Ohio's Model Curriculum | Mathematics

with Instructional Supports

Math 2 Course

Ohio | Department of Education

Mathematics Model Curriculum

with Instructional Supports

Math 2 Course

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Introduction

PURPOSE OF THE MODEL CURRICULUM

Just as the standards are required by Ohio Revised Code, so is the development of the model curriculum for those standards. Throughout the development of the standards (2016-17) and the model curriculum (2017-18), the Ohio Department of Education (ODE) has involved educators from around the state at all levels, Pre-K–16. The model curriculum reflects best practices and the expertise of Ohio educators, but it is not a complete a curriculum nor is it mandated for use. The purpose of Ohio's model curriculum is to provide clarity to the standards, a foundation for aligned assessments, and guidelines to assist educators in implementing the standards. The model curriculum is not a collection of lessons nor a full curriculum; it does not suggest pace, sequence, or amount of time spent on topics. It provides information about a topic related to the standards including ideas for examples, strategies for teaching, and possible connections between topics in addition to highlighting some misconceptions.

COMPONENTS OF THE MODEL CURRICULUM

The model curriculum contains two sections: Expectations for Learning and Content Elaborations.

Expectations for Learning: This section begins with an introductory paragraph describing the cluster's position in the respective learning progression, including previous learning and future learning. Following are three subsections: Essential Understandings, Mathematical Thinking, and Instructional Focus.

- **Essential Understandings** are the important concepts students should develop. When students have internalized these conceptual understandings, application and transfer of learning results.
- **Mathematical Thinking** statements describe the mental processes and practices important to the cluster.
- **Instructional Focus** statements are key skills and procedures students should know and demonstrate.

Together these three subsections guide the choice of lessons and formative assessments and ultimately set the parameters for aligned state assessments.

Content Elaborations: This section provides further clarification of the standards, links the critical areas of focus, and connects related standards within a grade or course.

Introduction, continued

COMPONENTS OF INSTRUCTIONAL SUPPORTS

The Instructional Supports section contains the **Instructional Strategies** and **Instructional Tools/Resources** sections which are designed to be fluid and improving over time, through additional research and input from the field. The **Instructional Strategies** are descriptions of effective and promising strategies for engaging students in observation, exploration, and problem solving targeted to the concepts and skills in the cluster of standards. Descriptions of common misconceptions as well as strategies for avoiding or overcoming them and ideas for adapting instructions to meet the needs of all students are threaded throughout. The **Instruction Tools/Resources** are links to relevant research, tools, and technology.

There are several icons that help identify various tips in the Instructional Strategies section:

= a common misconception

= a technology tip

= a career connection

TIP! = a general tip which may include diverse learner or English learner tips.

Standards for Mathematical Practice for Math 2

The Standards for Mathematical Practice describe the skills that mathematics educators should seek to develop in their students. The descriptions of the mathematical practices in this document provide examples of how student performance will change and grow as students engage with and master new and more advanced mathematical ideas across the grade levels.

MP.1 Make sense of problems and persevere in solving them.

Students persevere when attempting to understand the differences between quadratic functions and the linear and exponential functions they studied previously. They create diagrams of geometric problems to help make sense of the problems.

MP.2 Reason abstractly and quantitatively.

Quantitative reasoning entails habits of creating a coherent representation of the problem at hand; of considering the units involved; of attending to the meaning of quantities, not just how to compute them; and of knowing and flexibly using different properties of operations and objects.

MP.3 Construct viable arguments and critique the reasoning of others.

Students construct proofs of geometric theorems based relationships between sine and cosine of complementary angles.

MP.4 Model with mathematics.

Students apply their mathematical understanding of quadratic functions to real-world problems. They also discover mathematics through experimentation and by examining patterns in data from real-world contexts.

MP.5 Use appropriate tools strategically.

Students develop a general understanding of the graph of an equation or function as a representation of that object, and they use tools such as graphing calculators or graphing software to create graphs in more complex examples, understanding how to interpret the result.

MP.6 Attend to precision.

To avoid the extraneous solutions, students make use of the definition of the solution of the equation by asking, "Does this value make the equation a correct statement?"

MP.7 Look for and make use of structure.

Students develop formulas such as $(a \pm b)^2 = a^2 \pm 2ab + b^2$ by applying the distributive property. Students see that the expression $5 + (n - 2)^2$ takes the form of 5 plus "something squared," and because "something squared" must be positive or zero, the expression can be no smaller than 5.

Continued on next page

Standards for Mathematical Practice, continued

MP.8 Look for and express regularity in repeated reasoning.

Students understand that when figures are scaled by a factor of k, the effect on their lengths, areas, and volumes remain the same such that they are multiples of k, k^2, and k^3.

Modeling

Modeling links classroom mathematics and statistics to everyday life, work, and decision-making. Modeling is the process of choosing and using appropriate mathematics and statistics to analyze empirical situations, to understand them better, and to improve decisions. Quantities and their relationships in physical, economic, public policy, social, and everyday situations can be modeled using mathematical and statistical methods. When making mathematical models, technology is valuable for varying assumptions, exploring consequences, and comparing predictions with data.

A model can be very simple, such as writing total cost as a product of unit price and number bought, or using a geometric shape to describe a physical object like a coin. Even such simple models involve making choices. It is up to us whether to model a coin as a three-dimensional cylinder, or whether a two-dimensional disk works well enough for our purposes. Other situations—modeling a delivery route, a production schedule, or a comparison of loan amortizations—need more elaborate models that use other tools from the mathematical sciences. Real-world situations are not organized and labeled for analysis; formulating tractable models, representing such models, and analyzing them is appropriately a creative process. Like every such process, this depends on acquired expertise as well as creativity.

Some examples of such situations might include the following:

- Estimating how much water and food is needed for emergency relief in a devastated city of 3 million people, and how it might be distributed.
- Planning a table tennis tournament for 7 players at a club with 4 tables, where each player plays against each other player.
- Designing the layout of the stalls in a school fair so as to raise as much money as possible.
- Analyzing the stopping distance for a car.
- Modeling a savings account balance, bacterial colony growth, or investment growth.
- Engaging in critical path analysis, e.g., applied to turnaround of an aircraft at an airport.
- Analyzing risk in situations such as extreme sports, pandemics, and terrorism.
- Relating population statistics to individual predictions.

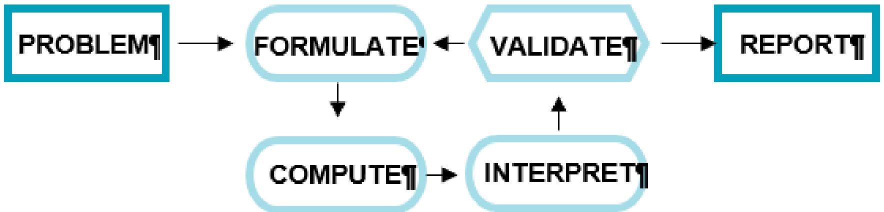
In situations like these, the models devised depend on a number of factors: How precise an answer do we want or need? What aspects of the situation do we most need to understand, control, or optimize? What resources of time and tools do we have? The range of models that we can create and analyze is also constrained by the limitations of our mathematical, statistical, and technical skills, and our ability to recognize significant variables and relationships among them. Diagrams of various kinds, spreadsheets and other technology, and algebra are powerful tools for understanding and solving problems drawn from different types of real-world situations.

Continued on next page

Modeling, continued

One of the insights provided by mathematical modeling is that essentially the same mathematical or statistical structure can sometimes model seemingly different situations. Models can also shed light on the mathematical structures themselves, for example, as when a model of bacterial growth makes more vivid the explosive growth of the exponential function.

The basic modeling cycle is summarized in the diagram. It involves (1) identifying variables in the situation and selecting those that represent essential features, (2) formulating a model by creating and selecting geometric, graphical, tabular, algebraic, or statistical representations that describe relationships between the variables, (3) analyzing and performing operations on these relationships to draw conclusions, (4) interpreting the results of the mathematics in terms of the original situation, (5) validating the conclusions by comparing them with the situation, and then either improving the model or, if it is acceptable, (6) reporting on the conclusions and the reasoning behind them. Choices, assumptions, and approximations are present throughout this cycle.



In descriptive modeling, a model simply describes the phenomena or summarizes them in a compact form. Graphs of observations are a familiar descriptive model—for example, graphs of global temperature and atmospheric CO_2 over time. Analytic modeling seeks to explain data on the basis of deeper theoretical ideas, albeit with parameters that are empirically based; for example, exponential growth of bacterial colonies (until cut-off mechanisms such as pollution or starvation intervene) follows from a constant reproduction rate. Functions are an important tool for analyzing such problems. Graphing utilities, spreadsheets, computer algebra systems, and dynamic geometry software are powerful tools that can be used to model purely mathematical phenomena, e.g., the behavior of polynomials as well as physical phenomena.

Modeling Standards

Modeling is best interpreted not as a collection of isolated topics but rather in relation to other standards. Making mathematical models is a Standard for Mathematical Practice, and specific modeling standards appear throughout the high school standards indicated by a star symbol (★).

Mathematics Model Curriculum

with Instructional Supports

Math 2 Course

STANDARDS	MODEL CURRICULUM (A.SSE.1-2)
Algebra **SEEING STRUCTURE IN EXPRESSIONS** **Interpret the structure of expressions.** **A.SSE.1**. Interpret expressions that represent a quantity in terms of its context. ★ **a.** Interpret parts of an expression, such as terms, factors, and coefficients. **b.** Interpret complicated expressions by viewing one or more of their parts as a single entity. **A.SSE.2** Use the structure of an expression to identify ways to rewrite it. *For example, to factor $3x(x - 5) + 2(x - 5)$, students should recognize that the "$x - 5$" is common to both expressions being added, so it simplifies to $(3x + 2)(x - 5)$; or see $x^4 - y^4$ as $(x^2)^2 - (y^2)^2$, thus recognizing it as a difference of squares that can be factored as $(x^2 - y^2)(x^2 + y^2)$.*	**Expectations for Learning** Students build expressions in grades K-5 with arithmetic operations. As they move into the middle grades and progress through high school, students build expressions with algebraic components, beginning with linear, exponential, and quadratic expressions. In later courses, they build algebraic expressions with polynomial, rational, radical, and trigonometric expressions. In this cluster, they focus on interpreting the components of linear, exponential, and quadratic expressions and their meaning in mathematical and real-world contexts. They also determine when rewriting or manipulating expressions is helpful in order to reveal different insights into a mathematical or real-world context. **ESSENTIAL UNDERSTANDINGS** • An expression is a collection of terms separated by addition or subtraction. • A term is a product of a number and a variable raised to a nonnegative integer exponent. • Components of an expression or expressions within an equation may have meaning in a mathematical context, e.g., $b^2 - 4ac$ in the quadratic formula indicates the number and nature of solutions to the equation. • Components of an expression may have meaning in a real-world context. • Expressions may potentially be rearranged or manipulated in ways to reveal different insights into the mathematical or real-world context. **MATHEMATICAL THINKING** • Attend to the meaning of quantities. • Use precise mathematical language • Apply grade-level concepts, terms, and properties. • Look for and make use of structure.

STANDARDS	MODEL CURRICULUM (A.SSE.1-2)
	Expectations for Learning, continued **INSTRUCTIONAL FOCUS** • Identify the components, such as terms, factors, or coefficients, of an expression and interpret their meaning in terms of a mathematical or real- world context. • Explain the meaning of each part of an expression, including linear, simple exponential, and quadratic expressions, in a mathematical or real-world context. • Analyze an expression and recognize that it can be rewritten in different ways. **Content Elaborations** **OHIO'S HIGH SCHOOL CRITICAL AREA OF FOCUS** • Math 2, Number 2, pages 4-5 **CONNECTIONS ACROSS STANDARDS** • Write expressions in equivalent forms (A.SSE.3).

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (A.SSE.1-2)

Instructional Strategies

sequence, or amount of time spent on topics. It provides information about a topic related to the standards including ideas for examples, strategies for teaching, common misconceptions, and possible connections between topics. In addition, the emphasis of the model curriculum is on <u>*instruction*</u> *not on assessment; therefore, the examples in the Instructional Strategies section are written for instructional purposes not necessarily as assessment items. Examples may need to be rewritten to accommodate the needs of each individual classroom.*

Standards for Mathematical Practice

This cluster focuses on but is not limited to the following practices:

MP.1 *Make sense of problems and persevere in solving them.*

MP.4 *Model with mathematics.*

MP.7 *Look for and make use of structure.*

The focus of this cluster is in "seeing" how each part of an algebraic expression interplays with the other parts of the expression to create meaning. Since the goal is on the interpretation of the components, the focus should not be simplifying expressions or solving equations. As the expressions become more complex, students should be able to see them built out of basic operations such as sums of terms or products of factors.

Development and proper use of mathematical language is an important building block for future content. For example, a student should recognize that in the expression $2x + 1$, "$2x$" and "1" are *terms* of the binomial, "2" is the coefficient, "2" and "x" are factors, and "1" is a constant.

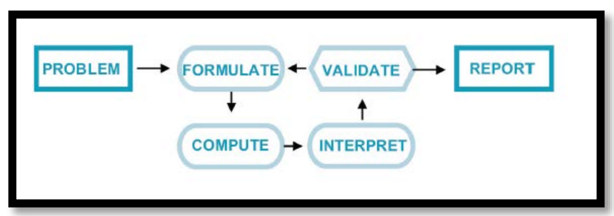
A student should also be able to see that more complicated expressions can be built from simpler ones. For example, the expression $3 + (y – 2)^2$ can be viewed as the sum of the constant term 3 and the squared term $(y – 2)^2$; therefore, viewing the expression in this manner allows a student to recognize that it is always greater than or equal to 3, because $(y – 2)^2$ is nonnegative and the sum of 3 and a nonnegative number is greater than 0. A student can also recognize that inside the squared term is the expression $y – 2$, so the square term is 0 when $y = 2$.

TIP!

Students may believe that the use of algebraic expressions is merely the abstract manipulation of symbols. To counter this idea, the use of real-world examples is very helpful. Students can be asked to explain the meaning of the parts of algebraic expression that represent the situation and provide a rationale for why one form of the expression is more beneficial than another.

MODELING

A.SSE.1 is a modeling standard. See page 12 for more information about modeling.



LINEAR, EXPONENTIAL, AND OTHER EXPRESSIONS AND EQUATIONS

Building upon prior knowledge of properties and structure of expressions, students can apply mental manipulations to solve problems.

EXAMPLE

For which values of *m* are the following inequalities true?

a. $m^2 + m < m^2 + m + 3$

b. $m^4 - m + 9 \geq -m + m^2 + m^4$

Discussion: For Part **a**., students should realize that $m^2 + m$ appears on both sides of the inequality. Therefore, the inequality is equivalent to $0 < 3$, which is a true statement, so any real number will make the inequality true. For Part **b**., students should realize that $m^4 - m$ is present on both sides of the inequality; therefore, the inequality is equivalent to $9 \geq m^2$ and $-3 \leq m \leq 3$.

EXAMPLE

Evaluate.

$$6.7 \times \frac{2}{3}\left(\frac{5.2 \times 4.3}{9.6^2}\right)(0)$$

Discussion: Students should recognize that the actual multiplication is pointless because the Zero Product Property can be applied. Therefore, the expression is equal to 0.

EXAMPLE

Solve for *b*.

a. $\frac{3}{5}(y-7)(y+2) = b(y-7)(y+2)$

b. $3(x-2) - b = 3x - 6$

Discussion: In Part **a**., students should recognize that $(y-7)(y+2)$ appears on both side of the equation, so $b = \frac{3}{5}$. In Part **b**., students should realize that $3(x-2) = 3x - 6$ by the Distributive Property, so $b = 0$.

Explore the nature of algebraic equations and systems of algebraic equations using mathematical and real-world contexts.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (A.SSE.1-2)

EXAMPLE

The equations below represent Mario's trip to the store to buy school supplies, where *m* represent binders and *n* represents folders.

$$m + n = 15$$
$$0.50n + 4.69m + 2.28 = 34.92$$

- **a.** Determine how much each binder costs.
- **b.** What could 2.28 represent in the second equation?
- **c.** What does $0.50n + 4.69m + 2.28$ in the second equation represent?
- **d.** How many different items did he buy?
- **e.** What does $0.50n$ represent?

EXAMPLE

Create two equivalent expressions. The first expression should have two terms, and the second expression should have three terms. Then, write an appropriate context for each equivalent expression and explain why the context best represents that form.

EXAMPLE

What information is true about the expression?

$$7 + (a - 1)^2$$

- **a.** It is always less than or equal to 1.
- **b.** It is always greater than or equal to 7.
- **c.** There will be two numbers for *a* that will make equivalent expressions.
- **d.** There will be no numbers for *a* that will make equivalent expressions.
- **e.** There will be three numbers for *a* that will make equivalent expressions.

Offer multiple real-world examples of exponential functions. For instance, students have to recognize that in an equation representing automobile cost $C(t) = 20{,}000(0.75)^t$, since the base of the exponential factor, 0.75, is positive and smaller than 1, it represents an exponential decay or a yearly 25% ($1 - 0.75 = 0.25$) depreciation of the initial \$20,000 value of automobile over the course of *t* years. On the contrary, in an exponential equation representing the amount of investment $A(t) = 10{,}000(1.03)^t$, over *t* years, since the base of exponential factor, 1.03, is greater than 1, it represents exponential growth or a yearly 3% increase of the initial investment of \$10,000 over the course of *t* years.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (A.SSE.1-2)

EXAMPLE

The equation below represents the amount of money Maria deposited in the bank at a fixed annual interest rate that is compounded monthly.

$$P = 2500\left(1 + \frac{0.06}{12}\right)^{12t}$$

- **a.** What does the "1 +" in the equation indicate? How would the equation change if it was "1 – "?
- **b.** What does 2500 represent in terms of the context of the situation?
- **c.** What does 0.06 represent in terms of the context of the situation?
- **d.** What does $\left(1 + \frac{0.06}{12}\right)$ represent in terms of the context of the situation?
- **e.** What does $12t$ represent in terms of the context of the situation?
- **f.** What is the contextual meaning of $\frac{0.06}{12}$?

EXTENDING TO QUADRATIC EXPRESSIONS AND EQUATIONS

This expressions in this cluster should be extended to quadratics. Quadratics should be in the various forms: standard form, factored form, and vertex form. Have students create their own expressions that meet specific criteria (e.g., number of terms factorable, difference of two squares, etc.) and verbalize how they can be written and rewritten in different forms. Additionally, pair/group students to share their expressions and rewrite one another's expressions.

Students may have a misconception that an expression cannot be factored because it does not fit into a form they recognize. Provide students with examples where they need to apply certain algebraic manipulations to transform the expression into a recognizable factorable form.

EXAMPLE

Factor the binomial completely.

$3x^2 - 12$
$3(x^2 - 4)$
$3(x - 2)(x + 2)$

EXAMPLE

Factor the trinomial completely.

$-2x^2 + 4x - 2$
$-2(x^2 - 2x + 1)$
$-2(x - 1)^2$

Discuss how to choose the most appropriate form of a quadratic equation to find the coordinate pair of the vertex of its graph; to determine an efficient method to obtain the form; and to use this form for determining the type of transformations applied to the graph of $y = x^2$ to attain a graph of the given quadratic equation.

Have students find the horizontal and vertical shift by rewriting a binomial using completing the square. Have them explain why the vertex form is useful.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (A.SSE.1-2)

Factoring expressions like $3x(x - 5) + 2(x - 5)$ is another opportunity to recognize the structure where the Distributive Property is applicable. Students should understand that the Distributive Property, $a(b + c) = ab + ac$ works in both directions and the expression $ab + ac$ factors as $a(b + c)$ because "a" is a common factor for terms "ab" and "ac". Similarly, in the expression $3x(x - 5) + 2(x - 5)$, the difference $(x - 5)$ is a common factor and the expression factors as $(x - 5)(3x + 2)$.

Technology may be useful to help students recognize that two different expressions represent the same relationship. For example, since $(x - y)(x + y)$ can be rewritten as $x^2 - y^2$, they can put both expressions into a graphing calculator (or spreadsheet) and have it generate two tables (or two columns of one table), displaying the same output values for each expression.

EXAMPLE

Multiply each pair of factors. Combine like terms where appropriate. Make a conjecture relating the terms in the resulting polynomials to the structure of the factors.

- **a.** $(h + 2)(h + 3)$; $(h + 2)(h - 3)$; $(h - 2)(h + 3)$; $(h - 2)(h - 3)$
- **b.** $(j + 5)(j + 7)$; $(j + 5)(j - 7)$; $(j - 5)(j + 7)$; $(j - 5)(j - 7)$
- **c.** $(w + 9)(w + 9)$; $(w + 9)(w - 9)$; $(w - 9)(w + 9)$; $(w - 9)(w - 9)$
- **d.** $(2g + 5)(g + 3)$; $(2g + 5)(g - 3)$; $(2g - 5)(g + 3)$; $(2g - 5)(g - 3)$

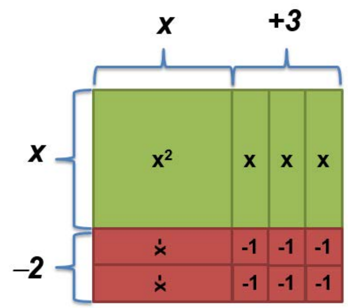
Discussion: Students should make the connection between the signs in the terms of the two binomials and the resulting trinomial. Once students make these connections, they should be able to apply that knowledge to factoring trinomials.

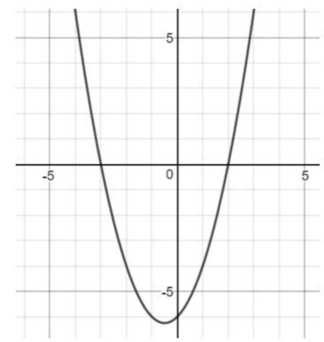
When a quadratic is factorable over integers, it can be modeled as a rectangle using algebra tiles. Explain patterns in factoring trinomials including those that can form squares—perfect square binomials and difference of squares. (See cluster A.SSE.3 for more information on factoring trinomials with algebra tiles.) After factoring a quadratic expression using algebra tiles, connect the factors represented by Algebra tiles to the graph of its corresponding parabola in order to make sense of the structure of a factored quadratic. Include examples that are perfect trinomials, a difference of squares, and un-factorable.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (A.SSE.1-2)

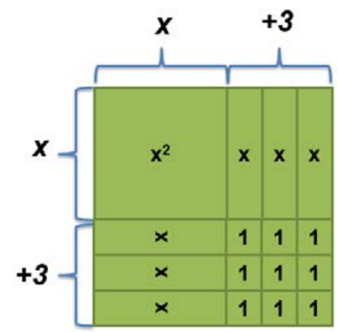
EXAMPLE

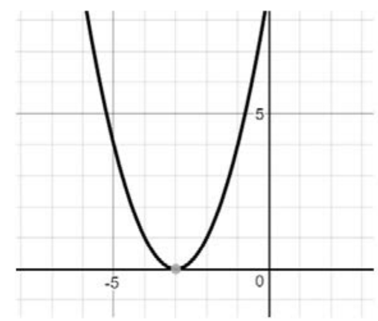
- Factor $x^2 + x - 6$ using algebra tiles, and then graph the quadratic. What connections do you see between the graph and the factored form of the quadratic?





- Factor $x^2 + 6x + 9$ using algebra tiles, and then graph the quadratic. What connections do you see between the graph and the factored form of the quadratic?





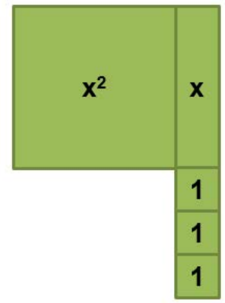
Example continued on next page

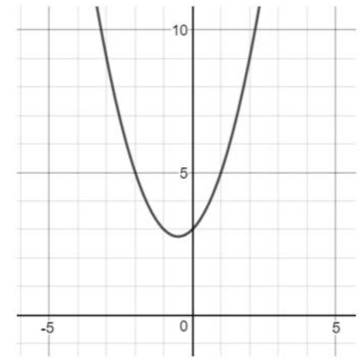
INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (A.SSE.1-2)

EXAMPLE, CONTINUED

- Factor $x^2 + x + 3$ using algebra tiles, and then graph the quadratic. What connections do you see between the graph and the factored form of the quadratic?

Discussion: In part **a.**, the students should write the factored expression $(x + 3)(x – 2)$. Connect how changing the structure of the quadratic trinomial will reveal the zeros, –3 and 2, in the graph. Students should see that the polynomial in factored form will allow one to “see” the solutions to a quadratic equation without graphing. In part **b.**, students should realize that since the quadratic forms a perfect square, there is only one zero, –3. In part **c.**, students should realize that the quadratic cannot be factored since the Algebra tiles cannot form a rectangle, and they can make a connection between the model and the parabola which does not have any zeros.





EXAMPLE

A baseball is thrown into the air which is illustrated by the equation below. Rewrite in vertex form, and then answer the questions.
$h = 75t + 5 – 16t^2$

- **a.** Which form is better to “see” the initial height?
- **b.** Determine the initial height of the rocket before launching.
- **c.** Which form is better to “see” the maximum height?
- **d.** Determine the maximum height of the rocket.

Factoring by grouping is another example of how students might analyze the structure of an expression. To factor $3x(x – 5) + 2(x – 5)$, students should recognize that the “$x – 5$” is common to both expressions being added, so it simplifies to $(3x + 2)(x – 5)$. Students should become comfortable with rewriting expressions in a variety of ways until a structure emerges.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (A.SSE.1-2)

Instructional Tools/Resources

These tools and resources may be helpful for instruction. Before use, they should be evaluated at the local level to determine their appropriateness for instruction.

Manipulatives/Technology

- Hands-on materials, such as algebra tiles, can be used to establish a visual understanding of algebraic expressions and the meaning of terms, factors and coefficients.
- Desmos is a free graphing calculator that is available to students as website or an app.
- GeoGebra is a free graphing calculator that is available to students as website.
- Wolframalpha is dynamic computing tool.

Factoring with Algebra Tiles

- Factoring Trinomials –With Algebra Tiles by Braining Camp is a YouTube video explain factoring trinomials with algebra tiles.
- Factoring Polynomials with Algebra Tiles (2) by Tom Horn is a YouTube video explain factoring trinomials with algebra tiles.
- Algebra Tiles 5: Factoring Trinomials by Simpson Math is a YouTube video explain factoring trinomials with algebra tiles.
- Algebra tile templates on the SMART Exchange has a variety of useful models that teacher can use if they have access to a SMART Board.
- Multiplying Binomials and Factoring Trinomials using Algebra Tiles and Generic Rectangle is a worksheet from West Contra Costa Unified School District about using algebra tiles.
- Advanced Algebra Tile Factoring is an applet by eMathLab.com that allows for factoring trinomials when *a* does not equal 1.

Interpreting Quadratic Expressions and/or Equations

- Seeing Dots is an Illustrative Mathematics task has students interpret two algebraic expressions in terms of quadratics and a geometric context. This task integrates modeling.
- Throwing Horseshoes is an Illustrative Mathematics task explores the structure of a real-world quadratic equation.
- Profit of a Company, Assessment is an Illustrative Mathematics task explores the structure of a real-world quadratic equation with respect to profit.
- Generating Polynomials from Patterns is a Formative Assessment Lesson from Mathematics Assessment Project that uses visual models to help students manipulate polynomials. This lesson should be discussed in context of structure of quadratic expressions with respect to which forms make the most sense in the context of the problem.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (A.SSE.1-2)

Interpreting Other Expressions and/or Equations

- Mixing Fertilizer is an Illustrative Mathematics task that applies ratio and proportions to mixture problems.
- The Physics Professor is an Illustrative Mathematics task has students drawing conclusions about expressions using information they already know.

Curriculum and Lessons from Other Sources

- EngageNY, Algebra 1, Module 4, Topic A, Lesson 1: Multiplying and Factoring Polynomial Expressions, Lesson 2: Multiplying and Factoring Polynomial Expressions, Lesson 3: Advanced Factoring Strategies for Quadratic Expressions, and Lesson 4: Advanced Factoring Strategies for Quadratic Expressions are lessons that pertain to this cluster.
- EngageNY, Algebra 1, Module 4, Topic B, Lesson 11: Completing the Square and Lesson 12: Completing the Square are lessons that pertain to this cluster.
- Mathematics Vision Project, Algebra 1, Module 6: Quadratic Functions has lessons that pertain to this cluster.
- Illustrative Mathematics, Algebra 1, Unit 6, Lesson 2: How Does It Change?, Lesson 3: Building Quadratic Functions from Geometric Patterns, Lesson 8: Equivalent Quadratic Expressions, Lesson 9: Standard and Factored Form are lessons that pertain to this cluster.
- Illustrative Mathematics, Algebra 1, Unit 7, Lesson 6: Rewriting Quadratic Expressions in Factored Form (Part 1), Lesson 7: Rewriting Quadratic Expressions in Factored Form (Part 2), Lesson 8: Rewriting Quadratic Expressions in Factored Form (Part 3), Lesson 10: Rewriting Quadratic Expressions in Factored Form (Part 4), Lesson 11: What are Perfect Squares?, Lesson 12: Completing the Square (Part 1), Lesson 14: Completing the Square (Part 3), Lesson 19: Deriving the Quadratic Formula, Lesson 22: Rewriting Quadratic Expressions in Vertex Form,
- Illustrative Mathematics, Geometry, Unit 6, Lesson 5: Squares and Circles, and Lesson 9: Equations of Lines are lessons that pertains to this cluster.

General Resources

- Arizona High School Progressions on Algebra is an informational document for teachers. This cluster is addressed on pages 4-6 and pages 11-12.
- Arizona High School Progression on Modeling is an informational document for teachers.
- High School Coherence Map by UnboundED is a map of the relationships among high school mathematics standards. Please note: This resource is aligned to the Common Core not Ohio's Learning Standards. Although the tool is useful, it needs to be checked with respect to Ohio's Learning Standards.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (A.SSE.1-2)

References

- Common Core Standards Writing Team. (2013, March 1). *Progressions for the Common Core State Standards in Mathematics (draft). High School, Algebra*. Tucson, AZ: Institute for Mathematics and Education, University of Arizona.
- Common Core Standards Writing Team. (2013, July 4). *Progressions for the Common Core State Standards in Mathematics (draft). High School, Modeling*. Tucson, AZ: Institute for Mathematics and Education, University of Arizona.
- Gurl, T., Artzt, A., Sultan, A., & Curcio, F. (2012). *Implementing the Common Core State Standards through Mathematical Problem Solving*. Reston, VA: National Council of Teachers of Mathematics.
- Kieran, C. (2007). Learning and teaching algebra at the middle school level through college levels. In F. K. Lester (Ed.), *Second Handbook of Research on Mathematics Teaching and Learning* (pp. 707-762). Charlotte, NC: Information Age Publishing.

STANDARDS	MODEL CURRICULUM (A.SSE.3)
Algebra **SEEING STRUCTURE IN EXPRESSIONS** **Write expressions in equivalent forms to solve problems.** **A.SSE.3** Choose and produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression. ★ **a.** Factor a quadratic expression to reveal the zeros of the function it defines. **b.** Complete the square in a quadratic expression to reveal the maximum or minimum value of the function it defines. **c.** Use the properties of exponents to transform expressions for exponential functions. *For example, 8^t can be written as 2^{3t}.*	**Expectations for Learning** Previously, students rewrite exponential expressions using properties of exponents. In Math 2, students rewrite quadratic expressions by factoring and completing the square, and they use these forms to analyze the graphs of the functions they define. In Math 3, students use these skills to analyze higher degree polynomial functions. **ESSENTIAL UNDERSTANDINGS** • Expressions may potentially be rearranged or manipulated in ways to reveal different insights into the mathematical or real-world context. • The factored form of a quadratic expression reveals the zeros of the function it defines. • The vertex form of a quadratic expression reveals the vertex and the maximum or minimum value of the function it defines. • Completing the square of a quadratic expression generates the vertex form of a quadratic expression. • Understanding the properties of exponents is essential for rewriting exponential expressions. **MATHEMATICAL THINKING** • Plan a solution pathway. • Determine the appropriate form of an expression in context. **INSTRUCTIONAL FOCUS** **For quadratic functions, students should work with expressions in which the leading coefficient can be any real number, but assessment questions should focus on expressions with leading coefficients of 1 with occasional questions using other simple leading coefficients. Also, exponential expressions should be limited to expressions with integer exponents.* • Determine the appropriate equivalent form of an expression for a given purpose. • Factor a quadratic expression so that the zeros of the function it defines can be identified. • Complete the square for a quadratic expression to identify the vertex and maximum or minimum value of the function it defines. • Rewrite exponential expressions by using properties of exponents. *Continued on next page*

STANDARDS	MODEL CURRICULUM (A.SSE.3)
	Content Elaborations **OHIO'S HIGH SCHOOL CRITICAL AREA OF FOCUS** • Math 2, Number 2, pages 4-5 **CONNECTIONS ACROSS STANDARDS** • Interpret key features of graphs (F.IF.4). • Interpret the structure of expressions (A.SSE.1-2). • Analyze functions using different representations (F.IF.8).

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (A.SSE.3)

Instructional Strategies

Note: The model curriculum is not a collection of lessons nor a full curriculum; it does not suggest pace, sequence, or amount of time spent on topics. It provides information about a topic related to the standards including ideas for examples, strategies for teaching, common misconceptions, and possible connections between topics. In addition, the emphasis of the model curriculum is on instruction not on assessment; therefore, the examples in the Instructional Strategies section are written for instructional purposes not necessarily as assessment items. Examples may need to be rewritten to accommodate the needs of each individual classroom.

This cluster extends the previous cluster (A.SSE.1-2) from analyzing structure to now using the most efficient equivalent form to solve a problem. It focuses specifically on transforming exponential equations using properties and rewriting quadratics in factored form and vertex form. Students are also connecting the various forms of an expression to the context of the problem and to the analysis of its corresponding graph.

Standards for Mathematical Practice

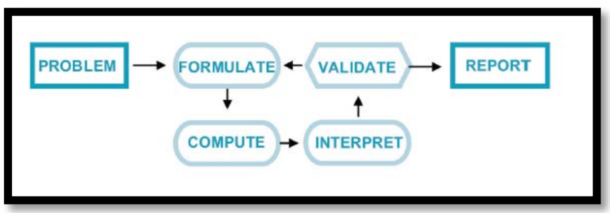
This cluster focuses on but is not limited to the following practices:

***MP.4** Model with mathematics.*

***MP.7** Look for and make use of structure.*

***MP.8** Look for and express regularity in repeated reasoning.*

Students must understand the idea that changing the forms of expressions, such as factoring, completing the square, or transforming expressions from one quadratic form to another are not independent algorithms that are learned for the sake of symbolic manipulations. They are processes that are guided by goals (e.g., investigating properties of families of functions, solving contextual problems, finding roots, and identifying maximum or minimum values). An expression can be written in many forms that may look different but are in fact equivalent, as each form still represents the given expression. Rewriting an expression in simplified form may not always be the best form for all situations. It is much more advantageous for students to think about which equivalent form would be the most useful for a particular context instead of always immediately simplifying.



MODELING

A.SSE.3 is a modeling standard. See page 12 for more information about modeling.

PROPERTIES OF EXPONENTS

In Math 1 students explored the properties of exponents. Now they need to expand their knowledge to more complicated situations.

Properties of Integer Exponents

For any nonzero real numbers *a* and *b* and integers *n* and *m*:

1. $a^0 = 1$

2. $a^{-n} = 1/a^n$

3. $a^n a^m = a^{n+m}$

4. $(a^n)^m = a^{nm}$

5. $a^n b^n = (ab)^n$

6. $\frac{a^n}{a^m} = a^{n-m}$

QUADRATIC EXPRESSIONS

Students should apply the understandings of the structure of polynomials in factored form and vertex form from the previous cluster (A.SSE.1-2) to this cluster and choose the best form of a quadratic expression needed to solve real-world and mathematical problems.

To give a student a clearer picture of what method to use and when to use it, stress that each chosen method results in different information:

- The solutions of quadratic equations solved by factoring are the x-intercepts of the parabola or zeros of quadratic functions.
- A pair of coordinates (h, k) from the vertex form $y = a(x - h)^2 + k$ represents the vertex of the parabola, where h is a horizontal shift and k is a vertical shift of the "parent" parabola $y = x^2$ from its original position at the origin.
- A vertex (h, k) is the minimum point of the graph of the quadratic function if $a > 0$ and is the maximum point of the graph of the quadratic function if $a < 0$. Understanding the algorithm of completing the square provides a solid foundation for deriving the quadratic formula.

Students may incorrectly think that the vertex (minimum) of the graph of $y = (x + 5)^2$ is shifted to the right of the vertex (minimum) of the graph $y = x^2$ due to the addition sign. Students should explore examples both analytically and graphically to overcome this misconception.

Comparing different forms of expressions, equations, and graphs helps students to understand key connections among arithmetic, algebra, and applications of geometry. Have students derive information about a function's equation, represented in standard, factored, or vertex form, by investigating its graph. The reverse thinking technique (a process that allows working backwards from the answer to the starting point) can be very effective.

Some students may incorrectly believe that the minimum of the graph of a quadratic function always occurs at the y-intercept. Students should be provided with plenty of opportunities to investigate graphs and make connections between the vertices, vertex forms, and transformations of parabolas.

Some students may believe that factoring and completing the square are isolated techniques within a unit of quadratic equations. Teachers should help students see the value of these skills for solving real-world problems and the connections between forms.

Factoring

For quadratic functions, students should work with expressions in which the leading coefficient can be any real number, but assessment questions should focus on expressions with leading coefficients of 1 with occasional questions using other simple leading coefficients.

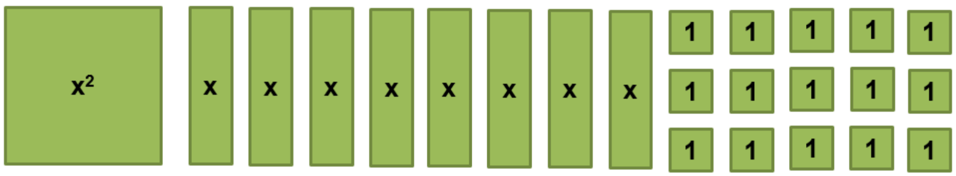
Factoring can be a challenging concept for students. Make connections between factoring of whole number, area models such as algebra tiles, and factoring of polynomials. Students should be familiar with multiplying and dividing using area models from earlier grades. Area models could also help students to make the connection between factoring a polynomial and division.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (A.SSE.3)

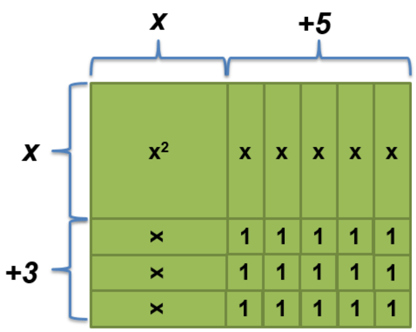
EXAMPLE

Factor $x^2 + 8x + 15$ using a model.

Represent $x^2 + 8x + 15$.



Arrange tiles into a rectangle and find its length and width.

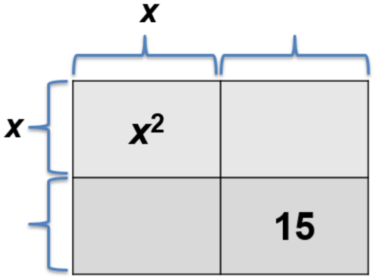


This can be modeled using algebra tiles. Since $x^2 + 8x + 15$ is the area, have students arrange the tiles into a rectangle.

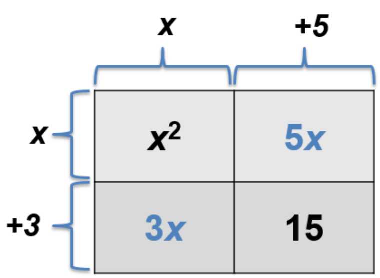
The length ($x + 5$) and the width ($x + 3$) of the rectangle are also the polynomial's factors.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (A.SSE.3)

Eventually students can move away from the actual tiles to rectangles, where they fill in the bottom left and top right corners.



They can then guess and check using factors of b in $ax^2 + bx + c$ to figure out the expressions in the empty of the corners of the rectangles.



15
5 · 3
3 · 5
15 · 1
1 · 15

More detailed explanations on factoring with algebra tiles as well as applets can be found in the Instructional Resources/Tools section.

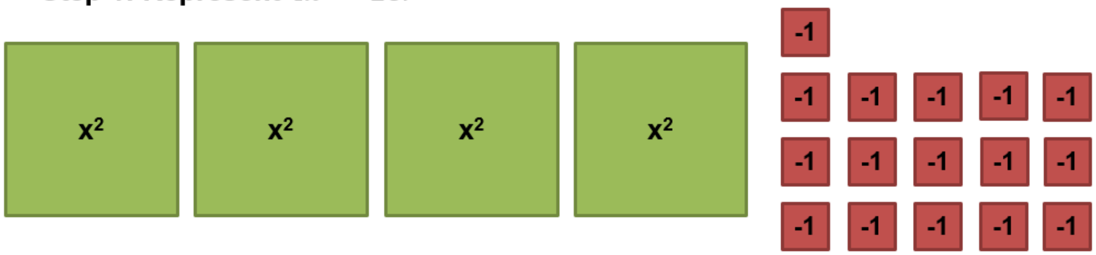
Students should also work with perfect square trinomials and differences of squares.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (A.SSE.3)

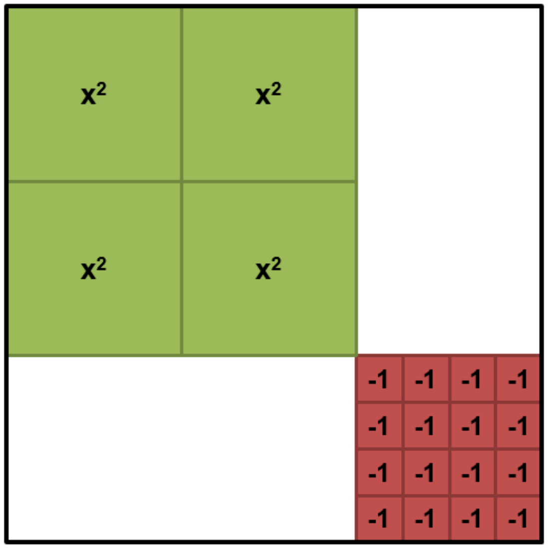
EXAMPLE

Factor $4x^2 - 16$ using a model.

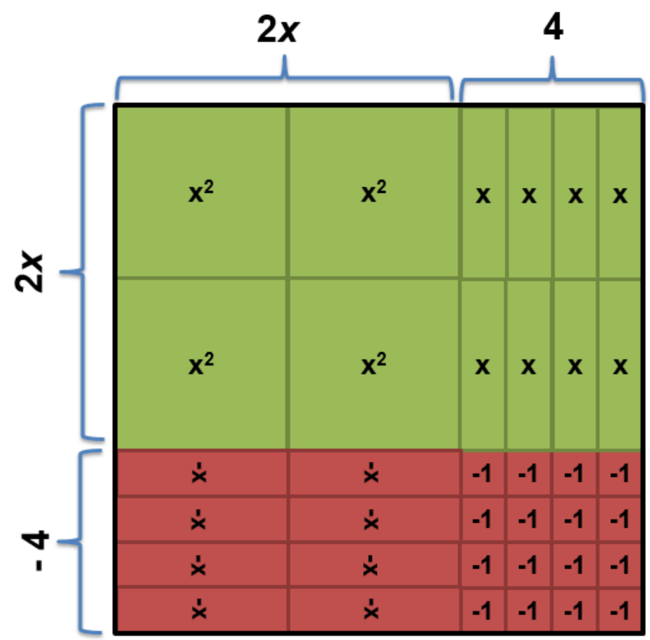
Step 1: Represent $4x^2 - 16$.



Step 2: Create a rectangle.



Step 3: Fill in the missing tiles and find the factors.

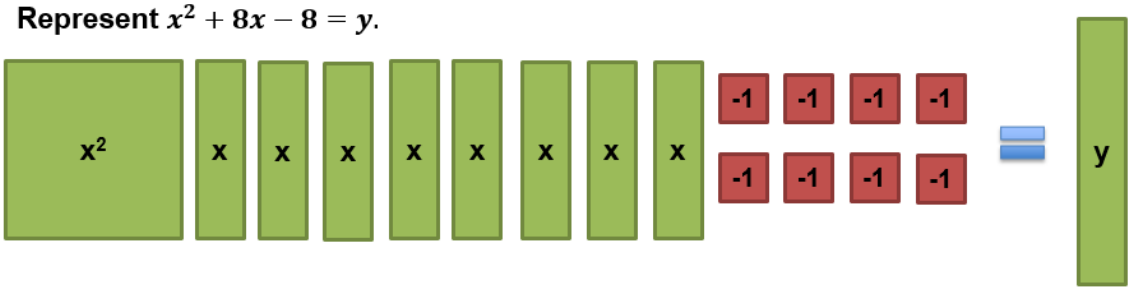


Ohio | Department of Education

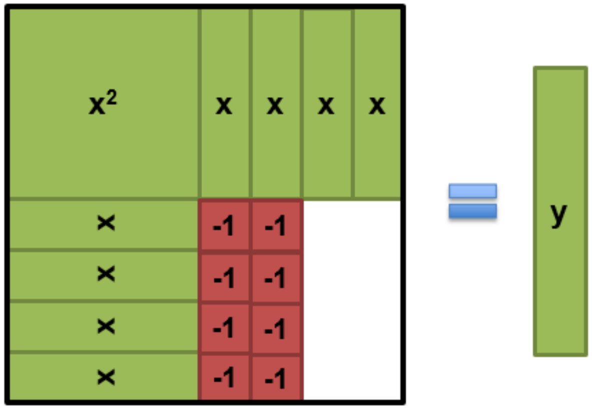
Completing the Square
Using algebra tiles for factoring should make an easy transition for students to conceptually understand completing the square.

EXAMPLE
Rewrite $x^2 + 8x - 8 = y$ in vertex form by completing the square.

Represent $x^2 + 8x - 8 = y$.



Then rearrange the tiles (excluding the y) to make a square.

