence of transformations, including a dilation that transforms one figure to another, results in figures that are similar. **MATHEMATICAL THINKING** • Explain mathematical reasoning. • Use mathematical vocabulary. • Use informal reasoning. • Make and modify a model to represent mathematical thinking. *Continued on next page*

STANDARDS	MODEL CURRICULUM

8.G.1-5, continued	**Expectations for Learning, continued**
	INSTRUCTIONAL FOCUS
	Transformations
	• Use transformation vocabulary and notation, e.g., image, pre-image, A, A'.
	• Use a variety of resources including physical models, transparencies, or geometry software to explore transformations to verify their properties using examples both with and without coordinates.
	    o Discover that rotations, reflections, and translations preserve angle measures and side lengths.
	    o Discover that dilations only preserve angle measure whereas corresponding side lengths are proportional.
	• Explore the relationships between the coordinates of figures before and after transformations.
	• Describe the effect of transformations on two-dimensional figures using coordinates.
	• Given two similar figures, describe a sequence of transformations that demonstrates the similarity between them using examples both with and without coordinates.
	• Investigate the sum of the interior angles of a triangle using physical models or geometry software.
	
	Angle Relationships
	• Explore the angle relationship between an exterior angle and the two remote interior angles of the triangle.
	• Investigate relationships of pairs of angles created by parallel lines cut by a transversal using transformations.
	
	Content Elaborations
	• Ohio's K-8 Critical Areas of Focus, Grade 8, Number 3, pages 53-54
	• Ohio's K-8 Learning Progressions, 6-8 Geometry, page 21
	
	CONNECTIONS ACROSS STANDARDS
	• Use similar triangles to explain why the slope, *m,* is the slope between any two distinct point on a non-vertical line (8.EE.6).

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (8.G.1-5)

Instructional Strategies

Note: The model curriculum is not a collection of lessons nor a full curriculum; it does not suggest pace, sequence, or amount of time spent on topics. It provides information about a topic related to the standards including ideas for examples, strategies for teaching, common misconceptions, and possible connections between topics. In addition, the emphasis of the model curriculum is on instruction not on assessment; therefore, the examples in the Instructional Strategies section are written for instructional purposes not necessarily as assessment items. Examples may need to be rewritten to accommodate the needs of each individual classroom.

The goal of this cluster is to develop a conceptual understanding of congruence and similarity; it interweaves the relationships of symmetry, transformations, and angle relationships to form these understandings. Transformations should include those done both with and without coordinates.

Standards for Mathematical Practice
This cluster focuses on but is not limited to the following practices:
MP.1 *Make sense of problems and persevere in solving them.*
MP.3 *Construct viable arguments and critique the reasoning of others.*
MP.4 *Model with mathematics.*
MP.5 *Use appropriate tools strategically.*

Students should be able to appropriately label figures, angles, lines, line segments, congruent parts, and images (primes or double primes). Students are expected to use logical thinking, expressed in words using correct terminology. They should also be using informal arguments, which are justifications based on known facts and logical reasoning. **However, they are not expected to use theorems, axioms, postulates or a formal format of proof such as two-column proofs.**

Students should solve mathematical and real-life problems based on understandings related to this cluster. Investigation, discussion, justification of their thinking, and application of their learning will assist them in the more formal learning of geometry standards in high school.

VAN HIELE CONNECTION

In Grade 8 students are expected to move from Level 1(Analysis) to Level 2 (Informal Deduction/Abstraction). In van Hiele Level 1 students use coordinates to examine transformations and recognize, use, and describe properties of transformations. Whereas van Hiele Level 2 can be characterized by the student doing some or all of the following:

- analyzing relationships within and between transformations;
- algebraically expressing transformations: reflections about the *x*- and *y*-axes, and rotations of 45°, 90°, and 180° about the origin *(Note: Grade 8 only requires 90° and 180° rotations about the origin or the vertex.)*
- understanding and justifying how to use perpendicular bisectors of corresponding points to locate the turn center of a rotation;
- understanding how the composition of two reflections can be a translation or a rotation; and/or
- following a rationale if shown proof.

See the van Hiele pdf on ODE's website for more information about van Hiele levels.

TRANSFORMATIONS

Without Coordinates

Transformational geometry is about the effects of rotations, reflections, translations and dilations on figures. This is the first time students work with transformations. For Grade 8, it is sufficient to say that a transformation moves (or maps) point A to another point A'.

Initial work should be presented in such a way that students understand the concept of each type of transformation and the effects that each transformation has on an object before working within the coordinate system. Provide opportunities for students to physically manipulate figures to discover properties of similar and congruent figures involving appropriate manipulatives, such as tracing paper, rulers, Miras, transparencies, and/or dynamic geometric software.

Time should be allowed for students to explore the figures for each step in a series of transformations, e.g., cutting out and tracing. Discussion should include the description of the relationship between the preimage (original figure) and its image(s) in regards to their corresponding parts (length of sides and measure of angles) and the description of the movement, (line of reflection, distance, and direction to be translated, center of rotation, angle of rotation, and the scale factor of dilation).

Although computer software is encouraged to be used in this cluster, it should not be used prematurely. Students need time to develop these geometric concepts with hands-on materials such as transparencies. *Note: Using transparencies will help students see that it is not really the figure that is moving, but it is the plane (transparency) that is moving. The concept of the plane moving will help students with vectors in later mathematics.*

With Coordinates

Work in the coordinate plane follows an intuitive understanding of the transformations and should involve the mapping of various polygons by changing the coordinates using addition, subtraction, and multiplication. For example, when translating, add 3 to x, subtract 4 from y, or any other combinations of changes to x and y.

Students should become fluid in both transforming a figure and describing its transformation given two figures.

CONGRUENCE

In Grade 6 students learned that when two shapes match exactly they have the same area. In Grade 7 they learn that two figures that "match up" or are put on top of each other are the same. In Grade 8, they learn the formal term of congruence and define it by using transformations. Students should also become familiar with the symbol for congruence ($\cong$).

Students should observe and discuss which properties of the polygons remained the same and which properties changed. Understandings should include generalizations about which transformations maintain size or maintain shape, as well as which transformations do not.

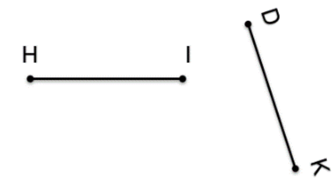
INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (8.G.1-5)

Students should be able to provide a sequence of transformations required to go from a preimage to its image. Provide opportunities for students to discuss the procedure used, whether different procedures can obtain the same results, and if there is a more efficient way that can be used instead. They need to learn to describe transformations using words, numbers, drawings, and expressions.

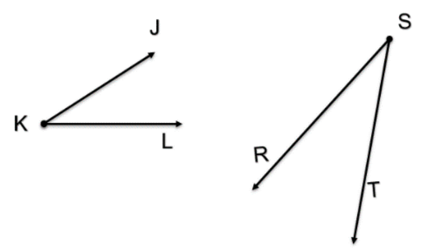
A discussion can be had about the meaning of congruence. Initially one can use the informal definition of congruence being the same size and shape, but the discussion should eventually move toward the definition in 8.G.2 "a two-dimensional figure is congruent to another if the second can be obtained from the first by a sequence of rotations, reflections, and translations." Use the word "mapping" when discussing the overlay of two figures.

EXAMPLE

Prove that segments $\overline{HI}$ and $\overline{DK}$ are the same length without measuring them. Explain your reasoning.



Discussion: Some students will come up with the idea of putting the line segments, angles, and figures on top of each other. Build on that idea to introduce the definition of congruence in terms of rigid motions. Explain that rigid motions preserve the distance and angle measures. Also, continue to emphasize that the students are "mapping" one figure onto the other.



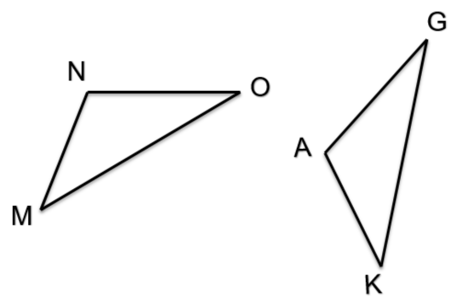
EXAMPLE

Prove that $\angle JKL$ is the same as $\angle RST$ without measuring them.

Discussion: Draw attention to the fact that the lengths of the rays do not affect angle size.

EXAMPLE

Prove that ΔMNO is the same size and shape as ΔKAG.



INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (8.G.1-5)

RIGID MOTIONS

A motion is rigid because the size of the figure does not change. Rigid motions do the following:

- map lines to lines, rays to rays, segments to segments, angles to angles, parallel lines to parallel lines;
- preserve distance; and
- preserve degrees.

Translations, reflections, and rotations preserve side lengths and angle measures, as well as distance from the line of reflection and distance from center of rotation. Therefore, they are rigid motions. Students should discover that congruence between two figures is explained by a sequence of rigid motions which may include translations, reflections, and rotations.

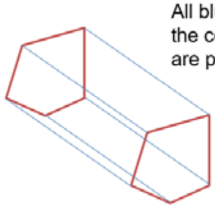
Translation

A translation is often described informally as a "slide." Teachers can use both the terms translation and slide simultaneously to reinforce the vocabulary.

Have students use transparencies or tracing paper to have figures slide onto one another. Emphasize that the plane (transparency or tracing paper) is moving and not the figure.

Practice using correct labeling such as prime notation. Ask students about similarities and differences between the preimage and image. They should notice that the sides and the angles are the same (congruent) since one figure can be superimposed over the other. They should also see that parallel lines within the figure are still parallel. There should also be a discussion about how a translation does not affect the orientation of a figure.

Have students connect the corresponding vertices of a translated figure by drawing lines between the vertices. Ask students what they notice about these lines. Students should realize that the lines connecting the vertices are all congruent and parallel to each other. Have students explain why this is the case.



All blue lines connecting the corresponding vertices are parallel and congruent.

As students solidify their understanding of translations, they should move to translating figures on a coordinate plane. Now, they should be more specific about describing the translation such as the image moved to the left 3 units and up 2 units from the preimage (or subtract 3 from x and add 2 to y). Grade 8 students are not required to use coordinate notation or vectors, but a teacher may introduce these concepts as an extension.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (8.G.1-5)

An image has **Translational Symmetry** if there is a translation (or series of translations) that can map the image to its preimage.

TIP! Connect the concept of translations to patterns on rugs, M.C. Escher's paintings, and tessellations.

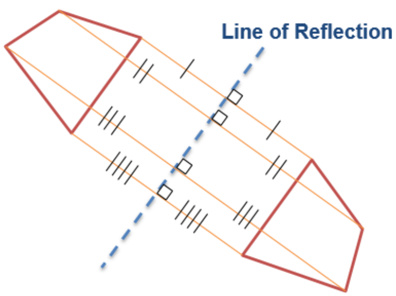
Dynamic Computer Software such as Geometer's Sketchpad or web based tools such as Desmos or Geogebra can be used to help students explore transformations.

Although students do not need to understand vectors in Grade 8, the Geogebra applet Translation of the Plane Along a Vector may be a good visual for translations.

Reflection

A reflection is often described informally as a "flip." Most students have an intuitive understanding of reflections because of their experiences with mirrors. They should realize that the figure retains the same size and shape, but that its orientation is different. Students should have practice literally flipping points, lines, and figures overs a line of reflection using transparencies or tracing paper. Lines of reflection should not be limited to horizontal and vertical lines but diagonal lines should be used as well (such as $y = x$ and $y = -x$). Also, students should have experiences where the line of reflection intersects a figure. In addition, they can explore reflections through constructions which can be shown in Constructing a Line Reflection from Mr. Burke's Constructions.

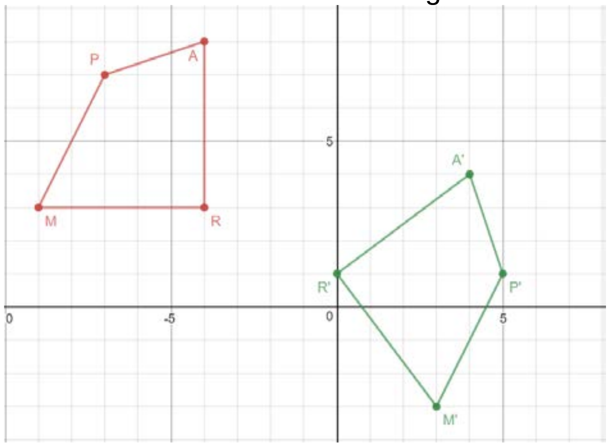
After students have had some experience with reflecting images by literally flipping them, have student connect the vertices of the preimage to its corresponding vertices on the image. Ask students what they notice. They should come to the realization that each vertex is the same distance from the line of reflection as its corresponding vertex and that each connecting line forms a right angle with the line of reflection. (This is easier to visualize when not using regular figures.) Have student justify why that is true. Ask them how they could use that information to draw reflections without actually flipping the figure. Also, students should also discover that a reflection is always its own inverse. Once students have a good grasp of these concepts they should move toward reflections on a coordinate plane.



INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (8.G.1-5)

EXAMPLE

Draw a line of reflection for the figure PARM and its reflected image P'A'R'M'.



Discussion: A task like this reinforces that the line of reflection is an equal distance from corresponding vertices of the two figures. To find the line of reflection students should be able to connect the vertices of the original figure and its image, and then find the midpoints of the connecting lines. (In Grade 8 students do not need to know the midpoint formula, but they can find the midpoint by counting or measuring or as a logical application of finding the distance between two points.) This concept can also be explored using constructions which can be shown in the video Constructing the Line of Reflection from Mr. Burke's Constructions.