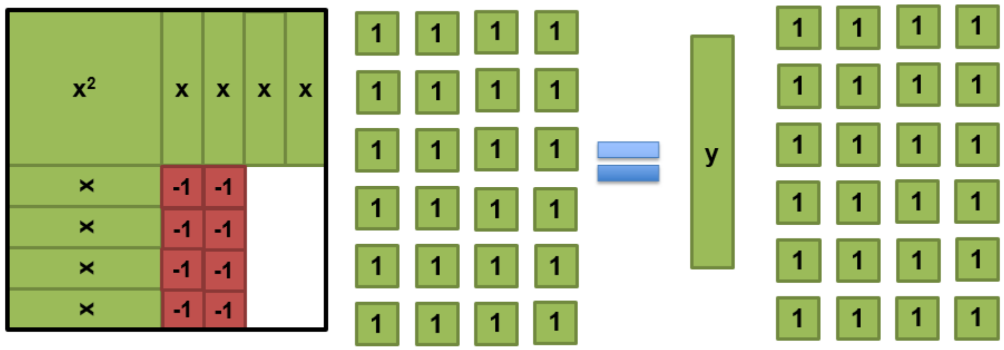


However, a problem arises because the units do not make a square. The student has to figure out how many more units are needed to make a square. The fact that the units that are present are negative is also a problem. For if all the x-term tiles are positive, the units also have to be positive in order to make a perfect square.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (A.SSE.3)

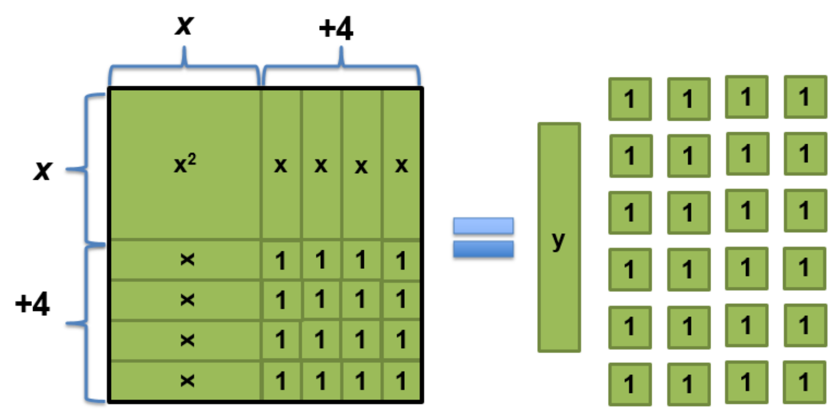
Students should realize that there needs to be 16 positive unit tiles to make a square; however, they also need to take into account the—8-unit tiles already present. Therefore, the student needs to add 16 + 8 or 24 unit tiles to the problem. But to maintain equivalency, 24 has to be added to both sides of the equation:

$x^2 + 8x - 8 + \boxed{24} = y + \boxed{24}$
$x^2 + 8x + 16 = y + 24$



Now he or she can finish factoring the equation.

So, the student gets $(x + 4)^2 = y + 24$ which can be rewritten in vertex form $y = (x + 4)^2 - 24$.



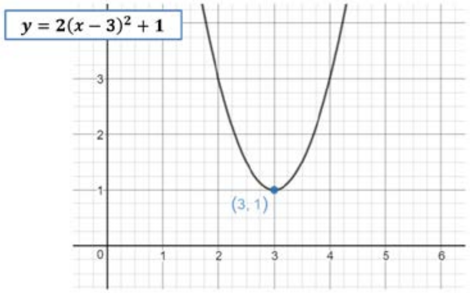
Discussion: After working with the algebra tiles, students should be able to generalize the rule of adding $\left(\frac{b}{2}\right)^2$ to both sides of the equation when completing the square in the form of $ax^2 + bx + c = 0$, where $a = 1$. Discuss with students how to approach problems when $a \neq 1$. Students may want to begin with dividing both sides by a to get a coefficient of 1 in front of the leading term to get $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$. This allows students to further generalize the rule that $\left(\frac{b}{2a}\right)^2$ is really the term that is being added to both sides of the equation. This concept could be further extended towards deriving the quadratic formula. See Completing the Square.gov which is an applet from Wikipedia Commons that shows how to complete the square in the form of $ax^2 + bx + c = 0$ when $a \neq 1$.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (A.SSE.3)

Students should be fluent in expanding binomials of the form $(a + b)^2$, $(a – b)^2$ and $(a – b)(a + b)$ which will help them recognize perfect square and difference of squares trinomials when they need to factor them.

Vertex Form

Connect completing the square to the vertex form of an equation. Draw attention to the fact that the vertex form, $y = a(x – h)^2 + k$, of a quadratic expression reveals the vertex (h, k) and the maximum or minimum of the function it defines.



Instructional Tools/Resources

These tools and resources may be helpful for instruction. Before use, they should be evaluated at the local level to determine their appropriateness for instruction.

Manipulatives/Technology

- Graphing utilities to explore the effects of parameter changes on a graph
- Desmos is a free graphing calculator that is available to students as website or an app.
- GeoGebra is a free graphing calculator that is available to students as website.
- Wolframalpha is dynamic computing tool.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (A.SSE.3)

Factoring with Algebra Tiles

- Factoring Trinomials –With Algebra Tiles by Braining Camp is a YouTube video that explains factoring trinomials with algebra tiles.
- Factoring Polynomials with Algebra Tiles (2) by Tom Horn is a YouTube video that explains factoring trinomials with algebra tiles.
- Algebra Tiles 5: Factoring Trinomials by Simpson_Math is a YouTube video that explains factoring trinomials with algebra tiles.
- Algebra tile templates on the SMART Exchange has a variety of useful models that teacher can use if they have access to a SMART Board.
- Multiplying Binomials and Factoring Trinomials using Algebra Tiles and Generic Rectangle is a worksheet from West Contra Costa Unified School District about using algebra tiles.
- Advanced Algebra Tile Factoring is an applet by eMathLab.com that allows for factoring trinomials when *a* does not equal 1.
- Lesson Plan 1: The X Factor – Trinomials and Algebra Tiles is a website that contains a lesson and a workshop that showcases ways that teachers can help students explore mathematical properties studied in Algebra. The activities use a variety of techniques to help students understand concepts of factoring quadratic trinomials.

Difference of Squares

- Difference of Squares from the National Council of Teachers of Mathematics, Illuminations is an activity that uses a series of related arithmetic experiences to prompt students to explore arithmetic statements leading to a factoring pattern for the difference of two squares. A geometric interpretation of the familiar formula is also included.

Completing the Square

- 3.3: Part 1, lesson 1 (Completing the Square using Algebra Tiles) is a YouTube video by martensmath that shows how to use algebra tiles to complete the square.
- Completing the Square-With Algebra Tiles is a YouTube video by Brainingcamp that shows how to use algebra tiles to complete the square.
- Using Algebra Tiles to Complete the Square is a YouTube video by Stacie Bender that shows how to use algebra tiles to complete the square.

Different Forms of Functions

- Graphs of Quadratic Functions is an Illustrative Mathematics task that explores graphing quadratics. It is most effective after students have graphed parabolas in vertex form but have not yet explored graphs in different forms.
- Profit of a Company is an Illustrative Mathematics Task that compares the usefulness of different forms of quadratic expressions.
- Forms of Exponential Expressions is an Illustrative Mathematics Task that investigates usefulness of different exponential expressions.
- Representing Quadratic Forms Graphically by Mathematics Assessment Project is a lesson where students interpret different algebraic forms of a quadratic function to interpret a graph.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (A.SSE.3)

Curriculum and Lessons from Other Sources

- Mathematics Vision Project, Algebra 1, Module 7: Structure of Expressions has many lessons that pertain to this cluster.
- Georgia Standards of Excellence Curriculum Frameworks, Algebra 1, Unit 3: Modeling and Analyzing Quadratic Functions has several tasks that are related to this cluster. This cluster is addressed on pages 32-61, 73-92, and 148-173.
- EngageNY, Algebra 1, Module 4, Topic A, Lesson 1: Multiplying and Factoring Polynomial Expressions, Lesson 2: Multiplying and Factoring Polynomial Expressions, Lesson 3: Advanced Factoring Strategies for Quadratic Expressions, Lesson 4: Advanced Factoring Strategies for Quadratic Expressions, Lesson 9: Graphing Quadratic Function from Factored Form, $f(x) = a(x - m)(x - n)$ are lessons that pertain to this cluster.
- EngageNY, Algebra 1, Module 4, Topic B, Lesson 11: Completing the Square, Lesson 12: Completing the Square, and Lesson 16: Graphing Quadratic Equations from the Vertex Form, $y = a(x - h)^2 + k$ are lessons that pertain to this cluster.
- Illustrative Mathematics, Algebra 1, Unit 6, Lesson 8: Equivalent Quadratic Expressions, Lesson 9: Standard and Factored Form, Lesson 10: Graphs of Functions in Standard and Factored Forms, Lesson 13: Graphing the Standard Form (Part 2) are lessons that pertain to this cluster.
- Illustrative Mathematics, Algebra 1, Unit 7, Lesson 9: Solving Quadratic Equations by Using Factored Form, Lesson 22: Rewriting Quadratic Expressions in Vertex Form, Lesson 23: Using Quadratic Expressions in Vertex Form to Solve Problems,

General Resources

- Arizona High School Progression on Algebra is an informational document for teachers. This cluster is address on pages 4-6 and pages 11-12.
- Arizona High School Progression on Modeling is an informational document for teachers. This cluster is addressed on pages 6-7 and 13.
- High School Coherence Map by UnboundED is a map of the relationships among high school mathematics standards. Please note: This resource is aligned to the Common Core not Ohio's Learning Standards. Although the tool is useful, it needs to be checked with respect to Ohio's Learning Standards.

References

- Common Core Standards Writing Team. (2013, March 1). *Progressions for the Common Core State Standards in Mathematics (draft). High School, Algebra*. Tucson, AZ: Institute for Mathematics and Education, University of Arizona.
- Common Core Standards Writing Team. (2013, July 4). *Progressions for the Common Core State Standards in Mathematics (draft). High School, Modeling*. Tucson, AZ: Institute for Mathematics and Education, University of Arizona.
- Davis, B. (August 2015). Exponentiation: A new basic? *Mathematics Teaching in the Middle School*, *21(1),* 34-41.
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<table>
<tr><th>STANDARDS</th><th>MODEL CURRICULUM (A.APR.1)</th></tr>
<tr><td>
<h2>Algebra</h2>
ARITHMETIC WITH POLYNOMIAL AND RATIONAL EXPRESSIONS
Perform arithmetic operations on polynomials.
A.APR.1 Understand that polynomials form a system analogous to the integers, namely, that they are closed under the operations of addition, subtraction, and multiplication; add, subtract, and multiply polynomials.
a. Focus on polynomial expressions that simplify to forms that are linear or quadratic. (A1, M2)
</td><td>
<h2>Expectations for Learning</h2>
In previous courses, students develop an understanding of the properties of integers as a number system under the operations of addition, subtraction, and multiplication. They also learn to combine like terms and simplify linear expressions. In this cluster, students explore the commonalities and differences between integers and polynomials regarding the operations of addition, subtraction, and multiplication. Students will also simplify linear and quadratic expressions, or those that simplify to linear or quadratic. In Math 3, students extend these ideas to include higher-degree polynomials.

ESSENTIAL UNDERSTANDINGS
Polynomials form a system (like the integers) in which addition, subtraction, and multiplication always result in another polynomial, but sometimes division does not.
MATHEMATICAL THINKING
Compute accurately and efficiently.
Use different properties of operations flexibly.
Recognize and apply mathematical concepts, terms, and their properties.
Draw a picture or create a model to represent mathematical thinking.
INSTRUCTIONAL FOCUS
Add, subtract, and multiply polynomial expressions, focusing on those that simplify to linear or quadratic expressions.
<h2>Content Elaborations</h2>
OHIO'S HIGH SCHOOL CRITICAL AREA OF FOCUS
Math 2, Number 2, pages 4-5
CONNECTIONS ACROSS STANDARDS
Interpret the structure of expressions (A.SSE.1).
</td></tr>
</table>

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (A.APR.1)

Instructional Strategies

Note: The model curriculum is not a collection of lessons nor a full curriculum; it does not suggest pace, sequence, or amount of time spent on topics. It provides information about a topic related to the standards including ideas for examples, strategies for teaching, common misconceptions, and possible connections between topics. In addition, the emphasis of the model curriculum is on instruction not on assessment; therefore, the examples in the Instructional Strategies section are written for instructional purposes not necessarily as assessment items. Examples may need to be rewritten to accommodate the needs of each individual classroom.

Standards for Mathematical Practice
This cluster focuses on but is not limited to the following practices:
MP.6 *Attend to precision.*
MP.8 *Look for and express regularity in repeated reasoning.*

The primary strategy for this cluster is to make connections between arithmetic of integers and arithmetic of polynomials. In order to understand this standard, students need to work toward both understanding and fluency with polynomial arithmetic. Furthermore, to talk about their work, students will need to use correct vocabulary, such as integer, monomial, polynomial, factor, and term.

CLOSURE

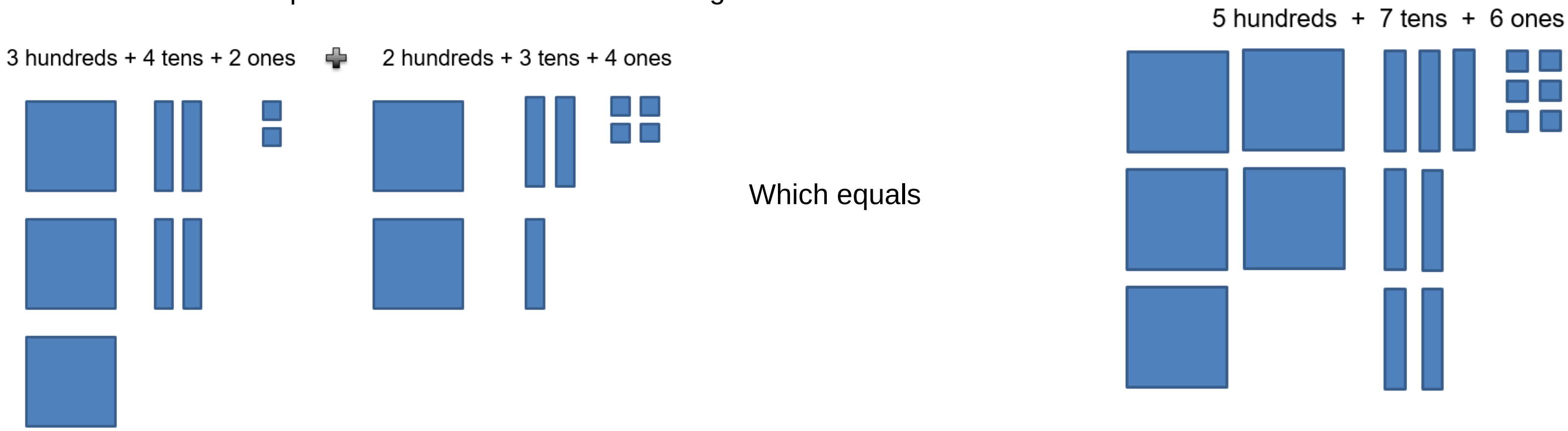
Closure is the new concept in this standard. A set of numbers is *closed* over a certain operation if an operation performed between two of the members of a set results in a member of the same set. For example, anytime two or more integers are added, the result is an integer. The same holds true for multiplication and subtraction of integers. It does not hold true for the division of integers since 3 ÷ 4 = 0.75 (a rational number). Therefore integers are closed under the operations of addition, subtraction, and multiplication, but they are not closed under division. Like integers, the system of polynomials behaves the same way. The emphasis here should not be on the term closure, but rather that a polynomial results if two or more polynomials, are added, subtracted, and/or multiplied together. Like integers, closure is not always true for the division of polynomials. For examples $(3x^2 + 6x – 2) + (4x – 9)$ can be added to obtain another polynomial $3x^2 + 10x – 11$, so polynomials are closed over addition. Students should use counterexamples to prove that polynomials are not closed under division. For example, $(x + 1) \div (2x^3 + x – 5)$ does not result in a polynomial, therefore polynomials are not closed under division. Of course, the quotient of two polynomials is sometimes a polynomial. For example, $(x^2 - 9) \div (x - 3) = x + 3$.

Students need to reason that the sum (difference or product) of any two polynomials is indeed a polynomial. At first, restrict attention to polynomials with integer coefficients. Later, students should consider polynomials with rational or real coefficients and reason that such polynomials are closed under these operations. Draw attention to the fact that rational expressions such as $\frac{2}{3}x^2 + \frac{1}{5x} - 6$ are not polynomials.

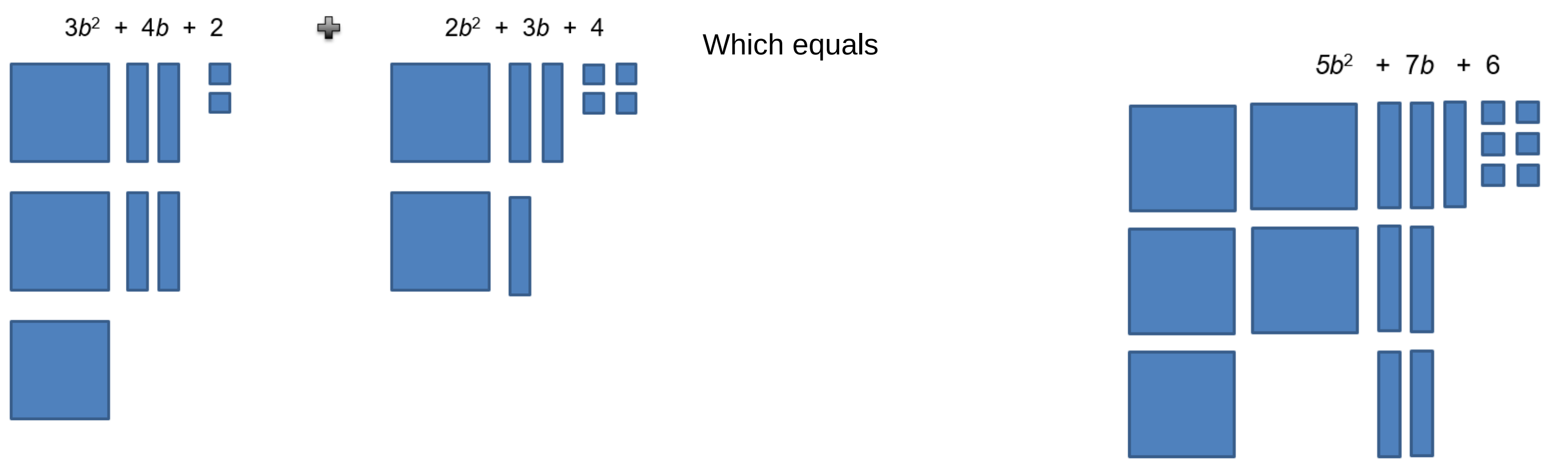
When dealing with fractional coefficients students may place the variable anywhere in the fraction (as can be done with the negative sign.) Draw attention to the fact that $\frac{2}{3}x^2 + \frac{1}{5}x - 6$ is not equivalent to $\frac{2}{3}x^2 + \frac{1}{5x} - 6$ since the second expression is a rational expression not a polynomial.

CONNECTING OPERATIONS IN BASE TEN TO POLYNOMIALS

In elementary school students learn to add and subtract whole numbers with base ten blocks. For example, to add 342 and 234 students could write the numbers in expanded form such as the following:



In Algebra, students can apply the understanding that they developed with respect to place value of whole numbers to polynomials that have integer coefficients using dimensionality. For example, a 1 can be represented by a unit square, a *b* can be represented by a stick (1 unit by *b* units in lengths), a b^2 can be represented by a square (*b* units by *b* units in lengths), and a b^3 can be represented by a cube (*b* units by *b* units by *b* units in lengths). In addition, *by* can be represented by a rectangle that is *b* units in width and *y* units in length.

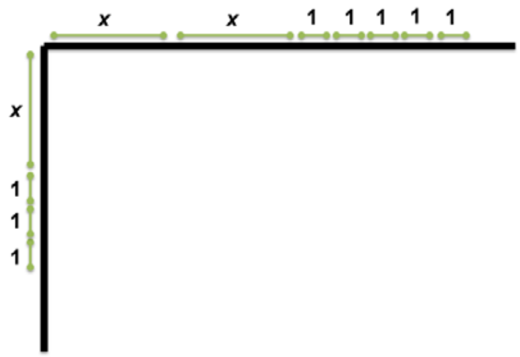


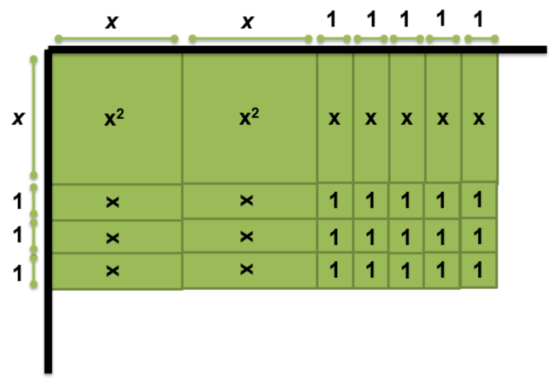
INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (A.APR.1)

The use of algebra tiles in adding and subtracting polynomials allows students to make a connection between whole numbers and polynomials. It also allows students to see each term as a distinct entity. For example, x-shaped pieces are different than x^2-shaped pieces which are also different then y^2-shaped pieces. This prevents the common error of misapplying exponent rules by adding x^2 and x. The dimensionality of the algebra tiles also allows for a geometric connection to area and even to volume. Negative integers can be represented by different color tiles just like integer chips.

Algebra tiles and/or area models can also be used to multiply polynomials. The benefit of using these models is that allows students to not only conceptualize multiplication of polynomials, but it also helps students make meaning out of factoring binomials by visualizing the connection to multiplication. For more information on factoring using algebra tiles, see A.SSE.3.

Example
Multiply $(2x + 5)(x + 3)$.





Discussion: This leads students to see that since there are two x^2-shaped pieces, eleven x-shaped pieces and fifteen square unit-shaped pieces, the product of $(2x + 5)(x + 3) = 2x^2 + 11x + 15$. This model can also be used with polynomials where x is multiplied by x^2 with the resulting piece being a cube or an x-term multiplied by a y-term with the resulting piece being a rectangle with an x-width and y-length.

TIP! Rearranging polynomials is a great opportunity to reinforce the Properties of Operations.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (A.APR.1)

Once students conceptually understand multiplying polynomials, they may choose to move more towards using a box instead of using the Algebra tiles. This follow's Bruner's levels of representation: concrete (enactive), pictorial (iconic), abstract. In addition this connects multiplying polynomials with the area models students used to understand multiplication in elementary school. Also, using the boxes as a strategy instead of algebra tiles allow for the use of coefficients involving rational numbers.

EXAMPLE

Multiply (y – 4)(2y + 3)

	y	–4
2y		
3		

=

	y	–4
2y	$2y^2$	$-8y$
3	$3y$	-12

Discussion: Therefore (y – 4)(2y + 3) equals $2y^2 - 5y - 12$. This model is not limited to the multiplication of binomials but can be used with the multiplication of any polynomials by extending the rectangle. After students have had experiences with the models, there should be a discussion about how the multiplication of polynomials can be done without using models. However, students should be able to use whichever method they feel the most comfortable.

In arithmetic of polynomials, emphasize that the central idea of using the Distributive Property is fundamental not only in polynomial multiplication but also in polynomial addition and subtraction. Do not use shortcuts such as FOIL since FOIL only applies to multiplying a binomial by a binomial. For example, when adding the monomials $3x$ and $2x$, the result can be explained with the Distributive Property as follows:
$3x + 2x = (3 + 2)x = 5x$.

TIP!

Students previously taught FOIL, may be confused when the polynomials are trinomials or others polynomials larger than a binomial. Therefore, emphasize the distributive property instead of the term FOIL.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (A.APR.1)

The connections between methods of multiplication can be generalized even further which can be seen by considering whole numbers in base ten place value to be polynomials in the base *b* = 10. For example, compare the product 213 x 47 with the product $(2b^2 + 1b + 3)(4b + 7)$:

$$\begin{array}{rrrrr} & & 2b^2 & +\ 1b & +\ 3 \\ \times & & & 4b & +\ 7 \\ \hline & & 14b^2 & +\ 7b & +\ 21 \\ + & 8b^3 & +\ 4b^2 & +\ 12b & \\ \hline & 8b^3 & +\ 18b^2 & +\ 19b & +\ 21 \end{array} \qquad \begin{array}{rrrrr} & & 200\ + & 10\ + & 3 \\ \times & & & 40\ + & 7 \\ \hline & & 1400\ + & 70\ + & 21 \\ + & 8000\ + & 400\ + & 120 & \\ \hline & 8000 & +\ 1800 & +\ 190 & +\ 21 \end{array} \qquad \begin{array}{rr} & 213 \\ \times & 47 \\ \hline & 1491 \\ + & 8520 \\ \hline & 10011 \end{array}$$

Discussion: Note how the distributive property is in play in each of these examples. In the left-most computation, each term in the factor $(4b + 7)$ must be multiplied by each term in the other factor, $(2b^2 + 1b + 3)$, just like the value of each digit in 47 must be multiplied by the value of each digit in 213, as in the middle computation, which is similar to "partial products methods" that some students may have used for multiplication in the elementary grades. The common algorithm on the right is merely an abbreviation of the partial products method.

Some students will apply the Distributive Property inappropriately. Emphasize that it is the *Distributive Property of Multiplication over Addition*. For example, the Distributive Property can be used to rewrite $2(x + y)$ as $2x + 2y$, because in this product the second factor is a sum (i.e., involving addition). But in the product $2(xy)$, the second factor, (xy), is itself a product, not a sum.

For Math 2, notice that the focus is on linear and quadratic expressions, but higher degrees polynomials may be used as long as the expression(s) simplify to a linear or quadratic. Therefore, simplifying an expression like $x^3 - 2x^2 - (x^3 + 8x)$ is appropriate for this course.

Instructional Tools/Resources

These tools and resources may be helpful for instruction. Before use, they should be evaluated at the local level to determine their appropriateness for instruction.

Manipulatives/Technology

- Computer Algebra Systems
- Algebra tiles
- Area models

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (A.APR.1)

Algebra Tiles

- Virtual Algebra Tiles is an applet from Michigan Virtual University that allows students to use algebra tiles. This applet allows for positive and negative representation of the tiles.
- CPM Tiles is an applet from the CPM Educational Program that allows students to use algebra tiles. The benefit of this applet is that students can change the dimensionality of *x* and *y*. However, it is limited by not allowing for a negative representation of the tiles.
- Algebra tile templates on the SMART Exchange has a variety of useful models that teacher can use if they have access to a SMART Board.
- Adding and Subtracting Polynomials Using Algebra Tiles is lesson from Buffalo State University.
- Adding and Subtracting Polynomials Using Algebra Tiles is a lesson from the Virginia Department of Education.
- Algebra Tiles Workbook from Learning Resources has lessons that use Algebra tiles to add, subtract, multiply, and divide polynomials.

Multiplying Polynomials

- The Astounding Power of Area by G'Day Math is a website that has a section that focuses on how to use an area model with polynomials.

Curriculum and Lessons from Other Sources

- EngageNY, Algebra 1, Module 1, Topic B, Lesson 8: Adding and Subtracting Polynomials and Lesson 9: Multiplying Polynomials are lessons that pertain to this cluster.
- EngageNY, Algebra 1, Module 4, Topic A, Lesson 1: Multiplying and Factoring Polynomial Expressions pertains to this cluster.
- Georgia Standards of Excellence Framework, Unit 1: Relationships Between Quantities has a couple of lessons that pertain to this cluster. This cluster is addressed on pages 16-29.
- High School Coherence Map by UnboundED is a map of the relationships among high school mathematics standards. Please note: This resource is aligned to the Common Core not Ohio's Learning Standards. Although the tool is useful, it needs to be checked with respect to Ohio's Learning Standards.

General Resources

- Arizona High School Progression on Algebra is an informational document for teachers. The cluster is addressed in the first two paragraphs on page 7.

Research

- Common Core Standards Writing Team. (2013, March 1). *Progressions for the Common Core State Standards in Mathematics (draft). High School, Algebra*. Tucson, AZ: Institute for Mathematics and Education, University of Arizona.
- Gurl, T., Artzt, A., Sultan, A., & Curcio, F. (2012). *Implementing the Common Core State Standards through Mathematical Problem Solving*. Reston, VA: National Council of Teachers of Mathematics.

STANDARDS	MODEL CURRICULUM (A.CED.1, 2, 4)
Algebra **CREATING EQUATIONS** **Create equations that describe numbers or relationships.** **A.CED.1** Create equations and inequalities in one variable and use them to solve problems. *Include equations and inequalities arising from linear, quadratic, simple rational, and exponential functions.* ★ **b.** Focus on applying simple quadratic expressions. (A1, M2) **A.CED.2** Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales. ★ **b.** Focus on applying simple quadratic expressions. (A1, M2) **A.CED.4** Rearrange formulas to highlight a quantity of interest, using the same reasoning as in solving equations. ★ **c.** Focus on formulas in which the variable of interest is linear or square. *For example, rearrange the formula for the area of a circle $A = (\pi)r^2$ to highlight radius r.* (M2)	**Expectations for Learning** In Math 1, students create linear and exponential equations and use them to solve problems. In this cluster, students extend this knowledge to include writing quadratic equations. Students also continue to rearrange formulas to highlight a particular variable. In Math 3, students model even more complicated situations. **ESSENTIAL UNDERSTANDINGS** • Regularity in repeated reasoning can be used to create equations that model mathematical or real-world contexts. • The graphical solution of a system of equations or inequalities is the intersection of the graphs of the equations or inequalities. • Solutions to an equation, inequality, or system may or may not be viable, depending on the scenario given. • A formula relating two or more variables can be solved for one of those variables (the variable of interest) as a shortcut for repeated calculations. **MATHEMATICAL THINKING** • Create a model to make sense of a problem. • Represent the concept symbolically. • Plan a solution pathway. • Determine the reasonableness of results. • Consider mathematical units and scale when graphing. *Continued on next page*

STANDARDS	MODEL CURRICULUM (A.CED.1, 2, 4)
	Expectations for Learning, continued **INSTRUCTIONAL FOCUS** • Given a mathematical or real-world context, express the relationship between quantities by writing an equation or inequality that must be true for the given relationship. Focus on situations where the equations will be linear, exponential, and quadratic. • For equations or inequalities relating two variables, graph the relationships on coordinate axes with proper labels and scales. Focus on situations where the equations will be linear, exponential, and quadratic. • Identify the constraints implied by the scenario and represent them with equations or inequalities. • Determine the feasibility (possibility) of a solution based upon the constraints implied by the scenario. • Solve formulas for a given variable. **Content Elaborations** **OHIO'S HIGH SCHOOL CRITICAL AREA OF FOCUS** • Math 2, Number 2, pages 4-5 **CONNECTIONS ACROSS STANDARDS** • Interpret the structure of expressions (A.SSE.1). • Build a function that models an exponential or quadratic relationship between two quantities (F.BF.1).

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (A.CED.1, 2, 4)

Instructional Strategies

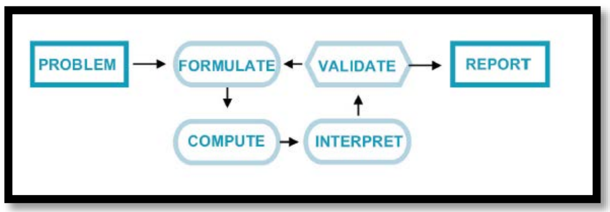
Note: The model curriculum is not a collection of lessons nor a full curriculum; it does not suggest pace, sequence, or amount of time spent on topics. It provides information about a topic related to the standards including ideas for examples, strategies for teaching, common misconceptions, and possible connections between topics. In addition, the emphasis of the model curriculum is on instruction not on assessment; therefore, the examples in the Instructional Strategies section are written for instructional purposes not necessarily as assessment items. Examples may need to be rewritten to accommodate the needs of each individual classroom.

Standards for Mathematical Practice
This cluster focuses on but is not limited to the following practices:
MP.2 *Reason abstractly and quantitatively.*
MP.4 *Model with mathematics.*
MP.5 *Use appropriate tools strategically.*
MP.6 *Attend to precision.*
MP.7 *Look for and make use of structure.*

MODELING

A.CED.1-4 is a modeling standard. See page 12 for more information about modeling.

The Arizona High School Progression on Algebra has a helpful statement with respect to modeling: "In high school, there is again a difference between directly representing the situation and finding a solution. The formulation of the equation tracks the text of the problem fairly closely, but requires more than a direct representation. The Compute node of the modeling cycle is dealt with in the next section, on solving equations. The Interpret node also becomes more complex. Equations in high school are also more likely to contain parameters than equations in earlier grades, and so interpreting a solution to an equation might involve more than consideration of a numerical value, but consideration of how the solution behaves as the parameters are varied. The Validate node of the modeling cycle pulls together many of the standards for mathematical practice, including the modeling standard itself." (Common Core Standards Writing Team, 2013)



CREATING EQUATIONS AND INEQUALITIES

Provide examples of real-world problems that can be modeled by writing an equation or an inequality. Students may believe that equations of linear, quadratic, and other functions are abstract and exist only "in a math book," without seeing the usefulness of these equations as modeling real-world phenomena. For example, use familiar contexts such as car depreciation to highlight linear and exponential equations. Begin with simple equations and inequalities and build up to more complex equations in two or more variables that may involve quadratic or exponential functions.

Lack of Fractional Knowledge

Research has shown a link between students' knowledge of fractions and their ability to write equations. Although there has been some work

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (A.CED.1, 2, 4)

done at the middle school level, some additional work may be needed for scaffolding at the high school level (See Grade 6 Model Curriculum 6.EE.5-8 and 6.EE.9 and Grade 7 Model Curriculum 7.EE.3-4 for scaffolding ideas.)

Key Words Strategy

Another issue in writing equations is that students are over reliant on key words in creating equations as they use key words in place of reasoning. This approach fails when problems become more complex and when there are several relationships between quantities. One of the problems with the Key Word approach is that it relies too heavily on numbers or values in the problem instead of the relationship between quantities, which ties into understanding the structure of an expression (A.SSE.1-2). To combat the misconceptions involving using key words, intentionally give students situations where aligning key words does not lend itself to writing equations that represents the situation. Drawing diagrams is one way to help students understand the structure of a problem. Another strategy is directing students to make sense of the situation by asking questions. Also, having students discuss their thinking in terms of quantities and relationships instead of values, calculations, and operations is another strategy for student success. Encourage students to explain "why" they did something in contrast to explaining "what" they did. It may be helpful for some students to write down all the quantities in the problem and to state the important aspects of a problem.

Writing Comparison Equations

Students also have a difficult time writing equations where the context is reversed such as "There are 8 times as many football players as cheerleaders." Many students incorrectly write $8F = C$ instead of correctly writing $F = 8C$. Studies show that this problem is not just limited to misunderstanding key words, but rather—

- incorrectly matching the order of the words in the situation to the equation;
- thinking that the larger number is placed next to the variable defining the larger group;
- treating variables as labels;
- treating variables as a fixed unknown rather than as a variable quantity; and/or
- treating the equal sign as representing equivalence but more like an association.

Students who are able to represent these problems correctly invent operations to establish equivalence thereby forcing unequal groups to be equal. They also look at the situation in terms of a function context instead of as two unrelated quantities. See Model Curriculum 8.F.4-5 for more information on writing comparison equations.

TIP!

Students should always identify variables when creating equations. A reader should never have to assume anything. Students should be precise when identify variables rather than just stating x = apples, they should state x = number of apples or x = price per apple.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (A.CED.1, 2, 4)

EXAMPLE

At the local steakhouse, for every 5 people who order steak, two order chicken. Write an equation to represent the number of steak entrees and the number of chicken entrees. Make sure to define the variables.

EXAMPLE

Write an inequality that describes the sides of a rectangular patio where the length is at least 4 meters longer than the width. Make sure to define the variables.

Guess and Check Strategy

Students often have difficulty writing equations and inequalities for given situations. Consider a strategy using guess and check as a process for writing an equation that must be true as described in Al Cuoco's blog entry in *Mathematical Musings*, "Teachers know that building is much harder for students than checking. The same practice of abstracting from numerical examples is useful here, too."

EXAMPLE

Rico launches a ball in the air and it lands 10 feet away. He released the ball from a height of 5 feet.

- **a.** Write an equation to represent the situation.
- **b.** Interpret the meaning of each coefficient in the equation.
- **c.** Analyze your resultant equation in the context of the situation.

Discussion: A student may think, "From previous experience I know that any quadratic that deals with gravity has to start with a leading coefficient of -16 which represent the gravitation constant for acceleration. In addition, I know that the *y*-intercept is $(0, 5)$. So, the equation will look like $y = -16x^2 + bx + 5$, where $y =$ height of the ball $x =$ time in seconds. That means I just have to find the coefficient in front of the second term. I also know that one of the zeros of the graph is $(10, 0)$. Since the zero is 10 and the other end of the parabola has to be in quadrant 2, the axis of symmetry has to be less than 5. I can use $\frac{-b}{2a}$ to find the axis of symmetry, which in this case would be $\frac{b}{-2(-16)}$ or $\frac{b}{32}$. So, I'm going to use my calculator to experiment with factors of 32. I tried substituting $32, 64, 96, 118, 150,$ and 182 into my equation for b. I realized that $y = -16x^2 + 150x + 5$ is really close with a zero of 9.4. Next I tried $y = -16x^2 + 160x + 5$ to see if its closer, and it gives me a zero of 10.03, which is really, really close, so I'll try $y = -16x^2 + 159.9x + 5$, which seems to get the zero of 10. That means the initial velocity is 159.9 ft/sec which seems really fast. It must have been launched by something that can go fast. I will use the calculator to find the maximum being 404 feet, which is really high, but the velocity is also very high." Eventually students should more from guessing and checking to more efficient strategies. *Note: This is an open-ended problem where students can write a variety of equations as long as they can justify their equation based upon the context of the situation.*

Using Arithmetic to Write a Generalized Equation Strategy
Students can use arithmetic to write an equation from a real-world context as a strategy for writing equations. They can start with an example using numbers and move towards a more general equation that is true.

EXAMPLE
Write an inequality that describes the area of a rectangular garden where the length is 8 meters longer than the width, and the area is at least 220 square meters.

Discussion: A student should understand that if the area is at least 220 square meters, then the area is either 220 or greater. A student may initially use numbers in the place for variables when writing an equation or inequality and then replace the numbers with variables to generalize the equation or inequality. In the example above, a student may use the numbers 10 and 18 to describe the width and length of the garden respectively. They could then write the inequality: $10 \cdot 18 = 180$, which is less than 220. The student should understand that the width is greater than 10 and the length is greater than 18. A student could then replace the width of 10 with the variable x, and the length of 18 with the variable $x + 8$, and the total area should be 220.

Using Tables to Write Equations
Students can use tables to help them notice patterns and write equations. Use quadratic contexts that lend themselves to tabular representations. They can create tables that represent this relationship simply by counting and use this table to write quadratic equations. See Model Curriculum F.BF.1-2 for more information about using tables with quadratics.

EXAMPLE
Nadia wants to sell water bottles as fundraiser. However, it would cost her $60 to join the bulk discount club. If she sells each bottle for $1.50, she can sell 485 bottles. For every $0.10 increase in price, she sells 10 less water bottles.

- **a.** Write an equation to represent the revenue?
- **b.** What is the maximum price she should sell the bottle of water for?

Discussion:

- **a.** The revenue is the selling price times the number of bottles sold. If she wanted to charge $1.50, she could sell 485 and have a revenue of $727.50. However, if she increased the price to $1.60, she would only sell 475 water bottles, but she would make $760. So if $x =$ the number of increases in price, then the selling price would be represented by $(1.5 + 0.1x)$, the number of bottles sold would be represented by $(485 - 10x)$, and the revenue per bottle is $R = (1.5 + 0.1x)(485 - 10x)$.
- **b.** She could use the table function on the graphing calculator to find that the maximum prices that she should sell the water for is $3.30. The selling price could be inserted into Y_1, the number of bottles sold could be inserted into Y_2, and the revenue could be inserted into Y_3. At that rate she would sell 305 water bottles to have a revenue of $1006.50. The table feature on a graphing calculator is useful because you can see the number of bottles in whole numbers, as the domain is restricted to whole numbers in this context.

REARRANGING FORMULAS

Explore examples illustrating when it is useful to rewrite a formula by solving for one of the variables in the formula. For example, the formula for the area of a trapezoid, $A = \frac{1}{2}h(b_1 + b_2)$, can be solved for *h* if the area and lengths of the bases are known, but the height needs to be calculated. This strategy of selecting a different representation has many applications in science and business when using formulas. *Note: In Math 2 exponential equations will not be rearranged, because this would introduce logarithms.*

TIP!

"Variable of interest" means the variable or quantity a person is interested in solving for or looking for a solution or relationship. It is not interest in the sense of banks and rate of growth.

Provide examples of real-world problems that can be solved by writing an equation, and have students explore the graphs of the equations on a graphing calculator to determine which parts of the graph are relevant to the problem context.

Use a graphing calculator to demonstrate how dramatically the shape of a curve can change when the scale of the graph is altered for one or both variables.

Give students geometric formulas such as area and volume or formulas from science or business contexts, and have students solve the equations for each of the different variables in the formula.

Solving equations for the specified letter with coefficients represented by letters (e.g., $A = \frac{1}{2}h(b_1 + b_2)$ when solving for b_1) is similar to solving an equation with one variable. Provide students with an opportunity to apply the same kind of manipulations to formulas as they did to equations.

EXAMPLE

The force of gravity is represented by the equation $F = \frac{Gm_1m_2}{r^2}$, where

- G is the gravitational constant,
- m_1 and m_2 are the masses of the two objects, and
- r^2 is the distance between the two objects.

Solve the formula for r to represent the distance between the two objects.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (A.CED.1, 2, 4)

TIP! Letters can be referred to as "variables," "parameters," or "constants," which can be helpful if they are used consistently as it may give insight into how students view a problem. However, for formulas such as Ohm's Law it may be best to avoid using those terms all together when working with this formula, because there are six different ways it can be viewed as defining one quantity as a function of the other with a third held constant. (Common Standards Core Writing Team, March 2013)

Students may believe that formulas are static, but formulas that are models may sometimes be readily transformed into functions that are models. For example, the formula for the volume of a cylinder can be viewed as giving volume as a function of area of the base and the height, or, rearranging, giving the area of the base as a function of the volume and height. Similarly, Ohm's law can be viewed as giving voltage as a function of current and resistance.

In Grade 7 students learned about proportional relationships and constants of proportionality: "7.RP.2 Recognize and represent proportional relationships between quantities." These concepts surface often in high school modeling situations. Students learn that many modeling situations begin with a statement like Ohm's Law or Newton's Second Law. In Ohm's Law, $V = IR$, V is the quantity of interest, V is directly proportional to R where I is the constant. The formula can be also rearranged to highlight a different quantity of interest, I, where $I = \frac{V}{R}$. In this case I is inversely proportional to R where V is the constant.

ADDITIONAL NOTES

To understand the differences among A.CED.1, A.CED.2, and A.CED.3, consider the following problem:

- We have 14 coins (nickels and dimes) and they are worth \$0.95. How many of each coin?

For A.CED.1, students can write an equation in one variable (as shown below) and then solve:
Value of nickels + value of dimes = 95 cents
$5n + 10(14 - n) = 95$

Alternatively, for A.CED.2 and A.CED.3, students can write two equations using two variables (A.CED.2), creating a system of equations in two variables (A.CED.3) and then solve:
$n + d = 14$
$5n + 10d = 95$

All students should be able to understand both of these approaches and should be able to use them as appropriate without requiring a particular approach on a given problem.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (A.CED.1, 2, 4)

For A.CED.4, when rewriting the formula for the area of a circle to highlight radius *r* (for example), first ask students to figure out what the radius would be if the area is 10 square units. Then ask them what the radius would be if the area is 20 square units. Then if the area is 23 square units. Eventually, students should understand the rewritten equation solved for *r* is a general formula for finding the radius given any area, instead of going through the several steps to find *r* every time. This process is a way to encourage students look for and express regularity in repeated reasoning (Mathematical Practice 8).

Instructional Tools/Resources

These tools and resources may be helpful for instruction. Before use, they should be evaluated at the local level to determine their appropriateness for instruction.

Manipulatives/Technology

- Graphing calculators
- Computer software that generate graphs of functions
- Examples of real-world situations that lend themselves to writing equations that model the contexts.
- Desmos is a free graphing calculator that is available to students as website or an app.
- GeoGebra is a free graphing calculator that is available to students as website.
- Wolframalpha is dynamic computing tool.
- Visual Patterns is a website that shows pictures of linear, exponential, and quadratic patterns.
- Patterns Posters for Algebra 1 from Finding Ways by Fawn Nguyen is a blog about a class activity where students are given patterns, and they have to make posters from them. She is the creator of the visual patterns link above.

Creating Equations

- Buying a Car is a task from Illustrative Mathematics where students create equations that involve different values for sales tax but move towards representing sales tax as a parameter.
- Paula's Peaches-Writing Quadratics is a task from Georgia Standards of Excellence Framework, Algebra 1, Unit 3 where students create a quadratic equation from the context of growing peaches.
- Interpreting Algebraic Expressions from Mathematics Assessment Project is a task where students translate algebraic expressions.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (A.CED.1, 2, 4)

Curriculum and Lessons from Other Sources

- EngageNY, Algebra 1, Module 4: Topic A, Lesson 7: Creating and Solving Quadratic Equations in One Variable is a lesson that pertains to this cluster.
- EngageNY, Algebra 1, Module 4: Topic C, Lesson 24: Modeling with Quadratic Functions is a lesson that pertains to this cluster.
- Mathematics Vision Project, Algebra 1, Module 4: Solving Equations has lessons that pertain to this cluster.
- Exploring Symbols by Burrill, Clifford, Scheaffer is the teacher's edition of a textbook in the Data-Driven Mathematics series published by Dale Seymour Publications. There are several lessons that pertain to this cluster. The student edition can be found here.
- Illustrative Mathematics, Algebra 1, Unit 6, Lesson 3: Building Quadratic Functions from Geometric Patterns is a lesson that pertains to this cluster.
- Illustrative Mathematics, Algebra 1, Unit 7, Lesson 1: Finding Unknown Inputs, Lesson 17: Applying the Quadratic Formula (Part 1), Lesson 18: Applying the Quadratic Formula,

General Resources

- Arizona High School Progression on Algebra is an informational document for teachers. This cluster is addressed on the last paragraph of pages 10-12.
- Arizona High School Progression on Modeling is an informational document for teachers. This cluster is addressed on the last paragraph of page 13 which continues on page 14 and is addressed under the Formulas as Models section on pages 16-17.
- High School Coherence Map by UnboundED is a map of the relationships among high school mathematics standards. Please note: This resource is aligned to the Common Core not Ohio's Learning Standards. Although the tool is useful, it needs to be checked with respect to Ohio's Learning Standards.