Use the Geogebra applet Reflection of the Plane About a Line by Sunil Koswatta to explore reflections.

Symmetry

This would also be a good place to discuss symmetry. As the former standard 4.G.3 (Recognize a line of symmetry for a two-dimensional figure as a line across the figure such that the figure can be folded along the line into matching parts. Identify line-symmetric figures and draw lines of symmetry.) is no longer explicitly in the standards, students may need to explore the concept of symmetry before exploring reflections.

A figure has **reflectional symmetry** (or bilateral symmetry or mirror symmetry) when a line can be drawn so that the figure can be folded exactly on to itself.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (8.G.1-5)

Rotation

A rotation is often described informally as a "turn." For a rotation the following is needed:

- center of rotation,
- direction, and
- number of degrees rotated.

Although students should be made aware that the center of rotation can be anywhere and a figure can be rotated any number of degrees, the expectation for Grade 8 is that rotations are in increments of 90 degrees. The direction is described as clockwise and counterclockwise, and the center of rotation is around the origin, a vertex, or the center of a figure.

Students should use transparencies or tracing paper to rotate figures to once again emphasize that the plane moves not the figure. They should connect rotations to circles. Connections can be made to real-life situations involving rotations such as doing a 180 or 360 on a skateboard.

Use the Geogebra applets Rotation of the Plane about the Point B in the Counterclockwise Direction by an Angle α and Rotation of the Plane about the Point B in the Clockwise Direction by Angle α by Sunil Koswatta to explore rotations.

EXAMPLE

a. Draw a circle with center O, and overlay point P on the circle using tracing paper or a transparency.
b. As you rotate the circle around the center O, what happens to point P?
c. Mark point P' any place on the circle. Using your protractor, measure your angle of rotation. Do not forget to state its direction—clockwise or counterclockwise.
d. What do you know about $\overline{OP}$ and $\overline{OP'}$? Explain.
e. What happens if you rotate P' 180 degrees clockwise from P?
f. What happens if you rotate P' 180 degrees counterclockwise from P?
g. After rotating P' 180 degrees in Part **e**., compare $\overline{PO}$ to $\overline{PP'}$.
h. Create an equivalent angle to $\angle POP'$ that you made in Part **b**. using the line segment $\overline{PO}$ as one of the angle rays. Call the new angle $\angle POR$. What is the difference between $\angle POP'$ and $\angle POR$?

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (8.G.1-5)

Students should explore images that result from rotations of 180 degrees. They should come to the conclusion that —

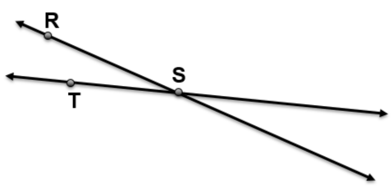
- Any line rotated 180 degrees around a point not on its line will be parallel to the preimage;
- Any line rotated 180 degrees around a point on its line will be the same line as the preimage;
- Any line segment rotated 180 degrees around its midpoint will be the same segment except the vertices (and all other points on the line) are switched.
- Any line segment rotated 180 degrees around its endpoint will create a line segment joining the image and preimage that is twice as long as the preimage.

This sets the foundation for understanding why vertical angles are congruent and understanding the relationships between corresponding angles and alternate interior and exterior angles.

EXAMPLE

a. Graph y = 3 on a coordinate plane.
b. Rotate the line around the origin.
c. How many degrees of rotation will it take for the line to be parallel?
d. Is this true for any line? Explain.

EXAMPLE



a. Rotate the intersecting lines 180° around the point S.
b. What do you notice about points R' and S'?
c. What do you notice about $\angle R'ST'$?
d. What do you notice about $\angle TSR'$ and $\angle RST'$?
e. Is that true for any pair of vertical angles? Justify your thinking.

Students should discover that a rotation is a rigid motion that preserves size, but does not preserve orientation. A figure has **rotational symmetry** if its image can be rotated back onto its original figure. The concept of rotation can also be developed using constructions. A video called Constructing a Rotation from Mr's Bruke's Constructions explains this concept.

Once students become familiar with rotations without a coordinate plane, they should move toward rotating figures on a coordinate plane.

SIMILARITY

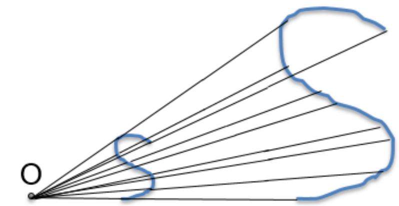
Review that in Grade 7 students learned that similar figures have sides that are proportional and angles that are congruent. Also, in 7th grade students used to say that figures are similar if they have the same shape but different size. Now they will be defining similarity in terms of transformations. Students should also become familiar with the symbol for similarity (~).

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (8.G.1-5)

Dilation

Introduce dilation by discussing a topic such as "How do we double the size of a wiggly curve?"

After much discussion lead students toward assigning an arbitrary point, O, on the plane, and pushing every point on the squiggly line twice as far away from O. Explain to students that this is a dilation. A dilation pushes out (or pulls in) every point of the figure from its center of dilation proportionally by the same amount. This can be easily modeled by pushing in or pulling out an image on an overhead projector or an image drawn on a flashlight. In this case the center of dilation is O, and it can be anywhere on the plane.



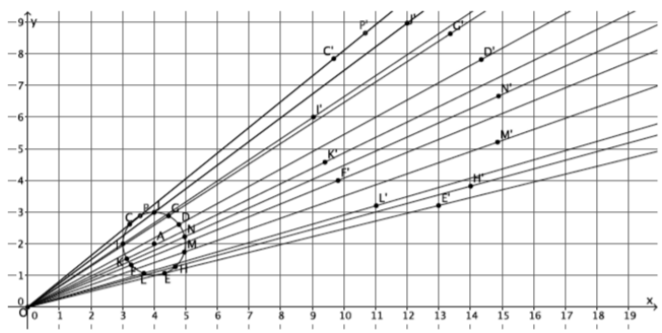
This can also be done by copying and pasting line segments using techonology such as Microsoft Word, Powerpoint, or Smartboard.

A dilation also has a scale factor. When doubling the size of a wiggly curve, the scale factor is 2, but a scale factor could be any number such as ½ or 3. Explain that in Grade 8 scale factors always have to be positive. Discuss what happens to a figure when the scale factor is less than one, compared to when the scale factor greater than one.

Although students should have experiences with the center of dilation being anywhere either inside or outside the figure, the expectation for Grade 8 is that they be proficient using centers of dilations at the origin and at a vertex of an image.

EXAMPLE

- Dilate the circle A with the center of dilation at the origin using a scale factor of $\frac{1}{2}$.
- Dilate the circle A with the center of dilation at the origin using a scale factor of 2.5.



Discussion: This is a challenging problem because a circle has no vertices. Students need to keep in mind the location of the points. One method is by using rays that originate at the center of dilation. Also, they need to think about how many points it would take for a circle to keep its circle shape. EngageNY, Grade 8, Module 3, Topic A, [Lesson 3: Lessons of Dilations] is a lesson that explore the dilations of circles. Another method could be discovering that they can increase or decrease the radius to get a dilated circle.

Dilations are not rigid motions. Although angle measure is preserved, length is not. Therefore, one cannot use dilations to show congruency, only similarity. Have students explore why angle measures are preserved in dilations.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (8.G.1-5)

Connect the idea of dilations to magnification (enlargement). An overhead projector or an image overlaid on a flashlight are good examples to show how light rays magnify an image, so the image can be seen on the wall. Another real-life example of dilations is shrinky dinks (reduction). Dilations can also be connected to microscope views, scale drawings, details on a blueprint, photos, and using a computer to zoom in on an image.

Another way to explore dilations is by using a compass and straight-edge to help students explore dilations. A Youtube video by LongoWTHS Construct a Dilation with a Scale Factor of 2 models this process. EngageNY, Grade 8, Module 3, Topic A, Lesson 2: Properties of Dilations and Lesson 3: Examples of Dilations are additional resources on constructing dilations. Students should also discover that to shrink (reduce) or magnify (enlarge) an image back to its original size, the multiplicative inverse of the scale factor on the transformed image needs to be applied.

Another way to explore constructing dilations without using a compass is to measure the distance from the center of dilation to the vertex. Then, multiply the distance by the scale factor and measure that far out on the line from the center of dilation to the vertex and make a mark there.

TIP!

If students ask if there is such a thing as a negative scale factor they should be encouraged to go online and explore. There are many videos of how to use a negative scale factor, e.g., Transformations: Enlargement by a Negative Scale Factor by mkyou2tube.

Similarity in Terms of Dilations

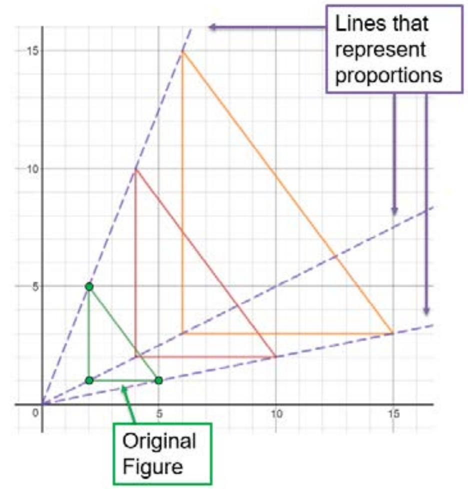
Dilations preserve angle measures and corresponding side lengths change proportionally. Students should discover that similarity between two figures is explained by the sequence of transformations that must include dilations. *Note: Congruence should be recognized as a special case of similarity—a scale factor of 1.*

Students can also view similarity in terms of dilations. If all the corresponding vertices of figures fall on a dilation line ($y = mx$) and the distance between the original vertices and the images' vertices are congruent, then the figures are similar. They should also discover that corresponding line segments are parallel in a dilation. (This informally touches on the Fundamental Theorem of Similarity which students will explore more formally at later grades.)

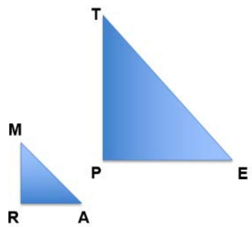
INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (8.G.1-5)

EXAMPLE

a. Graph three or four rays that represent a proportion ($y = mx$) on a coordinate plane.
b. Pick and label a point on each line.
c. Connect the dots to make a triangle or a quadrilateral.
d. Measure the distances from the origin and the vertices of your figure. Then multiply the distance of each segment by a scale factor of 2. Then connect your new points.
e. Measure the distances from the origin and the vertices of your figure. Then multiply the distance of each segment by a scale factor of 3. Then connect your new points.
f. What do you notice about the figures?
g. Explain and justify why your observations occur.



Discussion: Students should come to the realization that the figures are all similar. They should also realize that the corresponding vertices of the similar figures fall on the same ray that starts at the origin. They should also recognize that the center of dilation in this case is the origin for that is where the rays begin.



EXAMPLE

Informally prove whether the two figures are similar or not. Justify your thinking.

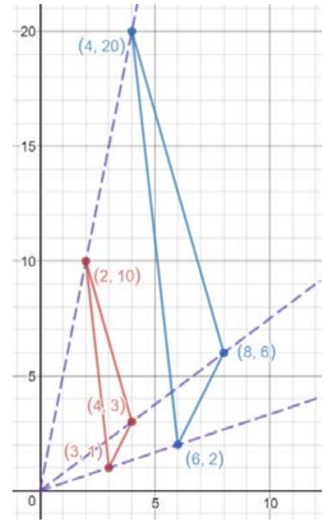
Discussion: Students should come to the realization that these figures are not similar. This is actually an example of using a nonexample to illustrate a concept

Draw students' attention to the *within* and *between* relationships of the figures and tie it back to proportional relationships of similar figures that students learned in Grade 7. For similar figures a *within* ratio compares two sides of one triangle and a *between* ratio compares the corresponding sides of two different triangles. For example there is a relationship *within* the coordinates that lie on the same line but there is also a relationship *between* the corresponding coordinates of the two figures.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (8.G.1-5)

Within Ordered Pairs on the Same Line:
(2, 10) (4, 20) and $y = 5x$;
(4, 3) (8, 6) and $y = \frac{3}{4}x$.
(3,1) (6, 2) and $y = \frac{1}{3}x$.

Between Corresponding Coordinates on the Same Line:
(2, 10) **x 2** = (4, 20)
(4, 3) **x 2** = (8, 6)
(3, 1) **x 2** = (6, 2)



ANGLE RELATIONSHIPS

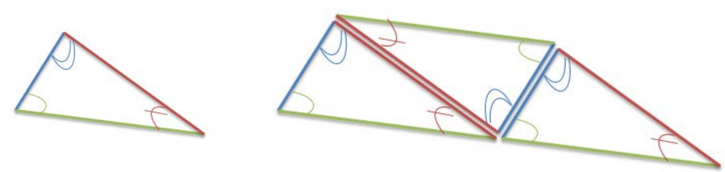
In Grade 7, students develop an understanding of the special relationships of angles and their measures (complementary, supplementary, adjacent, and vertical). Now in 8.G.5 the focus is on learning about the sum of the measures of the interior angles of a triangle and exterior angle of triangles by using transformations. Further, transformations are used to find the relationships of angles formed by parallel lines cut by a transversal. Transformations should be the primary tool for the discovery of angle relationships.

This might be a good time to introduce vocabulary of the types of angles, such as interior, exterior, alternate interior, alternate exterior, corresponding, same side interior, and same side exterior. Students are expected to recognize but not memorize this vocabulary.

Interior and Exterior Angles of a Triangle

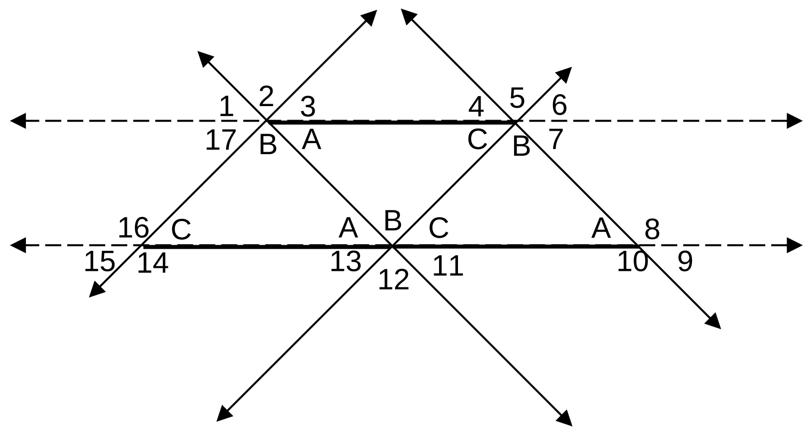
In Grade 7 students should have had some practice exploring that the sum of the angles inside a triangle equal 180 degrees. Now students use transformations to prove it.

Students can create a triangle and use rotations and transformations to line up all the angles to prove that the sum of the interior angles of a triangle equals 180 degrees. They need to be able to demonstrate and explain why the sum of the interior angles equals 180 degrees.



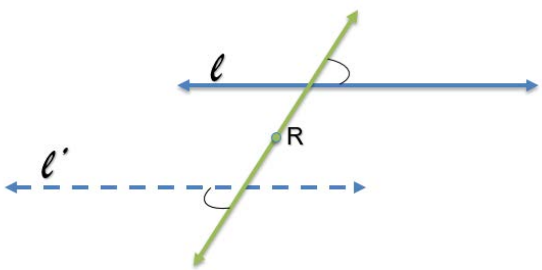
INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (8.G.1-5)

Students should build on this activity to explore exterior angle relationships in triangles. They can also extend this model to explorations involving other parallel lines, angles, and parallelograms formed. Students should be able to explain why two angles in a triangle have to be less than 180 degrees.



Angle-Angle (AA) Criterion for Similarity

Investigations should lead to the Angle-Angle criterion for similar triangles. For instance, groups of students should explore two different triangles with one, two, and three given angle measurements. Students observe and describe the relationship of the resulting triangles. As a class, conjectures lead to the generalization of the Angle-Angle criterion.



Relationships Between Angles Created by Parallel Lines Cut by a Transversal

Students can explore transforming line ℓ to create a parallel line ℓ'. They can do this by rotating line ℓ 180 degrees around point R. Then they can then map angle measures from line ℓ to line ℓ'. This can be used to prove alternate exterior angles are congruent. *Note: A transversal has no special properties; its only property is that it intersects at least two other lines.*

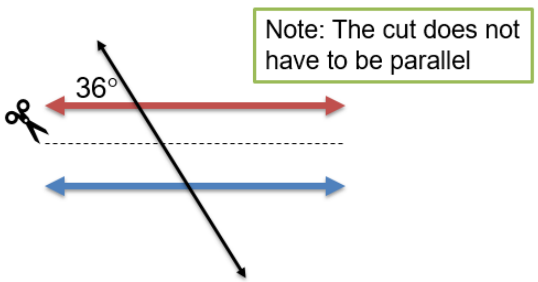
Students can also use transformations to prove the vertical angle congruency that they learned in 7th grade. Using vertical angles and alternate exterior angles, they can prove alternate interior angles and corresponding angles are congruent and can also reason through the converse.

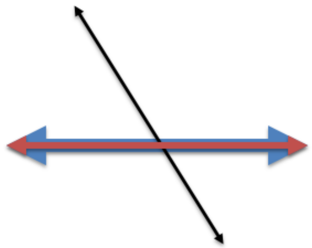
INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (8.G.1-5)

EXAMPLE

What information can be obtained by cutting between the intersections of two parallel lines and transforming one onto the other?

a. Find as many missing angle measures as you can.
b. Find and label as many pairs of congruent angles as you can. Justify why they are congruent using transformational arguments.





EXAMPLE

Given two lines that are cut by a transversal, explain why the lines are parallel if their corresponding angles at the point of intersection with the transversal are congruent. Justify your thinking.

Instructional Tools/Resources

These tools and resources may be helpful for instruction. Before use, they should be evaluated at the local level to determine their appropriateness for instruction.

Manipulatives/Technology

- pattern blocks or shape sets
- mirrors - Miras
- geometry software like Cabri Jr, GeoGebra or Desmos
- graphing calculators
- grid paper
- patty paper (tracing paper)
- transparencies

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (8.G.1-5)

Translations

- Translations by Andrew Stadel is a Desmos task that explores translations.
- Polygraph: Translations by Didi Andres is a Desmos activity about translations.

Reflections

- Point Reflection is a task by Illustrative Mathematics that has students reflect a single point.
- Reflecting Reflections is a task by Illustrative Mathematics that has students experiment with reflections.

Rotations

- Working with Rotations by J.J. Martinez is a Desmos activity that introduces rotations.
- Rotation by Math is Fun includes an applet that allows students to rotate a figure.

Dilations

- Dilations by Andrew Stadel is a Desmos task that explores dilations.
- Working with Dilations by Caleb Rothe is a Desmos activity that introduces dilations.
- Arts Impact—Art-Infused Institute Lesson Plan is an lesson that connects perspective and dilations.
- Transformations-Enlargements is an applet designed to explore what happens when applying dilations.

Series of Transformations

- Polygraph: Transformations by Mike Waechter is a Desmos activity about Transformations.
- Transformation Golf: Rigid Motion is a Desmos activity that has students use one or more transformation to transform the pre-image to the image.
- Des-Patterns is a Desmos activity where students will practice writing coordinate rules to transform figures to complete patterns.
- M.C. Escher 16 Facsimile Prints by the M.C. Escher Foundation is a site with M.C. Escher images that teachers could use to connect art to transformations.
- Transmographer 2 in an applet by Shodor where students can apply transformations to polygons.
- Linear Transformations is an applet by Lauren K. Williams form Mercyhurst University that allows students to explore different transformations.
- Flip-n-Slide: Exploring Transformations through Modeling and Computer Games is a lesson by NCTM Illuminations where students explore reflections, translations, and rotations. *NCTM now requires a membership to view their lessons.*

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (8.G.1-5)

Congruence and Similarity

- Triangle Congruence with Coordinates is a task by Illustrative Mathematics that has students explore rigid motions and congruence. *You must create an account to view Illustrative Mathematics lessons on OpenUp Resources.*
- Illustrating the Standards for Mathematical Practice: Congruence and Similarity by National Council of Supervisor's of Mathematics (NCSM) has teachers examine congruence and similarity through transformations.
- Illustrating the Standards for Mathematical Practice: Similarity, Slope, and Lines has teachers unpack the connections between similarity, slope, and graphs of linear functions.
- Different Areas is a task by Illustrative Mathematics that connects similarity and slope.

Angle Relationships

- Polygraph: Angle Relationships by mathycathy is a Desmos task that has students practice using angle vocabulary.
- Lines, Transversals, and Angles is a Desmos activity that reviews angle relationships formed by a transversal.

Curriculum and Lessons from Other Sources

- EngageNY, Grade 8, Module 2, Topic A, Lesson 1: Why Move Things Around?, Lesson 4: Definition of Reflection and Basic Properties, Lesson 5: Definition of Rotation and Basic Properties, Lesson 6: Rotations of 180 Degrees are lessons that pertain to this cluster.
- EngageNY, Grade 8, Module 2, Topic B, Lesson 12: Angles Associated with Parallel Lines, Lesson 13: Angle Sum of a Triangle, Lesson 14: More on the Angles of a Triangle are lessons that pertain to this cluster.
- EngageNY, Grade 8, Module 3, Topic A, Lesson 1: What Lies Behind "Same Shape"?, Lesson 2: Properties of Dilations, Lesson 3: Examples of Dilations, Lesson 4: Fundamental Theorem of Similarity, Lesson 5: First Consequences of FTS, Lesson 6: Dilations on the Coordinate Plane, Lesson 7: Informal Proofs of Properties of Dilations are lessons that pertain to this cluster.
- EngageNY, Grade 8, Module 3, Topic B, Lesson 8: Similarity, Lesson 9: Basic Properties of Similarity, Lesson 10: Informal Proof of AA Criterion for Similarity, Lesson 11: More About Similar Triangles, Lesson 12: Modeling Using Similarity are lessons that pertain to this cluster.
- Illustrative Mathematics, Grade 8, Unit 1: Rigid Transformations and Congruence has many lessons that pertain to this cluster. *You must create an account to view Illustrative Mathematics lessons on OpenUp Resources.*
- Illustrative Mathematics, Grade 8, Unit 2: Dilations, Similarity, and Introducing Slope, Lesson 1: Projecting and Scaling, Lesson 2: Circular Grid, Lesson 3: Dilations with No Grid, Lesson 4: Dilations on a Square, Lesson 5: More Dilations, Lesson 6: Similarity, Lesson 7: Similar Polygons, Lesson 8: Similar Triangles, Lesson 9: Side Length Quotients in Similar Triangles are lessons that pertain to this cluster. *You must create an account to view Illustrative Mathematics lessons on OpenUp Resources.*
- 2.3 Constructing Rotations, 2.1 Constructing Dilations, 2.2 Constructing Reflections, 2.5 Identifying a Series and Determining Congruence or Similarity are worksheets from Charleston Community Unit School District, Illinois that explores transformations.
- Course 3 Online is a website from Henrico County Public Schools, Virginia that has many resources for transformations.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (8.G.1-5)

General Resources

- Arizona High School Progression on Geometry
  This cluster is addressed on pages 2-3, 9-10.
- Coherence Map by Achieve the Core is an interactive website that shows the progression and interconnectedness of the Common Core standards. To be noted, there are some differences between Ohio's Learning Standards and the Common Core, so although the tool is useful, it needs to be checked with respect to Ohio's Learning Standards.
- High School Coherence Map by UnboundED is a map of the relationships among high school mathematics standards. Please note: This resource is aligned to the Common Core not Ohio's Learning Standards. Although the tool is useful, it needs to be checked with respect to Ohio's Learning Standards.
- van Hiele Model of Geometric Thinking is a pdf created by ODE that summarized the van Hiele levels.

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INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (8.G.1-5)

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- Wu, H. (2013). Teaching geometry according the common core standards. Retrieved from https://math.berkeley.edu/~wu/Progressions_Geometry.pdf
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<table>
<tr><th>STANDARDS</th><th>MODEL CURRICULUM (8.G.6-8)</th></tr>
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GEOMETRY

Understand and apply the Pythagorean Theorem.

8.G.6 Analyze and justify an informal proof of the Pythagorean Theorem and its converse.

8.G.7 Apply the Pythagorean Theorem to determine unknown side lengths in right triangles in real-world and mathematical problems in two- and three-dimensions.

8.G.8 Apply the Pythagorean Theorem to find the distance between two points in a coordinate system.

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Expectations for Learning

This is the introduction to the Pythagorean Theorem. It is the first time students are exposed to a conditional as a logical structure in relation to the Pythagorean Theorem and its converse.

Using manipulatives and drawings, students understand the Pythagorean Theorem and its converse. Students will learn in this cluster that in a right triangle the sum of the squares of the legs is equal to the square of the hypotenuse. It is imperative that students are fluent in applying the Pythagorean Theorem and its converse. These ideas will be developed in high school to include trigonometry and analytical geometry. The use of the converse of the Pythagorean theorem to determine whether a triangle is acute or obtuse is also explored in high school geometry. Informal proofs are introduced in eighth grade; formal proofs will begin in high school. Students will be able to apply the Pythagorean Theorem to the coordinate plane, in three-dimensional representations, and other real-world applications.

The student understanding of this cluster aligns with a van Hiele Level 2. (Informal Deduction/Abstraction).

ESSENTIAL UNDERSTANDINGS

- Side lengths need not be represented by rational numbers.
- The Pythagorean Theorem states that in a right triangle the sum of the squares of the lengths of the legs is equal to the square of the length of the hypotenuse.
- The Pythagorean Theorem is represented symbolically by $(leg\ a)^2 + (leg\ b)^2 = hypotenuse^2$
- The Pythagorean Theorem only applies to right triangles.
- The hypotenuse is the longest side of a right triangle and opposite the right angle.
- The legs of a right triangle are perpendicular.
- The distance between two non-vertical or non-horizontal points in the coordinate plane can be determined by creating a right triangle with vertical and horizontal legs and applying the Pythagorean Theorem.