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INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (A.CED.1, 2, 4)

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STANDARDS	MODEL CURRICULUM (A.REI.4)
Algebra **REASONING WITH EQUATIONS AND INEQUALITIES** **Solve equations and inequalities in one variable.** **A.REI.4** Solve quadratic equations in one variable. **a.** Use the method of completing the square to transform any quadratic equation in x into an equation of the form $(x - p)^2 = q$ that has the same solutions. **b.** Solve quadratic equations as appropriate to the initial form of the equation by inspection, e.g., for $x^2 = 49$; taking square roots; completing the square; applying the quadratic formula; or utilizing the Zero-Product Property after factoring. (+) **c.** Derive the quadratic formula using the method of completing the square.	**Expectations for Learning** In previous courses, students solve linear equations and inequalities, and they solve equations with numeric and letter coefficients. In this cluster, students solve quadratic equations (with real solutions) using a variety of methods. In other standards, students learn to factor quadratics; this cluster builds on that idea to solve quadratic equations with the Zero Product Property. In Math 3, students use these skills to solve more complicated equations. **ESSENTIAL UNDERSTANDINGS** • An appropriate solution path can be determined depending on whether the equation is linear or quadratic in the variable of interest. • Quadratic equations and expressions can be transformed into equivalent forms, leading to different solution strategies, including inspection, taking square roots, completing the square, applying the quadratic formula, or utilizing the Zero Product Property after factoring. • When the coefficients of the variable of interest are letters, the solving process is the same as when the coefficients are numbers. • The discriminant can show the nature and number of solutions a quadratic has. • (+) The quadratic formula is derived from the process of completing the square. **MATHEMATICAL THINKING** • Generalize concepts based on properties of equality. • Solve routine and straightforward problems accurately. • Plan a solution pathway. • Solve math problems using appropriate strategies. • Solve multi-step problems accurately. • (+) Use formal reasoning with symbolic representation. *Continued on next page*

STANDARDS	MODEL CURRICULUM (A.REI.4)

	Expectations for Learning, continued
	INSTRUCTIONAL FOCUS
	For quadratic functions, students should work with expressions in which the leading coefficient can be any real number, but assessment questions should focus on expressions with leading coefficients of 1 with occasional questions using other simple leading coefficients.
	• Recognize when an equation or inequality is linear or quadratic in one variable, and plan a solution strategy.
	• Recognize when an equation is quadratic in one variable, and choose an appropriate solution strategy:
	    o using inspection, e.g., $(x-3)^2 = 0$ or $x^2 = -5$;
	    o taking square roots, e.g., $x^2 = 8$;
	    o using Zero-Product Property after factoring;
	    o completing the square; or
	    o applying the quadratic formula.
	• Determine if a quadratic function has one solution, two solutions, or no real solutions based on the discriminant.
	• (+) Formally derive the quadratic formula using completing the square.
	
	Content Elaborations
	OHIO'S HIGH SCHOOL CRITICAL AREA OF FOCUS
	• Math 2, Number 2, pages 4-5
	
	CONNECTIONS ACROSS STANDARDS
	• Rearrange formulas to highlight a quantity of interest (A.CED.4).
	• Use the structure of an expression to identify ways to rewrite it (A.SSE.2).
	• Write expressions in equivalent forms to solve problems (A.SSE.3).

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (A.REI.4)

Instructional Strategies

Note: The model curriculum is not a collection of lessons nor a full curriculum; it does not suggest pace, sequence, or amount of time spent on topics. It provides information about a topic related to the standards including ideas for examples, strategies for teaching, common misconceptions, and possible connections between topics. In addition, the emphasis of the model curriculum is on instruction not on assessment; therefore, the examples in the Instructional Strategies section are written for instructional purposes not necessarily as assessment items. Examples may need to be rewritten to accommodate the needs of each individual classroom.

This cluster should not be taught in isolation; it should be joined with A.CED.1-4 where students create real-world problems and parameters for equations in context.

Standards for Mathematical Practice

This cluster focuses on but is not limited to the following practices:

***MP.1** Make sense of problems and persevere in solving them.*

***MP.2** Reason abstractly and quantitatively.*

***MP.6** Attend to precision.*

***MP.7** Look for and make use of structure.*

QUADRATICS

All students need to be exposed to solving quadratic equations by using inspection, taking square roots, factoring, completing the square, and the quadratic formula.

Note: Students in Math 2 will only be expected to solve for real solutions.

Highlight and compare different approaches to solving the same problem. Use technology to recognize that two different expressions or equations may represent the same relationship. For example, since $x^2 + 2x - 8 = 0$ can be rewritten as $(x - 2)(x + 4) = 0$ or $(x + 1)^2 - 9 = 0$, these are all representations of the same equation that have the solutions $x = -4, 2$. Demonstrate that they are equivalent by setting all expressions $x^2 + 2x - 8$, $(x - 2)(x + 4)$, and $(x + 1)^2 - 9$ equal to y and graph them using technology. Compare their graphs and tables, displaying the same output values for each expression or by looking at their y-intercepts.

INSPECTION

Students should have some practice solving quadratics by inspection. For example, the solution for $x^2 = 9$ is 3 and –3. *Note: this does not work for every quadratic.*

SQUARE ROOTS

Offer students examples of a quadratic equation, such as $x^2 - 9 = 0$. They could rewrite this in the form $x^2 = 9$ and square root both sides to get two solutions $x = 3$ and $x = -3$. Connect the solutions to the graph of its function. Then they should contrast their graph of $y = x^2 - 9$ to the graph of the quadratic function $y = x^2 + 9$ which is situated above the x–axis and opens upwards because it can be viewed as a vertical transformation of $y = x^2$. Since the graph of $y = x^2 + 9$ graph does not have x–intercepts, the quadratic equation does not have real solutions. This make sense since $x^2 = -9$ has no real solution. Students should realize that using square roots to solve quadratics may be used anytime an equation in x can be changed into the equation of the form $(x - p)^2 = q$ and then into the $x - p = \pm\sqrt{q}$ to be solved by taking a square root of both sides of the equation.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (A.REI.4)

Students may incorrectly think that $y = (x - 5)^2$ is the graph of $y = x^2$ shifted 5 units to the left. Use the technology to explore examples to counter this thinking, and have students justify why this is not the case.

EXAMPLE

Solve using square roots.

$(y - 3)^2 - 16 = 0$

Discussion: Students should notice that there are two perfect squares in the equation: $(y - 3)^2$ and 16. Therefore, it may be wise to isolate $(y - 3)^2$ by adding 16 to both sides of the equation to get $(y - 3)^2 = 16$. Then the student can take the square root of both sides to get $y - 3 = \pm 4$ or $y = 7$ and -1. Draw attention to the fact that there are two solutions for y.

COMPLETING THE SQUARE

Completing the square is usually introduced for several possible reasons:

- o to find the vertex of a parabola when an equation is written in standard or general form;
- o to look at the parabola through the lenses of transformations of a "parent" parabola $y = x^2$;
- o to derive the quadratic formula *(not required for all students).*

Connect completing the square to inspection:

- Start by inspecting equations such as $x^2 = 9$ that has two solutions, 3 and –3.
- Next, progress to equations such as $(x - 7)^2 = 9$ by substituting $x - 7$ for x.
- Then solve them either by inspection or by taking the square root on each side:

  $x - 7 = 3$ or $x - 7 = -3$

  $x = 10$ $\qquad$ $x = 4$
- Graph both pairs of solutions (–3 and 3, 4 and 10) on the number line and notice that 4 and 10 are 7 units to the right of –3 and 3.
- So, the substitution of $x - 7$ for x moved the solutions 7 units to the right.
- Next, graph the function $y = (x - 7)^2 - 9$, pointing out that the x-intercepts are 4 and 10, and emphasizing that the graph is the translation of 7 units to the right and 9 units down from the position of the graph of the parent function $y = x^2$ that passes through the origin (0, 0).
- Generate more equations of the form $y = a(x - h)^2 + k$ and compare their graphs using graphing technology.

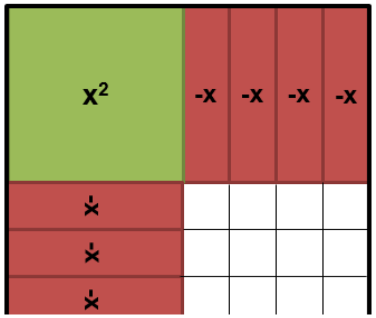
INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (A.REI.4)

EXPLORE

a. *Introduce completing the square using a visual model such as algebra tiles.*

- Create a perfect square trinomial with algebra tiles to represent $x^2 - 8x +$ _____.

Students should rearrange tiles to make a square. They should come to the realization that they cannot make a perfect square unless they add some unit tiles.



- How many more unit squares are needed to make a perfect square?
- Write the quadratic expression represented with the algebra tiles in standard form and factored form.

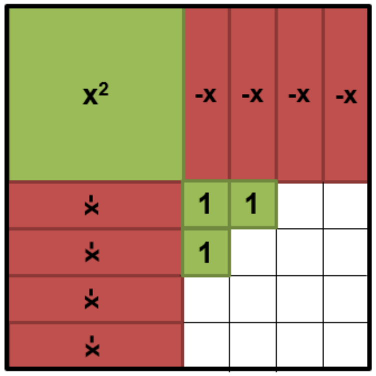
Give students several more examples such as $x^2 + 4x +$ ___, $x^2 - 6x +$ ___, and $x^2 + 2x +$ ___.

- Given a quadratic expression in the form of $x^2 + bx + c$, make up a rule to find c that makes a perfect square.

b. *Once a student can model creating a perfect square trinomial, they will be able to transform a quadratic equation into the equivalent vertex form $(x - p)^2 = q$.*

- Create an equivalent equation in vertex form of the equation $x^2 - 8x + 3 = 0$. First represent the equation in standard form with algebra tiles.

Note: A student could rearrange the unit squares differently.



- How many more unit squares will it take to make a square?

The student should answer 13.

Remember, according to the Addition Property of Equality we can add any number to both sides of an equation and still maintain equivalency, so we can write the following equation:

$x^2 - 8x + 3 + 13 = 0 + 13$
$x^2 - 8x + 16 = 13$

Now if we factor the left side, we will get an equivalent equation in vertex form $(x - 4)^2 = 13$.

After students have worked with several examples on completing the square using algebra tiles, have students generalize a rule for completing a square. Make sure to introduce examples where c is greater than b. A worksheet on Completing the Square using Algebra Tiles can be found here.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (A.REI.4)

A teacher can also guide students in transforming a quadratic equation in standard form, $0 = ax^2 + bx + c$, to the vertex form $0 = a(x - h)^2 + k$ by separating your examples into groups with $a = 1$ and $a \neq 1$ and have students guess the number that needs to be added to the binomials of the type $x^2 + 6x$, $x^2 - 2x$, $x^2 + 9x$, $x^2 - \frac{2}{3}x$ to form a complete square of the binomial $(x - m)^2$. Then they can generalize the process by showing the expression $\left(\frac{b}{2}\right)^2$ that has to be added to the binomial $x^2 + bx$ when $a = 1$. Completing the square for an expression where $a \neq 1$can be challenging for some students. Present multiple examples of the type $0 = 2x^2 - 5x - 9$ to emphasize the logic behind every step, keeping in mind that the same process will be used to complete the square in the proof of the quadratic formula. See A.SSE.3 for more information on completing the square.

QUADRATIC FORMULA

Discourage students from giving a preference to a particular method of solving quadratic equations. Students need experience in analyzing a given problem to choose an appropriate solution method before their computations become burdensome. Point out that the Quadratic Formula, $x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$, is a universal tool that can solve any quadratic equation; however, it is not the most efficient method when the quadratic equation is missing either a middle term, bx, or a constant term, c. When it is missing a constant term, (e.g., $3x^2 - 10x = 0$) a factoring method becomes more efficient. If a middle term is missing (e.g., $2x^2 - 15 = 0$), a square root method is usually more appropriate. Stress both the benefit of memorizing the Quadratic Formula and the flexibility of using a factoring strategy. Introduce the concept of discriminants and their relationship to the number and nature of the roots of quadratic equation.

Make connections between the form of the quadratic formula $x = -\frac{b}{2a} \pm \frac{\sqrt{b^2-4ac}}{2a}$, the formula for axis of symmetry, and the distance from the axis of symmetry to the zeros. Connect this to the concept of the discriminant and why the equation has one solution, no solution, or zero solutions.

Even though deriving the quadratic formula is not a requirement for all students, teachers may want to demonstrate the process.

FACTORING AND THE ZERO PRODUCT PROPERTY

Equations that are in factored form or can easily be written in factored form can be solved by using the Zero Product Property. Students should come with the understanding that any number when multiplied by 0 is 0 (See A.REI. 1); this includes expressions such as monomials, binomials and trinomials.

In equations where the product of two factors equals 0 ($a \cdot b = 0$), either a or b must equal 0, for the equation to be true. Although, a and b should not be limited to a single variable or number, since they can be represented by expressions such as binomials.

To make sense of solving quadratics by factoring, connect the graph to two linear equations: $f(x) = x + 3$ and $f(x) = x - 1$. Then multiply the solutions of two linear equations that make up the quadratic equation.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (A.REI.4)

EXAMPLE

Part 1

Exploring $y = (x + 3)(x - 1)$

	x	$x + 3$	$x - 1$	$(x + 3)(x - 1)$
Points 1	–4	–1	–5	5
Points 2	–3			
Points 3	–2			
Points 4	–1			
Points 5	0			
Points 6	1			
Points 7	2			

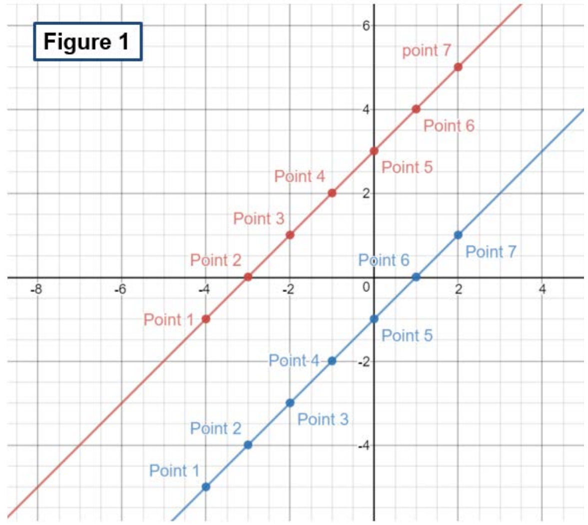


Figure 1

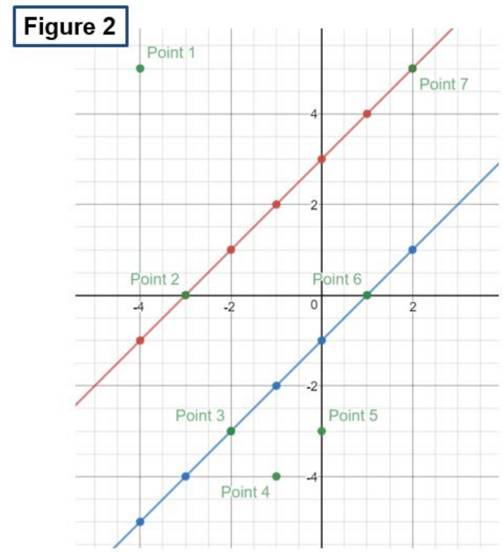


Figure 2

- Fill out Column 3 and Column 4 in the table for points 1-7.
- Use the points on the table to graph $y = x + 3$ in red and $y = x - 1$ in blue on the same coordinate plane (Figure 1).
- Looking at the graph multiply the y-coordinates of each point with the same x-coordinate. In green, plot the new points keeping the x-coordinate the same, but the y-value is the product of the y-values of the two lines (Figure 2).
- Fill out Column 5 in the table by multiplying the values in Column 3 and Column 4. How do the values in Column 5 connect to the green dots that you graphed? Why?
- How can you graph $y = (x + 3)(x - 1)$ using the binomials $(x + 3)$ and $(x - 1)$?
- Graph 3 more points on $y = (x + 3)(x - 1)$ using the binomials.
- Connect the green dots.
- What shape do you notice?

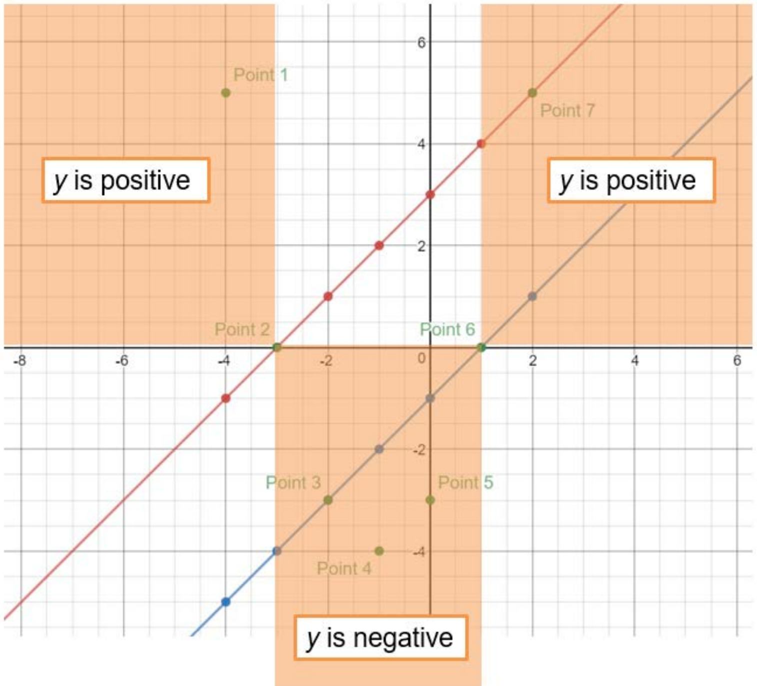
Discussion: The students should connect the multiplication of binomials to the solutions of the quadratic equation. Graph both linear equations on the same coordinate plan and have them reason about the lines to make conclusions about the graph of the quadratic equation. They should discover that the product of the two-corresponding y-values is the y-coordinate of the of the product of the linear equations.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (A.REI.4)

Part 2

Exploring $y = (x + 3)(x - 1)$

	x	$x + 3$	$x - 1$	Is the product of $(x + 3)(x - 1)$, positive, negative, or 0?	Which quadrant is $(x + 3)(x - 1)$ in?
Points 1	–4	–1	–5	Positive	Quadrant 2
Points 2	–3				
Points 3	–2				
Points 4	–1				
Points 5	0				
Points 6	1				
Points 7	2				



- Fill out the table.
- Shade in the regions of the graph where the equation $y = (x + 3)(x - 1)$ could possibly lie based on analyzing if the product of $(x + 3)$ and $(x - 1)$ is positive, negative, or 0.
- Why does the product of two linear equations have to be a U-shaped graph? Explain.
- Will the product of any two linear equations result in a U-shaped graph? Explain.

Discussion: Part 2 has students extend their understandings from Part 1 in order to make generalizations about all quadratics consisting of the product of two linear equations. Now they should see that the since the product of any two linear equations creates a U-shaped graph since the products of the *y*-coordinates at the ends of lines will be on the opposite top or bottom half of the figure compared to the products of the *y*-coordinate in the middle of the figure of the lines. Prompt students to explore a variety of quadratics consisting of the product of two linear equations such as $y = (3x + 1)(x + 3)$; $y = (2x - 2)(2x + 1)$; $y = (\frac{1}{3}x - 1)(2x - 1)$; $y = (-\frac{1}{2}x + 1)(x + 4)$; $y = (2x + 2)(-x - 3)$; $y = (-3x + 4)(-\frac{1}{2}x + 1)$. This example could also be extended to explore why larger coefficients lead to narrower parabolas. Students could discover that the as the coefficients get bigger the product becomes greater, therefore the parabola becomes narrower.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (A.REI.4)

Make sure students understand the Zero Product Property and its relationship to solving quadratic equations by factoring.

EXAMPLE

Given $x^2 - x = 2$

- First set the equation equal to zero: $x^2 - x - 2 = 0$.
- Factor the left side: $(x + 1)(x - 2) = 0$.
- Highlight the Zero Product Property.
- Set each factor equal to zero and solve.
- The solution is $x = -1$ or $x = 2$.

Discussion: Highlight the conceptual understanding behind the process of using the Zero Product Property instead of emphasizing procedures. Explain that, although, we could set an equation equal to any number, it is more important to set the equation equal to 0, because of the Zero Product Property. Since, if one of the binomials equals 0, the whole side equals 0.

One misconception is that students will factor $x^2 - x = 2$ as $x(x - 1) = 2$ instead of setting the equation equal to 0, and then state the solutions to be $x = 2$ or 1 because they think $2 \cdot 1 = 2$. However, infinitely pairs of numbers can give a product of 2.

CHOOSING A METHOD TO SOLVE QUADRATICS

Give students different situations where it is more efficient to choose a particular method for solving quadratics over another. Emphasize that they need to be strategic problem solvers.

Instructional Tools/Resources

These tools and resources may be helpful for instruction. Before use, they should be evaluated at the local level to determine their appropriateness for instruction.

Manipulatives/Technology

- Desmos is a free graphing calculator that is available to students as website or an app.
- GeoGebra is a free graphing calculator that is available to students as website.
- Wolframalpha is dynamic computing tool.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (A.REI.4)

Algebra Tiles

- Algebra Tiles Applet by NCTM Illuminations is a link to a virtual algebra tiles applet.
- Virtual Algebra Tiles is an applet from Michigan Virtual University that allows students to use algebra tiles. This applet allows for positive and negative representation of the tiles.
- CPM Tiles is an applet from the CPM Educational Program that allows students to use algebra tiles. The benefit of this applet is that students can change the dimensionality of *x* and *y*. However, it is limited by not allowing for a negative representation of the tiles.
- Algebra tile templates on the SMART Exchange has a variety of useful models that teacher can use if they have access to a SMART Board.

Inequalities

- Compound Inequalities on the Number Line is a Desmos activity that introduces compound inequalities.
- Best Buy Tickets by Mathematics Assessment Project is a task where students write and solve inequalities.

Completing the Square

- Completing the square with Algebra Tiles is an activity from Ms. Hennessey's Classroom blog that connects to a visual model for completing the square. In this activity students draw algebra tiles.

Interpreting Quadratic Equations

- Projectile Motion by PhET is an applet that allows exploration of projectile motion.
- Weightless Wonder is a NASA activity that has students solve quadratic function involving the parabolic flights of NASA's Weightless Wonder Jet.
- Quadratic Sequence 1 is an Illustrative Mathematics task that presents students with a sequence of figures for quadratic functions. Students are required to analyze the quadratics and rewrite them in different forms. Quadratic Sequence 2 and Quadratic Sequence 3 are part of this series.
- Why Does It Stay In Orbit? by YummyMath is a task that has students interpret quadratic equations.
- Throwing Up Again by Yummy Math is a task where students manipulate quadratic equations to establish a trajectory for snow throwing.
- My Teacher Says This Stream Is Parabolic. Is He Correct? by YummyMath has students rearrange an equation into vertex form.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (A.REI.4)

Curriculum and Lessons from Other Sources

- EngageNY, Module 1, Topic C, Lesson 10: True and False Equations, Lesson 11: Solution Sets for Equations and Inequalities, Lesson 12: Solving Equations, Lesson 13: Some Potential Dangers when Solving Equations, Lesson 14: Solving Inequalities, Lesson 15: Solution Sets of Two or More Equations (or Inequalities) Joined by "And" or "Or," Lesson 16: Solving and Graphing Inequalities Joined by "And" or "Or", Lesson 17: Equations Involving Factored Expressions are lessons that pertain to his cluster.
- EngageNY, Module 4, Topic A, Lesson 5: The Zero Product Property, Lesson 6: Solving Basic One-Variable Quadratic Equations, Lesson 7: Creating and Solving Quadratic Equations in One Variable are lessons that pertain to this cluster.
- EngageNY, Module 5, Topic B, Lesson 11: Completing the Square, Lesson 12: Completing the Square, Lesson 13: Solving Quadratic Equations by Completing the Square, Lesson 14: Deriving the Quadratic Formula, Lesson 15: Using the Quadratic Formula, Lesson 16: Graphing Quadratic Equations from the Vertex From, $y = a(x - h)^2 + k$ are lessons that pertain to this cluster. Note: Although some students may benefit from learning how to derive the quadratic formula, it is not required for all students.
- Mathematics Vision Project, Algebra 1, Module 4: Equations and Inequalities has several tasks that pertain to this cluster.
- Georgia Standards of Excellence Curriculum Frameworks, Algebra 1, Unit 3: Modeling and Analyzing Quadratic Functions has several tasks that pertain to this cluster. These tasks can be found on pages 62-84, 148-173, and 189-194.
- Illustrative Mathematics, Algebra 1, Unit 7, Lesson 2: When and Why Do We Write Quadratic Equations?, Lesson 3: Solving Quadratic Equations by Reasoning, Lesson 4: Solving Quadratic Equations with the Zero Product Property, Lesson 5: How Many Solutions?, Lesson 9: Solving Quadratic Equations by Using Factored Form, Lesson 10: Rewriting Quadratic Expressions in Factored Form (Part 4), Lesson 11: What are Perfect Squares?, Lesson 12: Completing the Square (Part 1), Lesson 13: Completing the Square (Part 2), Lesson 14: Completing the Square (Part 3), Lesson 16: The Quadratic Equation, Lesson 17: Applying the Quadratic Formula (Part 1), Lesson 18: Applying the Quadratic Formula, Lesson 19: Deriving the Quadratic Formula, Lesson 24: Using Quadratic Equations to Model Situations and Solve Problems are lessons that pertain to this cluster.

General Resources

- Arizona High School Progressions on Algebra is an informational document for teachers. This cluster is addressed on page 13, paragraph 3.
- High School Coherence Map by UnboundED is a map of the relationships among high school mathematics standards. Please note: This resource is aligned to the Common Core not Ohio's Learning Standards. Although the tool is useful, it needs to be checked with respect to Ohio's Learning Standards.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (A.REI.4)

Research

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STANDARDS	MODEL CURRICULUM (A.REI.7)
Algebra **REASONING WITH EQUATIONS AND INEQUALITIES** **Solve systems of equations.** **A.REI.7** Solve a simple system consisting of a linear equation and a quadratic equation in two variables algebraically and graphically. *For example, find the points of intersection between the line* $y = -3x$ *and the circle* $x^2 + y^2 = 3$.	**Expectations for Learning** In Math 1, students solve systems of linear equations in two variables graphically and algebraically, with a focus on the meaning of a solution to a system of equations. In Math 2, students extend this knowledge to solve systems of linear and quadratic equations in two variables. In Math 3, students solve systems of equations in three variables. Students who plan to take advanced mathematics courses (+) will represent systems of equations with matrices and use inverse matrices to solve the system. **ESSENTIAL UNDERSTANDINGS** • The graphical solution to a system of equations in two variables is the intersection of the equations when graphed. • The solution to a system of equations in two variables is the set of ordered pairs that satisfies both equations. • A system of a linear equation and a quadratic equation can have no solutions, one solution, or two solutions. **MATHEMATICAL THINKING** • Determine reasonableness of results using informal reasoning. • Solve multi-step problems accurately. • Plan a solution pathway. • Use technology strategically to deepen the understanding. **INSTRUCTIONAL FOCUS** *Note: For Math 2, students should work with systems of equations in two variables that include an equation of a line and an equation of a parabola, as well as an equation of a line and an equation of a circle.* • Solve a system of a linear equation and a quadratic equation in two variables algebraically using substitution and by graphing. • Discuss the efficiency and effectiveness of various methods of solving systems of equations.

STANDARDS	MODEL CURRICULUM (A.REI.7)
	Content Elaborations **OHIO'S HIGH SCHOOL CRITICAL AREA OF FOCUS** • Math 2, Number 2, pages 4-5 **CONNECTIONS ACROSS STANDARDS** • Solve linear and quadratic equations in one variable (A.REI.4). • Graph linear and quadratic models (F.IF.4, 7). • Rearrange formulas (A.CED.4). • Solve systems of equations and inequalities graphically (A. REI.11).

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (A.REI.7)

Instructional Strategies

Note: The model curriculum is not a collection of lessons nor a full curriculum; it does not suggest pace, sequence, or amount of time spent on topics. It provides information about a topic related to the standards including ideas for examples, strategies for teaching, common misconceptions, and possible connections between topics. In addition, the emphasis of the model curriculum is on instruction not on assessment; therefore, the examples in the Instructional Strategies section are written for instructional purposes not necessarily as assessment items. Examples may need to be rewritten to accommodate the needs of each individual classroom.

Standards for Mathematical Practice
This cluster focuses on but is not limited to the following practices:
MP.1 *Make sense of problems and persevere in solving them.*
MP.2 *Reason abstractly and quantitatively.*
MP.4 *Model with mathematics.*
MP.6 *Attend to precision.*

SYSTEMS OF LINEAR AND QUADRATIC EQUATIONS

Students should explore solving a system that involves a linear equation and a quadratic equation using the graphing method. Through exploration, they should realize that there could be 0, 1, or 2 solutions. Once they have an intuitive understanding of solving those types of systems by graphing, they should move toward solving a linear equation and a quadratic by substitution.

EXAMPLE
Find the intersection point(s) of graphs represented by the equations $y = x^2 + 4x + 1$ and $y = -x - 3$.

Discussion: The purpose of this example is to determine the ways a circle and a line can intersect. Build on this example so students can see that they can have 0, 1, or 2 common solutions.

Instructional Tools/Resources

These tools and resources may be helpful for instruction. Before use, they should be evaluated at the local level to determine their appropriateness for instruction.

Manipulatives/Technology
- Desmos is a free graphing calculator that is available to students as website or an app.
- GeoGebra is a free graphing calculator that is available to students as website.
- Wolframalpha is dynamic computing tool.

Systems of Equations
- Suit Yourself is a lesson designed by NASA where students use systems of equation to evaluate the oxygen use of two astronauts.
- Ground Beef is a lesson by Achieve the Core that ties in CTE. Students use systems of equations and Pearson's square to determine the profit with respect to selling meat.
- Those Horrible Coin Problems (And What We Can Do About Them) is a task from Dan Meyer on making value problems more interesting by showing the need for systems in computation.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (A.REI.7)

Systems of Equations, continued

- Quinoa Pasta 2 is a task from Illustrative Mathematics that integrates modeling and systems of equations in the context of a pasta made of both quinoa and corn. It continues with Quinoa Pasta 3.
- Find a System is a task from Illustrative Mathematics that gives two points and has students create a system given two points. It encourages critical thinking by reversing the typical process.
- Mix It Up and Don't Freeze the Engine are lessons from NCTM's Illuminations that have students write equations in the context of a concentration. "Mix It Up" is a tactile lesson where students develop and use a formula to determine the final percent mix from two mixtures. In "Don't Freeze the Engine" the students use systems to analyze the antifreeze in a particular cooling system.
- Algebra 1-Mixture Problems is a teaching channel video on teaching mixture problems. It ties an informal understanding of a fulcrum and inverse variation which she calls the see-saw method. This could be contrasted with solving the same problem using a system of equations.
- Students use matrices and technology to solve the Meadows or Malls problem, a linear programming problem with six variables.
- Piling Up Systems of Linear Equations: How Much Does Each Weigh? by Tap Into Teen Minds is a 3-act task that has students write systems of linear equations using the weight of office supplies.
- [Makeover] Systems of Equations from Dan Meyer's Blog discusses how to make systems of equation word problems more meaningful.
- A Linear and Quadratic System by Illustrative Mathematics is a task where students make connections between equations and the geometry of their graphs to find the intersection point of a line and a parabola.
- A Mixture of Problems by Laurie Riggs, et. al has a variety of problems including mixture problems using different conceptual methods to solve equations.

Curriculum and Lessons from Other Sources

- EngageNY, Algebra 1, Module 1, Topic C," Lesson 20: Solution Sets to Equations with Two Variables, Lesson 22: Solution Sets to Simultaneous Equation, Lesson 23: Solution Sets to Simultaneous Equation, and Lesson 24: Applications of Systems of Equations and Inequalities are lessons that pertain to this cluster.
- Illustrative Mathematics, Geometry, Unit 6, Lesson 13: Intersection Points is a lesson that pertains to this cluster.
- Mathematics Vision Project, Algebra 1, Module 5: Systems of Equations and Inequalities has many tasks that pertain to this cluster.
- Exploring Systems of Inequalities by Burrill and Hopfensperger is a pdf of the teacher's edition of the series Data-Driven Mathematics published by Dale Seymour Publications. It has many lessons that pertain to this cluster. The student edition can be found here.
- Illustrative Mathematics, Algebra 1, Unit 7, Lesson 24: Using Quadratic Equations to Model Situations and Solve Problems is a lesson that pertains to this cluster.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (A.REI.7)

General Resources

- Arizona High School Progression on Algebra is an informational document for teachers. This cluster is addressed on page 14 in paragraphs 3-6.
- High School Coherence Map by UnboundED is a map of the relationships among high school mathematics standards. Please note: This resource is aligned to the Common Core not Ohio's Learning Standards. Although the tool is useful, it needs to be checked with respect to Ohio's Learning Standards.

References

- Common Core Standards Writing Team. (2013, March 1). *Progressions for the Common Core State Standards in Mathematics (draft). High School, Algebra*. Tucson, AZ: Institute for Mathematics and Education, University of Arizona.
- Filloy, E., Rojan, T., & Solares, A. (2003). Two meanings of the 'equal' sign and senses of comparison and substitution methods. *International Group for the Psychology of Mathematics Education*, Paper presented at the 27th International Group for the Psychology of Mathematics Education Conference Held Jointly with the 25th PME-NA Conference: v4 p223-230: Honolulu, HI
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- McCaffrey, T. & Matthews, P. (October 2017). An emoji is worth a thousand words. *Mathematics Teacher*, *111(2),* 96-102.
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STANDARDS	MODEL CURRICULUM (A.REI.11)

Algebra
REASONING WITH EQUATIONS AND INEQUALITIES
Represent and solve equations and inequalities graphically.
A.REI.11 Explain why the *x*-coordinates of the points where the graphs of the equation *y* = *f(x)* and *y* = *g(x)* intersect are the solutions of the equation *f(x)* = *g(x)*; find the solutions approximately, e.g., using technology to graph the functions, making tables of values, or finding successive approximations
	Expectations for Learning
	In previous courses, students use graphs to solve equations with linear and simple exponential expressions. In Math 2, they extend this to solving equations with quadratic expressions. Then in Math 3, students will similarly study the relationship between the graph and solutions of equations with rational, radical, absolute value, polynomial, exponential, and trigonometric expressions.
	
	ESSENTIAL UNDERSTANDINGS
	• A point of intersection of any two graphs represents a solution of the two equations that define the two graphs.
	• An equation in one variable can be rewritten as a system of two equations in two variables, by thinking of each side of the equation as a function, i.e., writing *y* = left hand side and *y* = right hand side.
	   o The approximate solution(s) to an equation in one variable is the *x*-value(s) of the intersection(s) of the graphs of the two functions.
	   o Two-variable graphical and numerical (tabular) techniques to solve an equation with one variable always work and are particularly useful when algebraic methods are not applicable, e.g., $x^2 - 3x + 2 = 2^x$.
	
	MATHEMATICAL THINKING
	• Use technology strategically to deepen understanding.
	• Plan a solution pathway.
	• Create a model to make sense of a problem.
	
	INSTRUCTIONAL FOCUS
	• Rewrite a one-variable equation as two separate functions and use the *x*-coordinate of their intersection point to determine the solution of the original equation.
	• Approximate intersections of graphs of two equations using technology, tables of values, or successive approximations (focus on equations with linear, quadratic, and exponential expressions).
	Continued on next page

STANDARDS	MODEL CURRICULUM (A.REI.11)
	Content Elaborations **OHIO'S HIGH SCHOOL CRITICAL AREA OF FOCUS** • Math 2, Number 2, pages 4-5 **CONNECTIONS ACROSS STANDARDS** • Solve quadratic equations in one variable (A.REI.4). • Create equations in two variables (A.CED.2). • Graph functions expressed symbolically (F.IF.7). • Analyze functions using different representations (F.IF.9).

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (A.REI.11)

Instructional Strategies

Note: The model curriculum is not a collection of lessons nor a full curriculum; it does not suggest pace, sequence, or amount of time spent on topics. It provides information about a topic related to the standards including ideas for examples, strategies for teaching, common misconceptions, and possible connections between topics. In addition, the emphasis of the model curriculum is on instruction not on assessment; therefore, the examples in the Instructional Strategies section are written for instructional purposes not necessarily as assessment items. Examples may need to be rewritten to accommodate the needs of each individual classroom.

Standards for Mathematical Practice

This cluster focuses on but is not limited to the following practices:

MP.2 *Reason abstractly and quantitatively.*

MP.4 *Model with mathematics.*

MP.5 *Use appropriate tools strategically.*

MP.6 *Attend to precision.*

REPRESENTING SOLUTIONS OF EQUATIONS GRAPHICALLY

Begin with solving simple linear equations by tracing graphs and using tables on a graphing calculator. Then, advance students to nonlinear situations, so they can see that even complex equations that might involve quadratics, absolute value, or rational functions can also be solved fairly easily using this same strategy. While a standard graphing calculator does not simply solve an equation for the user, it can be used as a tool to approximate solutions.

Converting an Equation to Two Equations

An equation in one variable such as $2x + 3 = x – 7$ can be solved by converting an equation to a system of two equations in two variables: $y = 2x + 3$ and $y = x – 7$ and then graphing the functions $y = 2x + 3$ and $y = x – 7$. They should recognize that the intersection point of the lines is at (–10, –17). They should be able to verbalize that the intersection point means that when $x = –10$ is substituted into both sides of the equation, each side simplifies to a value of –17. Therefore, –10 is the solution to the equation. This same approach can be used whether the functions in the original equation are linear, nonlinear, or both.

EXAMPLE

Solve the equation $x^2 - 3x + 2 = 5$.

Discussion: In other standards, students have learned various strategies to solve this equation algebraically (factor, complete the square, quadratic formula). Notice this equation has only one variable, *x*, and there is no *y*. In A.REI.11, students gain another technique: Graph $y = x^2 - 3x + 2$ and $y = 5$ and find the *x*-values of the intersections of the graphs. Another approach is to set the equation equal to 0 to get $x^2 - 3x - 3 = 0$, and then have students graph the equations $y = x^2 - 3x - 3$ and $y = 0$ and connect the intersection points to the solutions (*x*-intercepts) of the quadratic.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (A.REI.11)

EXAMPLE

Solve the equation $x^2 - 3x + 2 = 2^x$.

Discussion: Algebraic techniques for solving equations require that equations can be manipulated into particular forms. Some equations like the example cannot be manipulated into forms that can be solved with algebraic techniques. The advantage of using the technique of setting a one-variable equation into two equations in two variables and finding the intersection points is that always works, even for equations like $x^2 - 3x + 2 = 2^x$, for which there are no algebraic techniques.

EXAMPLE

Compare the graphs of $x^2 - 3x + 2 = 2^x$ to $x^2 - 3x + 2 = 1.1^x$.

Students graphed the first equation in the previous example. Now graphing the second equation as a system of two equations $y = x^2 - 3x + 2$ and $y = 1.1^x$ appears to have two solutions in a traditional window. However, since the exponential function will eventually exceed the quadratic function, there is a third solution if an appropriate window is found.

Tables

Use the table function on a graphing calculator to solve equations. For example, to solve the equation $x^2 = x + 12$, students can examine the equations $y = x^2$ and $y = x + 12$ and determine that they intersect when $x = 4$ and when $x = -3$ by examining the table to find where the y-values are the same.

Students who make a table of values to find the solution to a system may start with evaluating each function at integer values to determine an approximate solution. Using technology, students can then zoom-in on a smaller window (more precise) of values that would include a solution of the system and make a zoomed-in table of values. They can continue this process, recognizing when the solution is exact, and when the solution is approximate.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (A.REI.11)

Instructional Tools/Resources

These tools and resources may be helpful for instruction. Before use, they should be evaluated at the local level to determine their appropriateness for instruction.

Manipulatives/Technology

- Desmos is a free graphing calculator that is available to students as website or an app.
- GeoGebra is a free graphing calculator that is available to students as website.
- Wolframalpha is dynamic computing tool.

Curriculum and Lessons from Other Sources

- EngageNY, Algebra 1, Module 3, Topic C, Lesson 16: Graphs Can Solve Equations Too is a lesson that pertains to this cluster. This lesson extends A.REI.11 to absolute value equations which can be used as an extension but is not required in Ohio.

General Resources

- Arizona High School Progression on Algebra is an informational document for teachers. This cluster is addressed on page 15.
- High School Coherence Map by UnboundED is a map of the relationships among high school mathematics standards. Please note: This resource is aligned to the Common Core not Ohio's Learning Standards. Although the tool is useful, it needs to be checked with respect to Ohio's Learning Standards.

References

- Common Core Standards Writing Team. (2013, March 1). *Progressions for the Common Core State Standards in Mathematics (draft). High School, Algebra*. Tucson, AZ: Institute for Mathematics and Education, University of Arizona.
- Davis, J. (August 2005). Connecting procedural and conceptual knowledge of functions. *Mathematics Teacher*, *99(1),* 36-39.
- Edwards, T. & Chelst, K. (February 1999). Promote systems of linear inequalities with real-world problems. *The Mathematics Teacher*, *92(2),* 118-123.
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STANDARDS	MODEL CURRICULUM (F.IF.4-5)
Functions **INTERPRETING FUNCTIONS** **Interpret functions that arise in applications in terms of the context.** **F.IF.4** For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship. *Key features include the following: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; and periodicity.* ★ (A2, M3) **b.** Focus on linear, quadratic, and exponential functions. (A1, M2) **F.IF.5** Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes. *For example, if the function h(n) gives the number of person-hours it takes to assemble n engines in a factory, then the positive integers would be an appropriate domain for the function.* ★ **b.** Focus on linear, quadratic, and exponential functions. (A1, M2)	**Expectations for Learning** Working with linear and exponential functions in Math 1, students interpret key features of graphs and tables. They also determine the domain of a function by looking at a graph or table. In a real-life scenario, students can find the restrictions on the domain. In this cluster, students apply these concepts to quadratic functions and compare them to the prior learning of linear and exponential functions from Math 1. In Math 3, students extend identifying and interpreting key features of functions to include periodicity. Students also have to select appropriate functions that model the data presented. Average rate of change over a specific interval will also be included in Math 3. *Note on differences between standards: In F.IF.4 and F.IF.5, the emphasis is on the context of the problem and on making connections among graphs, tables, and the context. In F.IF.7, the emphasis is on creating a graph given a symbolic representation, and then identifying the key features of the graph and connecting the key features to the symbols.* **ESSENTIAL UNDERSTANDINGS** • Key features (as listed in the standard) of a function can be illustrated graphically and interpreted in the context of the problem. • The sensible domain for a real-world context should be accurately represented in graphs, tables, and symbols. • A quadratic function is symmetrical about its axis of symmetry. • Functions can have continuous or discrete domains. More generally, writing a function in different ways can reveal different features of the graph of a function. **MATHEMATICAL THINKING** • Connect mathematical relationships to contextual scenarios. • Attend to meaning of quantities. • Determine reasonableness of results. Continued on next page

STANDARDS	MODEL CURRICULUM (F.IF.4-5)
	Expectations for Learning, continued **INSTRUCTIONAL FOCUS** **Remember, in this course, for exponential functions, assessments should focus on integer exponents only. For quadratic functions, students should work with expressions in which the leading coefficient can be any real number, but assessment questions should focus on expressions with leading coefficients of 1 with occasional questions using other simple leading coefficients.* • For quadratic functions, interpret intercepts; maximum or minimum; symmetry; intervals of increase or decrease; and end behavior given tables, graphs, and verbal descriptions. • Use written descriptions or inequalities to describe intervals on which a function is increasing/decreasing and/or positive/negative (neither interval notation nor set builder notation are required). • Determine whether to connect points on a graph based on the context (continuous vs. discrete domain). • Demonstrate understanding of domain in the context of a real-world problem. • Compare the key features of quadratic functions to linear and exponential functions. For example, o Linear functions are either always increasing, decreasing, or constant. o Exponential functions are either always increasing or decreasing. o Quadratic functions increase to a maximum then decrease or decrease to a minimum then increase. **Content Elaborations** **OHIO'S HIGH SCHOOL CRITICAL AREAS OF FOCUS** • Math 2, Number 3, pages 6-7 **CONNECTIONS ACROSS STANDARDS** • Create equations that describe numbers or relationships (A.CED.2b). • Graph quadratic functions and indicate maxima and minima (F.IF.7b). • Analyze functions using different representations (F.IF.9b).

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (F.IF.4-5)

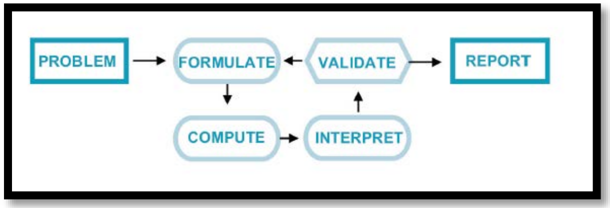
Instructional Strategies

Note: The model curriculum is not a collection of lessons nor a full curriculum; it does not suggest pace, sequence, or amount of time spent on topics. It provides information about a topic related to the standards including ideas for examples, strategies for teaching, common misconceptions, and possible connections between topics. In addition, the emphasis of the model curriculum is on instruction not on assessment; therefore, the examples in the Instructional Strategies section are written for instructional purposes not necessarily as assessment items. Examples may need to be rewritten to accommodate the needs of each individual classroom.

Standards for Mathematical Practice
This cluster focuses on but is not limited to the following practices:
MP.1 *Make sense of problems and persevere in solving them.*
MP.2 *Reason abstractly and quantitatively.*
MP.4 *Model with mathematics.*

Functions are often described in terms of their using key features. Graphs allow the behavior of the function to be more apparent.

MODELING



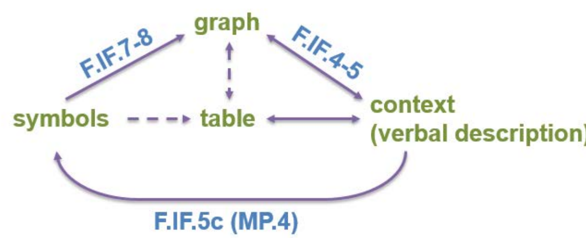
This cluster also includes the modeling standards. See page 12 for more information about modeling.