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STANDARDS	MODEL CURRICULUM (8.G.6-8)
8.G.6-8, continued	**Expectations for Learning, continued** **ESSENTIAL UNDERSTANDINGS, CONTINUED** • Converse is the reverse order of a hypothesis. • The converse of the Pythagorean Theorem states that if the sum of the squares of the legs is equal to the square of the hypotenuse then the triangle is a right triangle • The converse is used to determine whether or not a triangle is a right triangle • The converse of the Pythagorean Theorem works because of the uniqueness of a triangle. **MATHEMATICAL THINKING** • Use precise mathematical language • Draw a picture or create a model to make sense of a problem • Determine reasonableness of the results. • Solve real-world and mathematical problems accurately. • Describe mathematical reasoning using accurate mathematical vocabulary. **INSTRUCTIONAL FOCUS** • Identify the legs and hypotenuse of a right triangle. • Investigate and generalize the relationship between the areas of the squares created using the side lengths of a triangle. • Generalize an informal justification of the Pythagorean Theorem and its converse. • Use the Pythagorean Theorem to determine if a triangle is a right triangle. • Use the Pythagorean Theorem to find a missing side length of a right triangle. • Plot right triangles on a coordinate plane to investigate the relationship of side lengths. • Recognize that the distance between two coordinate points can be the length of the hypotenuse of a right triangle. • Apply the Pythagorean Theorem to find the distance between two points in the coordinate plane, • Model real-world problems including lengths that cannot be directly measured. *Continued on next page*

STANDARDS	MODEL CURRICULUM (8.G.6-8)
8.G.6-8, continued	**Content Elaborations** • Ohio's K-8 Critical Areas of Focus, Grade 8, Number 3, pages 53-54 • Ohio's K-8 Learning Progressions, 6-8 Geometry, page 21 **CONNECTIONS ACROSS STANDARDS** • Students will use rational and irrational numbers (8.NS.1, 2). • Students will use square roots (8.EE.2).

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (8.G.6-8)

Instructional Strategies

Note: The model curriculum is not a collection of lessons nor a full curriculum; it does not suggest pace, sequence, or amount of time spent on topics. It provides information about a topic related to the standards including ideas for examples, strategies for teaching, common misconceptions, and possible connections between topics. In addition, the emphasis of the model curriculum is on instruction not on assessment; therefore, the examples in the Instructional Strategies section are written for instructional purposes not necessarily as assessment items. Examples may need to be rewritten to accommodate the needs of each individual classroom.

Standards for Mathematical Practice

This cluster focuses on but is not limited to the following practices:

MP.1 *Make sense of problems and persevere in solving them.*

MP.2 *Reason abstractly and quantitatively.*

MP.3 *Construct viable arguments and critique the reasoning of others.*

MP.4 *Model with mathematics.*

MP.6 *Attend to precision.*

MP.8 *Look for and express repeated reasoning.*

Students should understand the Pythagorean theorem as an area relationship between the sum of the squares on the lengths legs and the square on the length of the hypotenuse. This can be represented as (leg a)^2 + (leg b)^2 = hypotenuse^2. The Pythagorean Theorem only applies to right triangles.

TIP! Exclusively using $a^2 + b^2 = c^2$ frequently leads to student errors in identifying the parts of the triangle. Use words like leg a, leg b, and hypotenuse c.

VAN HIELE CONNECTION

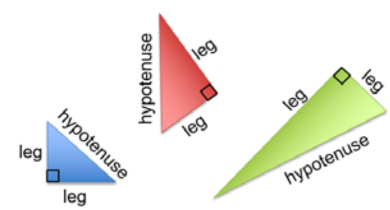
Van Hiele Level 2 can be characterized by the student doing some or all of the following:

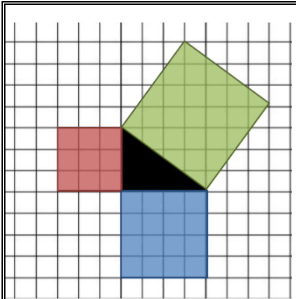
- noticing relationships within and between properties and figures;
- forming meaningful definitions;
- justifying reasoning using informal arguments;
- drawing logical maps and diagrams; and/or
- not yet understanding formal deduction.

See the van Hiele pdf on ODE's website for more information about van Hiele levels.

RIGHT TRIANGLES

A right triangle is a triangle that contains a right angle. The two shorter side lengths adjacent to the right angle are called legs and the longest side opposite from the right angle is called the hypotenuse. It is important for students to see right triangles in different orientations.





PROVING THE PYTHAGOREAN THEOREM

Previous understanding (such as the sum of two side measures in a triangle is greater than the third side measure and the sum of the angles in a triangle in addition to the area of squares) is furthered by the introduction of unique qualities of right triangles. Students should be given the opportunity to explore right triangles to determine the relationships between the measures of the legs and the measure of the hypotenuse. Experiences should involve using square grid paper to draw right triangles from given measures and representing and computing the areas of the squares on each side. Students can physically cut the squares on the legs apart and rearrange them, so they fit on the square along the hypotenuse. Data should be recorded in tables, allowing for students to conjecture about the relationship among the areas. An example of an area model that students can cut out to prove the Pythagorean Theorem is Proof of Pythagoras's Theorem by Math Is Good for You!

EXAMPLE

(Give students various triangles drawn on grid paper, so they can discover the relationships between the legs and the hypotenuse. Although a majority of the triangles should be right triangles include one or two non-right triangles for contrast as well.)

Red Triangle

	Length of Side	Area of Square on the Side
Side a		
Side b		
Side c		

Blue Triangle

	Length of Side	Area of Square on the Side
Side a		
Side b		
Side c		

Green Triangle

	Length of Side	Area of Square on the Side
Side a		
Side b		
Side c		

Orange Triangle

	Length of Side	Area of Square on the Side
Side a		
Side b		
Side c		

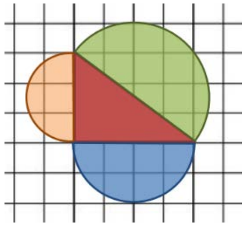
- Find the lengths of the sides of each triangle and the area of the square formed by the side of the triangle. Record your information in the table.
- What patterns do you notice?
- Does your pattern work for any triangle or just for right triangles? Explain.

Discussion: Students should come to the conclusion that the sum of the areas of the squares along the legs equals the area of the square along the hypotenuse. Also include triangles that are non-right triangles to prevent the misconception that the theorem applies to all triangles.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (8.G.6-8)

EXAMPLE (EXTENSION)

Prove that (leg a)^2 + (leg b)^2 = hypotenuse^2 using the area of semi-circles that are formed along the sides of a right triangle.



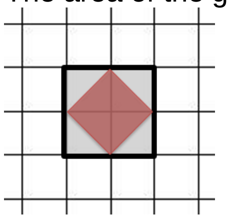
Discussion: Other shapes in addition to squares can also be used to prove the Pythagorean Theorem such as semi-circles, rectangles, or triangles.

Students should form a conjecture, and then explain and discuss their findings. Finally, the Pythagorean Theorem should be introduced and explained as the pattern they have explored.

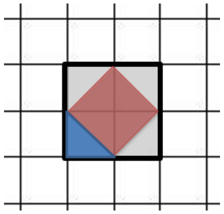
Time should be spent analyzing several proofs of the Pythagorean Theorem to develop a beginning sense of the process of deductive reasoning and the significance of a theorem. Students should be able to justify an informal proof of the Pythagorean Theorem or its converse. However, avoid algebra involving the expansion of $(a + b)^2$ since that is above the expectation of Grade 8.

EXAMPLE

The area of the gray square is four square units.

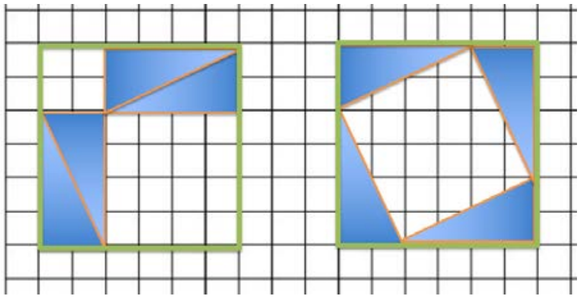


- What is the area of the red square?
- What is the side length of the red square?
- Notice each corner makes an isosceles right triangle (blue) with side lengths of 1.
- Therefore, what is the hypotenuse of a right triangle with side lengths of 1?



Discussion: Students should connect their knowledge that the side length of a square is the square root of the area (See 8.EE.2 for more information.) Therefore if the area of the red square is 2, then the side length of the square would have to be $\sqrt{2}$. So the right triangle formed on each corner would have legs that equal 1 unit and a hypotenuse that equals $\sqrt{2}$ units. This should then be extended to squares of other dimensions. After doing many example students should start to see that each time an isosceles right triangle is formed where the sum of the squared leg length equals the square root of the hypotenuse.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (8.G.6-8)



EXAMPLE

Explain the following proof of the Pythagorean Theorem.

Discussion: See Pythagoras in 60 Seconds by Antics Animation or Pythagoras Without Words by Tipping Point Math or Pythagoras Theorem-Learn Without Memorizing by BYJU's for more information about how to solve this proof. See the instructional resources/tools section for more resources using informal proofs of the Pythagorean Theorem.

Converse of the Pythagorean Theorem

Previously, students have discovered that not every combination of side lengths will create a triangle. Now they need to explore situations that involve the Pythagorean Theorem to test whether or not side lengths represent right triangles. This is an opportunity to remind students that the longest side is the only possibility for the hypotenuse. Students should be able to explain why a triangle is or is not a right triangle using the converse of the Pythagorean Theorem. This might be an opportunity for students to explore Pythagorean triples.

Applying the Pythagorean Theorem

Students can apply the Pythagorean Theorem to real-world situations involving two- and three-dimensions. Some examples of this may include designing roofs, ramp dimensions, etc. Students should sketch right triangles to model real-world situations. Challenge students to identify additional ways that the Pythagorean Theorem is or can be used in real-world situations or mathematical problems, such as finding the height of something that is difficult to physically measure, or the right triangle formed by the diagonal of a prism.

EXAMPLE

Jake wants to clean the gutters. If his house is 25 feet tall, and he has a 28 foot ladder, how far does he need to place the ladder from his house?

TIP! Students frequently neglect to apply the Pythagorean Theorem and instead add the leg lengths to find the length of the hypotenuse. Another frequent mistake is that students do not take the square root of their answer. Connecting the Pythagorean Theorem to pictures with squares could help students understand the formula. Also, students incorrectly assume the missing side is always the hypotenuse. It is important to make sure that they are given a variety of problems where different sides lengths are missing.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (8.G.6-8)

Students oftentimes round values prior to final calculations, which may result in inaccurate solutions especially when working with the converse of the Pythagorean. Advise against intermediate rounding.

TIP!

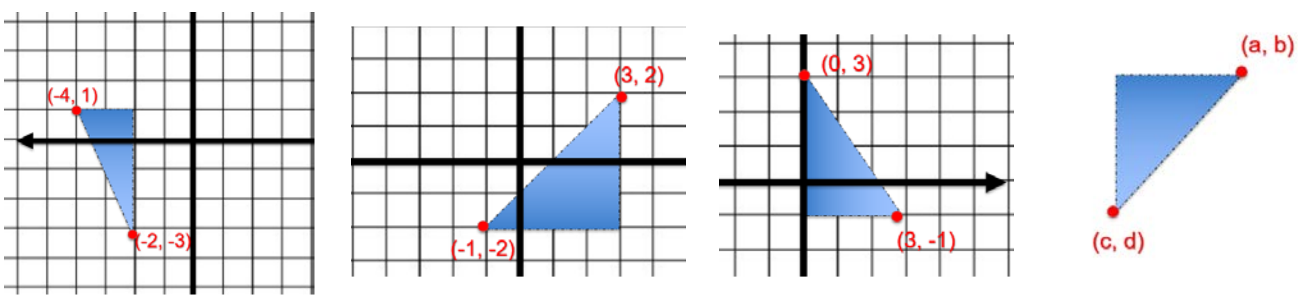
In construction, the Pythagorean Theorem is used to make sure buildings are square. The application of the theorem is used to accurately check ceilings, build connecting walls and staircases, and lay foundations to ensure 90° angles. As students are working with the Pythagorean Theorem consider creating opportunities for students to apply it to real-world situations that could arise in building homes and businesses. *For Example, the Harrell Construction Company is building a house and needs to make sure the building is up to code for their inspection. They need to check to ensure that the intersection of the adjoining walls forms a 90° angle. The length of the right wall measures 12 ft. and the adjacent wall measures 9ft. what must the distance between the two walls be to pass inspection?*

FINDING THE DISTANCE BETWEEN TWO POINTS

The Pythagorean Theorem can be used to find missing side lengths in a right triangle with or without coordinates. The Pythagorean Theorem should be applied to finding the lengths of segments on a coordinate grid, especially those segments that do not follow the vertical or horizontal lines, as a means of discussing the determination of distances between points. Students in Grade 8 should extend the use of the Pythagorean Theorem to find the distance between two points. Understanding how to determine distance by using vertical and horizontal lengths as legs of a right triangle is more important than deriving or memorizing a formula.

An extension could be having students understand how to find the midpoint as well as there is an intuitive connection to the finding the distance between two points.

EXAMPLE



- Use the Pythagorean Theorem to find the distance between (–4, 1) and (–2, –3).
- Use the Pythagorean Theorem to find the distance between (3, 2) and (–1, –2).
- Use the Pythagorean Theorem to find the distance between (0, 3) and (3, –1).
- Use the Pythagorean Theorem to find the distance between (a, b) and (c, d).
- How could you use the Pythagorean Theorem to find the distance between any two points? Explain.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (8.G.6-8)

Instructional Tools/Resources

These tools and resources may be helpful for instruction. Before use, they should be evaluated at the local level to determine their appropriateness for instruction.

Proofs of the Pythagorean Theorem

- Proof Without Words by Steve Phelps is a Geogebra activity with 30 questions that model the Pythagorean Theorem visually.
- Corner to Corner is a two-lesson unit by NCTM Illuminations. In the first lesson students use patter recognition to determine the length of a diagonal of a square is $\sqrt{2}$ using a baseball diamond. In the second lessons, students discover the Pythagorean Theorem by examining rectangle patterns. *This lesson requires a subscription to NCTM.*
- 6.5 Understanding the Pythagorean Relationship by NCTM Illuminations is an applet that shows a proof without words for the Pythagorean Theorem.
- Pythagorean Theorem is a lesson by Yummy Math that incorporates Geobebra and helps students discover the Pythagorean Theorem.
- Pythagorean Theorem by Rori Abernethy is a Desmos activity where students learn about the Pythagorean Theorem.
- Geogebra has many interactive activities about the Pythagorean Theorem by John Golden and Pythagoras by Stephen Jull and Proof Without Words by Steve Phelps.
- Pythagorean Review is an applet by NCTM Illuminations that explores the Pythagorean Theorem.
- Pythagorean Theorem is a lesson by Shodor that includes an applet and guiding questions about the Pythagorean Theorem.

Constructions

- Maya Constructions is a lesson by NCTM Illuminations where students use a compass and straight edge to construct rectangles. Then they find the diagonal of the square using the Pythagorean Theorem. *NCTM now requires a membership to view their lessons.*

Converse of the Pythagorean Theorem

- Converse of the Pythagorean Theorem is a task by Illustrative Mathematics that emphasizes the difference between the Pythagorean Theorem and its converse.

Application Using the Pythagorean Theorem

- Pipefitter and the Pythagorean Theorem is a video by PBS Learning Media that shows an apprentice pipefitter explain how he uses math and science in his every day work.
- Wrapping Presents on the Diagonal is a lesson by Yummy Math that discusses how to wrap presents more efficiently. It is a problem solving lesson that incorporates surface area and the Pythagorean Theorem.
- HDTV Surprise is a lesson by Yummy Math that has students apply the Pythagorean Theorem to find the length of a TV.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (8.G.6-8)

Application Using the Pythagorean Theorem, continued

- Fish Tale, Go Figure is a lesson by Yummy Math that has students apply the Pythagorean Theorem to find diagonal of a fish tank.
- Watson Save is a lesson by Yummy Math where students analyze a football play using the Pythagorean Theorem.
- Two Triangles' Area is a task by Illustrative Mathematics where students need to draw auxiliary lines and apply the Pythagorean Theorem to find the area of the triangles.
- Area of a Trapezoid is a task by Illustrative Mathematics where students need to use the Pythagorean theorem to find an unknown side length of a trapezoid.
- Shapes with the Same Perimeter is a task by Illustrative Mathematics where students use the Pythagorean Theorem to find the area of various shapes with the same perimeter.
- Spider Box is a task by Illustrative Mathematics that has students apply the Pythagorean Theorem to find diagonals of a spider box.
- Applications of the Pythagorean Theorem-Magnetism is a NASA activity where students apply the Pythagorean Theorem to three-dimensions. This could be used as an extension activity.

Finding the Distance Between Two Points

- Maritime Mysteries by KET has students determining the distances between points on a coordinate grid using the Pythagorean theorem and sea animals.
- Introduction to the Distance Formula by Lee-Anne Patterson is a Desmos Activity where students make sense of the distance formula.
- Finding Isosceles Triangles is a task by Illustrative Mathematics where students use the distance formula or Pythagorean Theorem to analyze triangles.
- Exploring Length with Geoboards is a Desmos activity where students explore length and use the Pythagorean Theorem.
- Finding the Shortest Route: A Schoolyard Problem is a task by Mathematics Assessment Project where students apply the Pythagorean Theorem to solve problems in the real-world.
- Tilted Squares is an activity Enriching Mathematics where students use the Pythagorean Theorem or the Distance Formula to find the area of tilted squares.
- Garden Shed is an activity Enriching Mathematics where students use the Pythagorean Theorem to find the lengths of beams.

Curriculum and Lessons from Other Sources

- Georgia Standards of Excellence, Grade 8, Unit 3: Geometric Applications of Exponents has many tasks that address this cluster. These tasks can be found on pages 113-114.
- EngageNY Grade 8, Module 2, Topic D, Lesson 15: Informal Proof of the Pythagorean Theorem, Lesson 16: Applications of the Pythagorean Theorem are lessons that pertain to this cluster.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (8.G.6-8)

Curriculum and Lessons from Other Sources, continued

- EngageNY Grade 8, Module 3, Topic C, Lesson 13: Proof of the Pythagorean Theorem, and Lesson 14: The Converse of the Pythagorean Theorem are lessons that pertain to this cluster.
- EngageNY Grade 8, Module 7, Topic C, Lesson 15: The Pythagorean Theorem Revisited, Lesson 16: Converse of the Pythagorean Theorem, Lesson 17: Distance on the Coordinate Plane, Lesson 18: Applications of the Pythagorean Theorem are lessons that pertain to this cluster.
- Illustrative Mathematics, Grade 8, Unit 8: Pythagorean Theorem and Irrational Numbers, Lesson 6: Finding Side Lengths of Triangles, Lesson 7: A Proof of the Pythagorean Theorem, Lesson 8: Finding Unknown Side Lengths, Lesson 9: The Converse, Lesson 10: Applications of the Pythagorean Theorem, Lesson 11: Finding Distances in the Coordinate Plane are lessons that pertain to this cluster. *You must create an account to view Illustrative Mathematics lessons on OpenUp Resources.*

General Resources

- Arizona 7-High School Progression on Geometry

  This cluster is addressed on pages 5, 11-12.
- Coherence Map by Achieve the Core is an interactive website that shows the progression and interconnectedness of the Common Core standards. To be noted, there are some differences between Ohio's Learning Standards and the Common Core, so although the tool is useful, it needs to be checked with respect to Ohio's Learning Standards.
- High School Coherence Map by UnboundED is a map of the relationships among high school mathematics standards. Please note: This resource is aligned to the Common Core not Ohio's Learning Standards. Although the tool is useful, it needs to be checked with respect to Ohio's Learning Standards.
- van Hiele Model of Geometric Thinking is a pdf created by ODE that summarized the van Hiele levels.

References

- Battista, M. (2007). The development of geometric and spatial thinking. In F. K. Lester (Ed.), *Second Handbook of Research on Mathematics Teaching and Learning,* (pp. 629-668). Charlotte, NC: Information Age Publishing.
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- Kinach, B. (March 2012). Fostering spatial vs. metric understanding in geometry. *Mathematics Teacher*, *105(7)*, 534-540.
- Van De Walle, J., Karp, K., & Bay-Williams, J. (2010). *Elementary and Middle School Mathematics* (7th ed.). Boston, MA: Pearson Education, Inc.

STANDARDS	MODEL CURRICULUM (8.G.9)
GEOMETRY **Solve real-world and mathematical problems involving volume of cylinders, cones, and spheres.** **8.G.9** Solve real-world and mathematical problems involving volumes of cones, cylinders, and spheres.	## Expectations for Learning Students have previously worked with finding the volume of right prisms and the areas of circles. The students add to their knowledge of volume by solving problems involving cones, cylinders, and spheres. These shapes were previously introduced by their attributes in past grades. The student understanding of this cluster aligns with a van Hiele Level 1 (Analysis) and moves toward van Heile Level 2 (Informal Deduction/Abstraction). **ESSENTIAL UNDERSTANDINGS** • The bases of cones and cylinders are circles. • The net of a cylinder is a rectangle with 2 circles. • Cones and pyramids have one base. • The point of a cone and pyramid is called the apex. • The height of a pyramid or cone is the perpendicular distance from the apex to the (possibly extended) base. • The slant height of a pyramid or cone is the distance measured along the lateral face from the apex to the base. • The volume of a pyramid is 1/3 of the volume of a prism with congruent bases and heights. • The volume of a cone is 1/3 of the volume of a cylinder with congruent bases and heights. **MATHEMATICAL THINKING** • Consider mathematical units involved in the problem. • Solve real-world problems accurately. • Solve mathematical problems accurately. • Draw a picture or create a model to make sense of a problem. • See structure in equations. *Continued on next page*

STANDARDS	MODEL CURRICULUM (8.G.9)
8.G.9, continued	**Expectations for Learning, continued** **INSTRUCTIONAL FOCUS** • Recognize the components of the formulas for the volume. • Compare the formulas for volume. • Given the volume, find the radius, diameter, height, or area of the base. • Given the dimensions, find the volume. • Differentiate between linear units, square units, and cubic units. • Use volume to model real-world situations. • Explore and make connections about the relationship between the volume of a cube and pyramid with the same base and height. • Explore and make connections about the relationship between the volume of a cylinder and cone with the same base and height. **Content Elaborations** • Ohio's K-8 Critical Areas of Focus, Grade 8, Number 3, pages 53-54 • Ohio's K-8 Learning Progressions, 6-8 Geometry, page 21 **CONNECTIONS ACROSS STANDARDS** • When given the volume, find the square or cube roots to determine the radius (8.EE.2).

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (8.G.9)

Instructional Strategies

Note: The model curriculum is not a collection of lessons nor a full curriculum; it does not suggest pace, sequence, or amount of time spent on topics. It provides information about a topic related to the standards including ideas for examples, strategies for teaching, common misconceptions, and possible connections between topics. In addition, the emphasis of the model curriculum is on instruction not on assessment; therefore, the examples in the Instructional Strategies section are written for instructional purposes not necessarily as assessment items. Examples may need to be rewritten to accommodate the needs of each individual classroom.

Standards for Mathematical Practice
This cluster focuses on but is not limited to the following practices:
MP.1 *Make sense of problems and persevere in solving them.*
MP.2 *Reason abstractly and quantitatively.*
MP.3 *Construct viable arguments and critique the reasoning of others.*
MP.4 *Model with mathematics.*
MP.5 *Use appropriate tools strategically.*
MP.6 *Attend to precision.*
MP.7 *Look for and make use of structure.*

In Grade 7 students explored circles and the surface area and volume of right prisms. In Grade 8, they are putting the two concepts together to explore right cylinders, right cones, and spheres. The focus in this cluster should be on relationships between solids and the real-world application of volume. Not only do students need to find the volume, but they should also be able to find a missing dimension given the volume.

Note: There is no place in the standards where students are explicitly asked to find the surface area of a cone or cylinder. If students have not explored these concepts in Grade 7, it might be a worthwhile extension in this cluster. Although, the volume of a pyramid is not explicitly mentioned until high school, some mathematicians consider a pyramid as a subset of a cone, so it is appropriate to teach volume of a pyramid in this cluster.

VAN HIELE CONNECTION

Level 1 (Analysis) can be characterized by the student doing some or all of the following:

- comparing length, area, or volume by manipulating and matching parts;
- visually comparing shapes by composing/decomposing;
- visualizing structured iteration of length, area, or volume units;
- organizing area and volume units into (2D, 3D) array structure without gaps or overlaps;
- using a single unit, row, or layer repeatedly (iterating) to correctly measure or construct length, area, or volume respectively;
- determining measurement without having to show every unit instead of using only numbers (no visible units or repeated units); and/or
- creating composite units, columns, rows, or layers to find length, area, or volume.

Level 2 (Informal Deduction/Abstraction) can be characterized by the student doing some or all of the following:

- fully understanding the procedures and formula for determining measurement of simple, familiar shapes;
- generalizing unit-measure enumeration to include fractional units;
- understanding and making unit conversions; and/or
- generalizing measurement to non-squares for area and non-cubes for volume.

See the van Hiele pdf on ODE's website for more information about van Hiele levels.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (8.G.9)

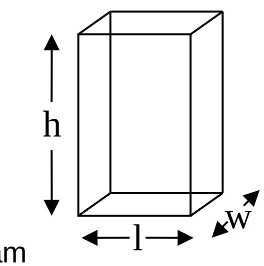
DRAWING

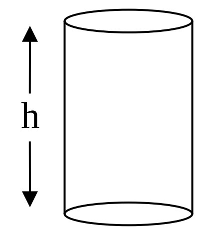
To develop students' spatial skills, they need practice learning how to draw three-dimensional solids such as cones, cylinders, pyramids, and spheres.

There are many times in life, where people need to represent three-dimensional solids as two-dimensional figures in presentations using technology. Have students practice creating three-dimensional solids on technology platforms.

CYLINDERS

Begin by recalling the formula, and its meaning, for the volume of a right rectangular prism: $V_{prism} = l \times w \times h$ and $V_{prism} = Bh$. Then ask students to make a conjecture about the volume formula for a cylinder.





circular

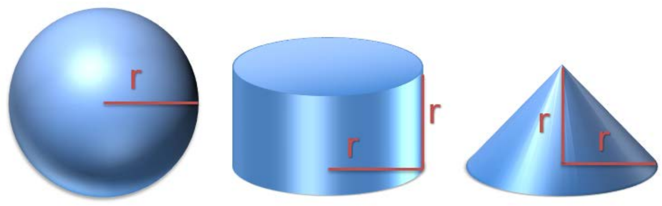
Most students can be readily led to the understanding that the volume of a right rectangular prism can be thought of as the area of a "Base" times the height, and so because the area of the base of a cylinder is a circle whose area equals πr^2 the volume of a cylinder is $V_{cylinder} = \pi r^2 h$ or $V = Bh$. Foam layers that have the height of 1 unit can be used to show how to build a cylinder of *h* height and reinforce Base × height as well.