Begin instruction from a modeling standpoint. Start with a context and ask, "Do one of the three functions—linear, exponential, quadratic—fit the behavior seen in the graph?" The answer sometimes needs to be "no," so that other function types can be explored within the context of the problem. For example, although not necessarily in Math 2, students should be aware that some scenarios are modeled with a periodic phenomenon that have graphs that repeat themselves after the particular interval along *x* – axis. Other situations are modeled by graphs with "wiggles" that are called polynomial functions.

Students should be given a formula that can be graphed using Desmos or other graphing technology and they should be able to reason about the graph after they can see it. Given a table of values students could then create a scatterplot, possibly fit a curve to it, and reason about it in the same way they reason about formulas. *Note: Draw attention to the fact that sometimes the function may not be able to be described by a formula; sometimes the best we can do to describe a function is by a graph or a table.*

INTERPRETING FUNCTIONS

Standards F.IF.4, F.IF.5, and F.IF.7 should be approached individually and in conjunction with one another. On a particular day, the focus could be on F.IF.4, F.IF.5, or F.IF.7 individually, but over the course of a few days, the standards should be woven together. See cluster F.IF.7-9 for an introductory activity about graphing stories. A follow-up activity should be connecting graphs to stories. Students should also be encouraged to write their own stories and then graph them. Then they could share their graphs with their classmates. Focus on graphs that are neither linear, quadratic, or exponential including piecewise scenarios.



INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (F.IF.4-5)

Investigate real-world data in which—

- several outputs may be paired with one input
- one output is paired with one input

Students should be able to reason about trends in the data.

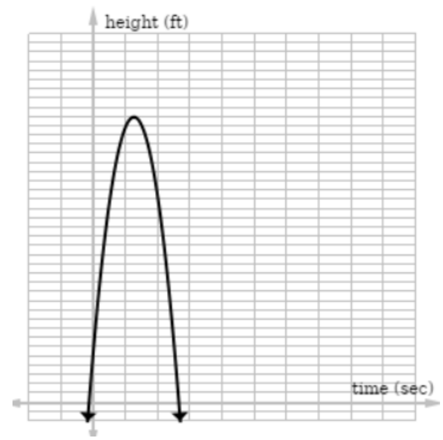
Have students flexibly move from examining a graph and describing its characteristics (e.g., intercepts, relative maximums, etc.) to using a set of given characteristics to sketch the graph of a function.

Some students incorrectly believe that a graph is a picture of the situation (such as the path of the basketball) rather than a representation of the relationship of two particular quantities in a situation. Attention needs to be paid to the quantities given in the context, such as time in seconds since the basketball was released and the height in centimeters of the basketball from the ground.

EXAMPLE

Using technology have students graph the relationship between the height of a basketball above the ground and the time the basketball is in the air: $y = -16x^2 + 40x + 6$. Have them complete a table using the trace button connecting key features of the graph to the quantities of the situation.

- What is the initial height of the ball above the ground? Why do you think the height does not start at 0?
- What is the maximum height of the ball?
- What time does the ball hit the 10ft hoop?
- Assuming there is no resistance, and the ball can follow its path, what time will the ball hit the ground?



Time (s)	Height (cm)

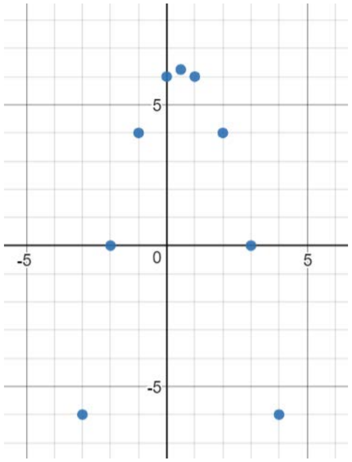
Draw attention to the fact that the height of the basketball is related to time. The horizontal access is not showing the distance of the hoop from the player nor does it trace the basketball's path. The shape would still be a parabola if the basketball was thrown straight up vertically into the air. Therefore, the graph and the function represent that the time is moving forward not the basketball. To break this misconception, have students jump straight up and down three times and graph their height and time. Point out that even though they did not move any distance horizontally, their graph creates 3 parabolas.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (F.IF.4-5)

EXAMPLE

The table represents a continuous function defined on the interval $-2 < x < 3$, where just some integer inputs being used are displayed. Identify the key features of a graph.

x	*f(x)*
-2	$\frac{2}{9}$
-1	$\frac{2}{3}$
0	2
1	6
2	18
3	54



EXAMPLE

The graph on the left represents a discrete quadratic function defined on the interval $-4 < x < 3$. Identify the key features of a graph.

KEY FEATURES OF FUNCTIONS

In Math 2, the majority of graphs representing functions may have the following key features:

- increasing intervals,
- decreasing intervals,
- relative maximums,
- relative minimums,
- *x*- and *y*-intercepts,
- symmetries,
- end behavior, and/or
- periodicity (not needed for Math 2)

There are limits to the key features depending on the function type. If the domain is restricted, additional key features are possible. Although the focus of Math 2 is on linear, exponential, and quadratic functions, students should be exposed to other function types and informally discuss their key features. Connect with concepts of parent functions and function families in F.IF.7-9. It might be helpful to start with a non-formula graph such as temperature over time.

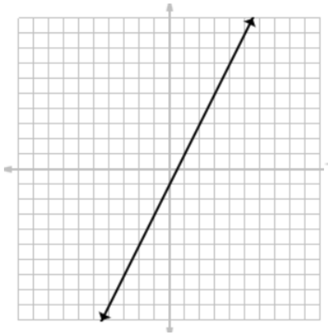
INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (F.IF.4-5)

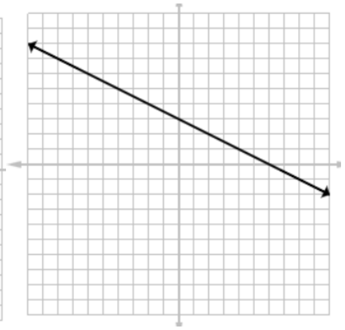
When discussing intervals, informal descriptions of end behaviors are acceptable. For example, "To the right, the graph goes to infinity and to the left, it is 'leveling off.'" (Interval notation and set builder notation are not necessary.) Also, written descriptions or inequalities are acceptable. For example,

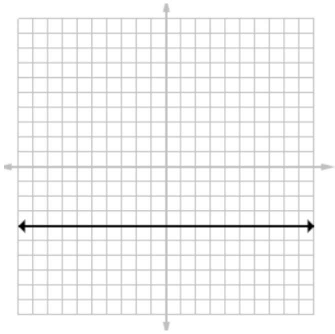
- all x-values greater than 3 or $x > 3$;
- the function is increasing between 3 and 7, or is increasing $3 < x < 7$.

Linear Functions

A graph of a linear function with an unrestricted domain may have *x*-intercept or *y*-intercept, or both. The graph may be increasing or decreasing, or neither. Regardless how the graph of a linear function looks, it does not have a minimum or maximum. See 8th Grade Model Curriculum cluster 8.F.4-5 for ideas about scaffolding with linear functions.

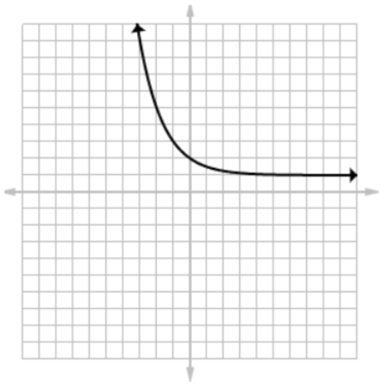


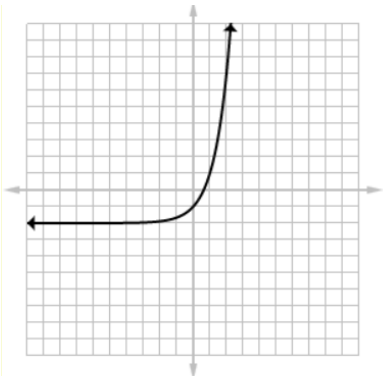




Exponential Functions

A graph of an exponential function may have an *x*-intercept or both *x*- and *y*-intercepts. If the definition of an exponential functions is that it is a function in which the values of the domain are exponents, then adding or subtracting a constant can make different *x*-intercepts possible. For example, a function such as $y = 2^x - 8$ has an *x*-intercept at 3 since the point (3, 0) is on the graph. The graph of an exponential function may be increasing or decreasing. It does not have relative maximums, minimums, or symmetry. However, it can be described by its end behavior.

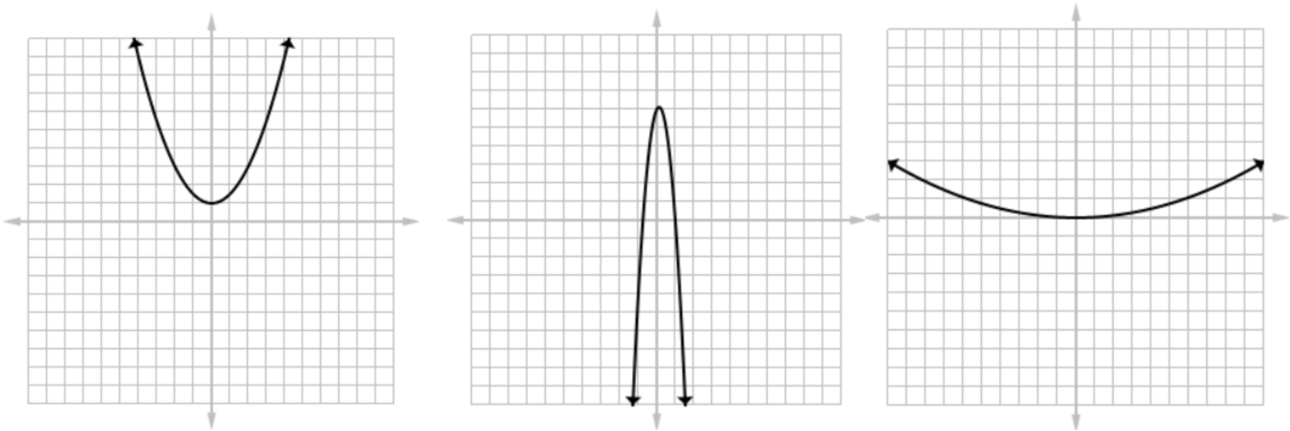




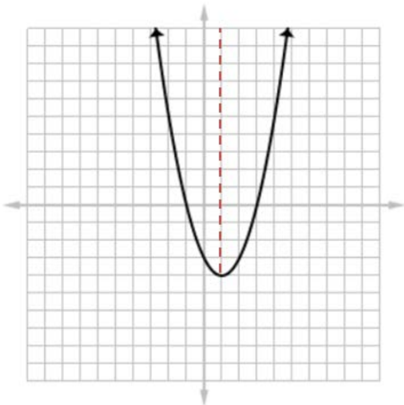
INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (F.IF.4-5)

Quadratic Functions

The graph of a quadratic function may have either 0, 1, or 2 *x*-intercepts but only 1 y-intercept. The graph has either a decreasing interval followed by an absolute minimum and then an increasing interval, or it has an increasing interval followed by an absolute maximum and a decreasing interval. The graphical representation of a quadratic function is also symmetrical and is described by its axis of symmetry. In addition, a quadratic can be described by its end behavior. If the domain is a restricted, a quadratic may not have both an increasing and decreasing interval or necessarily an absolute minimum/maximum.



Axis of Symmetry



To develop an intuitive sense of the vertex formula ($h = \frac{-b}{2a}$ for $f(x) = ax^2 + bx + c$ with vetex $(h, k)$and then evaluating *y* for *h* to find *k*) for the axis of symmetry, *x* = *h*, a student may find the mean of the two x-intercepts. However, to be noted this method will only apply to quadratics such as found in Math 2 that have real roots; See F.BF.3-4 for an example on using transformations to find the vertex formula.

DOMAIN OF A FUNCTION

When choosing a function family, be sure to ask whether that function family makes sense within the context. Sometimes the answer is no, and other times the answer may be yes over a restricted domain. For example, an entire roller coaster cannot be defined by a single quadratic equation, but one hill may be modeled by one quadratic function, and the next hill could be modeled by a different quadratic function.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (F.IF.4-5)

Recognize appropriate domains of functions in real-world settings. For example, when determining a weekly salary based on hours worked, the hours (input) could be a rational number, such as 25.5, but not a negative number. Furthermore, there must be a maximum number of hours worked, determined based on reasonable assumptions. If a function relates the number of cans of soda sold in a machine to the money generated, the domain must consist of whole numbers.

Make sure students are exposed to functions in mathematical and real-world contexts that have both continuous and discrete domains.

Students may incorrectly believe that it is reasonable to input any x-value into a function, not understanding that context determines the domain. Therefore, they will need to examine multiple situations in which there are various limitations to the domains.

Instructional Tools/Resources

These tools and resources may be helpful for instruction. Before use, they should be evaluated at the local level to determine their appropriateness for instruction.

Manipulatives/Technology

- Tables, graphs, and equations of real-world functional relationships
- Graphing calculators to generate graphical, tabular, and symbolic representations of the same function for comparison
- GeoGebra is a free graphing calculator that is available to students as website.
- Desmos is a free graphing calculator that is available to students as website or an app.
- Wolframalpha is dynamic computing tool.

Interpreting Functions

- Is the Gateway Arch a Parabola is a blog by Murray Bourne that discuss if the Gateway Arch is a parabola.
- Lifespan of a Meme, the Harlem Shake by Yummy Math is an activity where students interpret a graph to explore a viral video.
- The Queen's Reward is an activity where students help a young mathematician outwit the Queen's chief advisors by solving quadratic functions.
- Roller Coasting through Functions is a lesson by NCTM Illuminations where students use graphs and tables to analyze the falls of different roller coasters using a quadratic equation. *NCTM now requires a membership to view their lessons.*
- Protein Bar Toss Parts 1 and 2 is a task from the Georgia Standards of Excellence Framework where students find the maximums and minimums of quadratic functions. This task starts on page 93.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (F.IF.4-5)

Interpreting Functions, continued

- Egg Launch Contest by NCTM Illuminations is a lesson where students represent and interpret quadratic functions as a table, with a graph, and with an equation.
- Weightless Wonder by NASA is a lesson where students investigate the characteristics of quadratic functions in the context of parabolic flights involving NASA's Weightless Wonder Jet.
- Parabolic Pee by Yummy math has students interpret parabolic motion functions.

Domain and Range

- The Restaurant is a task from Illustrative Mathematics where students are asked to find the domain and range from a given context.

Curriculum and Lessons from Other Sources

- EngageNY, Module 1, Topic A, Lesson 1: Graphs of Piecewise Linear Functions, Lesson 2: Graphs of Quadratic Functions, Lesson 3: Graphs of Exponential Functions, Lesson 4: Analyzing Graphs—Water Usage During a Typical Day at School are lessons that pertain to this cluster.
- EngageNY Module 4, Topic A, Lesson 8: Exploring the Symmetry in Graphs of Quadratic Functions, Lesson 10: Interpreting Quadratic Functions from Graphs and Tables are lessons that pertain to this cluster.
- EngageNY, Module 5, Topic A, Lesson 1: Analyzing a Graph, Lesson 2: Analyzing a Data Set, Lesson 3: Analyzing a Verbal Description are lessons that pertain to this cluster.
- EngageNY, Module 5, Topic B, Lesson 9: Modeling a Context from a Verbal Description is a lesson that pertains to this cluster.
- Mathematics Vision Project, Algebra 1, Module 3: Features of Functions has many lessons that pertain to this cluster.
- Mathematics Vision Project, Algebra 1, Module 6: Quadratic Functions has many lessons that pertain to this cluster.
- Mathematics Vision Project, Algebra 1, Module 8: More Functions, More Features has many lessons that pertain to this cluster.
- Unit 3: Functions from eMATHinstruction has materials that could be used for intervention. These documents can be used for individual students or for the entire class.
- Illustrative Mathematics, Algebra 1, Unit 6, Lesson 6: Building Quadratic Functions to Describe Situations (Part 2), Lesson 7: Building Quadratic Functions to Describe Situations (Part 3), Lesson 14: Graphs That Represent Situations,
- Illustrative Mathematics, Algebra 1, Unit 7, Lesson 10: Rewriting Quadratic Expressions in Factored Form (Part 4), Lesson 17: Applying the Quadratic Formula (Part 1),

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (F.IF.4-5)

General Resources

- Arizona High School Progression on Functions is an informational document for teachers. This cluster is addressed on pages 8-9.
- Arizona's Progression on High School Modeling is an informational document for teachers. This cluster is addressed on page 12.
- High School Coherence Map by UnboundED is a map of the relationships among high school mathematics standards. Please note: This resource is aligned to the Common Core not Ohio's Learning Standards. Although the tool is useful, it needs to be checked with respect to Ohio's Learning Standards.

References

- Baron, L. (February 2015). An authentic task that models quadratics. *Mathematics Teaching in the Middle School*, *20(6),* 334-340.
- Common Core Standards Writing Team. (2013, March 1). *Progressions for the Common Core State Standards in Mathematics (draft). Grade 8, High School, Functions.* Tucson, AZ: Institute for Mathematics and Education, University of Arizona.
- Common Core Standards Writing Team. (2013, July 4). *Progressions for the Common Core State Standards in Mathematics (draft). High School, Modeling*. Tucson, AZ: Institute for Mathematics and Education, University of Arizona.
- Nebesniak, A. & Burgoa, A. (February 2015). Developing the vertex formula meaningfully. *Mathematics Teacher*, *108(6),* 429-433.
- Paoletti, T., Stevens, I., & Moore, K. (February 2017). Tricks may inhibit students' reasoning. *Mathematics Teacher*, *110(6),* 446-453.
- Weiss, M. (December 2016/January 2017). Solving and graphing quadratics with symmetry transformation. *Mathematics Teacher*, *110(5),* 394-397.

STANDARDS

Functions

INTERPRETING FUNCTIONS

Analyze functions using different representations.

F.IF.7 Graph functions expressed symbolically and indicate key features of the graph, by hand in simple cases and using technology for more complicated cases. Include applications and how key features relate to characteristics of a situation, making selection of a particular type of function model appropriate. ★

b. Graph quadratic functions and indicate intercepts, maxima, and minima. (A1, M2)

F.IF.8 Write a function defined by an expression in different but equivalent forms to reveal and explain different properties of the function.

a. Use the process of factoring and completing the square in a quadratic function to show zeros, extreme values, and symmetry of the graph, and interpret these in terms of a context. (A2, M3)

i. Focus on completing the square to quadratic functions with the leading coefficient of 1. (A1, M2)

Continued on next page

MODEL CURRICULUM (F.IF.7-9)

Expectations for Learning

In Math 1, students graph linear and exponential functions given a symbolic representation and indicate intercepts and end behavior. They compare linear and exponential functions given various representations.

In this cluster, students graph quadratic functions and indicate key features such as maxima/minima. They compare linear, quadratic, and exponential functions given various representations.

In Math 3, students graph polynomial, square root, cube root, trigonometric, piecewise-defined, (+) rational, and (+) logarithmic functions. Students identify and interpret key features (as applicable) including intercepts, end behavior, period, midline, amplitude, symmetry, asymptotes, maxima/minima, and zeros.

Note on differences between standards: In F.IF.4 and F.IF.5, the emphasis is on the context of the problem and on making connections among graphs, tables, and the context. In F.IF.7, the emphasis is on creating a graph given a symbolic representation, then identifying the key features of the graph and connecting the key features to the symbols.

ESSENTIAL UNDERSTANDINGS

- The graph of a quadratic function shows intercepts and maximum or minimum.
- The factored form of a quadratic function reveals the zeros of the function (i.e., the x-intercepts of the graph); the vertex form of a quadratic function reveals the maximum or minimum of the function; the standard form of a quadratic function reveals the y-intercept of the graph.
- Different representations (graphs, tables, symbols, verbal descriptions) illuminate key features of functions and can be used to compare different functions.

MATHEMATICAL THINKING

- Make connections between concepts, terms, and properties within the grade level and with previous grade levels.
- Analyze a mathematical model.

Continued on next page

STANDARDS	MODEL CURRICULUM (F.IF.7-9)
b. Use the properties of exponents to interpret expressions for exponential functions. *For example, identify percent rate of change*[G] *in functions such as* $y = (1.02)^t$, *and* $y = (0.97)^t$ *and classify them as representing exponential growth or decay.* (A2, M3) **i.** Focus on exponential functions evaluated at integer inputs. (A1, M2) **F.IF.9** Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions). *For example, given a graph of one quadratic function and an algebraic expression for another, say which has the larger maximum.* (A2, M3) **b.** Focus on linear, quadratic, and exponential functions. (A1, M2)	**Expectations for Learning, continued** **INSTRUCTIONAL FOCUS** **Remember, in this course, for exponential functions, assessments should focus on integer exponents only. For quadratic functions, students should work with expressions in which the leading coefficient can be any real number, but assessment questions should focus on expressions with leading coefficients of 1 with occasional questions using other simple leading coefficients.* • Given symbolic representations of quadratic functions, create accurate graphs showing all key features. • Identify the key features of the graph of a quadratic function by factoring, using the quadratic formula, or completing the square. • Compare and contrast linear, quadratic, and exponential functions given by graphs, tables, symbols, or verbal descriptions. • Determine the zeros of a quadratic function by factoring, using the quadratic formula, or completing the square. • Use different forms of quadratic functions (standard form, vertex form, factored form) to reveal different features. • Explore the relationship of the symbolic representation of a function and its graph by adjusting parameters. **Content Elaborations** **OHIO'S HIGH SCHOOL CRITICAL AREAS OF FOCUS** • Math 2, Number 3, pages 6-7 **CONNECTIONS ACROSS STANDARDS** • Interpret functions that arise in applications in terms of the context (F.IF.4) • Solve quadratic equations in one variable (A.REI.4). • Write expressions in equivalent forms to solve problems (A.SSE.3). • Build a function that models a relationship between two quantities (F.BF.1).

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (F.IF.7-9)

Instructional Strategies

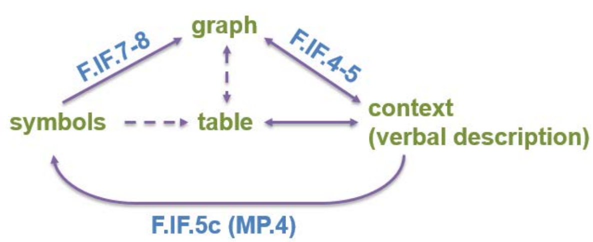
Note: The model curriculum is not a collection of lessons nor a full curriculum; it does not suggest pace, sequence, or amount of time spent on topics. It provides information about a topic related to the standards including ideas for examples, strategies for teaching, common misconceptions, and possible connections between topics. In addition, the emphasis of the model curriculum is on instruction not on assessment; therefore, the examples in the Instructional Strategies section are written for instructional purposes not necessarily as assessment items. Examples may need to be rewritten to accommodate the needs of each individual classroom.

Standards for Mathematical Practice
This cluster focuses on but is not limited to the following practices:
***MP.4** Model with mathematics.*
***MP.5** Use appropriate tools strategically.*
***MP.7** Look for and make use of structure.*

GRAPHING FUNCTIONS

Standards F.IF.4, F.IF.5, and F.IF.7 should be approached individually and in conjunction with one another. On a particular day, the focus could be on F.IF.4, F.IF.5, or F.IF.7 individually, but over the course of a few days, the standards should be woven together.

Introduce functions by having students create data from the walking and then graph the data to make a connection to distance/time graphs.



Some students may incorrectly believe that a function is a synonym for formula. Point out that some functions do not have formulas at all, and some formulas do not represent functions. For example, recoding the average daily temperature at the Columbus Airport cannot be represented by a formula, but can be represented by a table. There is a formula for the equation of circle, yet a circle is not a function.

Some students may incorrectly believe a piece-wise function is several different functions because it is represented by several different formulas. Emphasize that it is one function pieced together.

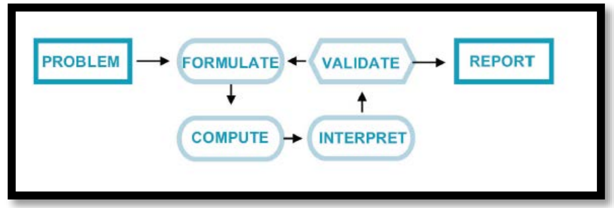
TIP!

Discuss the importance of using appropriate labels and scales on the axes when representing functions with graphs. Some students believe that the labels and scales on a graph are not important and can be assumed by a reader, and that it is always necessary to use the entire graph of a function when solving a problem that uses that function as its model.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (F.IF.7-9)

MODELING

This cluster is included in the modeling standards. See page 12 for more information about modeling.



FAMILIES OF FUNCTIONS

"Functions are often studied and understood as families, and students should spend time studying functions within a family, varying parameters to develop an understanding of how the parameters affect the graph of a function and formulas and its key features." (Common Core Standards Writing Team, March 2013)

A family of functions is a set of functions that are related by adjustable parameters. Students should develop an understanding of what the parameters in a family do. Explore various families of functions by graphing those families and helping students make connections in terms of the formulas and key features.

Have students explore and identify the function families: linear, quadratic, exponential, cubic, absolute value, square root, cubed root, sinusoidal among others but strive for fluency on linear, exponential, and quadratic functions with respect to representations and characteristics. However, for the other functions families, focus on shape. Introduce students to non-familiar functions to apply identification of key features. Also, some functions, such as piece-wise functions, may not have formulas. This lends to connections with the knowledge about parent functions to model data. Students must be able to differentiate between linear, exponential, and quadratic functions and identify the parent function and interpret its key features. This should be driven by applications for modeling. Use domain and range values that are appropriate to the context.

TIP!

In a quadratic function $f(x) = ax^2 + bx + c$, many students forget to specify that $a \neq 0$. But this is critical, for if $a = 0$, then the function is not quadratic and its graph is not a parabola.

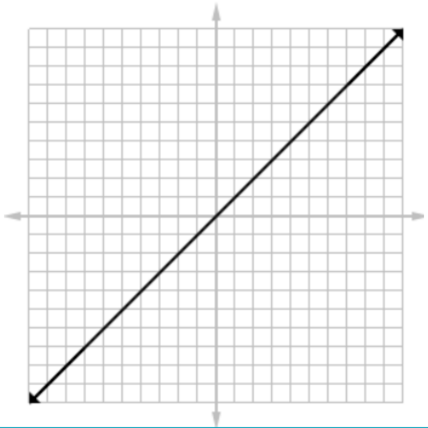
Real-world problems, such as maximizing the area of a region bound by a fixed perimeter fence, can help to illustrate applied uses of families of functions.

Students may believe that each family of functions (e.g., quadratic, square root, etc.) have no commonalities, so they may not recognize common aspects across the families of functions and their graphs such as *y*-intercepts and end behavior.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (F.IF.7-9)

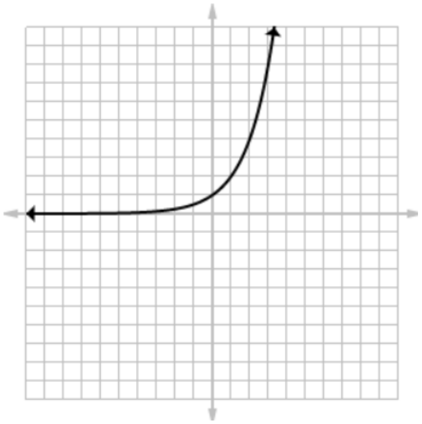
PARENT FUNCTIONS

Linear Functions



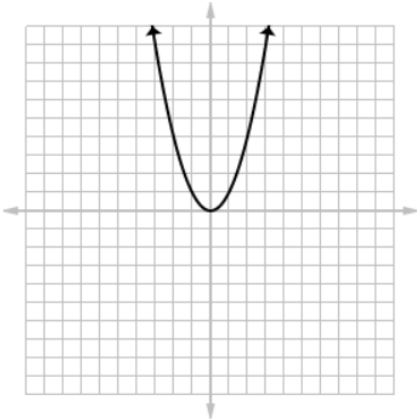
A graph of a linear function is a straight line with a constant rate of change and a y-intercept.

Exponential Functions



A graph of an exponential function is a curve that has a y-intercept and an end behavior of the curve approaching infinity at one end and approaching a number at the other end. It may or may not have an x-intercept.

Quadratic Functions



A graph of a quadratic function is a curve that is a U-shaped graph (or upside down U-shaped), y-intercepts, a maximum or minimum, and end behavior of the curve approaching toward either positive or negative infinity. It may or may not have an x-intercept(s).

EXAMPLE

Using technology have students create card pictures out of different function types. For example, give students a variety of exponential functions and determine what all exponential functions have in common. This allows students to connect the graphs of functions with their corresponding algebraic representations.

Students should graph simple cases of functions by hand, but use technology for more complicated graphing done by students. Make connections to algebra work (from A.APR.6) to functions. Sometimes displaying a good graph means getting a good "window."

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (F.IF.7-9)

EXAMPLE

Identify a family of functions whose graphs are parabolas that are symmetric about the y-axis.

Discussion: Students should come to the conclusion that $f(x) = ax^2 + c$, with $a \neq 0$ is a family of functions. This is a legitimate family, but many students may not recognize this. This family can be considered a "subfamily" of the family of quadratic functions. What makes it a family is that (1) the functions all have the same "form" and (2) a particular member of the family is chosen by specifying the parameters.

FUNCTIONS IN EQUIVALENT FORMS

Writing a function in different ways can reveal different features of the graph of a function. Think of $f(x) = 4x^2$as a vertical stretch of $y = x^2$and the equivalent $f(x) = (2x)^2$ as a horizontal shrink that yields the same graph. Also, think of $g(x) = 2x + 6$ as a vertical shift of $y = 2x$, and the equivalent $g(x) = 2(x + 3)$ as a horizontal shift that yields the same graph. Think of $k(x) = 3^{x+2}$ as a horizontal shift of $y = 3^x$ and the equivalent $k(x) = 3^2 \cdot 3^x = 9 \cdot 3^x$ as a vertical stretch. Think of $h(x) = 1.02^{3x}$ as a horizontal shrink of $y = 1.02^x$, and the equivalent $h(x) = (1.02^3)^x = (1.061208)^x$ as a change of base of the exponential, but is not a vertical stretch. Students should be given the opportunity to come up with equivalent forms of the same function, and then explore why the functions are the same both algebraically and graphically. The process of rewriting functions to reveal key features should be used to explain/reveal features in the context of real-world scenarios.

Students may believe that the process of rewriting functions into various forms is simply an algebraic symbol manipulation exercise. Focus on the purpose of allowing different features of the function to be exhibited.

TIP!

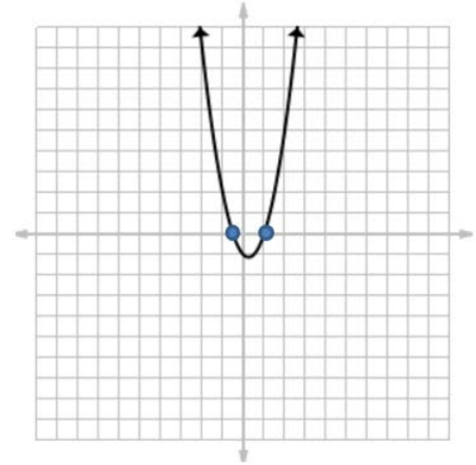
Use various representations of the same function to emphasize different characteristics of that function.

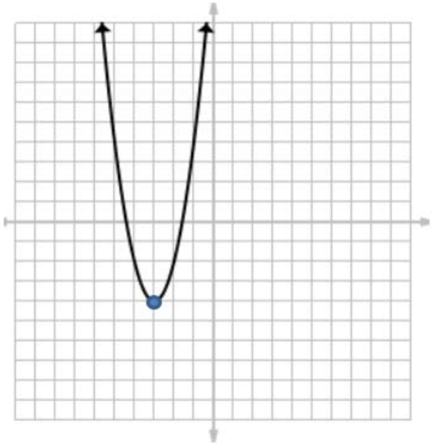
For quadratics in standard form, the y-intercept of the function $y = x^2 - 4x - 12$ is easy to recognize as (0, –12). However, rewriting the function as $y = (x - 6)(x + 2)$ reveals zeros at (6, 0) and at (–2, 0). Furthermore, completing the square allows the equation to be written as $y = (x - 2)^2 - 16,$ which shows that the vertex (and minimum point) of the parabola is at (2, –16) and reveals transformations applied to the graph of parent quadratic function $y = x^2$. The same can be true for the various forms of linear equations.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (F.IF.7-9)

FACTORING QUADRATICS

A quadratic expression written in factored form helps students recognize the zeros of a function without having to graph the function. Since $f(x) = 2x^2 - x - 1$ can be rewritten as $f(x) = (2x + 1)(x - 1)$, students can easily deduce that the zeros or x-intercepts are $x = -\frac{1}{2}$ and $x = 1$ by setting each factor equal to 0 and then solving for x. Connect with factoring in A.SSE.3.





COMPLETING THE SQUARE

Completing the square is useful to rewrite a quadratic expression in vertex form, $f(x) = a(x - h)^2 + k$. Vertex form allows students to easily determine the maximum or minimum point (h, k) without having to graph. Since $g(x) = 2x^2 + 12x + 14$ can be rewritten in vertex form $g(x) = 2(x + 3)^2 - 4$, students can easily deduce that the minimum is $(-3, -4)$ without having to graph or use $h = \frac{-b}{2a}$ and then evaluating function $g(x)$ at $x = h$ to find k. Connect with completing the square in A.SSE.3.

EXPONENTIAL FUNCTIONS

For exponential functions in Math 2, it is acceptable to use continuous graphs as a part of problem solving, even though students will not know what is really going on for the non-integer domain values (if they have evaluated the function only at integer inputs). A continuous graph allows students to see trends and to make claims such as "the city population will reach 1 million between years 7 and 8."

Examine multiple real-world examples of exponential functions so that students recognize that a base between 0 and 1 (such as an equation describing depreciation of an automobile, $f(x) = 15{,}000(0.8)^x$, represents the value of a $15,000 automobile that depreciates 20% per year over the course of x years) results in an exponential decay, while a base greater than 1 (such as the value of an investment over time $f(x) = 15{,}000(1.07)^x$ represents the value of an investment of $5,000 when increasing in value by 7% per year for x years) illustrates growth. Connect to properties of exponents in A.SSE.3.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (F.IF.7-9)

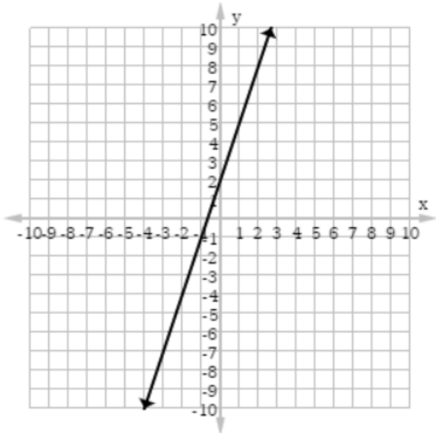
COMPARING FUNCTIONS

The purpose of F.IF.9 is so that students see key features across different representations of two functions.

EXAMPLE

Which function has a greater rate of change?

a. $y - 3 = -4(x + 2)$ **b.**



c.

x	*y*
5	19
8	28
9	31
11	37

Remind students that using a graphing tool such as a calculator or online applet does not always create an accurate graph. For example, technology may connect points when graphing a function that implies that the graph is continuous when in fact it is not (asymptotes drawn with lines, points of discontinuity are shown as complete points on graph unless traced). Window scale selection is key to show correct shape, features, and end behaviors of graphs.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (F.IF.7-9)

Instructional Tools/Resources

These tools and resources may be helpful for instruction. Before use, they should be evaluated at the local level to determine their appropriateness for instruction.

Manipulatives/Technology

- Graphing utilities on a calculator and/or computer can be used to demonstrate the changes in behavior of a function as various parameters are varied.
- GeoGebra is a free graphing calculator that is available to students as website.
- Desmos is a free graphing calculator that is available to students as website or an app.
- Wolframalpha is dynamic computing tool.

Graphing Functions

- Waterline by Desmos is a task that has students watch glasses filling with water and graph functions to uncover misconceptions about graphs.
- Marbleslides by Desmos is a good practice activity for functions and parameters but not the best initial instructional activity as it is not a delivery of content lesson.
- Should I Replace My Toilets? by Yummy Math has students create equations and graphs comparing a pre-1980s toilet and a high efficiency toilet.

Usefulness of Different Forms on a Function

- Which Function? is a task by Illustrative Mathematics that has students interpret different forms of a quadratic function.
- Graphs of Quadratic Functions is a task by Illustrative Mathematics that is an exploration of the usefulness of the different forms of a quadratic functions.

Factoring and Completing the Square

- Building Connections is a lesson by NCTM Illuminations that has students make connection among different classes of polynomial function by exploring their graphs. The last activity in the set, Higher Degree of Polynomials, could be an extension of the concepts.

Comparing Properties of Functions Represented in Different Forms

- Throwing Baseballs is a task by Illustrative Mathematics that has students compare different representations of two different quadratic functions.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (F.IF.7-9)

Curriculum and Lessons from Other Sources

- EngageNY, Module 4, Topic A, Lesson 9: Graphing Quadratic Functions in Factored Form is a lesson that pertains to this cluster.
- EngageNY, Module 4, Topic B, Lesson 16: Graphing Quadratic Equations from the Vertex Form, $y = a(x - h)^2 + k$, Lesson 17: Graphing Quadratic Functions from the Standard Form, $f(x) = ax^2 + bx + c$ are lessons that pertain to this cluster.
- EngageNY, Module 4, Topic C, Lesson 23: Modeling with Quadratic Functions and Lesson 24: Modeling with Quadratic Functions are lessons that pertain to this cluster.
- EngageNY, Module 5, Topic B, Lesson 4: Modeling a Context From a Graph, Lesson 5: Modeling From a Sequence, Lesson 6: Modeling a Context from Data, Lesson 8: Modeling a Context from a Verbal Description, Lesson 9: Modeling a Context from a Verbal Description are lessons that pertain to this cluster.
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- Illustrative Mathematics, Algebra 1, Unit 7, Lesson 23: Using Quadratic Expressions in Vertex Form to Solve Problems, Lesson 24: Using Quadratic Equations to Model Situations and Solve Problems are lessons that pertain to this cluster.

General Resources

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INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (F.IF.7-9)

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- Baron, L. (February 2015). An authentic task that models quadratics. *Mathematics Teaching in the Middle School, 20(6),* 334-340.
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STANDARDS	MODEL CURRICULUM (F.BF.1)
Functions **BUILDING FUNCTIONS** **Build a function that models a relationship between two quantities.** **F.BF.1** Write a function that describes a relationship between two quantities. ★ **a.** Determine an explicit expression, a recursive process, or steps for calculation from context. **ii.** Focus on situations that exhibit quadratic or exponential relationships. (A1, M2)	**Expectations for Learning** In Math 1, students write linear and exponential functions symbolically given the relationship between two quantities. They also write explicit and recursive rules for arithmetic and geometric sequences. In Math 2, students focus on quadratic situations and compare to linear and exponential relationships. Students also continue to work with sequences. In Math 3, students build functions from other functions allowing students to model more complex situations. This includes combining functions of various types using arithmetic operations or (+) composition. **ESSENTIAL UNDERSTANDINGS** • Functions (including quadratic functions) can be written as explicit expressions, recursive processes, and in other ways. • Some sequences may be defined recursively or explicitly while others cannot be defined by a formula. • The relationships between quantities can be modeled with functions that are linear, exponential, quadratic, or none of these. **MATHEMATICAL THINKING** • Make and modify a model to represent mathematical thinking. • Discern and use a pattern or structure. **INSTRUCTIONAL FOCUS** • Model relationships with linear, exponential, and quadratic functions using tables, graphs, symbols, and words in context. • Model relationships that are not linear, exponential, or quadratic using tables, graphs, symbols, and words in context. *Continued on next page*

STANDARDS	MODEL CURRICULUM (F.BF.1)
	Content Elaborations **OHIO'S HIGH SCHOOL CRITICAL AREAS OF FOCUS** • Math 2, Number 3, pages 6-7 **CONNECTIONS ACROSS STANDARDS** • Create equations that describe numbers or relationships (A.CED.2). • Analyze functions using different representations (F.IF.8).

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (F.BF.1)

Instructional Strategies

Note: The model curriculum is not a collection of lessons nor a full curriculum; it does not suggest pace, sequence, or amount of time spent on topics. It provides information about a topic related to the standards including ideas for examples, strategies for teaching, common misconceptions, and possible connections between topics. In addition, the emphasis of the model curriculum is on instruction not on assessment; therefore, the examples in the Instructional Strategies section are written for instructional purposes not necessarily as assessment items. Examples may need to be rewritten to accommodate the needs of each individual classroom.

Standards for Mathematical Practice

This cluster focuses on but is not limited to the following practices:

***MP.2** Reason abstractly and quantitatively.*

***MP.4** Model with mathematics.*

***MP.8** Look for and express regularity in repeated reasoning.*

Students should be given pattern tasks where the tasks are—

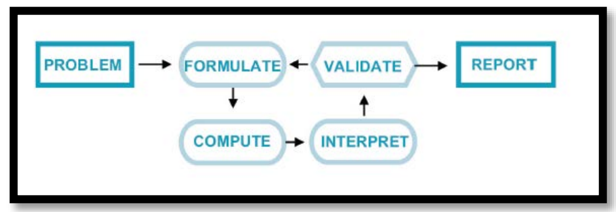
- able to be modeled recursively and explicitly (flexible tasks);
- more easily modeled explicitly; or
- more easily modeled recursively.

When using flexible tasks students should be asked the following:

- Which rule do you prefer and why?
- Does your preference depend on the situation? Explain.
- What advantages are there for using an explicit rule?
- What advantages are there for using a recursive rule?
- What are the connections between the recursive rule and the explicit rule?
- What are the different ways that slope is represented in the two rules?

MODELING

This cluster is included in the modeling standards. Geometric models that students can build are especially helpful to see patterns. See page 12 for more information about modeling.



EXPLICIT AND RECURSIVE FORMS OF FUNCTIONS

An *explicit* rule allows one to take any input and find the corresponding output, whereas a *recursive* rule requires the previous term(s).

TIP!

It may take some time for some students to realize that each term in the position *n* is defined by preceding term(s). Give students different sequences such as integers, odd integers, even integers, multiples of 3, etc., and have them pick any term as a starting point of the sequence and define the numbers going forward and backward.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (F.BF.1)

Provide real-world examples (e.g., a table showing how far a car has driven after a given number of minutes, traveling at a uniform speed). If time and distance are column headings, then examine the table by looking "down" the table to describe a relationship recursively, as well as "across" the table to determine an explicit formula to find the distance traveled if the number of minutes is known. (Changing the orientation of the table, swaps the "down" and the "across.")

Start with visual models (e.g., folding a piece of paper in half multiple times to compare the number of folds to the thickness of the paper), to generate sequences of numbers that can be explored and described with both recursive and explicit formulas. Perimeter and area problems that can be modeled with toothpicks or graph paper, could also be useful. As students are already familiar with function tables, use those to help build understanding.

TIP!

Students may believe that the best (or only) way to generalize a table of data is by using a recursive formula. Students naturally tend to look "down" a table to find the pattern but need to realize that finding the 100th term requires knowing the 99th term unless an explicit formula is developed.

Using the recursive formula has become easier with the use of technology and tables in graphing calculators and spreadsheets.

EXAMPLE

Your friend, Dominic, posts a meme to Facebook, and he asks you to not only share it with three people, but also to ask that the three people you share it with also share it with three people, and so on. Write a recursive and explicit rule for the situation. Discuss if this is an example of an arithmetic or geometric sequence and why.

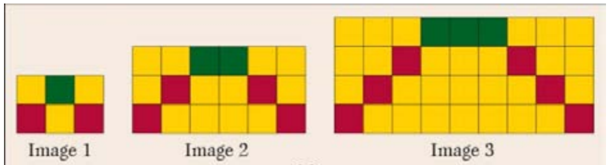
Students may also incorrectly believe that arithmetic and geometric sequences are the same. Students need experiences with both types of sequences (and sequences that are neither) to be able to recognize the difference and more readily develop formulas to describe them.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (F.BF.1)

EXAMPLE

In the examples below have students describe the patterns they see and then create two more images using the patterns. Have students create a table to describe the number of green, red, yellow, and total number of tiles including writing rules for describing the pattern. Then have students identify which patterns are linear or quadratic and explain why.

a.



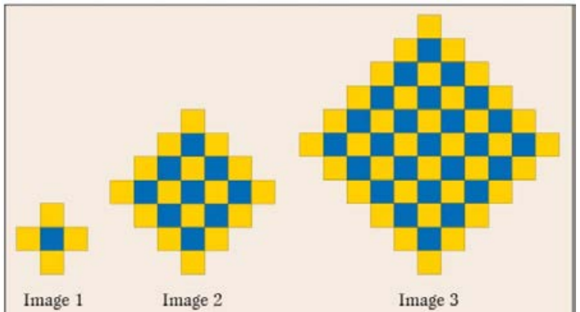
(a)

Number of Tiles

Image Number	Green	Red	Yellow	Total
1	1	2	3	6
2	2	4	12	18
3	3	6	27	36
4	4	8	48	60
x	x	$2x$	$3x^2$	$3x^2 + 3x$

(b)

b.



(a)

Number of Tiles

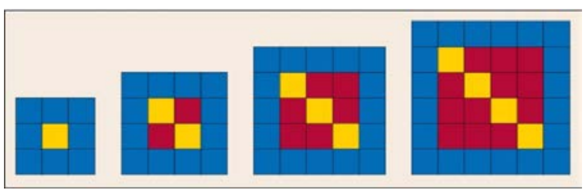
Image Number	Blue	Yellow	Total
1	1	4	5
2	9	16	25
3	25	36	61
4	49	64	113
x	$4x^2 - 4x + 1$	$4x^2$	$8x^2 - 4x + 1$

(b)

From "Finding Linear & Quadratic Equations in Linear Patterns by Terri Kurz published in Mathematics Teacher, Volume 110, No. 6, February 2017.

c.

d.



e. Have students create their own quadratic patterns accompanied by a table and a rule. Have them explain which components in their patterns are linear or quadratic.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (F.BF.1)

Give students examples of other functions represented in symbolic form that are not linear, exponential, or quadratic such as $V(s) = s^3$ or $f(n) = 1.04 \cdot f(n-1) + 500, f(0) = 500$.