CONES

To explore the formula for the volume of a cone, use cylinders and cones with the same radius and height. Fill the cone with rice or water and pour it into the cylinder. Students will discover/experience that 3 cones full are needed to fill the cylinder. This non-mathematical demonstration of the formula for the volume of a cone, $V_{cone} = \frac{1}{3}\pi r^3 h$ or $V_{cone} = \frac{1}{3}Bh$, will help students make sense of the formula. *Note: Teachers can extend this activity to explore the volume of a pyramid.*

TIP! Make sure to differentiate between the height of an object and slant height.

SPHERES



To explore the formula for the sphere, use spheres, cylinders, and cones with the same radius, whereas the height of the cone and the cylinder must be the same as the radius, but the height of the sphere will be twice the radius. Discuss the relationships between the solids. Fill the sphere with rice or water and pour into the cylinder. Students will discover/experience that there is water remaining in the sphere. This water/rice will fill the cone. The students should see that the volume of a sphere = the volume of a cylinder + volume of a cone. Because *r* = *h* (radius = height), by using substitution, the volume of the cylinder is $\frac{3}{3}\pi r^3$, and the volume of the cone is $\frac{1}{3}\pi r^3$, so the volume of the sphere is $V_{sphere} = \frac{4}{3}\pi r^3$, This non-mathematical demonstration of the formula for the volume of a sphere, $V_{sphere} = \frac{4}{3}\pi r^3$, will help students make sense of the formula.

REAL-WORLD APPLICATIONS

Students should experience many types of real-world applications using these formulas. They should be expected to explain and justify their solutions. Some examples include the following: finding the amount of space left over in a can with 3 tennis balls; finding total volume in a silo; finding how much ice cream in a cone, etc.

EXAMPLE

An ice cream shop that Jenni works at has a rule that every adult scoop should be the same size, so they have a standard ice cream scooper that has a diameter of 2 ¾ inches. A single scoop cone costs $2.79. Jenni is overgenerous with her scoops and makes her scoops have a diameter of 3 inches instead but still sells them for $2.79.

- Is Jenni a good employee?
- How much is she costing her owner per cone?
- If she scoops 100 cones per day, how much will she cost her owner?
- If she works 15 days that month and sells an average of 90 cones per day, how much will she cost her owner?

For more information on ice cream scoopers see Disher Scoop Sizes by Chef's Resources.

EXAMPLE

A half gallon (115.5 in^3) of ice cream in a cylindrical carton with a height of 5.75 inches used to cost $5.99. To save money the ice cream company shaved 0.2 inches off the radius and 0.3 inches off the height and still sold the ice cream carton for the same price.

- How much will the new ice cream carton hold?
- How much money will the company save per carton?
- How much money will the company save if they sell 1,000 cartons?

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (8.G.9)

EXAMPLE

An ice cream store buys a 3-gallon cylindrical ice cream tub with a diameter of 9.44 inches and a height of 10.63 inches including the cover. How many ice cream tubs can fit into a case that has 21"x 21" x 28" dimensions?

EXAMPLE

A cylindrical grain silo has a cone hopper at the bottom for efficient unloading. The height of the cylinder grain silo is 42 ft. with a diameter of 15 ft. In order to work properly, the volume of the cone hopper must not exceed half the volume of the upper cylinder part of the grain silo. What is the maximum height of the cone hopper?

Instructional Tools/Resources

These tools and resources may be helpful for instruction. Before use, they should be evaluated at the local level to determine their appropriateness for instruction.

Exploring Volume

- Volume Formulas is an activity by Annenberg Learner where students explore spheres and cylinders.
- Volume of a Sphere, Volume of a Cone, and Volume of a Cylinder are applets by Math Open Reference that explores the volume of a spheres, cones, and cylinders.
- Volume of Cylinder is an applet by Geogebra that uses sliders to explore the volume of a cylinder.
- The Cylinder Problem on Mathforum is a high school lesson plan where students build a family of cylinders to explore the relationship between their dimensions.
- Popcorn, Anyone? is a lesson by NCTM Illuminations where students discover the relationship between dimension and volume using rectangular prisms and cylinders. *NCTM now requires a membership to view their lessons.*
- Optical Illusion: Volume by MathFLIX is a video that explores the volume of a cylinder.

Applications of Volume

- Modeling: Making Matchsticks is a lesson by Mathematics Assessment Project where students have to compare the volume of matchsticks to the volume of a tree. This is a modeling task.
- Calculating Volumes of Compound Objects is a lesson by Mathematics Assessment Project where students have to compare volumes of various shaped glasses. This requires students to use several volume formulas for a single object.
- Greenhouse Management by AchievetheCore connects CTE standards to math. In this task, students determine the costs and expenses of Easter lilies. Students will need to use volume formulas to complete the task.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (8.G.9)

Applications of Volume, continued

- How Many Baseballs are in this Truck? is an activity from Yummy Math where students apply concepts of volume to a contest involving baseballs.
- Collecting the Most Candy! is an activity from Yummy Math where students apply the concepts of volume to collecting Halloween candy in different shaped buckets.
- Rescue of the Chilean Miners-The 33 is an activity from Yummy Math where students apply the concepts of volume to the real-life situation of miners trapped in a copper-gold mine in Chile.
- Analyzing Lego Fireman is a task from Yummy Math where students apply the concepts of volume to a life-sized Lego Fireman.
- Gumballs Galore is a task from Yummy Math where students apply the concepts of volume to find the number of gumballs in a machine.
- The Largest Cup of Coffee Ever! is a task from Yummy Math where students apply the concepts of volume to a huge cup of coffee.
- Flower Vases is a task by Illustrative Mathematics that has students apply volume concepts to a situation.
- How big is the 2010 Guatemalan Sinkhole? by Robert Kaplinsky is a real-world modeling problem that uses the volume of a cylinder.
- How Many Gumballs Fit in the Gumball Machine? by Robert Kaplinsky is a real-world modeling problem that uses the volume of a sphere.
- Ice Cream Puddle is a lesson by NCTM Illuminations that is open-ended with multiple entry points. Students use volume formulas of cones, cylinders, and spheres. *NCTM now requires a membership to view their lessons.*
- Grain Storage by Achieve the Core is a lesson that aligns math with CTE standards. They need to apply the Pythagorean Theorem and volume of a cylinder to solve the problem.
- Oh, Chute is a lesson from NASA where students use the Pythagorean Theorem and the volume of a cylinder to solve a problem involving a parachute and a space vehicle.

Curriculum and Lessons from Other Sources

- Comparing Spheres and Cylinders is a task from Georgia Standards of Excellence Framework, Grade 8, Unit 3. In this task students compare the effect of the volume of a cylinder and sphere. This task can be found on pages 115-121.
- Illustrative Mathematics, Grade 8, Unit 5: Functions and Volume, Lesson 11: Filling Containers, Lesson 12: How Much Will Fit?, Lesson 13: The Volume of a Cylinder, Lesson 14: Finding Cylinder Dimensions, Lesson 15: The Volume of a Cone, Lesson 16: Finding Cone Dimensions, Lesson 17: Scaling One Dimension, Lesson 18: Scaling Two-Dimensions, Lesson 19: Estimating a Hemisphere, Lesson 20: The Volume of a Sphere, Lesson 21: Cylinders, Cones, and Spheres are lessons that pertain to this cluster. *You must create an account to view Illustrative Mathematics lessons on OpenUp Resources.*
- EngageNY, Grade 8, Module 5, Topic B, Lesson 9: Examples of Functions from Geometry, Lesson 10: Volumes of Familiar Solids—Cones and Cylinders, and Lesson 11: Volume of a Sphere are all lessons that pertain to this cluster.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (8.G.9)

General Resources

- Arizona 7-High School Progression on Geometry
  This cluster is addressed on pages 4 and 12.
- Coherence Map by Achieve the Core is an interactive website that shows the progression and interconnectedness of the Common Core standards. To be noted, there are some differences between Ohio's Learning Standards and the Common Core, so although the tool is useful, it needs to be checked with respect to Ohio's Learning Standards.
- High School Coherence Map by UnboundED is a map of the relationships among high school mathematics standards. Please note: This resource is aligned to the Common Core not Ohio's Learning Standards. Although the tool is useful, it needs to be checked with respect to Ohio's Learning Standards.
- van Hiele Model of Geometric Thinking is a pdf created by ODE that summarized the van Hiele levels.

Research

- Common Core Standards Writing Team. (2016, March 24). *Progressions for the Common Core State Standards in Mathematics (draft). Grades 7–8, HS, Geometry*. Tucson, AZ: Institute for Mathematics and Education, University of Arizona.
- Wheatley, J. (August 2011). An investigation of three-dimensional problem solving and levels of thinking among high school geometry students. A Project Report Presented to the Graduate Faculty of Central Washington University. Retrieved from https://pdfs.semanticscholar.org/994a/d70f32b0dd2b38342da230b6275e08f9032e.pdf

STANDARDS	MODEL CURRICULUM (8.SP.1-4)

STATISTICS AND PROBABILITY
Investigate patterns of association in bivariate data.
8.SP.1 Construct and interpret scatterplot for bivariate[G] measurement data to investigate patterns of association between two quantities. Describe patterns such as clustering; outliers; positive, negative, or no association; and linear association and nonlinear association. (GAISE Model, steps 3 and 4)
8.SP.2 Understand that straight lines are widely used to model relationships between two quantitative variables. For scatterplot that suggest a linear association, informally fit a straight line, and informally assess the model fit by judging the closeness of the data points to the line. (GAISE Model, steps 3 and 4)
8.SP.3 Use the equation of a linear model to solve problems in the context of bivariate measurement data, interpreting the slope and intercept. *For example, in a linear model for a biology experiment, interpret a slope of 1.5 cm/hr as meaning that an additional hour of sunlight each day is associated with an additional 1.5 cm in mature plant height.* (GAISE Model, steps 3 and 4)
Continued on next page
	Expectations for Learning
	Building on the study of statistics using univariate data (one variable) in Grades 6 and 7 and using the framework of the GAISE Model, students are now ready to study bivariate data (two variables). Students will extend their descriptions and understanding of variation to the graphical displays and numerical analysis of bivariate data. In high school, students build on their experience from the middle grades with data exploration and summarization; randomization as the basis of statistical inference; and simulation as a tool to understand statistical methods.
	
	ESSENTIAL UNDERSTANDINGS
	Quantitative (numerical) variables
	• Scatterplots are used for bivariate quantitative data.
	• When two variables are represented on a scatterplot, an association may exist.
	• An association between two variables can be seen in the pattern created by the data:
	    o clusters;
	    o positive, negative, or no association; and/or
	    o linear or nonlinear association.
	• Outliers are bivariate points that do not fit the trend.
	• When a scatterplot suggests a linear association, a line can be informally fitted to the data.
	• Closeness of data points to the line can be judged visually.
	• When looking for a linear association, a line takes all of the points into consideration, and the prediction is based on an overall pattern rather than just one or two points.
	• The slope and *y*-intercept describe the linear association between two variables.
	• Linear functions can be used to describe contextual problems through the following:
	    o interpreting slope and intercept; and/or
	    o making predictions.
	Continued on next page

STANDARDS	MODEL CURRICULUM (8.SP.1-4)
8.SP.4 Understand that patterns of association can also be seen in bivariate categorical data by displaying frequencies and relative frequencies in a two-way table. Construct and interpret a two-way table summarizing data on two categorical variables collected from the same subjects. Use relative frequencies calculated for rows or columns to describe possible association between the two variables. *For example, collect data from students in your class on whether or not they have a curfew on school nights and whether or not they have assigned chores at home. Is there evidence that those who have a curfew also tend to have chores?*	**Expectations for Learning, continued** **ESSENTIAL UNDERSTANDINGS, CONTINUED** **Categorical variables** • Two-way tables are used for bivariate categorical data. • Two-way tables are used to display frequencies and relative frequencies. • Relative frequencies can be used to describe possible associations. • If row (or column) relative frequencies in the table are the same, there is little or no association. • If row (or column) relative frequencies in the table are different, there is some evidence of association. **MATHEMATICAL THINKING** • Represent concepts symbolically. • Construct and modify models to represent mathematical thinking. • Use informal reasoning. • Attend to precision in justifying mathematical reasoning. *Continued on next page*

STANDARDS	MODEL CURRICULUM (8.SP.1-4)
8.SP.1-4, continued	**Expectations for Learning, continued** **INSTRUCTIONAL FOCUS** **Quantitative** • Construct a scatterplot by choosing a scale, labeling axes, and plotting points. • Describe a scatterplot in terms of clusters, gaps, and unusual data points (outliers). • Describe a trend (linear, curved, positive, negative, strong association, weak association, no association). • Sketch a trend line through the "center" of the points. • Approximate the slope of the trend line. • Write an equation of the trend line. • Interpret the rate of change and intercept of the trend line in the context of the problem. • Interpret data points in relationship to the trend line (above, below, or on the line). • Answer questions and make predictions based on the trend line. • Determine how a change in data changes the trend line. • Measure the association between two quantitative variables using the Quadrant Count Ratio (QCR). **Categorical** • Construct a two-way frequency table. • Calculate relative frequencies. • Determine possible association between the two variables using row (or column) relative frequencies. **Content Elaborations** • Ohio's K-8 Critical Areas of Focus, Grade 8, Number 1, pages 50-51 • Ohio's K-8 Learning Progressions, Statistics and Probability, pages 22-23 • GAISE Model, pages 14 – 15 o Focus of 7th grade is Level A – B, pages 22-59 **CONNECTIONS ACROSS STANDARDS** • Apply concepts of linear functions (8.F.3-5).

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INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (8.SP.1-4)

Instructional Strategies

Note: The model curriculum is not a collection of lessons nor a full curriculum; it does not suggest pace, sequence, or amount of time spent on topics. It provides information about a topic related to the standards including ideas for examples, strategies for teaching, common misconceptions, and possible connections between topics. In addition, the emphasis of the model curriculum is on instruction not on assessment; therefore, the examples in the Instructional Strategies section are written for instructional purposes not necessarily as assessment items. Examples may need to be rewritten to accommodate the needs of each individual classroom.

Standards for Mathematical Practice
This cluster focuses on but is not limited to the following practices:
MP.4 *Model with mathematics.*
MP.6 *Attend to precision.*
MP.7 *Look for and make use of structure.*

Building on the study of statistics using univariate data in Grades 6 and 7, students are now ready to study bivariate data. They will extend their descriptions and understanding of variation to the graphical displays of bivariate data.

TIP! Explain to students *uni* means one, *bi* means two, and *variate* means variable, so univariate is data using one variable and bivariate is data using two variables.

Eighth graders apply their experience with the coordinate plane and linear functions in the study of association between two variables related to a question of interest. As in the univariate case, analysis of bivariate measurement data graphed on a scatterplot proceeds by describing shape, center, and spread. But now "shape" refers to a cloud of points on a plane, "center" refers to a line drawn through a cloud that captures the essence of its shape, and the "spread" refers to how far the data points stray from the central line.

Students extend their understanding of "cluster" and "outlier" from univariate data to bivariate data. They summarize bivariate categorical data using two-way tables of counts and /or proportions, and examine these for pattern associations.

GAISE MODEL

Students continue to use the GAISE model outlined in Grade 6 and 7: Formulate Questions, Collect Data, Analyze Data, and Interpret Results. In Grade 8 they formulate questions with two variables with the purpose of investigating a possible association between them. They begin to pose their own questions of interest that are not necessarily restricted to the classroom. Eighth graders can design and conduct nonrandom sample surveys; although they should begin to start thinking informally about random selection and what kind of sample best represents a population. They may also do comparative experiments. Although they will continue to use all four steps of the GAISE model, the focus on this cluster is on Analyzing Data and Interpreting Results. Students will start at Level A with respect to scatterplots and two-way frequency tables and will move toward Level B.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (8.SP.1-4)

QUANTITATIVE DATA

Scatterplots are the most common form of representations displaying bivariate data in Grade 8. Provide scatterplot of linear data and have students practice informally finding the *trend line*. Students could be given a scatterplot and a spaghetti noodle to determine the "best fit." Discussion should include "What does it mean for a data point to be above the line?" or "What does it mean for it to be below the line?"

As an extension to the spaghetti noodle activity, students could use the spaghetti noodle to consider the average vertical distance the points are from the line (noodle). This can also be connected to mean absolute deviation (MAD). *Note: Typically MAD is used to analyze univariate data not bivariate data, but since you are only looking at the vertical distance, MAD could be an appropriate tool.*

Some students may think there is only one correct answer and mistakenly think his or her line of best fit will be exactly the same as his or her classmates for the same set of data. It will be helpful for students to share their work, so they see a variety of trend lines.

By changing the data slightly, students can have a rich discussion about the effects of the change on the graph. The study of the trend line ties directly to the algebraic study of slope and *y*-intercept. Students should interpret the slope and *y*-intercept of the trend line in the context of the data. Then students can make predictions based on the trend line. Give students a variety of data sets that intersect the *y*-axis at various points, so students do not mistakenly think that all trend lines must go through the origin.

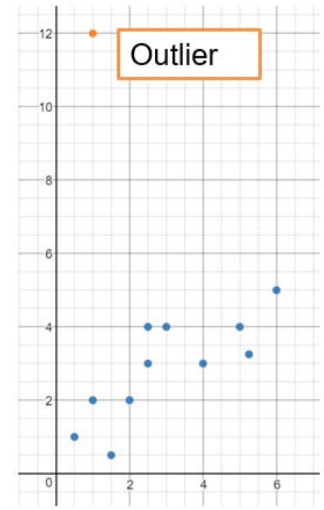
Students may not understand that the line of best fit may display predictions that do not make sense in the context of the problem.

Some students think that the trend line represents continuous data. Explain to students that the trend line describes the data, but it is not the data, as the points on the scatterplot are the data.

After a trend line is fitted through the data, the slope of the line is approximated and interpreted as a rate of change, in the context of the problem. If the slope is positive, then the two variables are positively associated. Similarly if the slope is negative, then the two variables are negatively associated. Students should also be exposed to data that do not have an association.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (8.SP.1-4)

Because students are informally drawing lines of best fit, the lines will vary slightly. After students are used to informally fitting trend lines to data, have students create a regression line using technology such as graphing calculators or Desmos. Use the equation $y_1 \sim mx_1 + b$ in Desmos to graph the trend line. On certain platforms, if you take data from an Excel spreadsheet and copy and paste into Desmos, it will graph the data. This allows students to explore larger data sets.



Students should create and interpret scatterplots, focusing on outliers, positive, or negative association, linearity, or curvature. Assuming the data are linear, students should informally draw a trend line on the scatterplot and informally evaluate the strength of fit. They should be able to interpret visually how well the trend line fits the "cloud" of points.

Students may not realize that a single data point on a scatterplot could represent multiple data points in a given data set, so give students examples where the data has two identical points.

Outliers are points substantially outside the general trend line. Discuss how outliers might have an influence on the positioning of the line. Students need to realize that an outlier now does not need to be at the end of the data, it can be a data point significantly higher or lower than all the other data anywhere along the line of best fit.

Quadrant Count Ratio

To move students from Level A to Level B, questions should move from "Is there an association?" to "How strong is the association?" The Quadrant Count Ratio (QCR) can help students informally determine the strength between two variables. This is an important building block in building the conceptual understanding of the correlation coefficient in high school.

EXAMPLE

Step 1: Formulate Questions A discussion in class might revolve around if technology causes people to be more or less social. Students may want to see if there is an association between the amount of text messages a person sends compared to the amount of time a person spends hanging out with friends. This is a good time to have a discussion about what defines an event and how the event will be measured. Are they measuring time in hours or minutes? What is the designated time frame for receiving text messages?

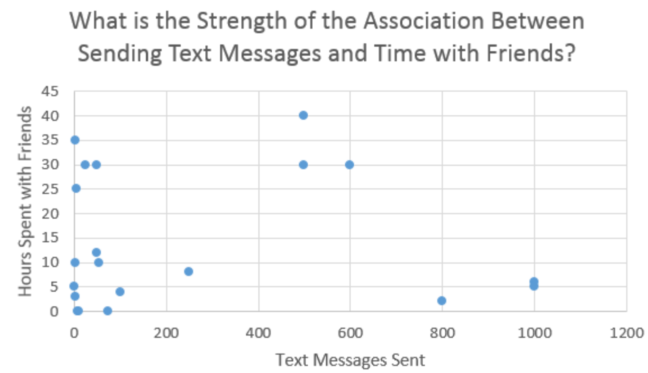
Step 2: Collect Data The students may want to conduct a census of a few classrooms. This may be a good time to have a discussion of bias in surveys. For example, if people know they are going to be surveyed, would they change their habits? Could the wording in the survey sway people to answer a certain way? This would also be a good time to informally discuss the concept of random selection. What is the population we want to know something about? Are certain classes better representations of the population than others? Students can also compare their data to similar student data across the nation or state using Census at School.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (8.SP.1-4)

Continue to discuss which variable is the independent variable compared to the dependent variable in a contextual situation.

TIP! Since students may interchange the independent and dependent variables emphasize the dependent variable depends on the independent variable, and the frame of the question indicates which one is which.

Step 3: Analyze Data Students may create a graph similar to the chart.



Discuss whether the data has a positive, negative, or no association. Look to see if there is any clustering and analyze why clustering is present or not. Look to see if there are any outliers. If there are, discuss whether to include them or not include them in the data, and the impact that each choice has.

Once an association has been determined, reframe the students' question in terms of strength: "What is the strength of the association between text messages sent and the hours spent with friends?"

Show students two scatterplots of data (or compare students' scatterplots). One should have a positive trend but the points are scattered, and another should have a positive trend but the points are closer together. Discuss which one has a stronger association.

TIP! Choosing how to display the data is a golden opportunity to discuss data that can be presented from different organizations to put forth different messages. For example, the union showing no pay increases vs. the employer showing the same data to show that the increase has been significant, but by using a different scale. Ask students questions such as "What message do you want to send to your audience?" and "How can the graph be manipulated to communicate your message?" It is important for people to be critical of the information that is presented to them statistically and graphically.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (8.SP.1-4)

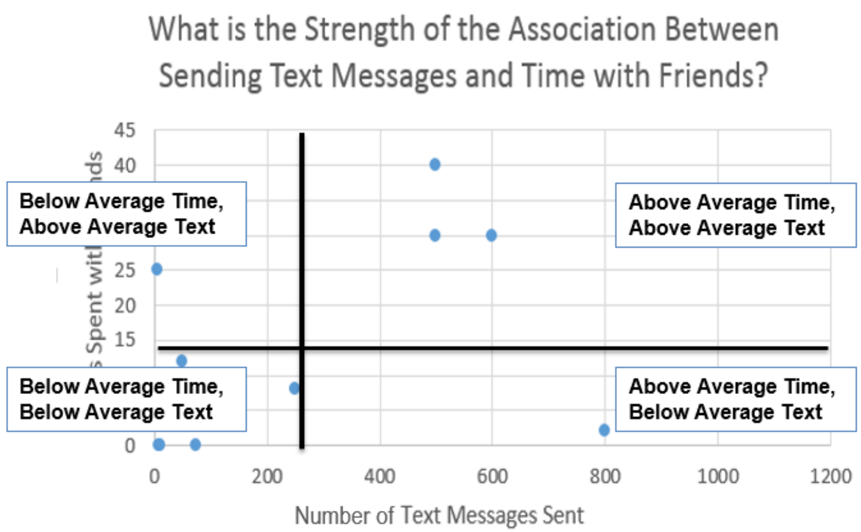
The Quadrant Count Ratio (QCR) informally measures the strength of the association.

Text Messages	Hours with Friends
0	5
600	30
7	0
3	35
50	12
2	10
1000	6
50	30
4	3
1000	5
25	30
75	0
10	0
55	10
800	2
100	4
250	8
500	30
500	40
6	25

1.) Find the mean of both the independent and dependent variable.

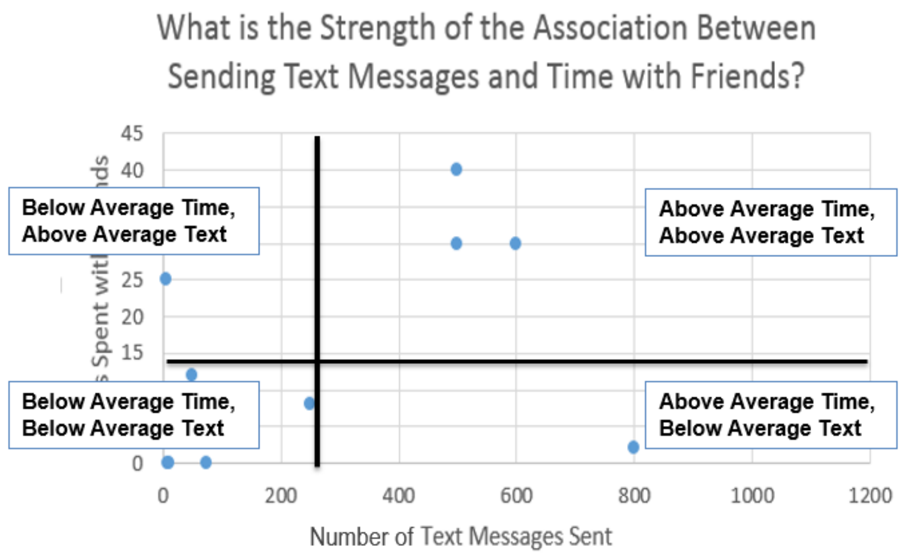
Mean Number of Text Messages = 251.85
Mean of Hours with Friends = 14.25

2.) Draw a vertical line through the mean of the independent variable and a horizontal line through the mean of the dependent variable.



INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (8.SP.1-4)

3.) The horizontal and vertical lines divide the scatterplot into four quadrants or regions. That divides that data into Above Average/Above Average (Q1), Below Average/Above Average (Q2), Below Average/Below Average (Q3), and Above Average/Below Average (Q4) quadrants. Students should notice that data with positive trends have more data points in Q1 and Q3 and data with negative trends have more data points in Q2 and Q4.



4.) Explain to students that the greater the number of points in quadrants 1 and 3 compared to number of points in quadrants 2 and 4, the stronger the positive association is. (The reverse is true for a negative association.) This relationship is called the Quadrant Count Ratio, which can be found as follows:

$$\frac{(Number\ of\ Points\ in\ Quadrants\ 1\ and\ 3) - (Number\ of\ Points\ in\ Quadrants\ 2\ and\ 4)}{The\ Total\ Number\ of\ Points\ in\ All\ Quadrants}$$

So,

$$\text{QCR} = \frac{13-7}{20} = 0.3$$

Explain to students that a QCR close to 1 or –1 means that the variables are strongly associated, and that a QCR close to 0 means the two variables are weakly associated. Since the QCR for these data is 0.3, it looks like there is a weak positive association. *Note: Your class data may have a strong positive association.*

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (8.SP.1-4)

Step 4: Interpret Results
Have a discussion about whether the sample represents the larger population, and what could be done to obtain better data. Discuss informally the difference between "cause and effect" and "association." Does sending text messages cause people to spend more time with their friends or could there be other factors that both groups have. For instance extroverted people may both like to send text messages and spend time with friends. However, there could be a cause and effect relationship if someone, for example, was sending lots of text messages to organize a party. In this case, since the association between the two events is fairly weak, sending text messages may not be a good predictor of time spent with friends.