DISCRETE VS CONTINUOUS DOMAINS

When creating graphs of functions within a context, it is important to discuss the usage of a discrete versus a continuous domain. Make sure to present examples where it does not make sense to have a continuous domain and therefore the points on the graph should not be connected. For example, the profit after selling *n* tickets at $8 each and deducting $100 facility rental fee would have a domain of only whole numbers, so the dots would not be connected. In comparison, if someone is selling fudge at $8 per pound, the dots would be connected because it is possible to sell non-whole number pounds of fudge.

Students may incorrectly believe that they can always "connect the dots" in a graph. Spend time interpreting the ordered pairs in between the dots. Provide examples where the context of the sequence can be modified to make a continuous domain so that students can "connect the dots."

Some situations that give rise to quadratic functions are as follows: heights of objects in projectile motion, scaling situations about area (including dot diagrams that are area-like arrays), sum of *n* consecutive terms of an arithmetic sequence, and the profit when the relationship between price and demand is linear.

Instructional Tools/Resources

These tools and resources may be helpful for instruction. Before use, they should be evaluated at the local level to determine their appropriateness for instruction.

Manipulatives/Technology

- Hands-on materials (e.g., paper folding, building progressively larger shapes using pattern blocks, etc.) can be used as a visual source to build numerical tables for examination.
- Desmos is a free graphing calculator that is available to students as website or an app.
- GeoGebra is a free graphing calculator that is available to students as website.
- Wolframalpha is dynamic computing tool.
- Visual Patterns is a website that shows pictures of linear, exponential, and quadratic patterns.
- Patterns Posters for Algebra 1 from Finding Ways by Fawn Nguyen is a blog about a class activity where students are given patterns and they have to make posters from them. She is the creator of the visual patterns link above.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (F.BF.1)

Continuous and Discrete Functions

- Continuous and Discrete Functions is a page by MathBitsNotebook that compare continuous and discrete functions.

Exponential Functions

- Game, Set, Flat by Desmos is an activity that helps students understand an exponential relationship to describe a "good" tennis ball. They will also construct an exponential equation.

Recursive Reasoning

- Snake on a Plane is a task from Illustrative Mathematics that approaches a function recursively and by algebraic definition.

Quadratic Functions

- Skelton Tower is a task from the Mathematics Assessment Project that has students use cubes to build a tower of different heights which will lead to quadratic sequences.
- Generalizing Patterns: Table Tiles by Mathematics Assessment Project has students explore a quadratic sequence.

Curriculum and Lessons from Other Sources

- Although EnageNY, Algebra 1, Module 3, Topic A has good problems in Lessons 1 and 2 that might be used, it emphasizes subscript notation which Ohio does not.
- The Mathematics Vision Project, Algebra 1, Module 1: Sequences. Sections 1.3-1.8 align to this cluster.
- Illustrative Mathematics, Algebra 1, Unit 5, Lesson 1: A Different Kind of Change, Lesson 5: Building Quadratic Functions to Describe Situations (Part 1), Lesson 6: Building Quadratic Functions to Describe Situations (Part 2), Lesson 7: Building Quadratic Functions to Describe Situations (Part 3),

General Resources

- Arizona High School Progressions on Functions is an informational document for teachers. This cluster is addressed on pages 11-12.
- Arizona High School Progressions on Modeling is an informational document for teachers. This cluster is addressed on pages 3 and 13.
- High School Coherence Map by UnboundED is a map of the relationships among high school mathematics standards. Please note: This resource is aligned to the Common Core not Ohio's Learning Standards. Although the tool is useful, it needs to be checked with respect to Ohio's Learning Standards.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (F.BF.1)

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STANDARDS	MODEL CURRICULUM (F.BF.3)

Functions
BUILDING FUNCTIONS
Build new functions from existing functions.
F.BF.3 Identify the effect on the graph of replacing $f(x)$ by $f(x) + k$, $kf(x)$, $f(kx)$, and $f(x + k)$ for specific values of k (both positive and negative); find the value of k given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. Include recognizing even and odd functions from their graphs and algebraic expressions for them. (A2, M3)
a. Focus on transformations of graphs of quadratic functions, except for $f(kx)$; (A1, M2)
	Expectations for Learning
	In eighth grade and Math 1, students attend to slope and intercepts for graphs of linear functions, without explicit attention to transformations of the graphs. In this cluster, students transform graphs of quadratic functions. Transformations of quadratic functions can be interpreted conveniently by observing the effect on the vertex and whether the parabola opens up or down. Students do not perform transformations of the form *f(kx).* In Math 3, students perform all types of transformations for various function families and recognize even and odd functions.
	
	ESSENTIAL UNDERSTANDINGS
	• Vertical and horizontal transformations of $y = x^2$ are as follows:
	    o horizontal shift: $g(x) = (x - h)^2$;
	    o vertical stretch/shrink: $g(x) = ax^2$ when $a > 0$;
	    o vertical shift: $g(x) = x^2 + k$;
	    o reflection across the x-axis: $g(x)= -x^2$; and
	    o a combination of transformations: $g(x) = a(x - h)^2 + k$.
	
	MATHEMATICAL THINKING
	• Explain mathematical reasoning.
	
	INSTRUCTIONAL FOCUS
	**Transformations occur in the quadratic expression rather than inside the function notation.*
	• Transform graphs of quadratic functions, and interpret the transformations geometrically.
	
	Content Elaborations
	OHIO'S HIGH SCHOOL CRITICAL AREAS OF FOCUS
	• Math 2, Number 3, pages 6-7
	
	CONNECTIONS ACROSS STANDARDS
	• Analyze functions using different representations (F.IF.7b).

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (F.BF.3)

Instructional Strategies

Note: The model curriculum is not a collection of lessons nor a full curriculum; it does not suggest pace, sequence, or amount of time spent on topics. It provides information about a topic related to the standards including ideas for examples, strategies for teaching, common misconceptions, and possible connections between topics. In addition, the emphasis of the model curriculum is on instruction not on assessment; therefore, the examples in the Instructional Strategies section are written for instructional purposes not necessarily as assessment items. Examples may need to be rewritten to accommodate the needs of each individual classroom.

Standards for Mathematical Practice
This cluster focuses on but is not limited to the following practices:
MP.2 *Reason abstractly and quantitatively.*
MP.4 *Model with mathematics.*
MP.8 *Look for and express regularity in repeated reasoning.*

TRANSFORMATIONS OF FUNCTIONS

Math 2 students need not use function notation to express transformation. Instead, for example, Math 2 students should see that $y = (x-h)^2$ is a horizontal transformation of a parent quadratic function $y = x^2$. Later, in Math 3, students should be able to see that the graph of the function $y = g(x-h)$ is a result of a horizontal transformation of the graph of $y = g(x)$.

Transformations with respect to linear functions should be postponed until after the study of transformations with quadratics because it is difficult to distinguish the horizontal and vertical transformations of a linear function. However, if choosing to discuss vertical translations of linear functions, relate them back to the *y*-intercept. In contrast to linear and exponential functions, transformations of quadratics are easier to observe and interpret because of the presence of the vertex of the parabola. Rather than asking students to memorize the connections between left/right, up/down, etc. encourage students to check their work with test points. The transformation $y = f(kx)$ is excluded until Math 3 because it is difficult to distinguish horizontal stretches from vertical stretches when dealing with quadratics. *Note: Fluency with absolute value functions is not expected until Math 3, so it is at the discretion of each district to determine how much emphasis to place on absolute value functions in Math 2.*

Use graphing calculators or computers to explore the effects of the position of a constant in an equation by the graph of its function. For example, students should be able to distinguish between the graphs of $y = x^2$, $y = 2x^2$, $y = x^2 + 2$, and $y = (x \pm 2)^2$. This can be accomplished by allowing students to work with a single parent function and examining numerous parameter changes in order to make generalizations. Websites such as Desmos Exploring Quadratic Transformations, Desmos Quadratics Explorations, Desmos, Quadratics Exploration 2, or WolframAlpha may be helpful.

Students may believe that the graph of $y = (x - 4)^2$ is the graph of $y = x^2$ shifted 4 units to the left (due to the subtraction symbol). Examples should be explored by checking a few points by hand or by using a graphing calculator to overcome this misconception.

EXAMPLE

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (F.BF.3)

Part 1
a. Using a graphing calculator or computer program, graph the following on the same grid (preferable in different colors if possible):
- $y = x^2$
- $y = 2x^2$
- $y = 3x^2$
- $y = 4x^2$

b. Predict what you think $y = 100x^2$ looks like.

Part 2
a. Graph the following on the same grid:
- $y = x^2$
- $y = -2x^2$
- $y = -3x^2$
- $y = -4x^2$

b. Predict what you think $y = -50x^2$ looks like.

Part 3
a. What do you predict $y = -\frac{1}{10}x^2$ will look like? Justify your prediction.
b. Graph the following on the same grid:
- $y = -\frac{1}{2}x^2$
- $y = -\frac{1}{3}x^2$
- $y = -\frac{1}{4}x^2$

c. Do you think your prediction in **a.** was correct? Try it and see.
d. What do you predict $y = \frac{2}{3}x^2$ looks like? Justify your prediction.
e. What do you predict $y = -\frac{3}{4}x^2$ looks like? Justify your prediction.

Part 4
a. Graph the following on the same grid:
- $y = x^2$
- $y = x^2 + 1$
- $y = x^2 + 2$
- $y = x^2 + 3$

b. What do you predict $y = x^2 + 100$ looks like? Explain how you came up with your prediction.
c. What do you predict $y = x^2 - 3$ looks like? Explain how you came up with your prediction.
d. What do you predict $y = -4x^2 + 5$ looks like? Explain how you came up with your prediction.

Part 5
a. Graph the following on the same grid:
- $y = x^2$
- $y = (x - 1)^2$
- $y = (x - 2)^2$
- $y = (x - 3)^2$

b. What do you predict $y = (x - 9)^2$ looks like? Explain how you came up with your prediction.
c. What do you predict $y = (x + 4)^2$ looks like? Explain how you came up with your prediction.

Using Properties of Transformations to Find the Vertex Formula
Students should also notice that a vertical shift does not affect the axis of symmetry, since the axis of symmetry is a vertical line. Because of this,

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (F.BF.3)

students should make connections between quadratics that are alike and their corresponding parabolas. For example, students should realize that $y = 3x^2 + 12x – 7$ is the same parabola as $y = 3x^2 + 12x$ except for the vertical shift; therefore, both equations have the same axis of symmetry. It is quite simple to factor the second equation in order to find its x-intercepts: $3x(x + 4) = 0$, so $x = 0, –4$. Since the axis of symmetry is in fact a line of symmetry for a parabola, students should understand that they can find they x-coordinate of the vertex (or the axis of symmetry) by finding the mean of the two x-intercepts of the parabola: $\frac{(0-4)}{2} = –2$. The student can then use $x = –2$ and substitute it into the original equation to find the vertex of the graph of the graph of $y = 3x^2 + 12x – 7$ as $y = 3(–2)^2 + 12(–2) – 7 = –19$, so the vertex is $(–2, –19)$. Using the structure of expressions and transformations allows students to find the axis of symmetry and therefore the vertex formula ($x = \frac{-b}{2a}$ where $x = h$ and then evaluating y for h to find k where (h, k) is the maximum or minimum) without complex computations. See F.IF.4-5 for more information on the vertex formula and the axis of symmetry.

INVERSE OF FUNCTIONS

Provide examples of inverse relations that are not purely mathematical to introduce the idea of inverse of functions. For example, given a function that names the capital of a state, f(Ohio) = Columbus, ask questions such as what is x, when $f(x)$ = Denver. Students can conclude that x = Colorado. Build on this concept by looking at numerical input and output values. Keep it simple, informal, and free of inverse function notation. Instead focus in on the idea of "going backwards." Ask question such as "What is the input when the output is known?"

The habit of immediately swapping x and y values may be confusing to some students. A better conceptual approach is to more clearly work backwards, keeping the letters the same. For example, suppose $f(x) = 3x + 5$. Call it $y = 3x + 5$ and solve for x to get $x = \frac{y-5}{3}$. Therefore if $x = g(y)$, and $g(y) = \frac{y-5}{3}$, then $f(x)$ and $g(y)$ are inverses of each other.

TIP! Swapping variables in an equation will lead to students misunderstanding notation in later mathematics courses when they will be required to use inverse function notation. For example, in later mathematics they may confuse the inverse of $y = f(x)$ with $y = f^{-1}(x)$. instead of the correct $x = f^{-1}(y)$. See "Inverse Functions: What Our Teachers Didn't Tell Us" article by Wilson, Adamson, Cox, and O'Bryan for further explanation.

For example, students might determine that folding a piece of paper in half 5 times results in 32 layers of paper. Then if they are given that there are 32 layers of paper, they can solve to find how many times the paper would have been folded in half.

EXAMPLE

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (F.BF.3)

Mark earns $9 an hour working at Best Buy. Write a function to describe the situation using *a* to represent the number of hours worked, and *b* to represent the total money earned.

Discussion: Discuss why both $b = 9a$ and $a = \frac{b}{9}$ could both describe the situation. Discuss when it would be more useful to have the hours be the independent variable and when it would be more useful to have the total money earned be the independent variable. Connect with rearranging equations and formulas in A.CED.4. In some circumstances, it is more useful to have the hours be assigned as the independent variable, and in other situations it is more useful to have the total money earned be assigned as the independent variable.

Instructional Tools/Resources

These tools and resources may be helpful for instruction. Before use, they should be evaluated at the local level to determine their appropriateness for instruction.

Manipulatives/Technology

- Use the book, *The Sneeches*, by Dr. Seuss to introduce students to the concept of inverse functions.
- Desmos is a free graphing calculator that is available to students as website or an app.
- GeoGebra is a free graphing calculator that is available to students as website.
- Wolframalpha is dynamic computing tool.

Transformations of Functions

- What's My Transformation? is a Desmos lesson that allows student to explore families of functions.
- Card Sort: Transformations is a Desmos lesson that has students match transformation of graphs to expressions using function notation.
- Marbleslides: Parabolas is a Desmos lesson that has students transform parabolas to send marbles through the stars.

Inverse of Functions

- Temperature in Degrees Fahrenheit and Celsius is an Illustrative Mathematics task that uses a real-world example of inverse functions.

Curriculum and Lessons from Other Sources

- EngageNY, Algebra, Module 4, Topic C, Lesson 19: Translating Graphs of Functions, Lesson 20: Stretching and Shrinking Graphs of

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (F.BF.3)

Functions, Lesson 21: Transformations of the Quadratic Parent Function, $f(x) = x^2$, Lesson 23: Modeling with Quadratic Functions, Lesson 24: Modeling with Quadratic Functions are lessons that align to this cluster. *Note: Some problems are above Ohio's expectations.*

- Georgia Standards of Excellence Framework, Algebra 1, Unit 3: Modeling and Analyzing Quadratic Functions has several lessons that address this cluster. These lessons can be found on pages 23-31, 62-72, 123-147, and 195-202.
- Illustrative Mathematics, Algebra 1, Unit 6, Lesson 12: Graphing the Standard Form (Part 1), Lesson 13: Graphing the Standard Form (Part 2), Lesson 15: Vertex Form, and Lesson 17: Changing the Vertex are lessons that pertain to this cluster.

General Resources

- Arizona High School Progressions on Functions is an informational text for teachers. This cluster is addressed on pages 12-13.
- High School Coherence Map by UnboundED is a map of the relationships among high school mathematics standards. Please note: This resource is aligned to the Common Core not Ohio's Learning Standards. Although the tool is useful, it needs to be checked with respect to Ohio's Learning Standards.

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- Wilson, F., Adamson, S., Cox, T., & O'Bryan, A. (March 2011). Inverse functions: What our teachers didn't tell us. *Mathematics Teacher*, *104(7),* 501-507.

STANDARDS	MODEL CURRICULUM (F.LE.3)
Functions **LINEAR, QUADRATIC, AND EXPONENTIAL MODELS** **Construct and compare linear, quadratic, and exponential models, and solve problems.** **F.LE.3** Observe using graphs and tables that a quantity increasing exponentially eventually exceeds a quantity increasing linearly or quadratically. ★ (A1, M2)	**Expectations for Learning** In eighth grade, students interpret the rate of change and initial value of a linear function in terms of the situation it models, and in terms of its graph or a table of values. Students also see examples of non-linear functions and learn and apply the properties of integer exponents. In Math 1, students compare linear and exponential functions. In Math 2, students compare across linear, exponential, and quadratic functions. **ESSENTIAL UNDERSTANDINGS** • The phrase "eventually exceeds" (F.LE.3) directs the focus towards large values in the domain and consideration of the base and *y*-intercept of the exponential function and the leading coefficient of the linear or quadratic function. • For large domain values, the growth of linear and quadratic functions is dominated by the leading term. **MATHEMATICAL THINKING** • Represent a concept symbolically. • Make and modify a model to represent mathematical thinking. • Make connections between concepts, terms, and properties within the grade level and with previous grade levels. **INSTRUCTIONAL FOCUS** • Use graphs, tables, and contexts to see that as the domain value increases, the values of an exponential function will eventually exceed the corresponding values of a linear or quadratic function. **Content Elaborations** **OHIO'S HIGH SCHOOL CRITICAL AREAS OF FOCUS** • Math 2, Number 3, pages 6-7 **CONNECTIONS ACROSS STANDARDS** • Build a function that models a relationship between two quantities (F.BF.1aii). • Interpret functions that arise in applications in terms of the context (F.IF.4b). • Analyze functions using different representations (F.IF.7b, 9b).

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (F.LE.3)

Instructional Strategies

Note: The model curriculum is not a collection of lessons nor a full curriculum; it does not suggest pace, sequence, or amount of time spent on topics. It provides information about a topic related to the standards including ideas for examples, strategies for teaching, common misconceptions, and possible connections between topics. In addition, the emphasis of the model curriculum is on instruction not on assessment; therefore, the examples in the Instructional Strategies section are written for instructional purposes not necessarily as assessment items. Examples may need to be rewritten to accommodate the needs of each individual classroom.

Standards for Mathematical Practice

This cluster focuses on but is not limited to the following practices:

MP.1 *Make sense of problems and persevere in solving them.*

MP.2 *Reason abstractly and quantitatively.*

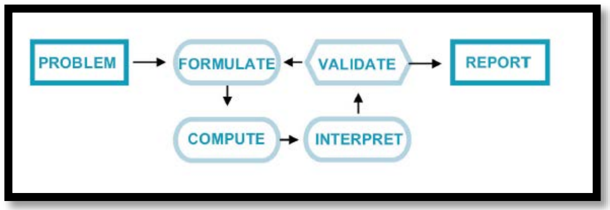
MP.4 *Model with mathematics.*

MP.7 *Look for and make use of structure.*

MP.8 *Look for and express regularity in repeated reasoning.*

MODELING

This cluster is included in the modeling standards. Geometric models that students can build are especially helpful to see patterns. See page 12 for more information about modeling.



COMPARING LINEAR AND EXPONENTIAL GROWTH

The phrase "increasing exponentially" in everyday language means "really fast." In mathematics, increasing exponentially means increasing using the model $y = ab^x$ (where $a > 0$ and $b > 1$), but it may be increasing incredibly slowly ($b = 1.01$). Because a graph of an exponential function eventually curves up, it will eventually have output values greater than a linear or quadratic (or polynomial) function. To understand this, students need to compare two graphs to see where the two graphs intersect. Then they will see that function behavior for values of *x* close to 0 is different than large (positive) values of *x*. Note: If $y = ab^x$ with $a > 0, 0 < b < 1$, then we have exponential decay.

EXAMPLE

a. Is it better to be paid a penny on the first day, and then double that amount each day thereafter for a month, or is it better to be paid $100 a day for the month?

b. If we add a third option to get $1 times the square of the day number, which of the three options would take? Explain.

Discussion: Have students make both a tabular representation and a graph of the situation before writing an equation. (An alternate introductory lesson could be on the fable "One Grain of Rice" by Demi. See Instructional Tools/Resources section for more resources on "One Grain of Rice.")

Explore simple linear and exponential functions by engaging in hands-on experiments. For example, students can measure the diameters and corresponding circumferences of several circles and discover that a function that relates the diameter to the circumference is a linear function

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (F.LE.3)

with a first common difference. Then they can explore the value of an investment for an account that will double in value every 12 years and see that it is an exponential function with a base of 2.

Use a graphing calculator or computer program to compare tabular and graphic representations of exponential and polynomial functions to show how the *y*- (output) values of the exponential function eventually exceed those of quadratic functions. A simple example would be to compare the graphs (and tables) of the functions $y = x^2$ and $y = 2^x$ to find that the *y*-values are greater for the exponential function when $x > 4$.

Students may also incorrectly believe that the end behavior of all functions depends on the situation and not on the fact that exponential function values will eventually get larger than those of any other polynomial function. Provide situations where students can discover this concept.

EXAMPLE

Suppose you wanted to join a gym that charges a $80 initiation fee, and then quotes you that it will cost you $230 for 6 months. Three price functions are given, all of which meet the quoted price, where *k* is the time in months and *P(k)* is the total cost. Which is the best model for you and why. Which is the best model for the gym?

Plan A: $P(k) = 80 + \frac{115}{3}k$

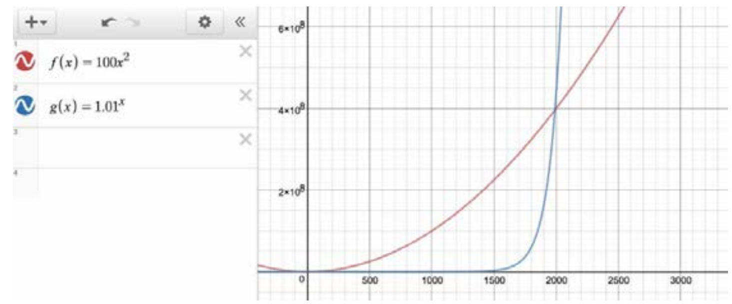
Plan B: $P(k) = 80 + \frac{97}{3}k + k^2$

Plan C: $P(k) = 80(1.25327)^k$

Discussion: Students should discover that the *y*-intercept of each graph represents the initiation fee. If you only needed a gym membership for less than 6 months and there is no cancellation fee, then Plan C would be best for the consumer. In contrast after 6 months, Plan C is significantly better for the gym. The best plan for the consumer who wants to be a member for 6 months or longer is Plan A.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (F.LE.3)

Think about comparing $f(x) = (1.01)^x$ to $g(x)=100x^2$. For these pairs, in a standard graphing window, it will not look as if *f(x)* will ever exceed *g(x).* With some strategic work, the graphing window can be adjusted to see that the exponential function *g(x)* will eventually exceed the quadratic function *f(x).* In this example, notice that for *x*-values between 1 and 10, the quadratic function appears much larger than the exponential function, but for *x*-values near 2,000, the exponential function becomes much larger than the quadratic (see the graph to the right). Students should be given the opportunity to compare the graphs of various functions by zooming out carefully and strategically, possibly by different horizontal and vertical ratios, in order to see what happens to the graphs of the functions for very large values of *x*. This example may be too hard for a first example, but it might be appropriate for a culminating task.



COMPARING LINEAR, QUADRATIC, AND EXPONENTIAL MODELS

Linear Function

x	f(x)	
-2	-9	
		+0.5
-1	-8.5	
		+0.5
0	-8	
		+0.5
1	-7.5	
		+0.5
2	-7	

Quadratic Function

x	f(x)		
-2	-6		
		+1	
-1	-5		+2
		+3	
0	-2		+2
		+5	
1	3		+2
		+7	
2	10		

Exponential Function

x	f(x)	
-2	$\frac{1}{2}$	
		×4
-1	2	
		×4
0	8	
		×4
1	32	
		×4
2	128	

Compare tabular representations of a variety of functions to show that over equal *x*-intervals linear functions have a constant first difference (equal differences over equal *x*-intervals), while exponential functions do not (instead function values grow by equal factors over equal *x*-intervals). Also, quadratic functions have a constant second difference over equal *x*-intervals. Have students explore these concepts instead of just telling them. Require them to explain why these patterns hold true and justify their thinking.

Students may believe that all functions have a constant first difference and need to explore in order to realize that, for example, a quadratic function will have equal constant second differences in a table. In addition, some students may believe that every function with a constant rate of change is linear. For example, exponential functions have a constant multiplicative (percent) rate of change.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (F.LE.3)

EXAMPLE

Give each student 3 tables and tell them to create an equation for each type of function. Then complete the table.

Linear:__________ **Exponential:**__________ **Quadratic:**__________

x	*y*	Patterns in *y*

x	*y*	Patterns in *y*

x	*y*	Patterns in *y*

a. What pattern did you notice about the linear functions?
b. What pattern did you notice about the exponential functions?
c. What pattern did you notice about the quadratic functions?
d. Share your equations with 5 other people in the class. How were the observations about your patterns similar or different than your classmates? Explain.
e. Will the patterns that you found, always hold true? Explain and justify you thinking.

Apply linear and exponential functions to real-world situations. For example, a person earning $10 per hour experiences a constant rate of change in salary given the number of hours worked, while the number of bacteria on a dish that doubles every hour will have equal factors over equal intervals.

Instructional Tools/Resources

These tools and resources may be helpful for instruction. Before use, they should be evaluated at the local level to determine their appropriateness for instruction.

Manipulatives/Technology

- Desmos is a free graphing calculator that is available to students as website or an app.
- GeoGebra is a free graphing calculator that is available to students as website.
- Wolframalpha is dynamic computing tool.
- Visual Patterns is a website that shows pictures of linear, exponential, and quadratic patterns.
- Patterns Posters for Algebra 1 from Finding Ways by Fawn Nguyen is a blog about a class activity where students are given patterns and they have to make posters from them. She is the creator of the visual patterns link above.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (F.LE.3)

One Grain of Rice and The King's Chessboard

Both are children's stories that can be used to teach exponential growth.

- *One Grain of Rice: A Mathematica Folktale* by Demi is a children's book about a rajah who takes his people's rice very year until a wise girl develops a clever plan using exponential growth.
- One Grain of Rice in an NCTM Illuminations lesson on exponential growth. *NCTM now requires a membership to view their lessons.*
- One Grain of Rice is another lesson from the Jim Wilson's University of Georgia's webpage using Demi's story and a spreadsheet.
- One Grain of Rice: Exponential Growth is a YouTube video on the story.
- *The Kings Chessboard,* by David Birch and Devis Grebu is a children's book about exponential growth as a wise man who refuses a king's reward for a favor instead takes a payment of rice.
- The Legend of a Chessboard: Teaser is a YouTube video that puts the quantity of rice in the context of different places such as a chessboard, a room, cities, and the country of Switzerland.
- The Legend of a Chessboard is a YouTube video based on the story.

Exponential Growth

- Exponential Models: Rhinos and M&M's is lesson from PBS that uses paperfolding, M&M's, and Rhinos to show exponential growth and decay.
- Exponential Growth & Decay (Ashby) is a lesson by Achieve the Core that uses several different representations to demonstrate exponential growth and decay.
- An Intro to Exponential Growth and Decay is a Desmos activity that also models exponential growth and decay using pennies and M&M's.

Exponential Growth, continued

- Growth and Decay is a Desmos activity that uses March Madness brackets to illustrate exponential growth and decay.
- Overrun by Skeeters-Exponential Growth and Skeeter Populations and Exponential Growth are lessons from Annenberg Learner where students model functions that represent exponential growth about skeeters (mosquitoes).
- Predicting your Financial Future is an NCTM Illumination's lesson about compound interest. *NCTM now requires a membership to view their lessons.*
- Fry's Bank is a 3-Act Math Task by Dan Meyer that introduces exponential growth.
- Pixel Pattern is a 3-Act Math Task by Dan Meyer that explores patterns.
- Identifying Exponential Functions is a task by Illustrative Mathematics that introduces exponential functions by experimenting with the parameters of the function.
- Two Points Determine an Exponential Function I and Two Points Determine an Exponential Function II are tasks by Illustrative Mathematics where students have to find the values of *a* and *b* in an exponential function given two points.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (F.LE.3)

Comparing Linear, Exponential and/or Quadratic Growth

- Piles of Paper is a CPalms activity where students fold paper to demonstrate linear and exponential growth.
- Rainforest Deforestation-Problem or Myth? is a lesson by NCTM Illuminations that allows students to explore deforestation as an exponential function. Students will use first or second differences to determine whether data models are linear, quadratic, or exponential. *NCTM now requires a membership to view their lessons.*
- National Debt and Wars is a lesson by NCTM Illuminations where student collect information about the National Debt, plot the data by decade, and decide whether an exponential curve is a good fit. *NCTM now requires a membership to view their lessons.*
- Shrinking Candles, Running Waters, Folding Boxes is a lesson by NCTM Illuminations that has students determine which function type best fits the data. Skip the "Weather, It's a Function" section as it is above grade-level. *NCTM now requires a membership to view their lessons.*
- Birthday Gifts and Turtle Problem is a Mathematics Design Collaborative lesson from the State of Georgia Department of Education that explores the rates of changes of linear functions versus exponential functions.
- Representing Linear and Exponential Growth by Mathematics Assessment Project is a lesson that has students interpret exponential and linear functions.

Curriculum and Lessons from Other Sources

- EngageNY Algebra 1, Module 3, Topic A, Lesson 5: The Power of Exponential Growth, Lesson 6: Exponential Growth—U.S. Population and World Population, Lesson 7: Exponential Decay are lessons that pertain to this cluster.
- EngageNY Algebra 1, Module 3, Topic D, Lesson 21: Comparing Linear and Exponential Models Again, Lesson 22: Modeling an Invasive Species Population, Lesson 23: Newton's Law of Cooling are lessons that pertain to this cluster.
- The Georgia Standards of Excellence Curriculum Frameworks for Algebra 1, Unit 5: Comparing and Contrasting Functions compares and contrasts functions. There are many tasks in this document that align with this cluster.
- The Mathematics Vision Project Secondary Math 1, Module 2: Linear and Exponential Functions has many task that align with this cluster.
- A lesson on Exponential Modeling developed by the Virginia Department of Education that uses a graphing calculator. It has the following activities: Who Wants to be a Millionaire?, Paper Folding, M&M Decay, Decaying Dice Game, Population Growth, and Baseball Players' Salaries.
- Illustrative Mathematics, Algebra 1, Unit 5, Lesson 3: Building Quadratic Functions from Geometric Patterns, Lesson 4: Comparing Quadratic and Exponential Functions,

General Resources

- Arizona High School Progression on Functions is an informational document for teachers. This cluster is addressed on pages 16-17.
- Arizona High School Progression on Modeling is an informational document for teachers. This cluster is addressed on page 5.
- High School Coherence Map by UnboundED is a map of the relationships among high school mathematics standards. Please note: This resource is aligned to the Common Core not Ohio's Learning Standards. Although the tool is useful, it needs to be checked with respect to Ohio's Learning Standards.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (F.LE.3)

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- Kirwan, J. (April 2017). Using visualization to generalize on quadratic patterning tasks. *Mathematics Teacher*, *110(8),* 588-593.
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STANDARDS	MODEL CURRICULUM (G.SRT.1-3)
Geometry **SIMILARITY, RIGHT TRIANGLES, AND TRIGONOMETRY** **Understand similarity in terms of similarity transformations.** **G.SRT.1** Verify experimentally the properties of dilations[G] given by a center and a scale factor: **a.** A dilation takes a line not passing through the center of the dilation to a parallel line and leaves a line passing through the center unchanged. **b.** The dilation of a line segment is longer or shorter in the ratio given by the scale factor. **G.SRT.2** Given two figures, use the definition of similarity in terms of similarity transformations[G] to decide if they are similar; explain using similarity transformations the meaning of similarity for triangles as the equality of all corresponding pairs of angles and the proportionality of all corresponding pairs of sides. **G.SRT.3** Use the properties of similarity transformations to establish the AA criterion for two triangles to be similar.	**Expectations for Learning** The standards in this cluster make more precise the informal notion of "same shape, different size." In middle school, students represent proportional relationships within and between similar figures; create scale drawings; describe the effect of dilations on two-dimensional figures; and understand similarity transformations as a sequence of basic rigid motions and dilations. In this cluster, students verify the properties (given center and scale factor) of dilations and use those properties to establish the AA criterion for triangles. They also explore the relationships among corresponding parts of similar figures. The student understanding of this cluster begins at van Hiele Level 1 (Analysis) and moves to Level 2 (Informal Deduction/Abstraction). **ESSENTIAL UNDERSTANDINGS** • A dilation requires a center and a scale factor. • A similarity transformation often requires a sequence of basic rigid motions, in addition to a dilation. • A scale factor is a ratio of corresponding lengths between figures. • A similarity transformation with a scale factor of 1 is a special case, which is a congruence transformation. • While the definition of similarity applies to polygons, it also applies to non-polygonal shapes, e.g. circles, parabolas, etc. • The AA criterion is equivalent to the AAA criterion because the angle sum in a triangle is 180 degrees. • The AA criterion and the AAA criterion apply only to triangles. **MATHEMATICAL THINKING** • Use accurate mathematical vocabulary to represent geometric relationships. • Use formal reasoning with symbolic representation. • Determine reasonableness of results. • Recognize, apply, and justify mathematical concepts, terms, and their properties. • Make connections between terms and properties. *Continued on next page*

STANDARDS	MODEL CURRICULUM (G.SRT.1-3)
	Expectations for Learning, continued **INSTRUCTIONAL FOCUS** • Use basic rigid motions and dilations to map similar figures onto one another. • Given a figure, carry out a similarity transformation, and then verify its properties. • Know the precise definitions of dilation and similarity. • Identify center and scale factor of a dilation. • Determine the scale factor. • Establish that triangles with two pairs of corresponding congruent angles are similar. **Content Elaborations** **OHIO'S HIGH SCHOOL CRITICAL AREAS OF FOCUS** • Math 2, Number 4, page 8 **CONNECTIONS ACROSS STANDARDS** • Prove theorems involving similarity (G.SRT.4-5). • Apply right triangles trigonometry (G.SRT.6-8). • Apply the understanding that all circles are similar (G.C.1). • Understand the relationships between lengths, areas, and volumes in similar figures (G.GMD.6). • Use coordinates to prove simple geometric theorems algebraically (G.GPE.6). • Apply geometric concepts in modeling situations (G.MG.1-3).

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.SRT.1-3)

Instructional Strategies

Note: The model curriculum is not a collection of lessons nor a full curriculum; it does not suggest pace, sequence, or amount of time spent on topics. It provides information about a topic related to the standards including ideas for examples, strategies for teaching, common misconceptions, and possible connections between topics. In addition, the emphasis of the model curriculum is on instruction not on assessment; therefore, the examples in the Instructional Strategies section are written for instructional purposes not necessarily as assessment items. Examples may need to be rewritten to accommodate the needs of each individual classroom.

The concept of similarity builds on the concept of congruence. This is one of the reasons for teaching congruence prior to teaching similarity.

It may be helpful to teach this cluster in conjunction with G.SRT.4-5 and G.GMD.5-6 especially with respect to the Side Splitter Theorem and the Fundamental Theorem of Similarity.

Standards for Mathematical Practice

This cluster focuses on but is not limited to the following practices:

MP.3 *Construct viable arguments and critique the reasoning of others.*

MP.4 *Model with mathematics.*

MP.5 *Use appropriate tools strategically.*

MP.7 *Look for and make use of structure.*

MP.8 *Look for and express regularity in repeated reasoning.*

VAN HIELE CONNECTION

In Math 2 students are expected to move from Level 1(Analysis) to Level 2 (Informal Deduction/Abstraction). In van Hiele Level 1 students use coordinates to examine transformations and recognize, use, and describe properties of transformations. Whereas van Hiele Level 2 can be characterized by the student doing some or all of the following:

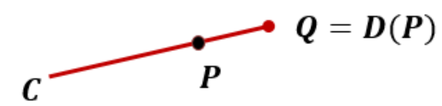
- analyzing relationships within and between transformations;
- algebraically expressing transformations: reflections about the *x*- and *y*-axes, and rotations of 45°, 90°, and 180° about the origin; *Note: In high school, students should be given an opportunity to experience rotations around any point (not just the origin) and onto any angle, not just multiples of 90. Also, reflections are not limited to the x-and y-axes but could be about any line.*
- understanding and justifying how to use perpendicular bisectors of corresponding points to locate the turn center of a rotation;
- understanding how the composition of two reflections can be a translation or a rotation; and/or
- following a rationale if shown proof.

See the van Hiele pdf on ODE's website for more information about van Hiele levels.

VERIFYING PROPERTIES OF DILATIONS EXPERIMENTALLY

Students build upon the informal idea of dilation that they learned in Grade 8. Now in high school students use a more formal definition of a dilation as a function on a plane. A dilation is a rule that expands or contracts the plane about a center. The center of dilation is a fixed point in the plane about which all distances from the center are expanded or contracted. A scale factor informs the magnitude or the amount of expansion or contraction. A dilation must name a center and a scale factor. *Note: Throughout the Model Curriculum different notations are used such a function notation and prime notation etc. As each districts' resources are different, it is up to each district to determine the notation that students need to use. It may be helpful to show students a variety of notations, so they can be mathematically literate when seeing different notations in different resources/courses/schools.*

Definition of Dilation



The dilation D with center C and positive scale factor r, leaves C unchanged and takes every point P to the point $Q = D(P)$ on the ray CP whose distance from C is $r|CP|$ so that $|CQ| = r|CP|$.

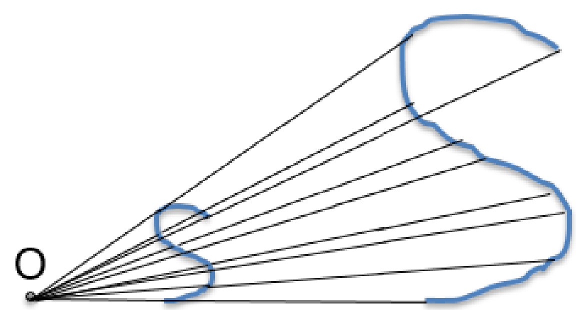
Students should recognize that—

- A scale factor between 0 and 1 $(0 < r < 1)$ pulls (contracts) every point on the plane proportionally closer to the center;
- A scale factor greater than 1 $(r > 1)$ pushes (expands) every point on the plane proportionally the same amount away from the center; and
- A scale factor of 1 leaves every points' position unchanged $(r = 1)$.

TIP! Emphasize to students that the entire plane dilates not just the image as the center of dilation remains fixed. The image not only increases (or decreases) in size but the image is pushed farther away from (or pulled closer to) the pre-image because the plane has changed.

Introduce dilation by discussing a topic such as "How do we triple the size of a wiggly curve?"

After discussion, lead students toward assigning an arbitrary point, O, on the plane, and pushing every point on the squiggly line three times as far away from O. Explain to students that this is a dilation. A dilation pushes out (expands) or pulls in (contracts) every point in the plane as well as the figure from its center of dilation proportionally. This can be easily modeled by pushing in or pulling out a light source that displays an image such as an overhead projector or a flashlight. In this case the center of dilation is O, however the center of dilation can be located anywhere on the plane.

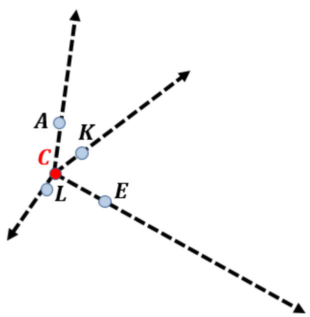


See Model Curriculum Grade 7.G.1-3 and Grade 8.G.1-5 for scaffolding ideas.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.SRT.1-3)

TIP! Use rubber bands to explore dilations. Tie two rubber bands together with a knot in between. Given a figure and a point of dilation, hold the end of the rubber band at the dilation point. With a writing utensil at the far end, make the knot trace the original figure. This should create a dilated image of the original figure. See the YouTube video called Rubber Band Dilation, by Robin Betcher for an illustration of the process.

EXAMPLE

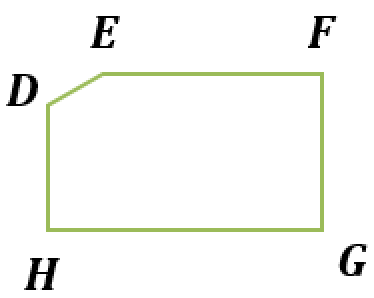


a. Using a ruler or compass map the points $L, A, K,$ and E using a dilation about Point C where the scale factor is $k = 3$.
b. Write four equations to describe how the distance between the original point and Point C relates to the distance between its image and Point C.

EXAMPLE

Draw a dilation image of polygon $DEFGH$.

a. Pick a center of dilation, and label it C.
b. Draw a ray from your center of dilation through each vertex on the figure.
c. Use a compass or ruler to map the points D', E', F', G', and H' using a dilation by a scale factor of $\frac{1}{3}$.
d. Then connect the corresponding images of the original vertices creating line segments.
e. How does your dilated figure compare to a scale drawing?
f. How does your dilated figure compare to your classmates?



Have students focus on the properties of dilations. In Grade 8 students explored the basic properties of dilations. Now students need to explain *why* the properties of dilations are true by using reasoned arguments. Since, in general, a dilation changes the distances between two fixed points, dilations do not preserve congruence (except when the scale factor is 1); they do, however, preserve similarity.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.SRT.1-3)

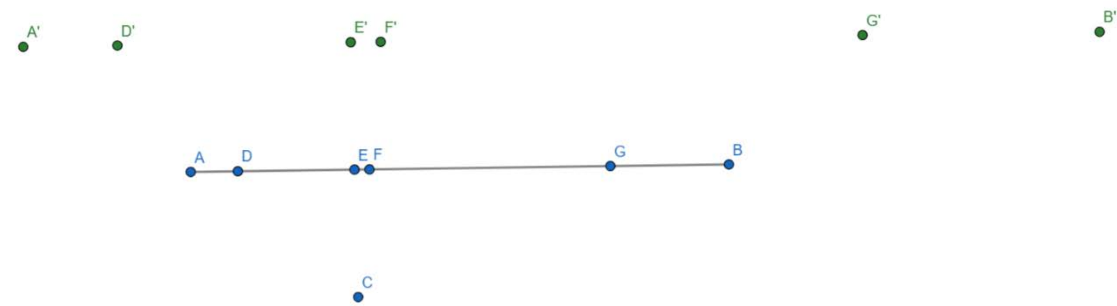
Dilations—

- Map line to lines, rays to rays, and segments to segments;
- Change a distance by a factor of *r*, where *r* is the scale factor of the dilations;
- Map every line passing through the center of dilation to itself, and map every line not passing through the center of dilations to a parallel line; and
- Preserve angle measure, betweenness, and collinearity. (Distance is not preserved.)

Note: Discussion of the Fundamental Theorem of Similarity with respect to dilations and parallel lines will be discussed in G.GMD.5-6.

EXAMPLE

Use colored pencils, ruler, and a compass or dynamic geometric software to perform experiments exploring the properties of dilations.



Part 1: Collinearity, Betweenness and Distance

Draw 6 collinear points (not equally spaced). Then draw Point C which is not collinear and let C be the center of dilation with a scale factor of 2.

- **a.** Find the images of the 6 points.
- **b.** Compare the relationships between corresponding points. Does distance appear to be preserved? Give rationale to support your answer.
- **c.** Make a conjecture based on your discovery for part **b**.
- **d.** Does collinearity appear to be preserved? Explain.
- **e.** Does betweenness appear to be preserved? Support your answer with reasons.
- **f.** Does the distance between a pre-image point and its image appear to be the same for all points? What is it about a dilation that impacts the distance in this way?

Discussion: Students should come to the conclusion that distance between the points is not preserved. For example, the distance between Point A and Point D in not the same as the distance between Point A' and Point D'. Students should see that the collinearity is preserved since the image points also make a straight line. Betweenness is also preserved. For example, just as Point D is between A and E, Point D' is between Point A' and E'. The distance between Point A and Point A' is not the same as the distance between Point D and Point D'.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.SRT.1-3)

Part 2: Segments

Draw a segment $\overline{AB}$ with a length of at least 3 inches. Draw a point C, which is not on $\overleftrightarrow{AB}$ and let C be the center of dilation with a scale factor of $\frac{2}{3}$. Use as many points as necessary to convince you what the image of $\overline{AB}$ looks like.

- **a.** Describe the resulting figure. What relationship exists between the distance from $A' = D(A)$ and $B' = D(B)$ to the distance of AB.
- **b.** Is the distance between $D(A)$ and $D(B)$ equal to AB?
- **c.** Does the dilation preserve either betweenness or collinearity? Support your response.
- **d.** Predict what you think the image of $\overrightarrow{AB}$ or $\overleftrightarrow{AB}$ would look like after a dilation?
- **e.** What is the easiest way for you to determine $D(\overline{AB})$?
- **f.** Does $D(\overline{AB})$ intersect $\overline{AB}$? Explain.

Discussion: In Part 2, ask the question, "How can we be sure that the dilation maps all the points between A and B to the image points between A' and B'? Students may need to use the coordinate plane to make the argument, but then ask them how they would know that its true without a coordinate plane. Students should also come to the conclusion that $D(\overline{AB})$ and $\overline{AB}$ are parallel.

Part 3: Angles

Draw an obtuse angle $\angle ABX$. Let O be a point in the interior of $\angle ABX$ and let D be a dilation with center O and scale factor of $\frac{1}{3}$. Find the images of A, B, X and as many other points as necessary to convince you what the image of $\angle ABX$ looks like.

- **a.** What kind of figure is $D(\angle ABX)$?
- **b.** Does the dilation preserve angle measure? Explain
- **c.** How are $\overrightarrow{BX}$ and $D(\overrightarrow{BX})$ related?
- **d.** True or False? If $D(B) = B'$ and $D(A) = A'$, then $\frac{|A'B'|}{|AB|} = k$, where k is the scale factor. Explain.
- **e.** What is the minimum number of points of $\angle ABX$ needed to find $D(\angle ABX)$? Explain.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.SRT.1-3)

Part 4: One Triangle, Two Transformations

Draw a triangle with side lengths of 2 inches, 1 inch, and 1.5 inches, and label it $\triangle PQR$. Draw Point A in the interior of $\triangle PQR$ and point E in exterior of $\triangle PQR$. Let D be the dilation of $\triangle PQR$ with Point A as the center of dilation and a scale factor of $\frac{3}{2}$. Then let d be the dilation of with Point E as the center of dilation and a scale factor of $\frac{3}{2}$.

- **a.** What is the relationship between $D(\triangle PQR)$ and $d(\triangle PQR)$?
- **b.** Compare the corresponding angles of $\triangle PQR$ and $D(\triangle PQR)$. What is the relationship between them?
- **c.** Compare the corresponding angles of $\triangle PQR$ and $d(\triangle PQR)$. What is the relationship between them?
- **d.** What is the minimum number of points needed to create the image of the triangle under either one of these dilations? Explain.