Students should explore the differences between variations and errors. Teachers can draw attention to naturally occurring errors that happen in the classroom and discuss the impact of the errors on the results. These notions can be used to explain outliers, clusters, and gaps.

Misleading Statistics
Students should be given the opportunity to analyze misleading bivariate statistics. Discuss how it is important for the intervals in scales to be consistent and then if the scale jumps from 0, a break needs to be shown. Explore how changing the scale can make data seem more favorable. They should also be critical of the samples that supposedly represent the population. Discuss what makes a sample a good representation of a population. (Connect with 7.SP.1.)

CATEGORICAL DATA
Building on experience with decimals, percents, and the ideas of association between measurement variables, students now take a more careful look at possible association between categorical variables.

Categorical bivariate data is typically displayed using two-way tables. Discuss the meaning of each individual cell frequency and the sums of the rows and columns in the context of a question of interest.
- What do the cells individually mean in relation to each other?
- What do the row/column sums mean in relation to each other?

Students may believe bivariate data is only displayed in a scatterplot. The standard 8.SP.4 provides the opportunity to display bivariate, categorical data in a table.

Note: This standard is very similar to the high school standard S.ID.5. In Grade 8 the marginal frequencies are usually given and tables are limited to 2 by 2 tables. There should be an emphasis on converting back and forth between two-way frequency tables and two-way relative frequency tables. Although the terminology joint, marginal, and relative frequencies are not used in this standard, the expectation is that they find these relative frequencies.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (8.SP.1-4)

TIP! One cannot use frequency counts to compute a mean or median for categorical variable. The frequency counts are the numerical summary for the categorical data.

EXAMPLE

Step 1: Formulate Questions

Students may have read an article about the relationship between the quality of sleep a person has and whether or not a person sleeps with a cellphone in their room. Studies have found that it takes longer for people to fall asleep with cell phones in their rooms and that the quality of sleep for those people is poorer and the length of sleep is shorter. Students could come up with a question such as the following: Is there a relationship between someone sleeping with a cell phone in their room and the time it takes to fall asleep?

Step 2: Collect Data

Students may decide to give a survey to several classes in the school. A discussion should be had about the wording of the survey, so that it leads to two discrete answers such as the following:

- Do you sleep with a cell phone in your room: yes or no?
- Does it take you more than 60 minutes to fall asleep: yes or no?

Discuss why discrete questions are preferable to open ended questions such as "How long does it take you to fall asleep?" or questions that do not include all possible answers such as "How long does it take you to fall asleep: less than 60 minutes or more than 60 minutes?" This type of question will exclude people who take exactly 60 minutes to fall asleep. Also discuss the importance of accurately recording the responses and how not doing so could affect the data.

Step 3: Analyze Data

Display survey results in a two-way frequency table or a relative frequency table. Whereas frequency tables display counts, relative frequency tables display the data in the form of ratios, decimals, or percents. The ratio of the joint or marginal frequencies to the total number of subjects define relative frequencies (and percentages), respectively. Notice that the total in a relative frequency table (bottom right-hand corner) is 1 or 100%. Students should be able to convert back and forth between these two types of tables, increasingly becoming fluent in the relative frequency tables. Discuss how it is possible to switch the row and column headers. *Note: The words joint, marginal, and conditional frequency are used in this document for informational purposes. Students are not responsible for this terminology.*

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (8.SP.1-4)

Frequency Table

	More than 60 min	60 min or less	TOTAL
Phone	61	48	109
No Phone	11	40	51
TOTAL	72	88	160

Relative Frequency Table

	More than 60 min	60 min or less	TOTAL
Phone	$^{61}/_{160} \approx 0.38 =$ 38%	$^{48}/_{160} \approx 0.30 =$ 30%	$^{109}/_{160} \approx 0.68 =$ 68%
No Phone	$^{11}/_{160} \approx 0.07 =$ 7%	$^{40}/_{160} \approx 0.25 =$ 25%	$^{51}/_{160} \approx 0.32 =$ 32%
TOTAL	$^{72}/_{160} \approx 0.45 =$ 45%	$^{88}/_{160} \approx 0.55 =$ 55%	$^{160}/_{160} = 1.00 =$ 100%

Marginal Frequency

Row totals and column totals constitute the marginal frequencies. These are found in the margins of the table. These can also be found by adding across columns or rows.

	More than 60 min	60 min or less	TOTAL
Phone	61	48	109
No Phone	11	40	51
TOTAL	72	88	160

Add

Add

	More than 60 min	60 min or less	TOTAL
Phone	61	48	109
No Phone	11	40	51
TOTAL	72	88	160

Joint Frequency

Joint frequency is where the two variables "join" such as keeping the phone in the bedroom and taking longer than 60 minutes to fall asleep. These can be found in the body of the table.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (8.SP.1-4)

Conditional Frequency

Conditional relative frequencies are determined by focusing on a specific row or column of the table. They are particularly useful in determining any associations between the two variables. For example, if a person sleeps with their phone, what is the relative frequency it will take him or her more than 60 minutes to fall asleep? ($^{61}/_{109} \approx 56\%$.)

	More than 60 min	60 min or less	TOTAL
Phone	61	48	109
No Phone	11	40	51
TOTAL	72	88	160

Notice the example above is different than "What is the relative frequency that a person that takes more than 60 minutes to fall asleep has a phone in their bedroom?" ($^{61}/_{72} \approx 85\%$)

Students may think that they can use frequency counts to compute a mean or median for categorical variable, but the frequency counts are the numerical summary for the categorical data.

Step 4: Interpret results

Column or row conditional relative frequencies can be used to determine if two variables are associated.

- If a person sleeps with their phone, what is the relative frequency it will take he or she more than 60 minutes to fall asleep. ($^{61}/_{109} \approx 56\%$)
- If a person does not sleep with their phone, what is the relative frequency it will take him or her more than 60 minutes to fall asleep. ($^{11}/_{51} \approx 22\%$)

Or

- What is the relative frequency that a person who takes more than 60 minutes to fall asleep has a phone in their bedroom? ($^{61}/_{72} \approx 85\%$.)
- What is the relative frequency that a person who takes 60 minutes or less to fall asleep has a phone in their bedroom? ($^{48}/_{88} \approx 55\%$)

Since the conditional probabilities are different, there is an association between a person sleeping with their phone in their room and how long it takes them to fall asleep. The larger the difference in the conditional probabilities, the stronger the association is.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (8.SP.1-4)

Instructional Tools/Resources

These tools and resources may be helpful for instruction. Before use, they should be evaluated at the local level to determine their appropriateness for instruction.

Manipulatives/Technology

- Graphing calculator
- Desmos is a free online graphing utility and app.
- Census at School is an international classroom project that provides real data for classrooms. *(Note: On a PC this data can be copied from their Excel spreadsheet right into Desmos.)*

Scatterplots

- Scatter It! (Using Census Results to Help Predict Melissa's Height) by Susan Haller is a lesson from amstat's STEW (Statistics Education Web) that has students create a scatterplot and approximate a line of best fit explicitly tied to the GAISE model.
- You and Michael by Stephen Miller is a lesson from amstat's STEW (Statistics Education Web) that has students create a scatterplot and approximate a line of best fit explicitly tied to the GAISE model.
- What Fits by Anna Bargagliotti and Stephanie Casey is a lesson from amstat's STEW (Statistics Education Web) that has students explore a line of best fit explicitly tied to the GAISE model.
- Fitting a Line to Data—Earning and Educational Achievement is a lesson by the United States Census Bureau using real census data where students create a scatterplot and add an approximate line of best fit.
- Patterns of Associations—Quality of English Spoken by People Who Speak Spanish in Their Homes is a lesson by the United States Census Bureau using real census data where students study the associations between data.
- Animal Brains by Illustrative Mathematics is a task where students create scatterplots and think critically about association and outliers in data.
- Bird Eggs by Illustrative Mathematics is a task where students interpret data points on a scatterplot and make predictions based on a trend line.
- Hand Span and Height by Illustrative Mathematics is a task where students create a scatterplot by taking measurements of and spans and height.
- Texting and Grades I by Illustrative Mathematics is a task where students address the form, direction, and strength of data on a scatterplot.
- Laptop and Battery Charge by Illustrative Mathematics is a task where students make a scatterplot and informally create a line of best fit to represent the data.
- US Airports, Assessment Variation by Illustrative Mathematics is a task where students must interpret a line of best fit.
- Positive Correlation is a task by Open Middle where students have to create data points that represent a positive, negative, and no correlation.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (8.SP.1-4)

Scatterplots, continued

- Line of Best Fit is a task by Open Middle where students have to create data points that could generate a line of best fit with the equation $y = -x + 8$.
- Interpreting and Using Data: Setting Taxi Fares by Mathematics Assessment Project is a taks that has students analyze a scatterplot.
- Exploring Linear Data by NCTM Illuminations is a lesson where students model linear data in settings that range from car repair costs to sports to medicine. *NCTM now requires a membership to view their lessons.*
- Scatter Diagram by Inside Mathematics is a Performance Assessment Task where students analyze a scatterplot.
- House Prices by Inside Mathematics is a Performance Assessment Task where students analyze a scatterplot.
- Scatterplot Capture by Desmos has students make predictions about future points in a scatterplot.
- Opening Weekend the Black Panther is a task by YummyMath where students create scatterplots.
- The Star Wars Phenomena Continues by Yummy Math is paired with Star Wars Earnings by Desmos is a task that uses real data to explore a scatterplot with a linear model.
- Bivariate Data and Analysis: Anthropological Studies by PBS Learning has students investigate how skeletal populations can help determine the impact of slavery on the height and health of African American males. There is an accompanying video.
- Contingency Tables by Annenberg Learner has a lesson that connects the Quadrant Count Ratio (QCR) to a two-way table.

Two-Way Frequency Table

- Two-Way Tables—Walking and Bicycling to Work is a lesson by the United States Census Bureau using real census data to calculate relative frequencies and conditional relative frequency using two-way tables.
- Music and Sports by Illustrative Mathematics is a task where students summarize data in a two-way frequency table.
- What's Your Favorite Sport? by Illustrative Mathematics is a task where students determine association based on data in a two-way frequency table.
- Using Data: Testing a New Product by Mathematics Assessment Project is a task that allows students to organize data by constructing two-way frequency tables.
- Ice Cream Scoop by Youcubed is a task where students can apply using a two-way frequency table.

Curriculum and Lessons from Other Sources

- Illustrative Mathematics, Grade 8, Unit 6: Associations in Data has many lessons that pertain to this cluster. *You must create an account to view Illustrative Mathematics lessons on OpenUp Resources.*
- EngageNY, Grade 8, Module 6, Topic B, Lesson 6: Scatterplot, Lesson 7: Patterns in Scatterplot, Lesson 8: Informally Fitting a Line are lessons that pertain to this cluster.
- EngageNY, Grade 8, Module 6, Topic C, Lesson 9: Determining the Equation of a Line Fit to Data, Lesson 10: Linear Models, Lesson 11: Using Linear Models in Data Contexts are lessons that pertain to this cluster.
- EngageNY, Grade 8, Module 6, Topic D, Lesson 12: Nonlinear Models in Data Context, Lesson 13: Summarizing Bivariate Categorical Data in a Two-Way Table, and Lesson 14: Association Between Categorical Data are lessons that pertain to this cluster.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (8.SP.1-4)

Curriculum and Lessons from Other Sources, continued

- Mathematics in a World of Data by Burrill, Clifford, Errthum, Kranendonk, Mastromatteo, O'Connor is a textbook by Dale Seymour Publications has some lessons that may pertain to this cluster with respect to two-way frequency and relative frequency tables.
- Georgia Standards of Excellence Curriculum Framework, Grade 8, Unit 6: Linear Models and Tables has many lessons that pertain to this cluster. This cluster is mostly addressed on pages 79-199.
- LOCUS (Levels of Conceptual Understanding in Statistics) has various statistical assessment items by grade level.

General Resources

- Arizona 6-8 Progression on Statistics and Probability
  This cluster is addressed on pages 11-12.
- Coherence Map by Achieve the Core is an interactive website that shows the progression and interconnectedness of the Common Core standards. To be noted, there are some differences between Ohio's Learning Standards and the Common Core, so although the tool is useful, it needs to be checked with respect to Ohio's Learning Standards.
- High School Coherence Map by UnboundED is a map of the relationships among high school mathematics standards. Please note: This resource is aligned to the Common Core not Ohio's Learning Standards. Although the tool is useful, it needs to be checked with respect to Ohio's Learning Standards.
- Statistics Teacher is an online journal published by the American Statistical Association (ASA) and NCTM. It includes articles, lesson plans, and professional development opportunities.
- Significance is a magazine that demonstrate the practical use of statistics and shows how statistics benefits society.
- Chance is a magazine about statistics for anyone who is interested in using data to advance science, education, and society.
- LOCUS is an NSF Funded project focused on developing assessments of statistical understanding. Teachers must create an account to access the assessments.
- K-12 Statistics Education Resources is a collection of websites put together by the American Statistical Association for teachers.

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