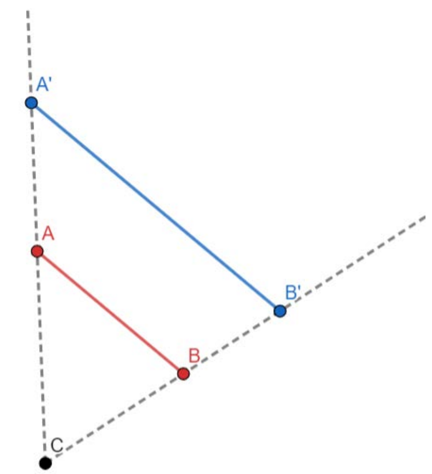
Part 5: Parallels and Perpendiculars

Draw a rectangle $ABCE$ with vertices (2,2), (6,4), (6,2), and (2,4). Draw its image under a dilation with a center of (0,0) and a scale factor of 2.5.

- **a.** What are the coordinates of the vertices of the image?
- **b.** Does the image seem to be a rectangle? Explain.
- **c.** Does the dilation preserve parallelism?
- **d.** Does the dilation preserve perpendicularity?
- **e.** Does the dilation preserve orientation?

Example take from Coxford and Usiskin, 1971, *Geometry: A Transformational Approach.*

According to the standards, students need to verify that under a dilation, a line (or line segment) is parallel to its image. This could be done using geometric software or using coordinate geometry. Depending on the sequencing of concepts, more advanced students could prove this theorem assuming that in the classroom, students start with the postulate that dilations preserve angle measure.



EXAMPLE

Given $\overline{AB}$ has been dilated about Point C. Prove that $\overline{AB} \parallel \overline{A'B'}$.

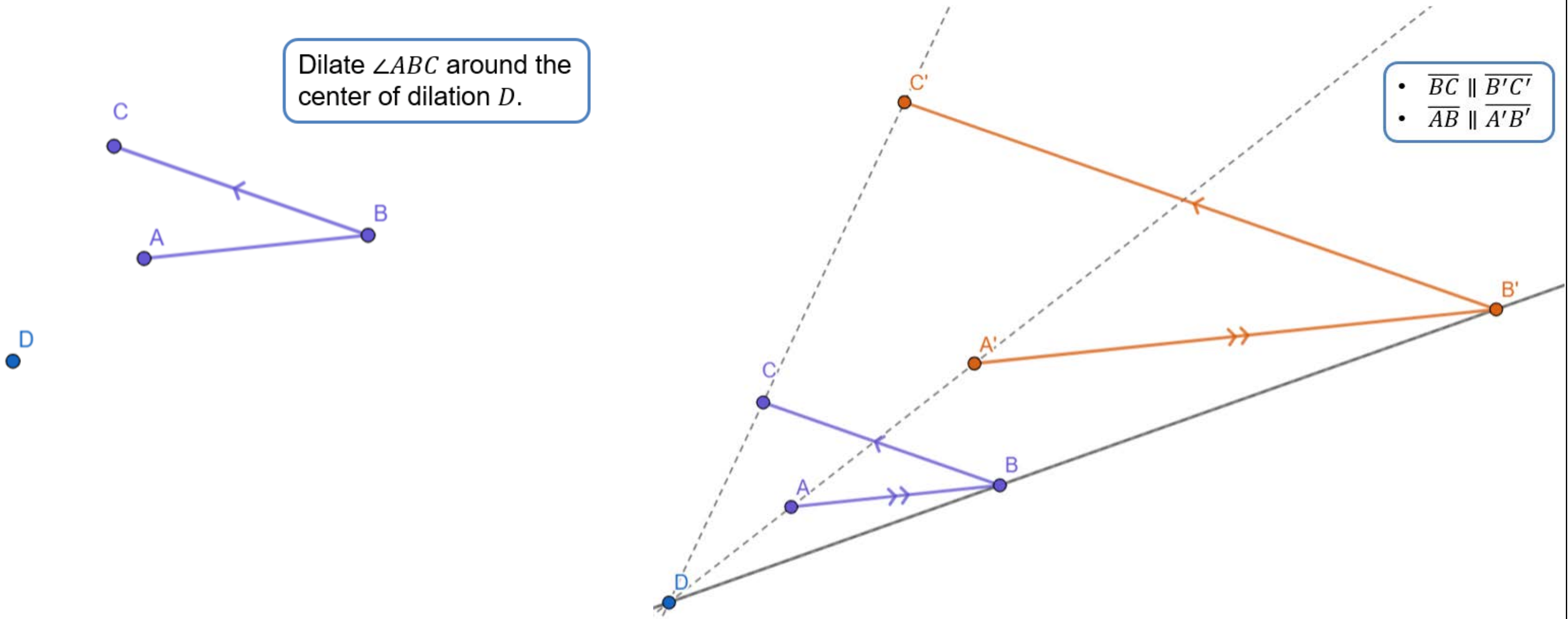
Discussion: Under the dilation with the center C, the image of point C is C, the image of point B is B', points C, B, and B' are collinear and $m\angle ABC = m\angle A'B'C$ because dilations preserve angle measure. Since angles ABC and $A'B'C$ are also corresponding angles that are formed by line segments $\overline{AB}$, $\overline{A'B'}$ and the transversal ray CB', line segments $\overline{CB}$ and $\overline{C'B'}$ must be parallel.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.SRT.1-3)

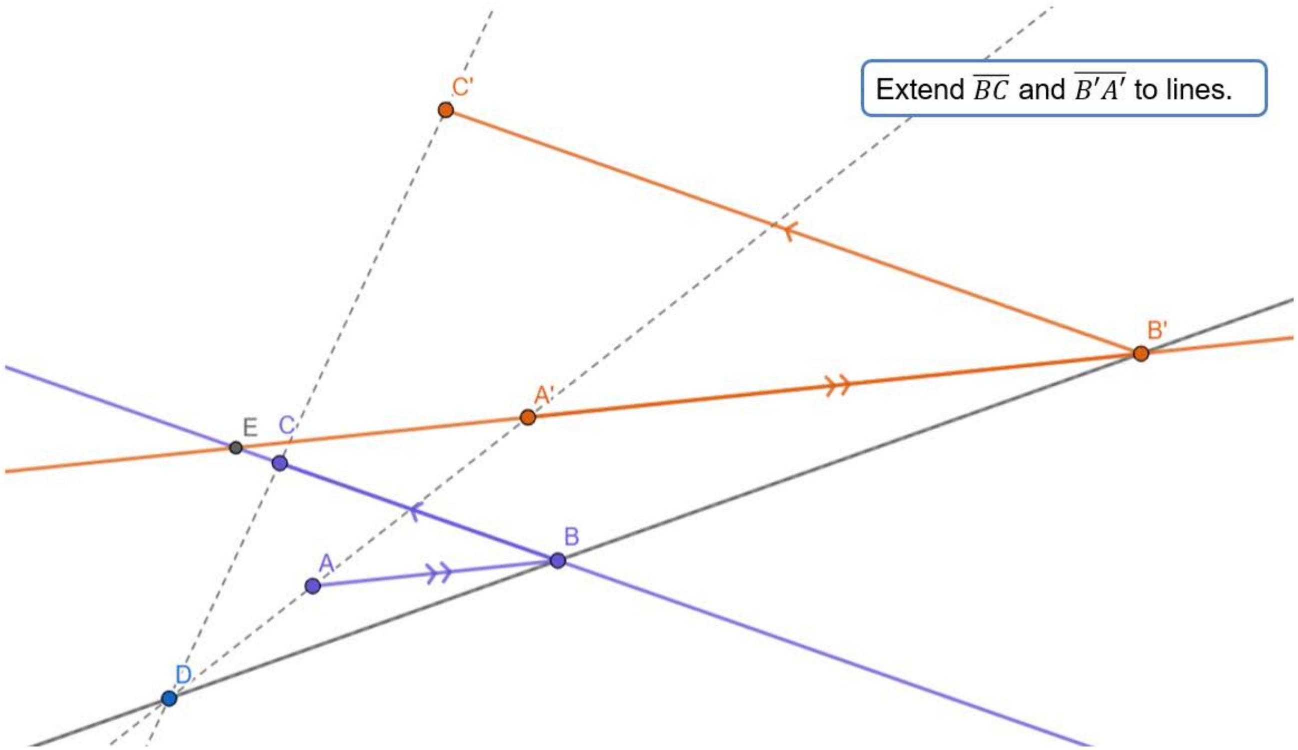
In contrast if the sequence of instruction has students start with the postulate that under a dilation, a line is parallel to its image (verified experimentally in G.SRT.1.b), they can prove the dilations preserve angle measure.

EXAMPLE

Prove that dilating $\angle ABC$ preserves angle measure after undergoing a dilation.

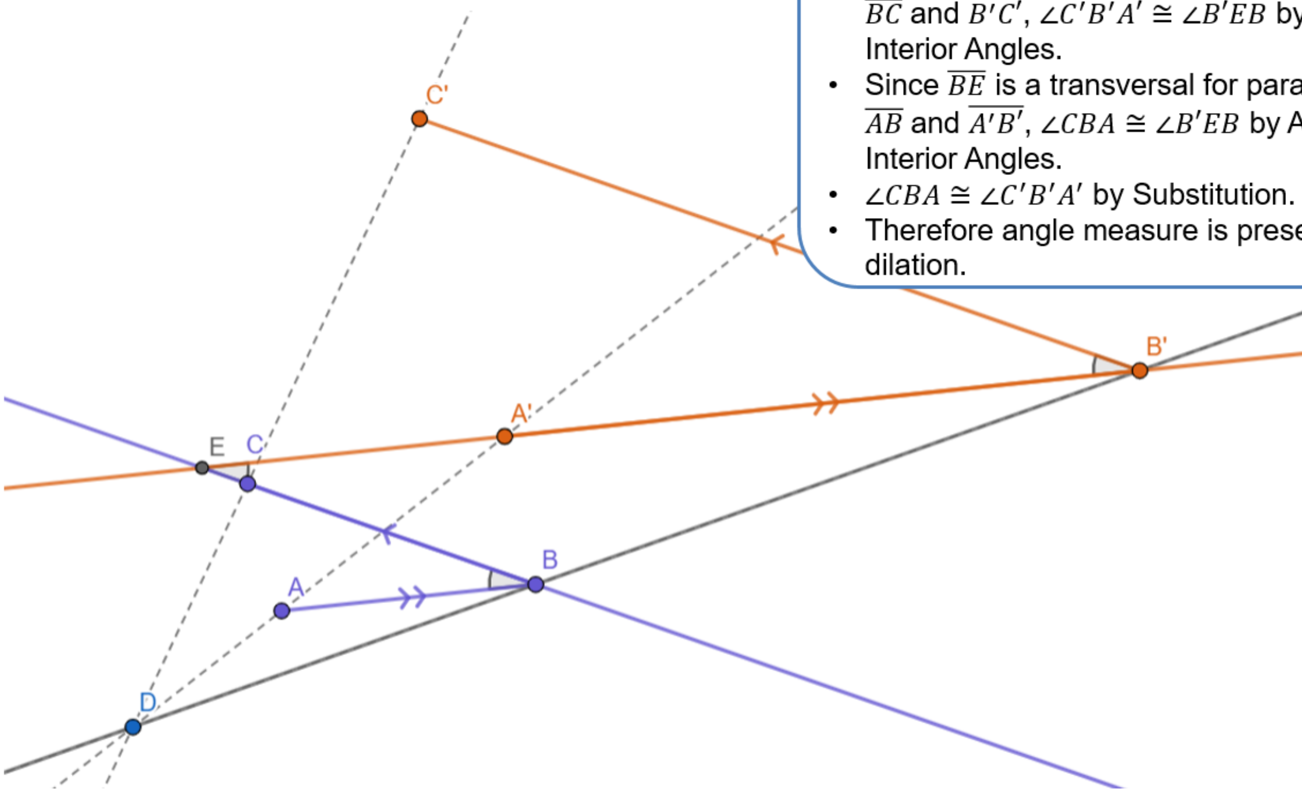


INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.SRT.1-3)



INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.SRT.1-3)

- Since $\overline{B'E}$ is a transversal for parallel lines $\overline{BC}$ and $\overline{B'C'}$, $\angle C'B'A' \cong \angle B'EB$ by Alternative Interior Angles.
- Since $\overline{BE}$ is a transversal for parallel lines $\overline{AB}$ and $\overline{A'B'}$, $\angle CBA \cong \angle B'EB$ by Alternative Interior Angles.
- $\angle CBA \cong \angle C'B'A'$ by Substitution.
- Therefore angle measure is preserved under a dilation.



Use graph paper and a ruler or dynamic geometric software to obtain images of a given figure under dilations having specified centers and scale factors. Carefully observe the images of lines passing through the center of dilation and those not passing through the center, respectively. A line segment passing through the center of dilation will simply be shortened or elongated but will lie on the same line, while the dilation of a line segment that does not pass through the center will be parallel to the original segment (this is intended as a clarification of Standard G.SRT.1a).

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.SRT.1-3)

EXAMPLE

Part 1

Dilate each of the following figures around the center of dilation X and a scale factor of 2.

Figure 1

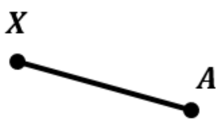


Figure 2

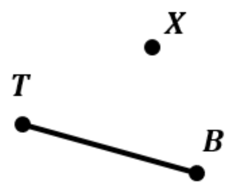


Figure 3

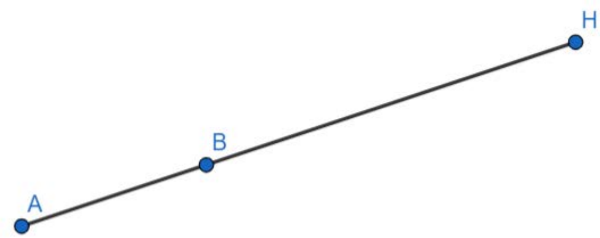
a. What is the same about all three dilations?

b. What is different about all three dilations?

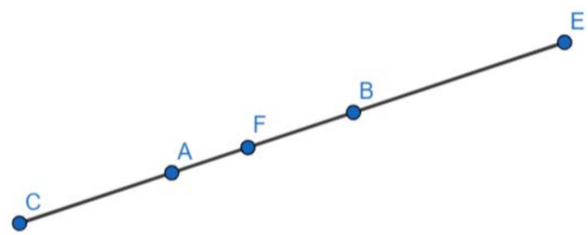
Part 2

Name the center of dilation.

a. If there is a dilation that maps $\overline{AB}$ to $\overline{AH}$, what is the center of dilation? Using the center of dilation that you found, how would the dilation of $\overline{AB}$ change if $\overline{AB}$ were $\overleftrightarrow{AB}$ or $\overrightarrow{AB}$?

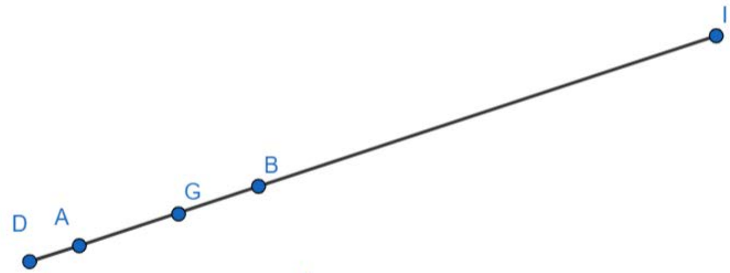


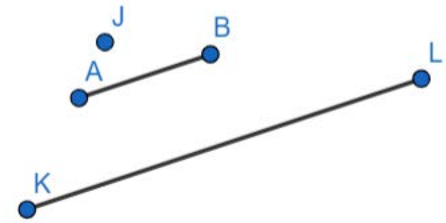
b. If there is a dilation that maps $\overline{AB}$ to $\overline{CE}$, what is the center of dilation? Using the center of dilation that you found, how would the dilation of $\overline{AB}$ change if $\overline{AB}$ were $\overleftrightarrow{AB}$ or $\overrightarrow{AB}$?



INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.SRT.1-3)

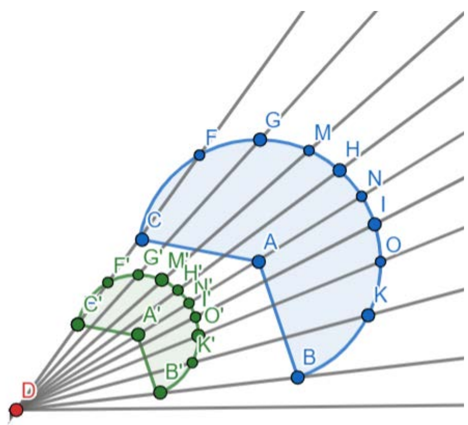
c. If there is a dilation that maps $\overline{AB}$ to $\overline{GI}$, what is the center of dilation? Using the center of dilation that you found, how would the dilation of $\overline{AB}$ change if $\overline{AB}$ were $\overleftrightarrow{AB}$ or $\overrightarrow{AB}$?





d. If there is a dilation that maps $\overline{AB}$ to $\overline{KL}$, what is the center of dilation? Using the center of dilation that you found, how would the dilation of $\overline{AB}$ change if $\overline{AB}$ were $\overleftrightarrow{AB}$ or $\overrightarrow{AB}$?

Discussion: Part 2 of this example is intended to be a progression for students to understand that the center of dilation can be an endpoint of the segment, in the middle of the segment, or outside of a line segment of which it is dilated.



Introduce students to examples of dilations that include curved lines to show that dilations are not limited to polygons. This will reinforce the idea that all the points in the pre-image undergo a dilation not just the vertices.

Connect dilations with perimeter and area in G.GMD.6. Include examples where orientation is changed.

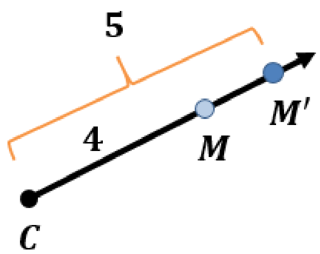
Students do not need to prove G-SRT.1b but specific cases may be appropriate for advanced students for example when the scale factor is 2.

Scale Factor

The scale factor is the ratio of corresponding lengths in an image to the corresponding lengths in the preimage. In case of circles, it is the ratio between two corresponding arc lengths or two radii. Give students scale factors in various forms: fraction, decimal, and percent, as oftentimes in real-life scale factor is expressed in various forms. For example, a photocopier typically uses percents and other computer software uses decimals.

Scale factor (or magnitude), usually represented by *k* or *r*, can be any factor except 0; however, in high school geometry concepts, $k > 0$.

EXAMPLE



Part 1

a. How far is M away from C?
b. How far is M' away from C?
c. How many times as far is M' away from C compared to M?
d. How many times as far is M away from C compared to M'?
e. Fill in the blank $M'C =$ ___MC to make the statement true.
f. Fill in the blank ___$M'C = MC$ to make the statement true.

Part 2

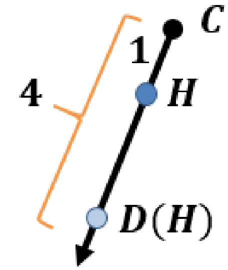
Use the results from Part 1e. to fill in the blank: Draw a picture showing all the points that are _____ times as far from C as M is.

Discussion: This example lends itself to a discussion of scale factor in relation to distance. Students can see that M' is $\frac{5}{4}$ times away from C compared to M and that M is $\frac{4}{5}$ times away from C compared to M. Draw attention to the Multiplicative Inverse Property and its relationship to the coefficients. Students should write the following equations: $M'C = \frac{5}{4}MC$ and $\frac{4}{5}M'C = MC$. Writing comparison equations is difficult for some students. Have discussion about why the two equations are equivalent. See Model Curriculum Instructional Supports A.CED.1-3 for more information about writing comparison equations. In Part 2, students come to understand that there is an infinite amount of points that can be drawn and that they in fact make a circle.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.SRT.1-3)

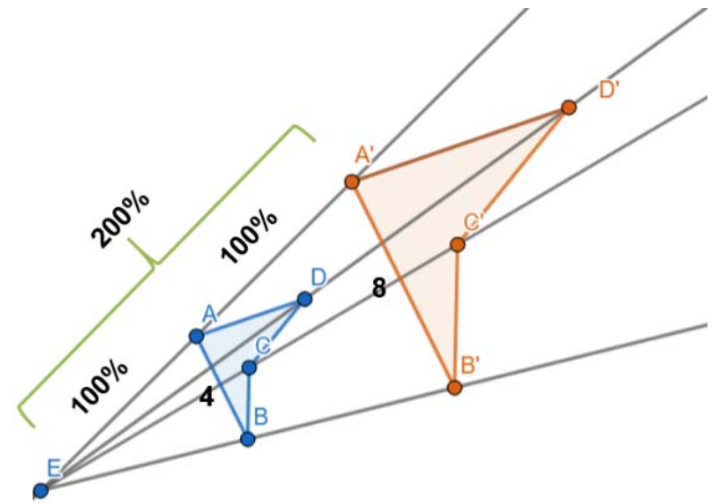
EXAMPLE

If C is center of dilation, find the scale factor for the given Point H and its image $D(H)$.



Some students may incorrectly think that when a scale factor is applied to a figure, the scale factor also affects the size of the angles. In actuality, the scale factor only affects the side lengths of a figure, the angles stay the same regardless of the scale factor.

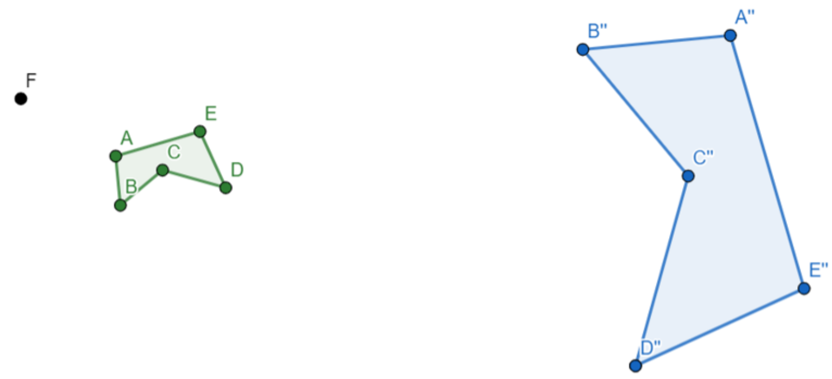
Students may incorrectly think that scale factor is the distance added onto the original distance as they focus on the ratios of the amounts of increase in the length of the corresponding sides rather than the ratio of the lengths of the sides. Draw attention to the lengths of the sides. For example, in the image to the right, students may think that the image has scale factor of 100% but rather it has a 100% increase where the scale factor is 200% because the image is twice as far from the center of dilation. Direct students who struggle with this concept to focus on the side lengths of the figure. They should notice that $\overline{A'B'}$ is twice as long as $\overline{AB}$.



SIMILARITY TRANSFORMATIONS

There are two ways to look at similarity. The traditional definition of similarity occurs between two figures if corresponding angles are congruent and corresponding line segments are proportional; however, this definition only applies to polygons. Another definition defines similarity in terms of transformations which is called a similarity transformation. Two figures are similar if there is a series of transformations including a dilation that maps one figure onto the other. Similarity cannot be defined by dilations alone because otherwise, for example, dilated figures that are also rotated would not be similar; therefore, congruence must be introduced. Therefore, two figures are similar if one figure is congruent to the dilation of the other. Once similarity is defined in terms of transformations and congruence, the traditional definition of similarity (and its converse) becomes a consequence of the definition instead of the definition itself. The benefit of using the transformational definition of similarity is that it can extend to all figures such as circles, ellipses, or even open figures and is not limited to polygons.

Similar figures that have been rotated and dilated.



TIP! Some students find it tempting to use the word dilation instead of similarity transformation. A dilation has a fixed center and scale factor. Many similarity transformations also require basic rigid motions in addition to the dilation.

Every rigid motion is also a similarity transformation, since technically congruence is a special case of similarity. If the image and pre-image are congruent figures, then there is a dilation with a scale factor of 1. Therefore, if two figures are congruent, they are also similar. When a scale factor equal to 1 is applied to a figure, it is called an identity transformation because each point coincides with its image, and the image is identical to the preimage. Figures also have inverse transformations that return a dilated point back to itself. Inverse transformations "undo" each other. The inverse transformation of a dilation D with center C and scale factor r, would be a dilation D with center C and a scale factor of $\frac{1}{r}$.

GeoGebra has an applet titled Dilation and Inverse by Alfred Estberg that allows students to explore inverse dilations

This cluster could be taught in parallel with G.C.1: Proving that all circles are similar.

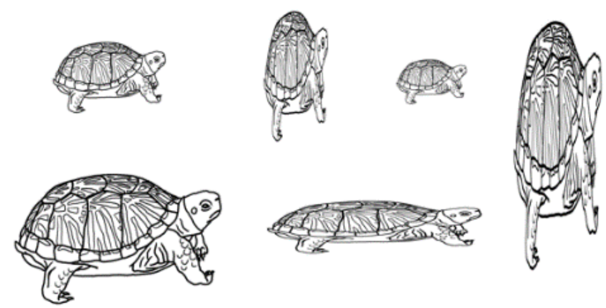
INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.SRT.1-3)

Use technology such as Microsoft Word, PowerPoint, Smartboard etc. to show how stretching or shrinking a figure in one direction does not maintain similarity but stretching or shrinking a figure in all directions does. Allow adequate time and hands-on activities for students to explore dilations visually and physically.

Some students may incorrectly think that a stretch in one-dimension results in similarity. Show examples of figures such as turtles that are stretched in only one dimension compared to those that are stretched in both dimensions.

Use graph paper and rulers or dynamic geometric software to obtain the image of a given figure under a series of transformations: a dilation followed by a sequence of rigid motions (or rigid motions followed by dilation). [Similar Shapes & Transformations](#) by Khan Academy illustrates this process. Students should experience similarity transformations with and without coordinates. Only using coordinates is too restrictive and causes students to create misconceptions.

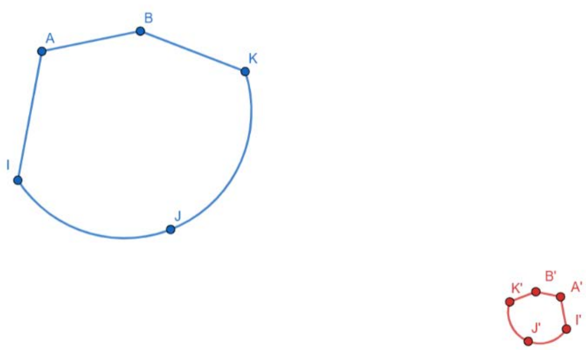
Identify the similar turtles.



Given two figures, students could translate, rotate, and/or reflect an image to line up a corresponding angle of the image and preimage and then dilate to show similarity.

EXAMPLE

Perform a similarity transformation(s) to show that $ABKJI$ is similar to $A'B'K'J'I'$.



Discussion: Students may find it helpful to line up a corresponding angle and then dilate to show similarity.

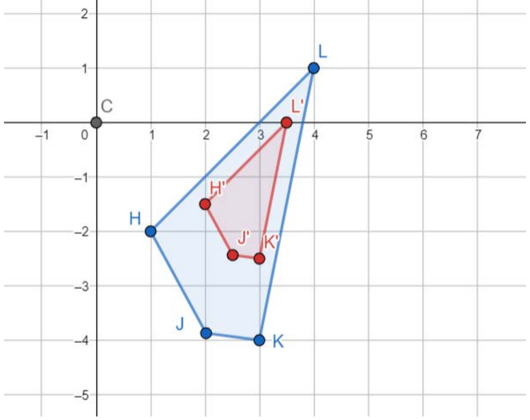
INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.SRT.1-3)

TIP! Students can use shadow puppets to explore properties of similar figures. They can also connect constructing hexagons and similarity. See the References section for NCTM articles for lessons surrounding these concepts.

Work backwards – given two similar figures that are related by dilation, determine the center of dilation and scale factor. Given two similar figures that are related by a dilation followed by a sequence of rigid motions, determine the parameters of the dilation and rigid motions that will map one onto the other.

EXAMPLE

Name the specific transformations that will map $H'J'K'L'$ to $HJKL$.



Discussion: Make sure students mention both a dilation and a translation. The dilation should include the center of dilation and the scale factor for the dilation. The translation should state the distance and direction.

Students may not realize that similarities preserve shape of the figure, but not size. Angle measures stay the same, but side lengths change by a constant scale factor.

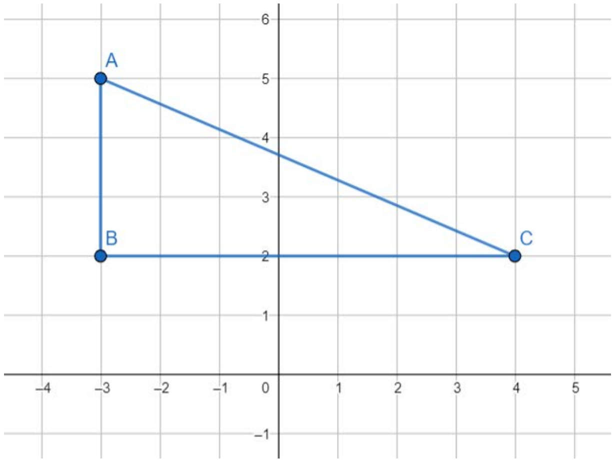
The Reflexive, Symmetric, and Transitive Properties also hold true for similarity. Have students form informal arguments about why these properties hold true for similarity.

Similar Triangles

Similar triangles should be thought of as an extension of other similar figures. Students should use similarity transformations to show that in similar triangles corresponding angles are congruent and corresponding sides are proportional. Students can measure the corresponding angles and sides of the original figure and its image to verify that the corresponding angles remain unchanged and the corresponding sides are proportionally stretched or shrunk by the same scale factor.

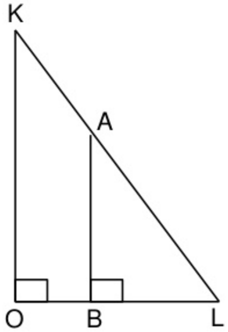
EXAMPLE

a. Using a series of transformations, create two similar triangles.
b. Using your two triangles, make a conjecture about corresponding parts of similar triangles? Justify your reasoning.
c. Compare your conjecture in part **b.** to several of your classmates' examples of similar triangles. Does your conjecture in part **b.** hold true? Explain.
d. Does your statement in part **b.** hold true for any pair of similar triangles? Explain.
e. Would your conjecture hold true for figures other than triangles? Explain.



EXAMPLE

Determine if $\triangle KOL \sim \triangle ABL$. Justify your steps.



Discussion: Given two triangles with a pair of congruent angles, students translate, rotate, and/or reflect one of the triangles in order to map one angle onto its corresponding angle so that the opposite sides from this angle are parallel. Then a dilation completes the map of one triangle onto the other.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.SRT.1-3)

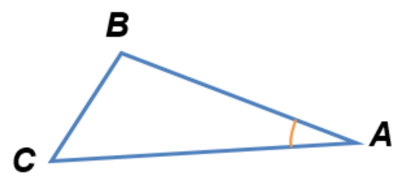
Some students often do not list the vertices of similar triangles in order. However, the order of corresponding vertices is especially important for similar triangles so that proportional sides can be correctly identified.

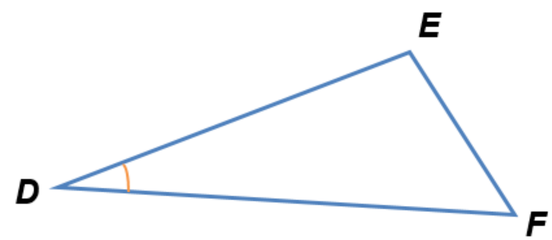
EXAMPLE

Part 1

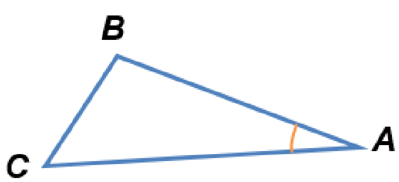
If $\angle A \cong \angle D$, determine if two given triangles, $\triangle ABC$ and $\triangle DEF$, are similar using transformations. Justify your steps.

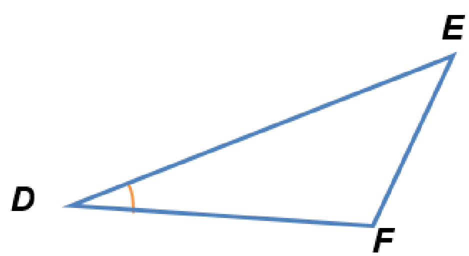
Discussion: Given that $\angle A \cong \angle D$, using patty paper, tracing paper, or geometric software, students should discover that a reflection followed by a translation will map $\angle A$ onto $\angle D$. Then a dilation of $\triangle ABC$ will map onto $\triangle DEF$. Then students can see $\angle B \cong \angle E$ and $\angle C \cong \angle F$, so the triangles are similar.





Part 2





If $\angle A \cong \angle D$, determine if two given triangles, $\triangle ABC$ and $\triangle DEF$, are similar using transformations. Justify your steps.

Discussion: Given that $\angle A \cong \angle D$, using patty paper, tracing paper, or geometric software, students should discover that a reflection followed by a translation will map $\angle A$ onto $\angle D$. This would be a good opportunity to discuss why A-Similarity Criteria only works with right triangles

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.SRT.1-3)

TIP!

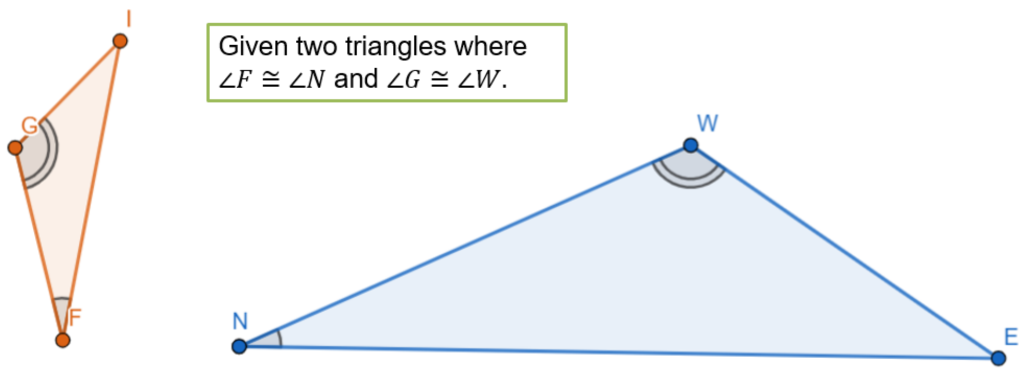
Optical range-finding golf scopes or similar hand-made devices could be used to illustrate similar triangles. However, a dilation of $\triangle ABC$ will not map onto $\triangle DEF$ as student can see $m\angle B \neq m\angle E$ and $m\angle C \neq m\angle F$, so the triangles are not similar.

Angle-Angle (AA) Criterion

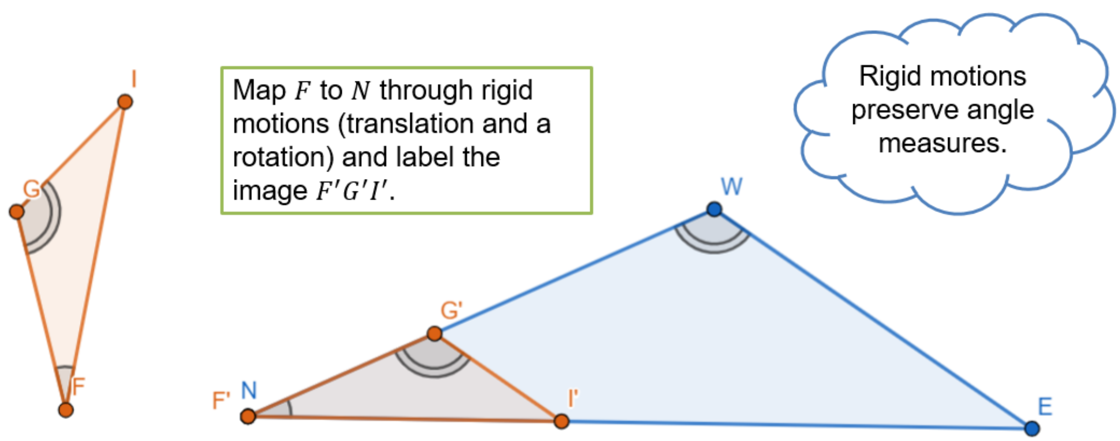
Prove AA similarity criteria for triangle, given two triangles that have two pairs of corresponding angles by using rigid motions to map a vertex of one triangle onto the corresponding vertex of the other in such a way that their corresponding sides are aligned. Then show that dilation will complete the mapping of one triangle onto the other.

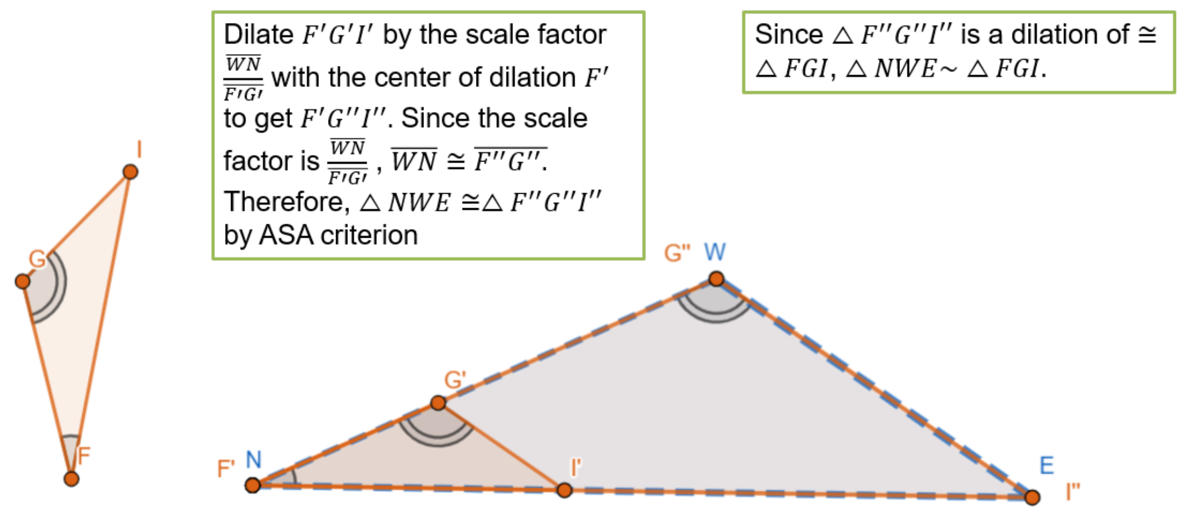
EXAMPLE

Prove two triangles are similar using only two pairs of corresponding congruent angles (AA criterion.)



INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.SRT.1-3)





INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.SRT.1-3)

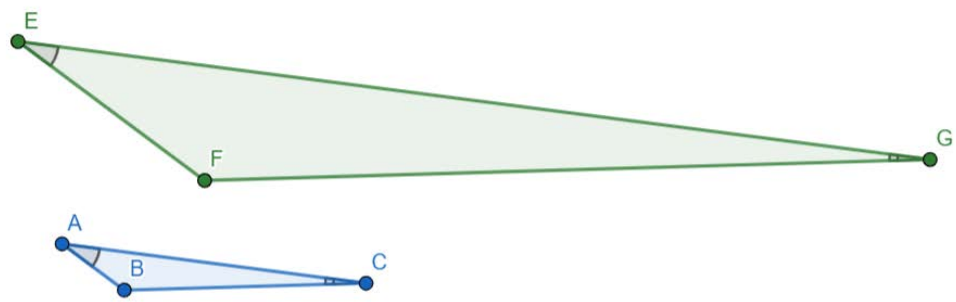
Using the theorem that the sum of the angles is a triangle is 180°, verify that the AA Similarity Theorem is equivalent to the AAA-Similarity Criteria.

EXAMPLE

Prove that only two angles are needed to prove similarity.

Given: $\angle A \cong \angle E$ and $\angle C \cong \angle G$.

Prove: $\triangle ABC \sim \triangle EFG$.



Discussion:

- When all corresponding angles in a triangle are congruent, the triangles are similar.
- $\angle A + \angle B + \angle C = 180°$ because the sums of the angles in a triangle = $180°$.
- $\angle E + \angle F + \angle G = 180°$ because the sums of the angles in a triangle = $180°$.
- $\angle A + \angle F + \angle C = 180°$ by substitution.
- $\angle A + \angle B + \angle C = \angle A + \angle F + \angle C$ by substitution.
- $\angle B = \angle F$.
- Therefore if 2 pairs of corresponding angles are congruent, then the third angle must also be congruent. By default, if AA criteria has been met, then AAA criteria has also been met and $\triangle ABC \sim \triangle EFG$. Therefore, only 2 angles are needed to prove that two triangles are similar.

Note: This example presupposes that students know the theorem that if all corresponding angles in a triangle are congruent, then the figures are similar.

Students should investigate the SAS and SSS similarity criteria for triangles.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.SRT.1-3)

Instructional Tools/Resources

Manipulatives/Technology

- Dot paper
- Patty paper
- Graph paper
- Rulers
- Protractors
- Pantograph
- Photocopy machine
- Computer dynamic geometric software (Geometer's Sketchpad®, Desmos, Cabri®, or GeoGebra®).
- Web-based applets that demonstrate dilations, such as those at the National Library of Virtual Manipulatives.

Dilations

- Dilating a Line by Illustrative Mathematics is a task that gives students the opportunity to verify the properties of dilations experimentally.
- Working with Dilations by Caleb Rothe is a beginning Desmos activity that introduces students to dilations.
- Pantry Dilations by Mathycathy is a Desmos activity that uses dilated scaled images on items found in a pantry.
- The Shadow Knows Dilations (and Transformations) by Ivan Cheng is a Desmos activity that explores dilations in the context of shadows.
- Dilations by Victor Bartodej is a webpage that has many GeoGebra dilations activities.
- Enlarging Figures by Dilations by Shasta County Office of Education is a student handout that has students use rubber bands to explore dilations.
- Embracing Transformational Geometry in CCSS-Mathematics by Jim Short has several explorations on dilations.
- Dilations by Hand2Mind has students explore dilations through Geoboards.
- Key Visualizations: Geometry by The Mathematics Common Core Toolbox has an applet that pushes students to explore dilations of lines and circles.
- Exploring Dilations of the Plane by Michael Andrejkovics is a GeoGebra applet designed to help you visualize how dilations affect both the object in the plane and the plane itself. The applet also allows the center of dilation to be anywhere on the plane.
- Dilations around a Point and their Dilation Factor by Juan Castaneda from Online Math Tutor has three interactive dilations applets.

Similar Figures

- Similar Quadrilaterals is task by Illustrative Mathematics that has students explore how the AA criterion for triangles is to similar criteria for quadrilaterals.
- Episode 1: Similarity—Project Mathematics by Caltech is a YouTube video that explores similarity.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.SRT.1-3)

Similar Triangles

- Are They Similar? by Illustrative Mathematics is a task where students are given two triangles that appear to be similar but whose similarity cannot be proven without further information.
- Similar Triangles by Illustrative Mathematics is a task that examines the similarity of triangles from the viewpoint of proportional sides.
- Congruent and Similar Triangles by Illustrative Mathematics is a task where students understand similarity as a natural extension of congruence.

AA Similarity Criteria Theorem

- AA Similarity Theorem by Tim Brzenzinski is a GeoGebra applet that explores AA Similarity criteria.
- AA Similarity Exploration by BIM is a GeoGebra exploration about AA Similarity.

Curriculum and Lessons from Other Sources

- EngageNY, Geometry, Module 2, Topic A, Lesson 1: Scale Drawings, Lesson 2: Making Scale Drawing Using the Ratio Method, Lesson 3: Making Scale Drawing Using the Parallel Method are lessons that pertain to this cluster.
- EngageNY, Geometry, Module 2, Topic B, Lesson 6: Dilations as Transformations of the Plane, Lesson 7: How Do Dilations Map Segments?, Lesson 8: How Do Dilations Map Lines, Rays, and Circles?, Lesson 9: How Do Dilations Map Angles?, Lesson 10: Dividing the King's Foot into 12 Equal Pieces, Lesson 11: Dilations from Different Centers are lessons that pertain to this cluster.
- EngageNY, Geometry, Module 2, Topic C, Lesson 12: What are Similarity Transformations, and Why Do We Need Them?, Lesson 13: Properties of Similarity Transformations, Lesson 14: Similarity, Lesson 15: The Angle-Angle (AA) Criterion for two Triangles to Be Similar, Lesson 16: Between-Figure and Within-Figure Ratios, Lesson 17: The Side-Angle-Side (SAS) and Side-Side=Side (SSS) Criteria for Two triangles to Be Similar are lessons the pertain to this cluster.
- Mathematics Vision Project, Geometry, Module 4: Similarity and Right Triangle Trigonometry has several tasks that pertain to this cluster.
- Georgia Standards of Excellence Curriculum Framework, Geometry, Unit 2: Similarity, Congruence, and Proofs has many tasks that pertain to this cluster.
- Illustrative Mathematics, Geometry, Unit 3: Similarity, Lesson 1: Scale Drawings, Lesson 3: Measuring Dilations, Lesson 4: Dilating Lines and Angles, Lesson 6: Connecting Similarity and Transformations, Lesson 7: Reasoning about Similarity with Transformations, Lesson 8: Are They All Similar, and Lesson 9: Conditions for Triangle Similarity are lessons that pertain to this cluster.

General Resources

- Arizona 7-12 Progression on Geometry is an informational resource for teachers. This cluster is addressed on pages 16-17.
- High School Coherence Map by UnboundED is a map of the relationships among high school mathematics standards. Please note: This resource is aligned to the Common Core not Ohio's Learning Standards. Although the tool is useful, it needs to be checked with respect to Ohio's Learning Standards.
- van Hiele Model of Geometric Thinking is a pdf created by ODE that summarizes the van Hiele levels.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.SRT.1-3)

References

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STANDARDS	MODEL CURRICULUM (G.SRT.4-5)
Geometry **SIMILARITY, RIGHT TRIANGLES, AND TRIGONOMETRY** **Prove and apply theorems both formally and informally involving similarity using a variety of methods.** **G.SRT.4** Prove and apply theorems about triangles. *Theorems include but are not restricted to the following: a line parallel to one side of a triangle divides the other two proportionally, and conversely; the Pythagorean Theorem proved using triangle similarity.* **G.SRT.5** Use congruence and similarity criteria for triangles to solve problems and to justify relationships in geometric figures that can be decomposed into triangles.	**Expectations for Learning** In middle school, students draw, construct, and describe geometric figures; use informal arguments to establish facts about similar triangles; and explain a proof of the Pythagorean Theorem and its converse. In this cluster, students prove theorems and solve problems involving similarity of triangles. They will also solve problems by applying these theorems to geometric figures that can be decomposed into triangles. The student understanding of this cluster begins at van Hiele Level 2 (Informal Deduction/Abstraction) and moves to Level 3 (Deduction). **ESSENTIAL UNDERSTANDINGS** • The altitude to the hypotenuse divides a right triangle into two triangles that are similar to the original triangle. • A line parallel to the side of a triangle makes similar triangles and divides the other two side lengths proportionally. • Two right triangles are similar if they have another congruent angle. • Polygons can be divided into congruent and/or similar triangles. **MATHEMATICAL THINKING** • Use accurate mathematical vocabulary to represent geometric relationships. • Recognize, apply, and justify mathematical concepts, terms, and their properties. • Use formal reasoning with symbolic representation. • Make conjectures. • Plan a solution pathway. • Justify relationships in geometric figures. • Determine reasonableness of results. • Create a drawing and add components as appropriate. *Continued on next page*

STANDARDS	MODEL CURRICULUM (G.SRT.4-5)
	Expectations for Learning, continued **INSTRUCTIONAL FOCUS** • Form conjectures and construct a valid argument for why the conjecture is true or not true, both formally and informally. • Recognize when polygons are divided into congruent and/or similar triangles. • Justify relationships in geometric figures that can be decomposed into triangles. • Solve problems using triangle congruence and similarity. **Content Elaborations** **OHIO'S HIGH SCHOOL CRITICAL AREAS OF FOCUS** • Math 2, Number 4, page 8 **CONNECTIONS ACROSS STANDARDS** • Understand similarity (G.SRT.1-3). • Define trigonometric ratios, and solve problems involving right triangles (G.SRT.6-8). • Understand the relationships between lengths, areas, and volumes in similar figures (G.GMD.5-6). • Use coordinate geometry (G.GPE.4). • Use coordinates to prove simple geometric theorems algebraically (G.GPE.6). • Find arc lengths and areas of sectors of circles (G.C.5). • Apply geometric concepts in modeling situations (G.MG.2-3). • Write equations in one variable and use them to solve problems (A.CED.1). • Solve quadratic equations in one variable (A.REI.4).

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.SRT.4-5)

Instructional Strategies

Note: The model curriculum is not a collection of lessons nor a full curriculum; it does not suggest pace, sequence, or amount of time spent on topics. It provides information about a topic related to the standards including ideas for examples, strategies for teaching, common misconceptions, and possible connections between topics. In addition, the emphasis of the model curriculum is on instruction not on assessment; therefore, the examples in the Instructional Strategies section are written for instructional purposes not necessarily as assessment items. Examples may need to be rewritten to accommodate the needs of each individual classroom.

Standards for Mathematical Practice
This cluster focuses on but is not limited to the following practices:
MP.3 *Construct viable arguments and critique the reasoning of others.*
MP.5 *Use appropriate tools strategically.*
MP.6 *Attend to precision.*
MP.7 *Look for and make use of structure.*

This cluster focuses on providing a logical argument that demonstrates the truth about theorems involving triangles. A proof is typically a series of justifications presented either formally or informally. They may use different proving methods such as deductive, inductive, two-column, paragraph, flow chart, visual, counter-examples, and others. The Pythagorean Theorem and its converse are proved and applied in Grade 8. In high school, another proof, based on similar triangles, is presented.

The Fundamental Theorem of Similarity will be addressed in G.GMD.5-6.

VAN HIELE CONNECTION

In Math 2 students are expected to move from Level 2 (Informal Deduction/Abstraction) to Level 3 (Deduction). Whereas in Level 2 students use informal arguments, Level 3 requires a more formal approach to thinking.

Level 3 can be characterized by the student doing some or all of the following:

- constructing proofs and understanding the necessity of proofs;
- understanding the role and relationships between of axioms, theorems, definitions, and postulates;
- differentiating between necessary and sufficient conditions; and/or
- making logical conclusions.

PROVING AND APPLYING THEOREMS ABOUT TRIANGLES

Here is some general information about proof instruction:

- The main concept that should be gleaned from these standards is that students need to be able to explain reasons for their thinking and why/how something is true, much like in the ELA Writing Standards where *evidence* must be included for any claim.
- Teachers should vary the level of formality that is appropriate for the content, and the reader of the proof. However, every level of formality includes students' ability to formally/informally reference the appropriate source—a definition, a property, a law, an axiom, a theorem, etc. Make sure that the level of formality does not distract from the main idea of the proof.
- The emphasis should be on a progression toward proof and not on formal proof. Students need to be able to come up with their own conjectures and then provide mathematically sound justification for the conjectures' validity. Ultimately students should construct a complete argument, in a variety of formats, to move from given information to a conclusion.
- Direct instruction is not the best way to introduce formal proof. Instead the focus should "be on justifying ideas while helping students build the visual and empirical foundation for higher levels of geometric thought, we can lead students to appreciate the need for formal proof." (Battista and Clements, 1995)
- Students continue to use precise language and relevant vocabulary to justify steps in their work to construct viable arguments that defend their method of solution.
- All statements need to be examined on their own merit—conditionals and their converse statements are not always both true. For the times when both conditionals and their converses are true, biconditional statements can be written.
- Proof methods could include but are not limited to the following: deductive, inductive, two-column, paragraph, flow chart, visual, and/or counterexample.
- The concepts of inverse and contrapositive are not emphasized but can be explored as an extension.

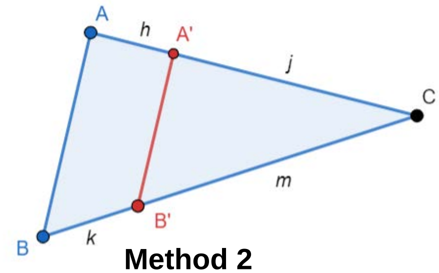
Triangle Side-Splitter Theorem

Another important application of similarity is the Side-Splitter Theorem: If a line is parallel to one side of a triangle and intersects the other two sides in distinct points, it splits these sides into proportional segments. Another way to phrase it is that a line segment splits two sides of triangle proportionally if and only if it is parallel to the third side. Students should also be able to prove its converse.

EXAMPLE

Given: $\overline{A'B'}$ is a dilation of $\overline{AB}$ with scale factor r and a point C as the center of dilation. The side lengths are as labeled.

Prove: $\frac{h}{j} = \frac{k}{m}$.



Method 1

This approach uses the definition of dilation (scale factor).
According the definition of a dilation, $|CA'| = r|CA|$ and $|CB'| = r|CB|$.

Therefore, $j = r(j+h)$ and $m = r(m+k)$, so

$r = \frac{j}{j+h}$ and $r = \frac{m}{m+k}$

$\frac{m}{m+k} = \frac{j}{j+h}$ by substitution

$$m(j+h) = j(m+k)$$

$$mj + mh = jm + jk$$

$$mh = jk$$

$$\frac{mh}{mj} = \frac{jk}{mj}$$

$$\frac{h}{j} = \frac{k}{m}$$

Method 2

This approach uses a property of dilation (parallel lines). Since $\overline{A'B'}$ is a dilation of $\overline{AB}$, $\overline{A'B'} \parallel \overline{AB}$ because a dilation takes a line to a parallel line if its center is not on the line, then $\angle CA'B' \cong \angle CAB$ and $CB'A' \cong \angle CBA$ by corresponding angles formed by a parallel lines and a transversal. Thus, $\Delta CA'B' \sim \Delta CAB$ by AA similarity criteria. Since corresponding sides of similar figures are proportional,

$\frac{j+h}{j} = \frac{m+k}{m}$, so

$$m(j+h) = j(m+k)$$

$$mj + mh = jm + jk$$

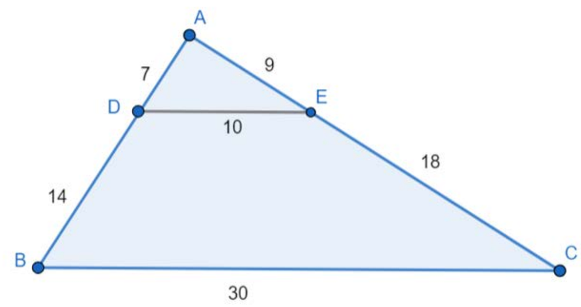
$$mh = jk$$

$$\frac{mh}{mj} = \frac{jk}{mj}$$

$$\frac{h}{j} = \frac{k}{m}$$

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.SRT.4-5)

TIP! Using the AA Similarity Criteria to prove the Side-Splitter Theorem uses circular reasoning. More advanced students can confront this idea by using the formula for the area of a triangle to explain the Side-Splitter Theorem. See Parallels in Geometry by Snapp and Findell pages 91-99 and EngageNY, Geometry, Module 2, Topic A, Lesson 4: Comparing the Ratio Method with the Parallel Method for more information.



When a line parallel to one side of a triangle divides the other two proportionally, students will sometimes misapply the Side-Splitter Theorem to situations with similar triangles. It is important to give students problems where they not only find missing parts as in the Side-Splitter Theorem but also give situations where one of the missing measures is one of the parallel sides. Sometimes students incorrectly say $part_1$:$part_2$ (sides divided proportionally) = $whole_1$:$whole_2$ (parallel side lengths), instead of correctly applying the Side-Splitter Theorem by stating $\frac{1\text{st part of side } 1}{2\text{nd part of side } 1} = \frac{1\text{st part of side } 2}{2\text{nd part of side } 2}$ or $\frac{1\text{st part of side } 1}{1\text{st part of side } 2} = \frac{2\text{nd part of side } 1}{2\text{nd part of side } 2}$ and differentiating between similar triangles by stating $\frac{1\text{st side of litte triangle}}{1\text{st side of big triangles}} = \frac{2\text{nd side of litte triangle}}{2\text{nd side of big triangles}}$ or $\frac{1\text{st side of litte triangle}}{2\text{nd side of little triangles}} = \frac{1\text{st side of big triangle}}{2\text{nd side of big triangles}}$. Decomposing the diagram into two similar triangles may help students recognize corresponding parts.

$\frac{Part_1}{Part_2} \neq \frac{Whole_1}{Whole_2}$, so $\frac{7}{14} \neq \frac{10}{30}$

Instead of using the Side-Splitter Theorem

$\frac{Part_1 \text{ of } \overline{AB}}{Part_2 \text{ of } \overline{AB}} = \frac{Part_1 \text{ of } \overline{AC}}{Part_2 \text{ of } \overline{AC}}$ such as $\frac{7}{14} = \frac{9}{18}$ or

$\frac{Part_1 \text{ of } \overline{AB}}{Part_1 \text{ of } \overline{AC}} = \frac{Part_2 \text{ of } \overline{AB}}{Part_2 \text{ of } \overline{AC}}$ such as $\frac{7}{9} = \frac{14}{18}$

Compared to using Corresponding Parts of Similar Triangles are Proportional (CPSTP)

$\frac{little_1}{little_2} = \frac{big_1}{big_2}$ such as $\frac{7}{10} = \frac{21}{30}$ or

$\frac{little_1}{big_1} = \frac{little_2}{big_2}$ such as $\frac{7}{21} = \frac{10}{30}$

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.SRT.4-5)

EXAMPLE

Given:

Figure 1: $\overline{BC} \parallel \overline{DF}$

Figure 2: $\overline{OK} \parallel \overline{PE}$

a. Find x in Figure 1.

b. Find y in Figure 2.

c.

- Rihanna said that she solved both problems the same way and got $x = 6$ for Figure 1 and $y = \frac{12}{5}$ for Figure 2.
- Marcus said he solved them both the same way (but differently from Rihanna) and got $x = 6$ for Figure 1 and $y = \frac{32}{5}$ for Figure 2.
- Lamar said that he did them different ways, but also got $x = 6$ for Figure 1 and $y = \frac{12}{5}$ for Figure 2.

How did each person get their result? Who did the problem correctly?

Figure 1

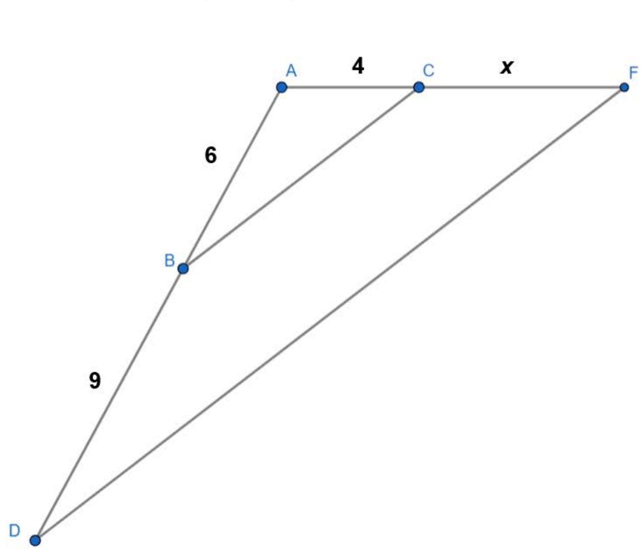
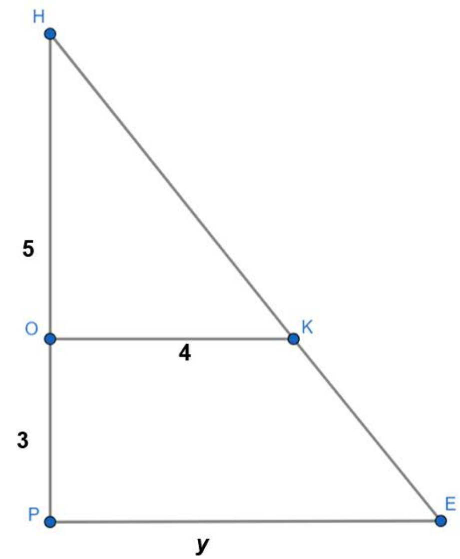


Figure 2



Midsegment (or Midpoint-Connector) Theorem

Students should recognize that the Midsegment Theorem "The segment joining midpoints of two sides of a triangle is parallel to the third side and half the length." is a special case of the Side-Splitter Theorem. (The theorem from Math 1 can be proved in Math 2.) The Midsegment Theorem can be proved using SAS Similarity Criteria for Triangles and the Alternate Interior Angle Theorem.

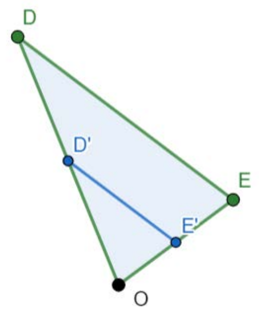
An optional application of the Midsegment Theorem is to prove that the figure formed by joining consecutive midpoints of sides of an arbitrary quadrilateral is a parallelogram. (This result is known as the Midpoint Quadrilateral Theorem or Varignon's Theorem.)

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.SRT.4-5)

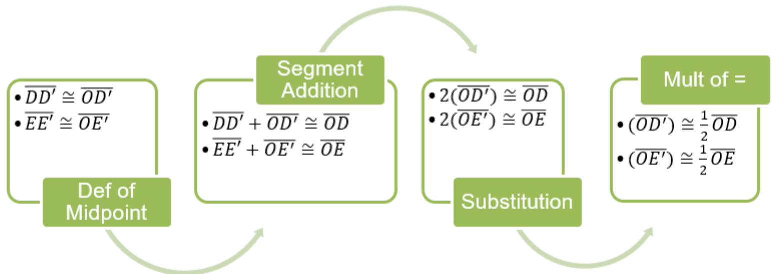
EXAMPLE

Given: $D'$is the midpoint of $\overline{OD}$, and $E'$is the midpoint of $\overline{OE}$.

Prove: $\overline{D'E'} = \frac{1}{2}\overline{DE}$



Discussion:



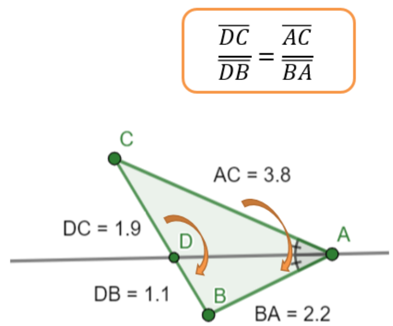
Therefore a dilation with the scale factor of $\frac{1}{2}$ has been performed. This means all figures on the plane have undergone a dilation with a scale factor of $\frac{1}{2}$ around the center of dilation O including $\overline{DE}$, therefore $\overline{D'E'} = \frac{1}{2}\overline{DE}$

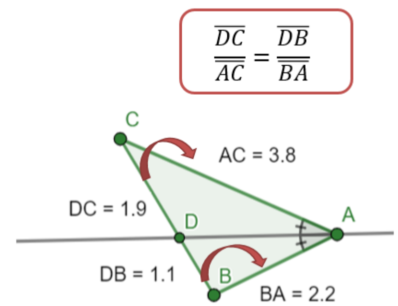
Note: This theorem could be approached from the perspective of similar triangles or could also be tied into G.GPE.4 as a coordinate proof.

Midsegment Theorem by Brad Findell is an interactive proof that explores the Midsegment Theorem: The segment joining the midpoints of two sides of a triangle is parallel to and half the length of the third side.

Angle Bisector Theorem

The Angle Bisector Theorem states the angle bisector of the interior angle of a triangle, divides the opposite side into two segments that are proportional to the other two sides of the triangle. It rests on the Angle Bisector Conjecture stating that any point on the bisector of an angle is equidistant from its sides (See G.CO.9-11).





INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.SRT.4-5)

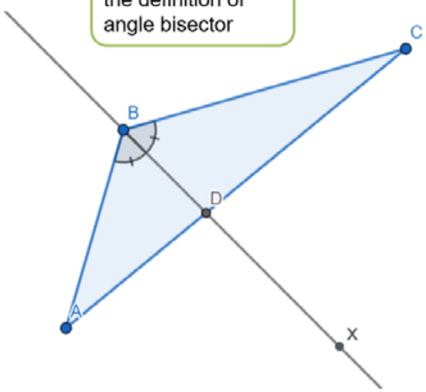
EXAMPLE

Given: $\overleftrightarrow{BX}$ bisects $\angle ABC$ in $\triangle ABC$

Prove: $\frac{\overline{DC}}{\overline{DA}} = \frac{\overline{BC}}{\overline{AB}}$ or $\frac{\overline{DC}}{\overline{BC}} = \frac{\overline{AD}}{\overline{AB}}$

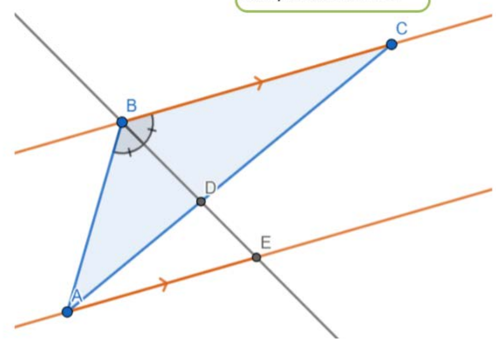
Step 1:

$\angle ABD \cong \angle CBD$ by the definition of angle bisector

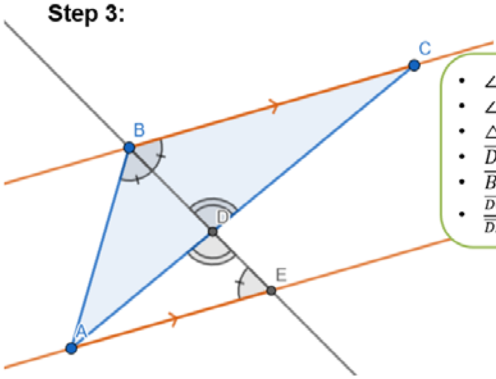


Step 2:

Construct a line $\overleftrightarrow{AE}$ parallel to $\overleftrightarrow{BC}$.

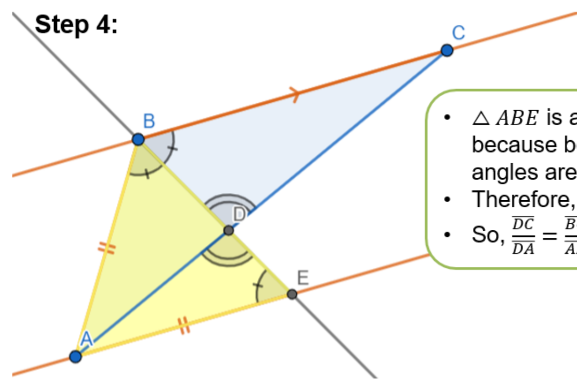


Step 3:



- $\angle CBE \cong \angle BEA$ by Alternate Interior Angles
- $\angle BDC \cong \angle EDA$ by Vertical Angles
- $\triangle BDC \sim \triangle EDA$ by AA Similarity Criteria
- $\overline{DC}$ is proportional to $\overline{DA}$ by CPSTP.
- $\overline{BC}$ is proportional to $\overline{AE}$ by CPSTP.
- $\frac{\overline{DC}}{\overline{DA}} = \frac{\overline{BC}}{\overline{AE}}$

Step 4:



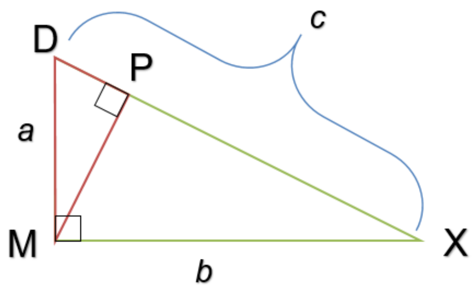
- $\triangle ABE$ is an isosceles triangle because both of the base angles are $\cong$.
- Therefore, $\overline{AB} \cong \overline{AE}$.
- So, $\frac{\overline{DC}}{\overline{DA}} = \frac{\overline{BC}}{\overline{AB}}$ by Substitution.

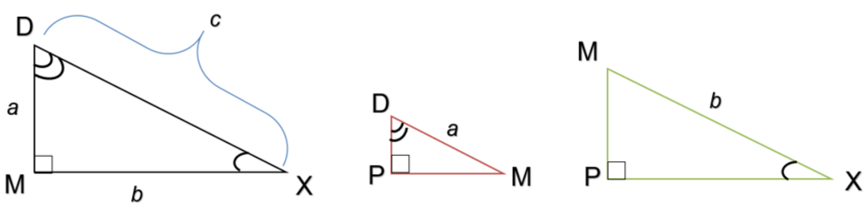
Pythagorean Theorem

It is important for students to understand an altitude drawn to the hypotenuse of a right triangle creates two triangles that are similar to the original triangle. Use physical models such as cardboard cutouts to illustrate that the altitude to the hypotenuse divides a right triangle into two triangles that are similar to the original triangle. Then use AA Similarity Criteria to prove this theorem. This is used as a proof of the Pythagorean Theorem and a basis of Trigonometry. *Note: The geometric mean is discussed later in this section.*

EXAMPLE

Prove that if an altitude drawn to the hypotenuse of a right triangle creates two triangles, then the new right triangles are similar to the original triangle.





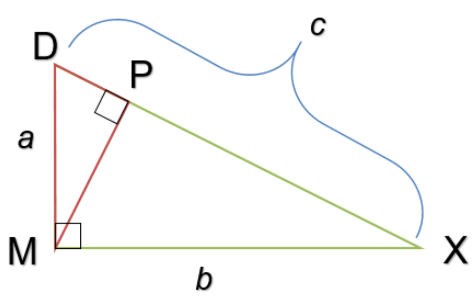
$\triangle MDX \sim \triangle PDM$
and
$\triangle MDX \sim \triangle PMX$
by AA similarity criteria

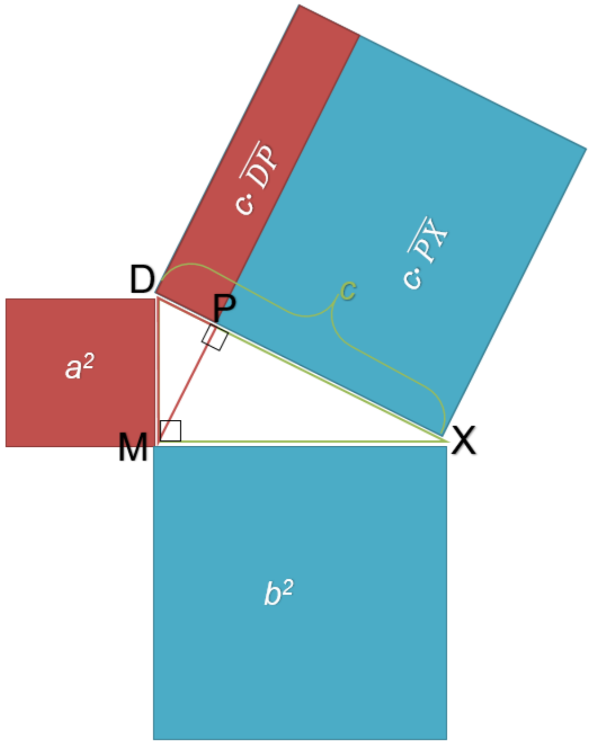
Discussion: Then, use this result to establish the Pythagorean relationship among the sides of a right triangle ($a^2 + b^2 = c^2$), and thus obtain an algebraic proof of the Pythagorean Theorem. See the next example.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.SRT.4-5)

EXAMPLE

Prove that $a^2 + b^2 = c^2$ for a right triangle using concepts of similarity.





Discussion:

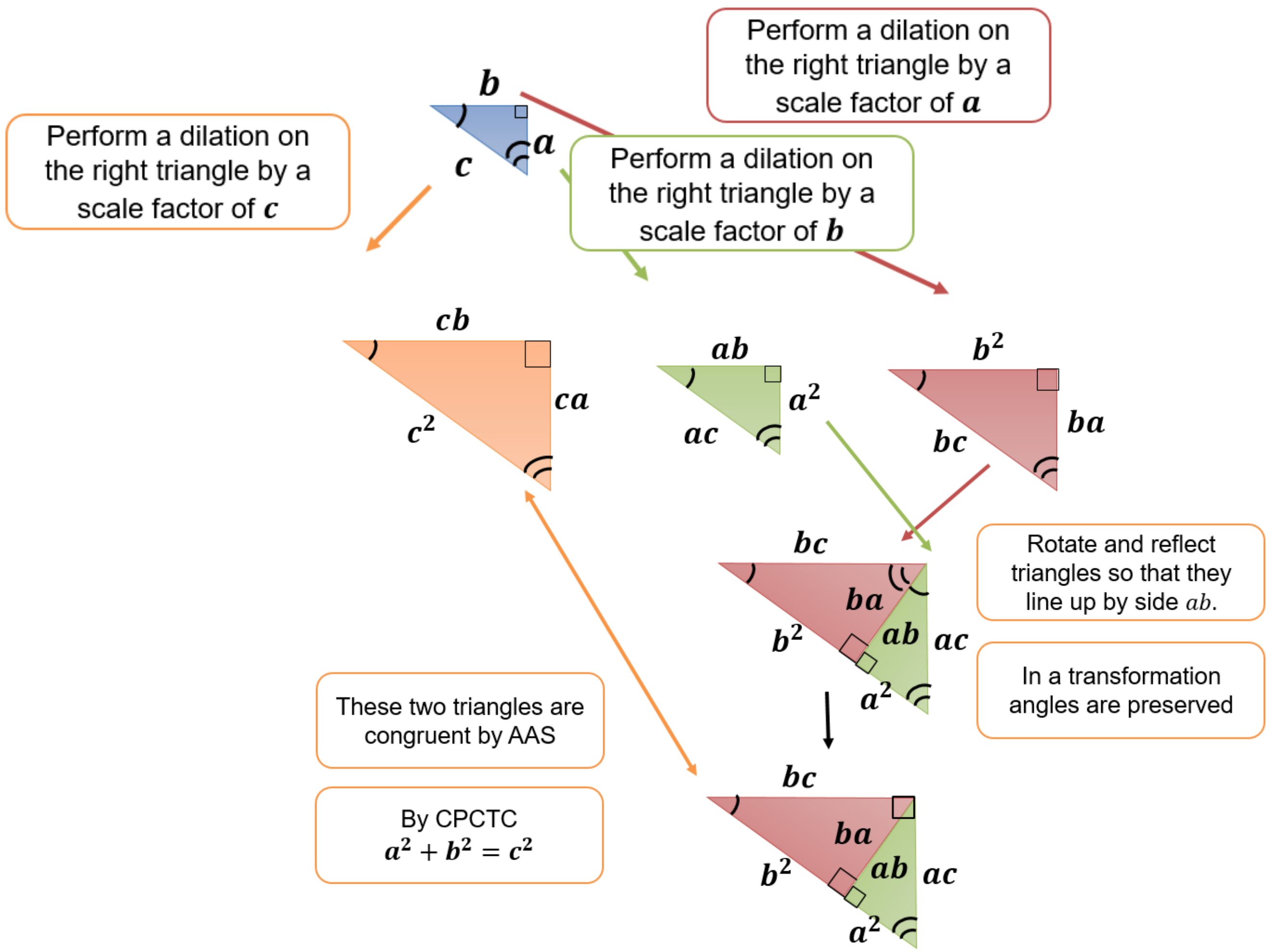
- By AA criterion students should see that $\Delta MDX \sim \Delta PDM$ and $\Delta PMX \sim \Delta MDX$ by AA-Similarity Criteria (See pervious example).
- Since corresponding sides are proportional in similar triangles $\frac{\overline{DP}}{\overline{DM}} = \frac{\overline{DM}}{\overline{DX}}$ and $\frac{\overline{XM}}{\overline{DX}} = \frac{\overline{PX}}{\overline{XM}}$.
- Then the equations could be rewritten as $\overline{DM}^2 = \overline{DP} \cdot \overline{DX}$ and $\overline{XM}^2 = \overline{PX} \cdot \overline{DX}$.
- Combine the two equations: $\overline{DM}^2 + \overline{XM}^2 = \overline{DP} \cdot \overline{DX} + \overline{PX} \cdot \overline{DX}$.
- Factor out $\overline{DX}$ to get $\overline{DM}^2 + \overline{XM}^2 = \overline{DX}\,(\overline{DP} + \overline{PX})$.
- Since $\overline{DX}$ is $\overline{DP} + \overline{PX}$, the equation can be rewritten as $\overline{DM}^2 + \overline{XM}^2 = \overline{DX}^2$.
- Then by substituting lengths $a, b,$ and c back into the equation we get $a^2 + b^2 = c^2$ which is the Pythagorean Theorem for triangle ΔMDX.

Pythagorean Theorem Proof for Similar Right Triangles is a video by bikes4fish explaining the proof a little more-in depth.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.SRT.4-5)

EXAMPLE

Prove the Pythagorean Theorem using transformations.

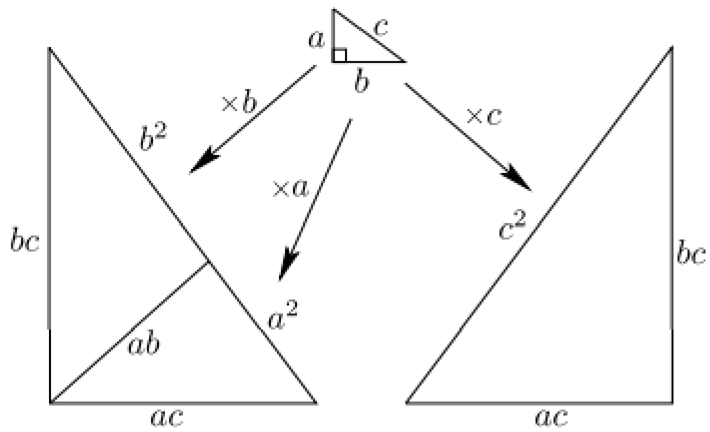


INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.SRT.4-5)

Similar Right Triangles by Brad Findell is an interactive proof where students can explore proving triangles created by dropping a perpendicular line to the hypotenuse are also similar.

EXAMPLE (EXTENSION)

To challenge students, give them the image below as a "Proof without Words." Have them explain how it proves the Pythagorean Theorem. *Note: To be an extension, it should be in place of the previous example.*



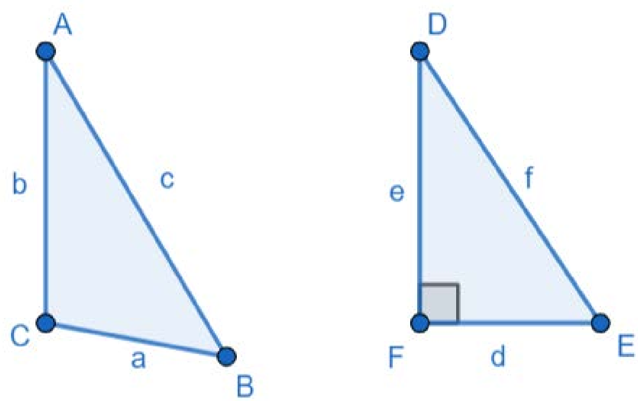
Taken from Snapp & Findell, 2016, Parallels in Geometry

Prove the converse of the Pythagorean Theorem, using the theorem itself as one step in the proof.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.SRT.4-5)

EXAMPLE

Prove that if $a^2 + b^2 = c^2$, then a triangle is a right triangle so $\angle C$ must be a right angle. To use a method of contradiction, assume that in $\triangle ABC, c^2 = a^2 + b^2$, and $\angle C$ not a right angle.



- Construct another triangle $\triangle DEF$ that is a right triangle where $\overline{FE} \cong \overline{CB}$ and $\overline{DF} \cong \overline{AC}$, and $FE = d$ and $DF = e$.
- Since $\triangle DEF$ is a right triangle, by the Pythagorean Theorem, $d^2 + e^2 = f^2$,
- Since $a = d$ and $b = e$, then by substitution, $a^2 + b^2 = f^2$,
- Since $a^2 + b^2 = c^2$ and $a^2 + b^2 = f^2$, then by substitution $c^2 = f^2$ or $c = f$,
- Since $a = d$, $b = e$, and $c = f$, then by SSS, $\triangle ABC \cong \triangle DEF$,
- By CPCTC, $\angle C \cong \angle F$,
- By definition of congruence, then $m\angle C = m\angle F = 90°$,
- This means that $\angle C$ is a right angle and contradicts our assumption,
- Therefore for $a^2 + b^2 = c^2$ to be true, $\angle C$ must be a right angle and $\triangle ABC$ must be a right triangle.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.SRT.4-5)

TIP! Have students create a flipbook animation to demonstrate the Pythagorean Theorem or have students create a fractal tree using the Pythagorean Theorem. How to Draw Fractal Tree by Dearing Wang illustrate how to draw the tree.

TIP! Some students may confuse theorems and their converses such as the Alternate Interior Angle Theorem and its converse or the Pythagorean Theorem and its converse.

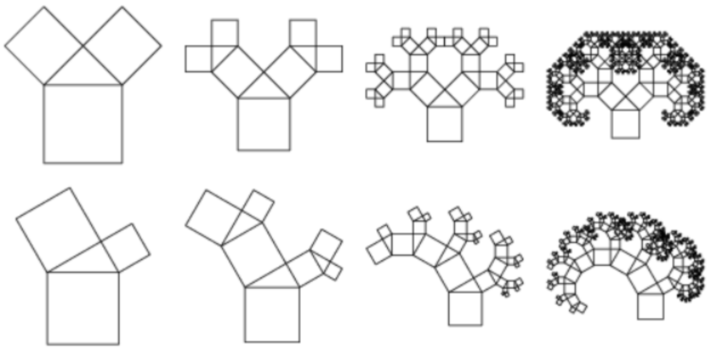


Image taken from
http://mathworld.wolfram.com/PythagorasTree.html

Connecting Irrational Roots to the Pythagorean Theorem

Give students opportunities to conceptually understand irrational numbers. One way is for students to draw a square that is one unit by one unit and find the diagonal using the Pythagorean Theorem. The diagonal drawn has an irrational length of $\sqrt{2}$. Other irrational lengths can be found using the same strategy by finding diagonal lengths of rectangles with various side lengths. Students can also explore the Wheel of Theodorus to explore irrational numbers without using a number line. See the Instructional Resources/Tools for more ideas.

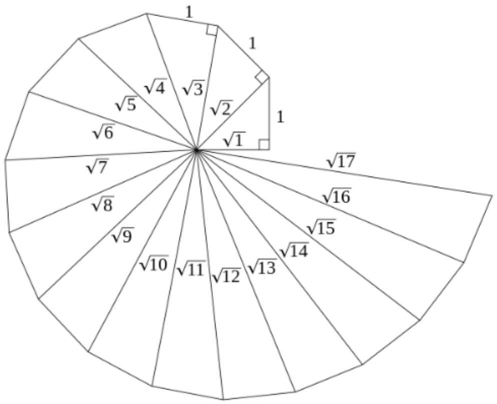


Image taken from
https://en.wikipedia.org/wiki/Spiral_of_Theodorus

SOLVE PROBLEMS USING TRIANGLE CONGRUENCE AND SIMILARITY CRITERIA

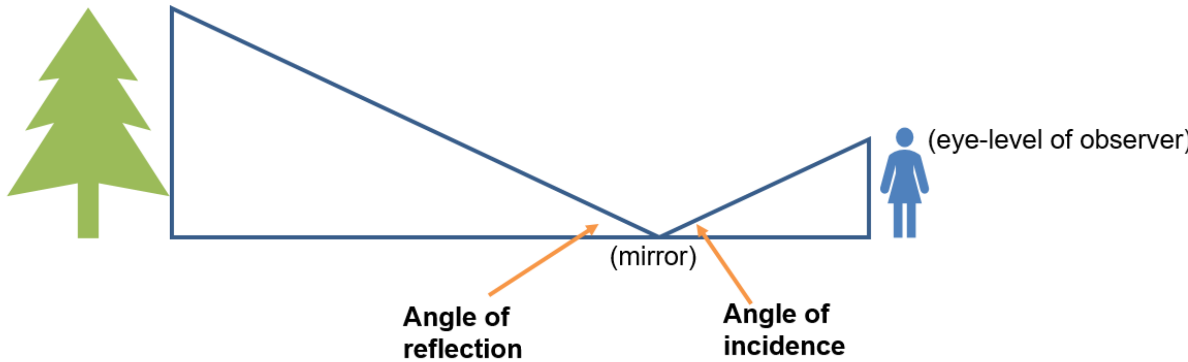
Students should apply their knowledge of congruence and similarity to solve mathematical and real-world problems surrounding triangles.

TIP! Students sometimes are dependent on given triangles presented with the same orientation instead of using corresponding parts for similarity. Give students shapes with different orientations to confront this issue.

EXAMPLE

Indirectly find the height of a nearby tree or telephone pole using a mirror and a meter stick.

Discussion: Students should apply their knowledge of similar triangles and the idea that the angle of incidence equals the angle of reflection to find the height of a nearby object.



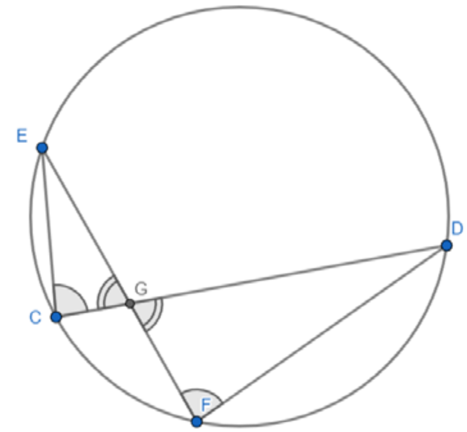
TIP! The concept of using angles to find images can be tied to Physics applications of light reflecting. [The Physics Classroom] has some information about images, light, reflections, and angles.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.SRT.4-5)

EXAMPLE

A tree has a shadow of 15 feet. Ray who is 6'1" has a shadow of 4'2" feet. How tall is the tree in feet?

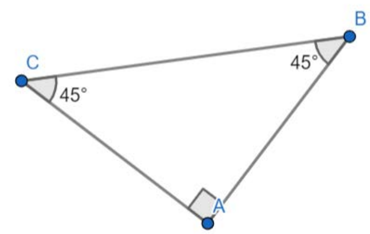
Students can also discover that triangles formed by angles inscribed in circles can be an application of similarity.

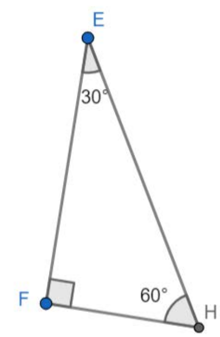


$\triangle GCE \sim \triangle GFD$ because
- $\angle EGC \cong \angle DGF$ by vertical angles, and
- $\angle ECG \cong \angle DFG$ are angles that intercept the same arc, so
- $\triangle GCE \sim \triangle GFD$ by AA criteria.

Special Right Triangles

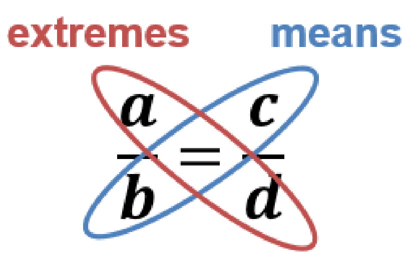
Students should explore special right triangles such as 30°-60°-90° triangles and 45°-45°-90° triangles.





Pythagorean Triples

Pythagorean triples should be taught as an application of similar triangles, for example a (a 6:8:10 right triangle is similar to a 3:4:5 right triangle). They can be used as an algebraic extension and an opportunity to explore patterns.



Geometric Mean

Traditionally middle school students have solved proportions using cross products also known as the Means-Extremes Property. The Means-Extreme Property states that in a true proportion, the product of the means equals the product of the extremes or "If $\frac{a}{b} = \frac{c}{d}$, then $ad = bc$." However, Ohio's Learning Standard for Mathematics in the middle grades now deemphasize this method because students at that level do not understand why it works, and therefore oftentimes misapply it. Instead they emphasize solving proportions using within and between relationships, common denominator method, the unit rate method, and graphing proportions, so students may not be familiar with solving proportions this way. See Model Curriculum 7.RP.1-3 for more information on solving proportions.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.SRT.4-5)

Means Exchange Property

If $\frac{a}{b} = \frac{c}{d}$, then $\frac{a}{c} = \frac{b}{d}$.

Reciprocals Property

If $\frac{a}{b} = \frac{c}{d}$, then $\frac{b}{a} = \frac{d}{c}$.

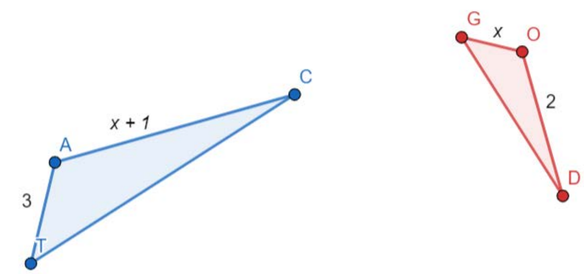
In a proportion the means (or the extremes) can be changed to form an equivalent true proportion or the reciprocals can be used to create an equivalent true proportion. These properties, although not formerly named, should have been explored in middle school. They can be proved through algebraic methods.

EXAMPLE

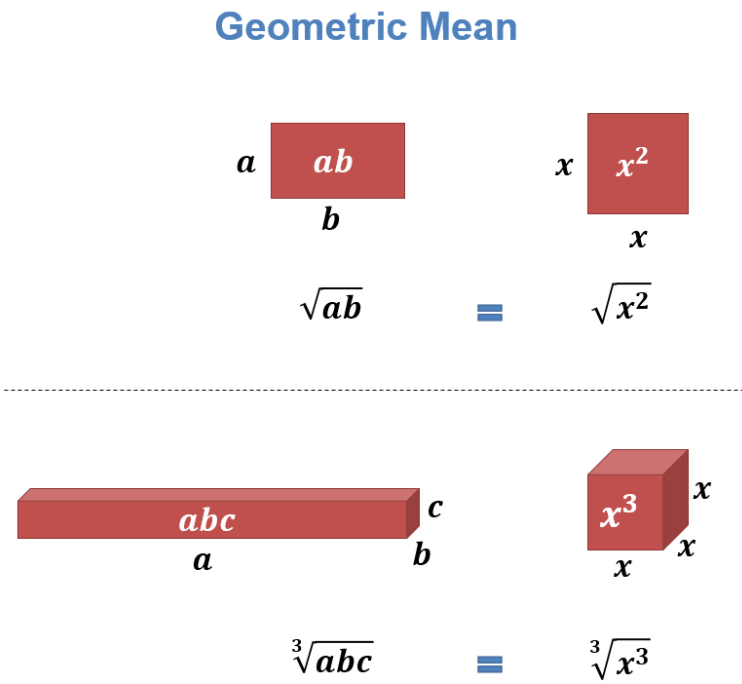
$\Delta CAT \sim \Delta DOG$. Find the missing side lengths of the similar triangles. Triangles are not drawn to scale.

When the means of a proportion are identical, the identical segment or number is called the geometric mean.

In $\frac{a}{m} = \frac{m}{d}$, m is the geometric mean



INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.SRT.4-5)

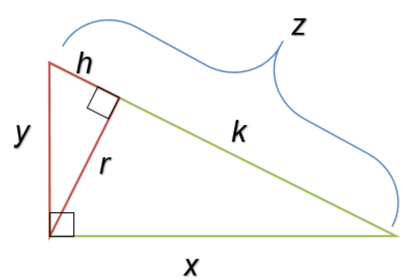


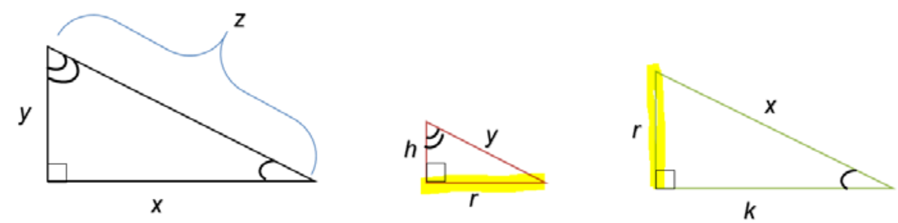
Although the geometric mean is often used in the contexts of proportions, it is not limited to proportional situations. It is a special type of mean where the factors are multiplied together, and then the root (corresponding to the number of factors) is taken. The nth root of the product for n numbers. It is useful when comparing things with different properties. Whereas the arithmetic mean sums all the values in a data set and then divides the sum by the number of terms, the geometric mean multiplies all the values in a data set, and takes the root corresponding to the number of terms. The geometric mean is useful when comparing values with different units. The concept of geometric mean should be connected to right triangles.

The geometric mean can also be viewed in light of geometric figures. The geometric mean of two numbers, a and b, is the length of one side of a square whose area is equal to the area of the rectangles with sides of lengths a and b. Similarly, the geometric mean of three numbers, a, b, and c is the length of one side of a cube whose volume is equal to the volume of the prism with side lengths, a, b, and c.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.SRT.4-5)

EXAMPLE





Part 1

- **a.** How many triangles are in the picture?
- **b.** Draw all the triangles you see in the same orientation keeping the labels.
- **c.** What is the relationship between the triangles? Explain.
- **d.** What is the same on the two smaller triangles? Explain.
- **e.** Why is this important?

Part 2

- **a.** Use the information from Part 1 to find the missing side in figures 1 and 2.
- **b.** Create a rule describing the relationship between *h*, *k*, and *r*.
- **c.** The rule you created in part **b.** illustrates the geometric mean. What other rules can you create using the side lengths of the three right triangles.

Figure 1:

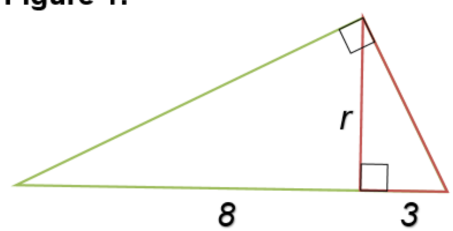
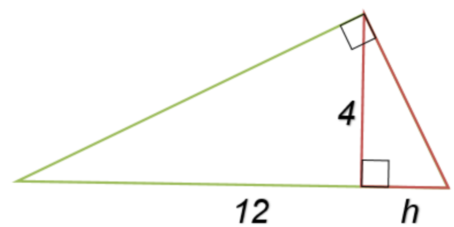


Figure 2:



Discussion: Students should notice that there are three right triangles in the diagram, and that they are all similar by AA Triangle Similarity Criteria. They should be able to come up with the rule: The altitude to the hypotenuse is the geometric mean of the two segments into which it divides the hypotenuse or $r = \sqrt{hk}$. They should also be able to deduce that each leg is the geometric mean of the hypotenuse, and the segment of the hypotenuse adjacent to the leg or $x = \sqrt{zk}$ and $y = \sqrt{zh}$. This is the Right Triangle Altitude Theorem.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.SRT.4-5)

The geometric mean can also be connected to concepts in Algebra.

EXAMPLE

Ted wants to book a hotel and is looking at two different websites. The Cabana Inn is ranked 3.7 stars on the first website and 8.63 on the second site. The Holiday Hotel is ranked 4.2 on the first website and 8.12 on the second website. Which hotel has the best ratings?

Discussion: Although these examples are algebraic instead of geometric, they lay the foundation for understanding the geometric mean and help students make connections between algebraic and geometric concepts. The idea of geometric mean will be connected to right triangles. Since the Cabana Inn has a geometric mean of $\sqrt{3.7 \cdot 8.63} \approx 5.65$, and the Holiday Hotel has a geometric mean of $\sqrt{4.2 \cdot 8.12} \approx 5.84$, the Holiday Hotel has better overall rankings.

EXAMPLE

A stock grows by 15% one year, declines 25% the second year, and then grows by 35% the third year. Use the geometric mean to calculate the average growth rate (known as the compounded annual growth rate).

Discussion: This can be calculated by $\sqrt[3]{(1+0.15)(1-0.25)(1+0.35)} - 1$ which is approximately 0.052 or 5.2% annually.

Have students explore how a right triangle's altitude connects with the geometric mean.

EXAMPLE

Find the three missing terms in the geometric sequence 2, ____, ____, ____, 10.125…

Discussion: Students can find the 3rd term by finding the geometric mean of 2 and 10.125: $b = \sqrt{2 \cdot 10.125} = 4.5$. Then, they can find the 2nd term by finding the geometric mean of 2 and 4.5: $a = \sqrt{2 \cdot 4.5} = 3$. Finally, they can find the 4th term by finding the geometric mean of 4.5 and 10.125: $c = \sqrt{4.5 \cdot 10.125} = 6.75$. If desired, they could also then take the concept a step further and find the common ratio of the sequence which is 1.5.

Decomposing Polygons into Triangles to Justify Relationships

Students should justify relationships in geometric figures that can be decomposed into triangles. For example, given a trapezoid, students can discover that the diagonals create two similar triangles.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.SRT.4-5)

Instructional Tools/Resources

These tools and resources may be helpful for instruction. Before use, they should be evaluated at the local level to determine their appropriateness for instruction.

Manipulatives/Technology

- Cardboard models of right triangles.
- Dynamic geometric software (Geometer's Sketchpad®, Desmos®, Cabri®, or GeoGebra®).
- Mirrors
- Tape Measures
- Protractors

Applying Theorems about Triangles

- Solving Problems with Circles and Triangles by Mathematics Assessment Project is a lesson where students solve problems by determining the lengths of the sides in right triangles.

Midpoints of a Sides of a Triangle

- Joining Two Midpoints of Sides of a Triangle by Illustrative Mathematics is a task where students solve a problem involving midpoints of sides of a triangle.

Pythagorean Theorem

- Pythagorean Theorem by Illustrative Mathematics is a task where students prove the Pythagorean Theorem using similar triangles.
- Pythagorean Theorem by Davis Associates, Inc. has an animated proof of the Pythagorean Theorem.
- Pythagorean Theorem by Cut the Knot has 18 approaches to the Pythagorean Theorem.
- Video: *The Theorem of Pythagoras* from Project MATHEMATICS!
- Lunar Rover by NASA is a lesson where students apply the Pythagorean Theorem to a situation involving a lunar rover.
- Wheel of Theodorus is also called the spiral of Theodorus, the Square Root Spiral, the Einstein Spiral, or the Pythagorean Spiral. Starting with a right triangle with legs the length of one, each succeeding right triangle is formed with one of the legs being one and the second leg being the hypotenuse of the preceding triangle.
    - Root Spiral of Theodorus is a YouTube video by bikes4fish that shows creating the Wheel of Theodorus.
    - Pythagorean Spiral Video by Jendar40 is a YouTube video that shows creating the Wheel of Theodorus.
    - Create the Wheel of Theodorus and Find its Pattern of Irrational Numbers by Using the Iterations of the Pythagorean Theorem is a lesson by LearnZillion.
    - Wheel of Theodorus by Bill Lombard is a GeoGebra animation that shows the Wheel of Theodorus.
    - Wheel of Theodorus Art Project Part 1 by Sara Scholes is a YouTube Video that shows examples of the Wheel of Theodorus in nature and shows you how to make it and examples of student work with a corresponding grading rubric. Wheel of Theodorus Calculation Chart by Sara Scholes is the second part of the first video. Students calculate the measures of each of the legs of the right triangles in the wheel to determine the patterns in the triangles.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.SRT.4-5)

Geometric Mean

- Geometric Mean and Right Triangles by Jeffery P. Smith is a GeoGebra applet where students can explore the geometric mean using a right triangle.
- Similar Right Triangle Side Lengths by Math Warehouse has a video, applet, and examples of using the geometric mean to solve problems involving similar right triangle.

Using Similarity Criteria to Solve Problems

- Bank Shot by Illustrative Mathematics is a task where students use similarity to solve a problem in the context of a pool table.
- Tangent Line to Two Circles by Illustrative Mathematics is a task where students use similarity to calculate a side length.
- Points from Directions by Illustrative Mathematics is a task where students use similarity and the Pythagorean Theorem to solve a problem using directions.
- Extensions, Bisections and Dissections in a Rectangle by Illustrative Mathematics is a task where students use an application of similar triangles.
- Folding a Square Into Thirds by Illustrative Mathematics is a task where students apply knowledge about similar triangles to an origami construction.
- How Far Is the Horizon? by Illustrative Mathematics is a modeling task where students use similarity and the Pythagorean Theorem in the context of looking at a horizon in Alaska. Students will need to make reasonable assumptions as part of the modeling process and seek out information for themselves.
- Mirror, Mirror, on the Ground by CPalms is a lesson that applies similar triangles to situations that use the angle of incidence and angle of reflection.
- Similar Triangle Applications by Passy World of Mathematics is a blog that has many examples of similar triangle applications.
- Similar Triangle Applications by Rochester City School District is a worksheet that has many similar triangle application problems.
- Real Life Real World Activity: Forestry Similar Triangles and Trigonometry by Texas Instruments is a lesson where students solve forestry problems using similar triangles and trigonometry ratios.

Special Right Triangles

- Covering the Plane with Rep-Tiles is a lesson by NCTM Illuminations where students use rep-tiles (a geometric figure whose copies can fit together to form a larger similar figure) to create patterns. These can be used to illustrate isosceles right triangles, 30-60-90 triangles, equilateral triangles, parallelograms, trapezoids etc. *NCTM now requires a membership to view their lessons.*

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.SRT.4-5)

Curriculum and Lessons from Other Sources

- EngageNY, Geometry, Module 2, Topic A, Lesson 4: Comparing the Ratio Method with the Parallel Method is a lesson that pertains to this cluster.
- EngageNY, Geometry, Module 2, Topic C, Lesson 18: Similarity and the Angle Bisector Theorem, Lesson 19: Families of Parallel Lines and the Circumference of the Earth, Lesson 20: How Far Away Is the Moon? are lessons that pertain to this cluster.
- EngageNY, Geometry, Module 2, Topic D, Lesson 21: Special Relationships Within Right Triangles—Dividing into Two Similar Sub-Triangles and Lesson 24: Prove the Pythagorean Theorem Using Similarity are lessons that pertain to this cluster.
- Mathematics Vision Project, Geometry, Module 4: Similarity and Right Triangle Trigonometry has many tasks that pertain to this cluster.
- Georgia Standards of Excellence Curriculum Frameworks, Geometry, Unit 2: Similarity, Congruence, and Proofs has several lessons that pertain to this cluster.
- Georgia Standards of Excellence Curriculum Frameworks, Geometry, Unit 3: Right Triangle Trigonometry has several lessons that pertain to this cluster.
- Illustrative Mathematics, Geometry, Unit 3, Lesson 5: Splitting Triangle Sides with Dilation, Part 1, Lesson 6: Connecting Similarity and Transformations, Lesson 11: Splitting Triangle Sides with Dilation, Part 2, Lesson 12: Practice with Proportional Relationships, Lesson 13: Using the Pythagorean Theorem and Similarity, Lesson 14: Proving the Pythagorean Theorem, Lesson 15: Finding All the Unknown Values in Triangles, and Lesson 16: Bank Shot are lessons that pertain to this cluster.

General Resources

- Arizona 7-12 Progression on Geometry is an informational resource for teachers. This cluster is addressed on page 17.
- High School Coherence Map by UnboundED is a map of the relationships among high school mathematics standards. Please note: This resource is aligned to the Common Core not Ohio's Learning Standards. Although the tool is useful, it needs to be checked with respect to Ohio's Learning Standards.
- van Hiele Model of Geometric Thinking is a pdf created by ODE that summarizes the van Hiele levels.
- Not Sure About MCC9-12.GSRT.5 and Others is a blog entry in Bill McCallum's website Mathematical Musings that discusses the standard. Note: Bill McCallum was one of the writers of the Common Core.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.SRT.4-5)

References

- Common Core Standards Writing Team. (2016, March 24). *Progressions for the Common Core State Standards in Mathematics (draft). Grades 7–8, HS, Geometry*. Tucson, AZ: Institute for Mathematics and Education, University of Arizona.
- Snapp, B. & Findell, B. (2016). Parallels in Geometry. Creative Commons. Retrieved from: https://people.math.osu.edu/findell.2/ParallelsInGeometry2016.pdf
- Cirillo, M. & Hummer, J. (April 2019). Addressing misconception in secondary Geometry proof. *Mathematics Teacher*, *112, (6),* 410-417.
- Coxford, A. & Usiskin, Z. (1971). *Geometry: A Transformational Approach*. Laidlaw Brothers Publishers: Riverforest, IL.
- Coxford, A., Usiskin, Z., & Hirschorn, D. (1991). *The University of Chicago Mathematics Project: Geometry, Teacher's Edition.* Scott Foresman and Company, Glenview: IL
- Van De Walle, J., Karp, K., Bay-Williams, J. (2010). *Elementary and Middle School Mathematics* (7th ed.). Boston, MA: Pearson Education, Inc.

STANDARDS	MODEL CURRICULUM (G.SRT.6-8)
Geometry **SIMILARITY, RIGHT TRIANGLES, AND TRIGONOMETRY** **Define trigonometric ratios, and solve problems involving right triangles.** **G.SRT.6** Understand that by similarity, side ratios in right triangles are properties of the angles in the triangle, leading to definitions of trigonometric ratios for acute angles. **G.SRT.7** Explain and use the relationship between the sine and cosine of complementary angles. **G.SRT.8** Solve problems involving right triangles.★ **a.** Use trigonometric ratios and the Pythagorean Theorem to solve right triangles in applied problems if one of the two acute angles and a side length is given. (G, M2)	**Expectations for Learning** In middle school, students draw, construct, and describe geometric figures; use informal arguments to establish facts about similar triangles; and explain a proof of the Pythagorean Theorem and its converse. In this cluster, students use similarity to define trigonometric ratios and then solve problems using right triangles (excluding inverse trigonometric functions). The student understanding of this cluster aligns with van Hiele Level 2 (Informal Deduction/Abstraction). **ESSENTIAL UNDERSTANDINGS** • Because right triangles with the same acute angle are similar, within-figure ratios are equal. Three of these possible ratios are named sine, cosine, and tangent. • The sine of an acute angle is equal to the cosine of its complement and vice versa. • Given an angle and a side length of a right triangle, the triangle can be solved, which means finding the missing sides and angles. **MATHEMATICAL THINKING** • Use accurate mathematical vocabulary to represent geometric relationships. • Recognize, apply, and justify mathematical concepts, terms, and their properties. • Discern and use a pattern or structure. • Plan a solution pathway. • Justify relationships in geometric figures. • Determine reasonableness of results. • Create a drawing and add components as appropriate. • Use technology strategically to deepen understanding. • Solve routine and straightforward problems accurately. • Connect mathematical relationships to real-world encounters. *Continued on next page*

STANDARDS	MODEL CURRICULUM (G.SRT.6-8)
	Expectations for Learning, continued **INSTRUCTIONAL FOCUS** • Define trigonometric ratios for acute angles (sine, cosine, tangent). • Explain and apply the relationship between sine and cosine of complementary angles. • Solve problems involving right triangles (excluding inverses of trigonometric functions). • Use the Pythagorean Theorem to explore exact trigonometric ratios for 30, 45, and 60-degree angles (fluency not required). • Use triangle similarity criteria to define trigonometric ratios. • Given the sine, cosine, or tangent of an angle, find other trigonometric ratios in the triangle. • Solve mathematical and real-world problems given a side and an angle (or the sine, cosine, or tangent of an angle) of a right triangle. **Content Elaborations** **OHIO'S HIGH SCHOOL CRITICAL AREAS OF FOCUS** • Math 2, Number 4, page 8 **CONNECTIONS ACROSS STANDARDS** • Understand similarity (G.SRT.1-3). • Prove and apply theorems involving similarity (G.SRT.4-5). • Apply geometric concepts in modeling situations (G.MG.1-3). • Use coordinates to prove simple geometric theorems algebraically (G.GPE.4). • Write equations in one variable and use them to solve problems (A.CED.1). • Solve quadratic equations in one variable (A.REI.4).

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.SRT.6-8)

Instructional Strategies

Note: The model curriculum is not a collection of lessons nor a full curriculum; it does not suggest pace, sequence, or amount of time spent on topics. It provides information about a topic related to the standards including ideas for examples, strategies for teaching, common misconceptions, and possible connections between topics. In addition, the emphasis of the model curriculum is on instruction not on assessment; therefore, the examples in the Instructional Strategies section are written for instructional purposes not necessarily as assessment items. Examples may need to be rewritten to accommodate the needs of each individual classroom.

In Math 2, students will only be finding missing *side lengths* of triangles or shapes that can be decomposed into triangles through trigonometry and Pythagorean Theorem. In Math 3, students will use trigonometric functions to find missing angles, because inverses of trigonometric functions will not be introduced until Math 3.

Standards for Mathematical Practice

This cluster focuses on but is not limited to the following practices:

MP.1 *Make sense of problems and persevere in solving them.*

MP.4 *Model with mathematics.*

MP.5 *Use appropriate tools strategically.*

MP.6 *Attend to precision.*

MP.7 *Look for and make use of structure.*

Trigonometry comes froom the Greek words *trigonon* meaning triangle and *-metria* meaning measurement. However, the word trigonometry was not always used when describing the concepts behind trigonometry. Originally the concepts behind trigonometry were used in relation to circles and spheres as the concepts were used primarily for studying astronomy. Tangents were treated seperately as they were originally used in relation to studying shadows. It took a long time for these concepts to be combined and then applied to right triangles without a direct connection to circles.

Right triangle trigonometry (a geometry topic) has implications when studying algebra and functions concepts in later math courses. For example, students learn that trigonometric ratios are functions of the size of an angle. Then, the trigonometric functions will be revisited in later courses after radian measure has been studied, and the Pythagorean Theorem will be used to show that $(\sin A)^2 + (\cos A)^2 = 1$.

Students should be proficient in using the trigonometric functions on a calculator.

Some students may incorrectly believe that right triangles must be oriented a particular way. When applying trigonometric ratios to situations, show examples of right triangles in different orientations to confront this misconception.

Note: The Geometry standards no longer include inverse trigonometric functions; however, some schools such as career technical schools may want to include this content.

VAN HIELE CONNECTION

In Math 2 students are expected to be at Level 2 (Informal Deduction/Abstraction) and move towards Level 3 (Deduction).

Level 2 can be characterized by the student doing some or all of the following:

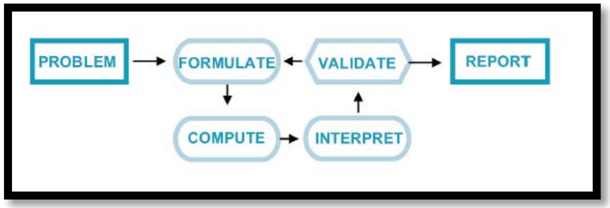
- noticing relationships within and between properties and figures;
- forming meaningful definitions;
- justifying reasoning using informal arguments;
- drawing logical maps and diagrams; and/or
- not yet understanding formal deduction.

Level 3 can be characterized by the student doing some or all of the following:

- constructing proofs and understanding the necessity of proofs;
- understanding the role and relationships between of axioms, theorems, definitions, and postulates;
- differentiating between necessary and sufficient conditions; and/or
- making logical conclusions.

See the van Hiele pdf on ODE's website for more information about van Hiele levels.

MODELING



This cluster is included in the modeling standards. Geometric models that students can build are especially helpful to see patterns. See page 12 for more information about modeling.

DEFINITION OF TRIGONOMETRIC RATIOS

Have students make their own diagrams of a right triangle with labels showing the trigonometric ratios. Make cutouts or drawings of right triangles or manipulate them on a computer screen using dynamic geometric software and ask students to measure side lengths and compute side ratios. Observe that when two or more triangles satisfy the similarity criteria, corresponding side ratios are equal. Although students like mnemonics such as SOHCAHTOA, these are not a substitute for conceptual understanding. As an extension some students may investigate the reciprocals of sine, cosine, and tangent to discover cosecant, secant, and cotangent.

Some students incorrectly believe the trigonometric ratio is the answer when finding the measure of an angle or length of a side. Instead, the ratio shows the relationship between the sides' lengths and the measure of the specified angle in the right triangle. They are also functions of angles which students will discover in Math 3.

EXAMPLE

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.SRT.6-8)

a. Draw 5 different right triangles with an angle of $60°$. Label the $60°$ angle θ. Measure the lengths of the corresponding sides of each triangle. What do you notice?
b. Draw 5 different right triangles with an angle of $45°$. Label the $45°$ angle θ. Measure the lengths of the corresponding sides of each triangle. What do you notice?
c. Draw 5 different right triangles with an angle of $15°$. Label the $15°$ angle θ. Measure the lengths of the corresponding sides of each triangle. What do you notice?
d. Draw 5 different right triangles with an angle of $20°$. Label the $20°$ angle θ Measure the lengths of the corresponding sides of each triangle. What do you notice?
e. Given a specified angle measure for a right triangle, what do you know about all triangles that are similar to the given triangle? Explain.

Discussion: Students should understand that any acute angle in the right triangle has several fixed ratios (trigonometric ratios) related to that specified angle. This is true because based on AA-Similarity Criteria of Triangles (or A-Similarity Criteria for Right Triangles) the triangles are similar; therefore, sides are proportional. It may help some students to demonstrate this using dynamic geometric software.

TIP! Right Triangle Similarity stems for similarity criteria of general triangles. For example, A-Similarity Criteria of Right Triangles stems from AA-Similarity Criteria of Triangles, since in a right triangle, one angle is already known; HL stems from SSS.

Students often confuse the sine or cosine of an angle as an angle measure instead of a ratio of side lengths for a particular angle.

Sine and Cosine

Although the emphasis of trigonometric study in Math 2 is not on the unit circle, some informal workings with circles and the unit circle (where the radius is 1) can lay the foundation for the conceptual understanding of sine beyond simply memorizing the mnemonic SOHCAHTOA, which has no meaning for students. Students can apply concepts of similar triangles to compare any right triangle to a similar triangle with the hypotenuse of 1.

Sine comes from the word "half-chord" or "bowstring."
A sine is—

- A half-chord on a circle;
- Part of a right triangle when the radius (or hypotenuse) is 1; and
- A ratio comparing the lengths of 2 sides of a right triangle.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.SRT.6-8)

Avoid using the notation "$\sin x$" until more advanced courses since it can lead students to think that "$\sin x$" is short for "sin $\cdot\, x$" and incorrectly divide out the variable $\frac{\sin x}{x} = sin$. Instead at this level, refer to the sine of an angle as ($\sin \angle A$) or sine of an angle measure ($\sin \theta$). *Note the parenthesis are used for clarity instead of representing a function.*

EXAMPLE

Note: The word "sine" will be used throughout this example in place of the abbreviation "$sin\ \theta$." The abbreviation will be introduced after students comprehend that $sin\ \theta$ is the sine-to-radius ratio of a right triangle whose central angle is θ which can be described as $\frac{length\ of\ opposite\ side\ of\ \theta}{length\ of\ hypotenuse}$. The abbreviation of $sin\ \theta$ will be introduced at the end of this example.

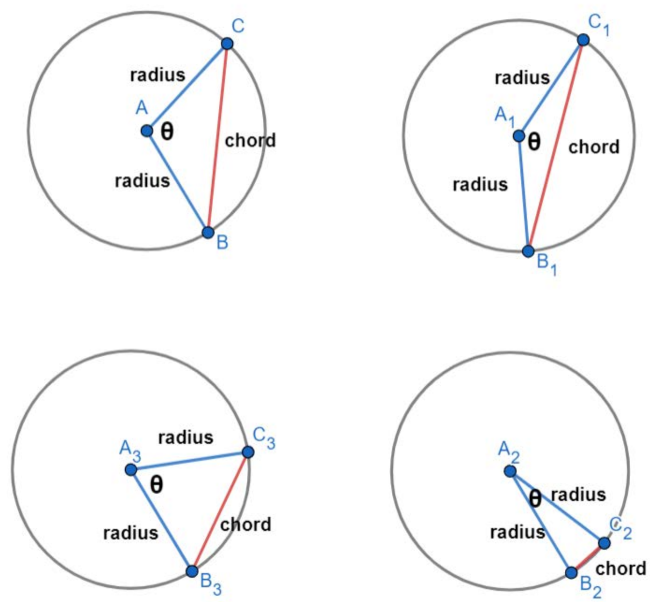
Part 1

a. Use geometric software or a compass and straight edge to construct several triangles inscribed in circles where two radii of the circle are the sides of the triangle.

b. Label the sides of the triangle according to its relationship with the circle.

c. What kind of angles are present in each of the triangles?

d. What is different about each of the triangles? What is the same about each of the triangles?

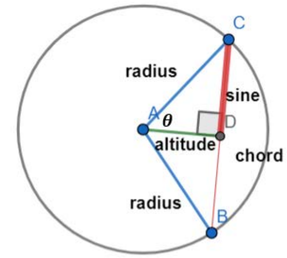


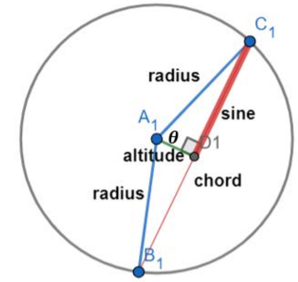
Discussion: Students should be able to label the radius and chords of each circle and realize that they are also parts of the sides of a triangle. Students should also notice that all the triangles contain a central angle. They should realize that all the triangles are isosceles because the radii create two congruent sides, but the central angle measures may be different than the base angle measures in each triangle.

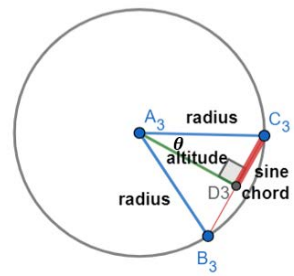
INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.SRT.6-8)

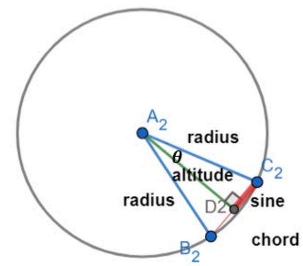
Part 2

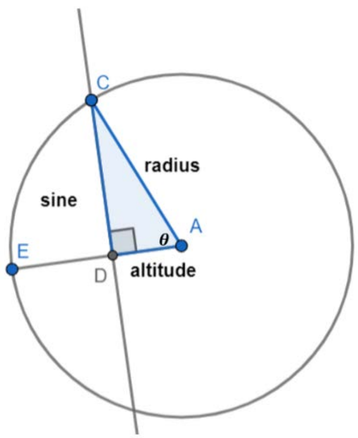
a. Draw or create a circle and a chord.
b. Create a triangle by connecting the endpoints of the chord to the center of the circle using the radii of the circle. Make the length of the radii 1 unit.
c. Draw an altitude from the center of the circle to the chord and label the altitude.
d. What kind(s) of triangles are formed? How do you know?
e. Sine is also known as a half-chord (or more accurately the sine of a central angle θ is half the chord of twice the angle or 2θ). Label sine and theta (θ) in each one of your triangles.
f. How are the θs and the central angles in Part 1 related?
g. Notice that any half-chord (sine) and the radius (which is the endpoint of the half-chord on the circle), and the altitude (from the center to the half-chord) form a right triangle. What part of the right triangle is the radius? (If using geometric software, drag the intersection point of the chord and the radius that is located on the circle to illustrate the concept.)











Discussion: Draw attention to the fact that the sine of the central angle θ is half the chord. When the chord is not cut in half, the central angle intercepting the chord is 2θ. Both θ and 2θ are central angles. Students should conclude that the radius is the hypotenuse of the right triangle. Using half-chords instead of chords forces the triangles to be right triangles.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.SRT.6-8)

Part 3

a. Create a right triangle in a circle with the same central angle θ that you used in Part 2 where the radius (which is also the hypotenuse of your triangle) is 1. Label the half-chord sine and the radius 1 (Figure 1).

b. Create another right triangle in a circle where the radius of your choosing is the hypotenuse of your triangle. Label the half-chord *x* and the radius *r* (Figure 2).
- How does your circle compare to the circle in part **a.**? Explain.
- How does your triangle compare to the triangle in part **a.**? Explain.

c. Sine θ can be thought of a sine-to-radius ratio of a right triangle whose central angle is θ. Write a proportion comparing the sine-to-radius ratios of both triangles.

d. Why is the half-chord only needed to express sine, when the radius is 1?

e. Why do we need both the half-chord and the radius to express the ratio, when the radius $\neq 1$?

f. Find sine θ for the triangle in Figure 3.

g. How does that relate to a similar triangle inscribed in a circle with radius of 1? Explain. See Figure 4.

Figure 1

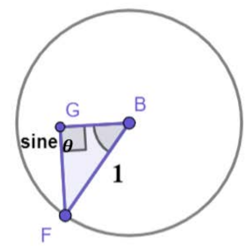


Figure 2

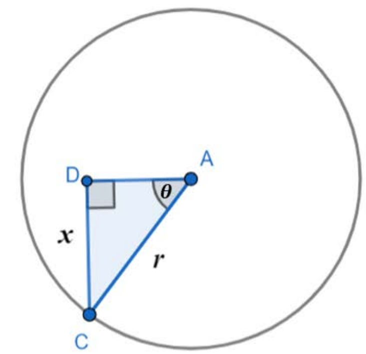
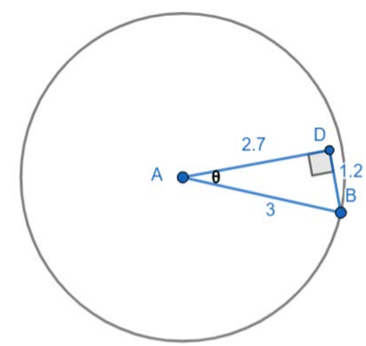
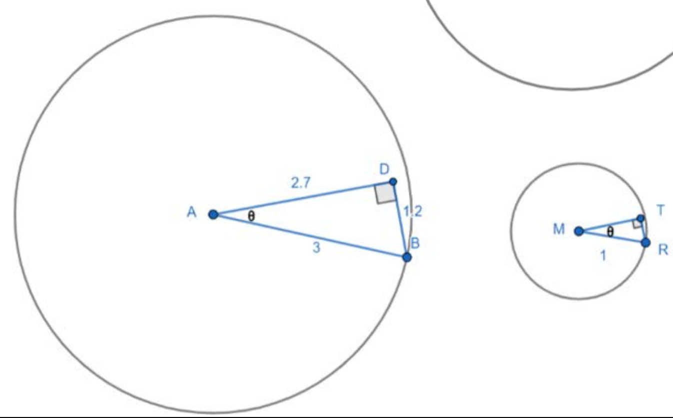


Figure 3



Discussion: Students should realize that both circles are similar since all circles are similar (G.C.1) and that the triangles are similar by AA-Triangle Similarity Criteria. Students should be able to write the proportion $sine\ of\ the\ central\ angle\ \theta = \frac{sine}{1} = \frac{x}{r}$, which can be rewritten as $sine\ \theta = \frac{x}{r}$. Since $\frac{sine}{1} = sine$, the sine of the central angle θ is the length of the half-chord when the radius equals 1. When the radius does not equal 1, the sine-to-radius ratio needs to be described by both the half-chord and the radius. However, when the ratio is simplified it describes the half-chord of a similar circle (and triangle) with a radius of 1. Students should recognize that the sine-to-radius ratio in Figure 3 equals the half-chord (sine) in Figure 4 because the radius is 1 and the figures are similar. Therefore, all sine ratios can be thought of as a description of the half-chord in a circle with a radius of 1.

Figure 4



INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.SRT.6-8)

Part 4

a. Consider two circles with the same radius and triangles formed by central angles and half chords (Figure 1).
b. What do you notice about both triangles? Explain.
c. What do you notice about the central angles θ and α? (Figure 2)
d. Write a statement describing the relationship between the altitude and the sine of both triangles.
e. A complementary angle's sine is referred to as the cosine. Therefore, what is another name for the altitude? Label cosine on your diagrams.
f. Fill in the blanks:
- $sine\ \theta = cosine$ ____ and $sine\ \alpha = cosine$ ____.
- $sine$ and $cosine$ of ___________ angles are _____.

g. In a right triangle with a hypotenuse (radius) of 1, what determines which side is the sine and which side is the cosine?

Discussion: Students should recognize both triangles are congruent because the radii are equal, and the two angles are congruent. They are transformations of one another (a rotation and a translation). The central angle of $\triangle ADC$ is the complement to the central angle of $\triangle JIH$. The reflection also causes the sines and the altitudes to interchange. $The\ sine\ of\ \triangle ADC = the\ altitude\ of\ \ \triangle JIH$ and vice versa (since the radius is 1). Students should note that the cosine is the altitude when the radius is 1. Therefore $the\ sine\ of\ \triangle ADC = the\ cosine\ of\ \ \triangle JIH$ or $sine\ \theta = cosine\ \alpha$ and $sine\ \alpha = cosine\ \theta$. Therefore, the sine and cosine of complementary angles are equal. The students should understand that position relative to the central angle determines which side of the triangle is sine and which is cosine. The sine is the half-chord, which is opposite from the central angle of the circle and the cosine is the altitude, which is adjacent to the central angle. Ask students how they would know which is which if the circle is removed. That discussion will help them transition to Part 5.

Figure 1:

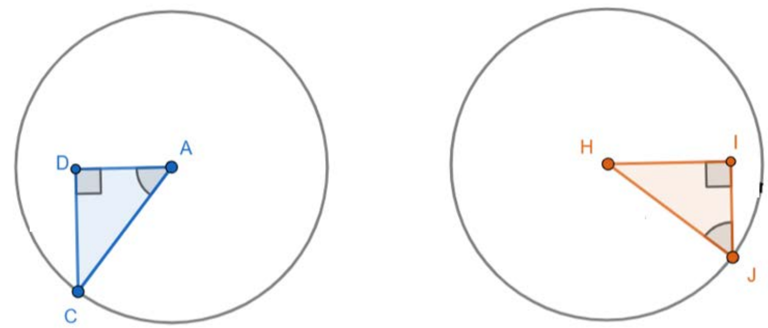


Figure 2:

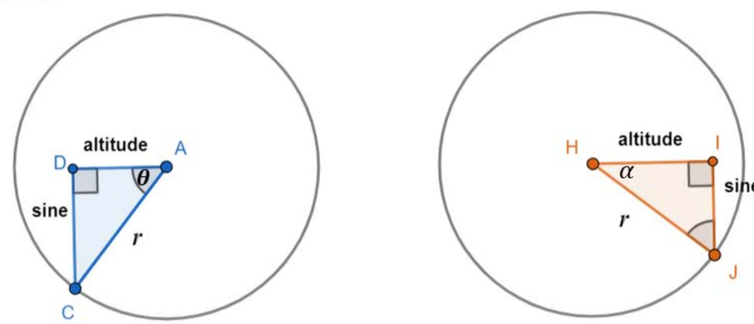
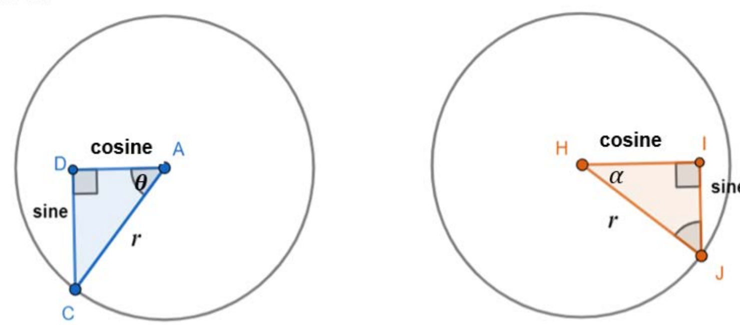


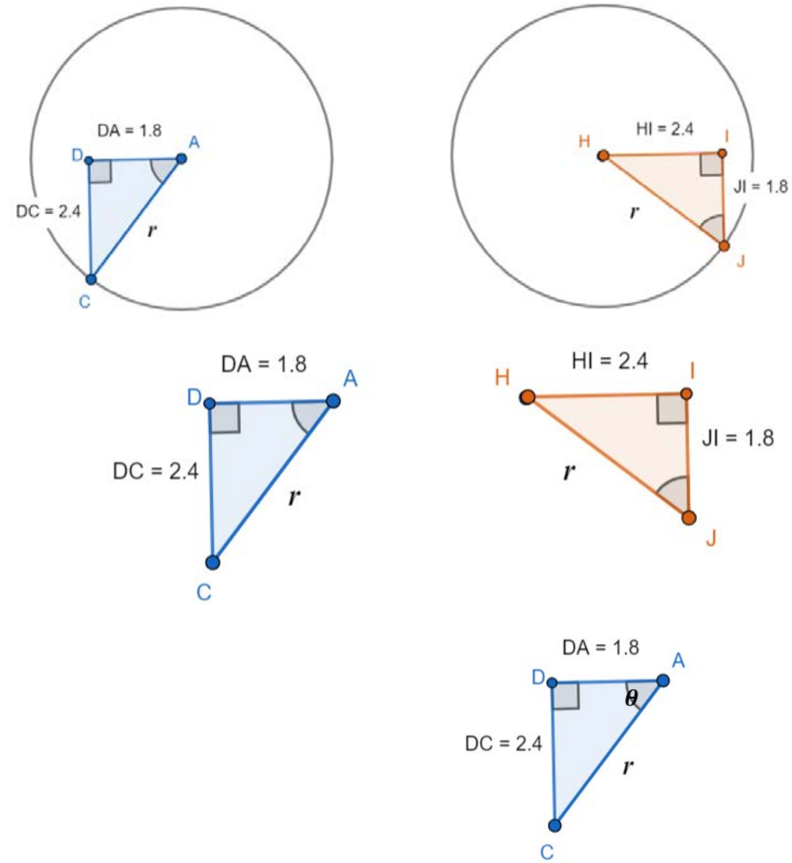
Figure 3:



INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.SRT.6-8)

Part 5

a. Now if you remove the circles, how do you determine which angle is the central angle and which side length is the half-chord (sine)?
b. Look back at the position of the sine (half-chord) in relation to the central angle in each of the triangles. What do you notice?
c. When the circle is removed, what is the name for the radius in relation to the rest of the right triangle?
d. Use words to describe the sine-to-radius ratio for the angle θ that can be consistently seen when right triangles are not part of the radius and half-chord inside a circle.
e. The cosine of an angle is the sine of its complementary angle. So what words can be used to describe the cosine of θ?
f. Write the sine and cosine ratio of the angle θ for the triangle in Figure 1.
g. What is the sine and cosine of the other acute angle in the triangle (Figure 2).



Discussion: Once the circles are removed it may be difficult to determine which angle is the central angle θ and which side length is the half-chord (sine) of the right triangle. As one can see either of the acute angles can be θ depending on the position of the triangle in the relationship to the circle. Therefore, when triangles are not inside circles, it makes more sense to talk about the ratios of the side lengths in relation to θ. Since, sine (half-chord) is always the side opposite of the θ (central angle), hopefully students will come up with the idea that the sine θ can be described as $\frac{opposite}{hypotenuse}$. Then students may use the words $\frac{side\ next\ to\ the\ angle}{hypotenuse}$ or $\frac{beside}{hypotenuse}$ or $\frac{adjacent}{hypotenuse}$ to describe cosine θ, which is the same as sine of the angle that is the complementary angle of α. In Figure 1 $sine\ \theta = \frac{3}{\sqrt{34}}$ and $cosine\ \theta = \frac{5}{\sqrt{34}}$, and in Figure 2 $sine\ \alpha = \frac{5}{\sqrt{34}}$ and $cosine\ \alpha = \frac{3}{\sqrt{34}}$. Students should note that the sine of one angle is the cosine of the other and vice versa. Eventually students can move towards abbreviating sine and cosine.

Figure A:

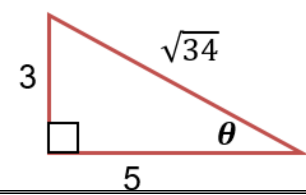
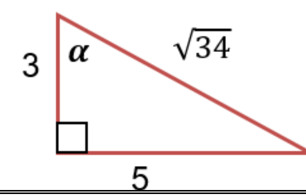


Figure B:



INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.SRT.6-8)

Understanding Trigonometric Notation

It is just a shorthand!

The sine-to-radius ratio of a right triangle whose central angle is θ is $\frac{length\ of\ opposite\ side\ from\ \theta}{length\ of\ hypotenuse}$.

~~The~~ sine-to-radius ratio ~~of a right triangle whose central angle is~~ θ is $\frac{\cancel{length\ of}\ opposite\ side\ from\ \theta}{\cancel{length\ of}\ hypotenuse}$.

sine-~~to-radius ratio~~ of θ is $\frac{opposite\ side\ from\ \theta}{hypotenuse}$

sine ~~of~~ θ = $\frac{opposite\ \cancel{side\ from\ \theta}}{hypotenuse}$

sin θ = $\frac{opposite}{hypotenuse}$

In the past, trigonometric tables listed only sines of angles. To find the cosine of an angle, one would have to look up the sine of its complementary angle. Investigate sines and cosines of complementary angles, and guide students to discover that they are equal to one another. Trigonometry tables can still be used to emphasize the complementary nature of sine and cosine.

EXAMPLE

Use a trigonometric table to list the sine of each of the following angle measures:

- 30°
- 60°
- 20°
- 70°
- 10°
- 80°
- 25°
- 65°
- 15°
- 75°

Now list the cosine of each angle measure. What do you notice?

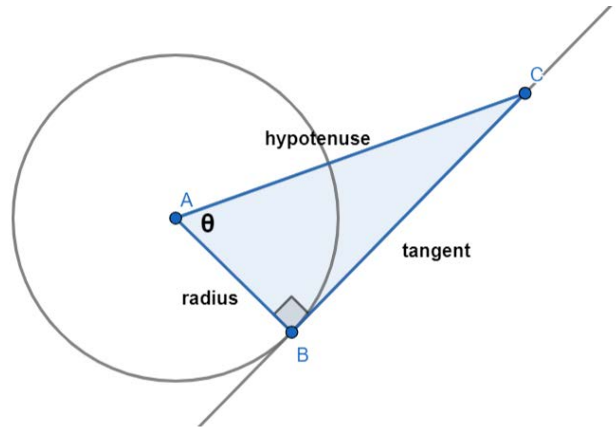
Tangent

Tangent means "touching line." Students should be able to relate the tangent of a circle to the tangent ratio in a right triangle. The concepts of trigonometric ratios as lines in a unit circle will be extended to the reciprocal ratios secant, cosecant, and cotangent in later courses.

EXAMPLE

Part 1

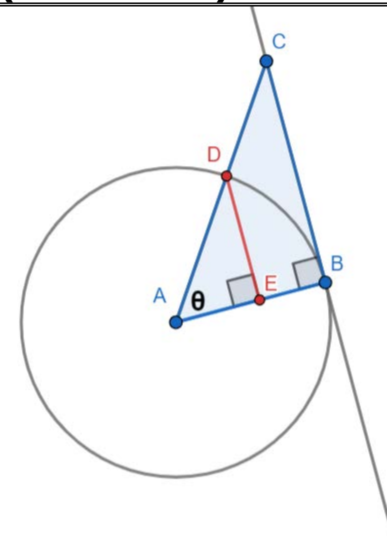
- **a.** What do you know about tangent lines?
- **b.** Draw a circle (or create one using dynamic geometric software) where the radius intersects the tangent line. Then use the radius and the tangent (in addition to a third line) to create a right triangle. Label the tangent line, radius, and the hypotenuse.
- **c.** There is also a tangent-to-radius ratio of a right triangle whose central angle is θ. Measure your triangle and write the tangent-to-radius ratio.
- **d.** Now draw a triangle and circle that is geometrically similar to what you drew in part **b.** This time fix the radius at 1. Write a proportion comparing the two tangent-to-radius ratios.
- **e.** Why can the word *tangent* be both used to describe the tangent-to-radius ratio and the segment tangent to the radius of the circle when the radius equals 1?
- **f.** If the radius $\neq 1$, what is needed to describe tangent θ?



Discussion: Students should recall the definition of a tangent line: a line which intersects a circle at exactly one point. They should also recall the theorem that if a line is tangent to a circle, then it is perpendicular to the radius drawn to the point of tangency. Students should recognize that the tangent-to-radius ratio equals the length of the tangent when the radius is 1 because $\frac{length\ of\ tangent\ segment}{1} = \frac{length\ of\ tangent\ segments}{length\ of\ radius}$ or $length\ of\ tangent = \frac{length\ of\ tangent\ segments}{length\ of\ radius}$. Therefore, all tangent ratios can be thought of as a description of the ratio of the length of tangent line segment to the length of the radius of a circle when the radius is 1. Make connections between this example and the sine-to-radius ratios and the complement of sine-to-radius ratio.

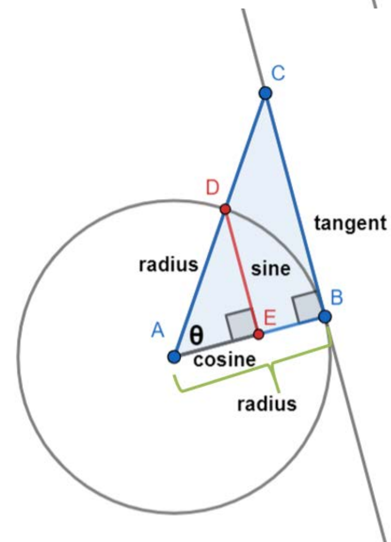
INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.SRT.6-8)

Part 2



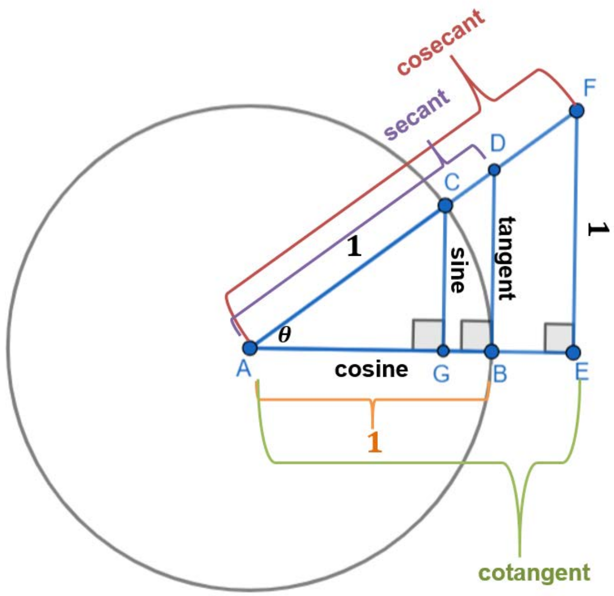
- **a.** In the diagram circle A has a radius of 1. Label the tangent and radius.
- **b.** What is the relationship between $\triangle ABC$ and $\triangle AED$? Explain.
- **c.** What implication does your answer in part **b.** have on the side length of the triangles?
- **d.** Use the tangent-to-radius ratio to write a proportion that is true for the two triangles.
- **e.** What other geometric term is $\overline{DE}$ known by?
- **f.** What other geometric term is $\overline{AE}$ known by?
- **g.** Fill in the blank using the terms for $\overline{DE}$ and $\overline{AE}$ in parts **e.** and **f.** $\frac{tangent}{radius} = \underline{\qquad\qquad}$
- **h.** Since the radius of the circle is 1, how can you define tangent?
- **i.** Instead of defining tangent as $\frac{sin\theta}{cos\theta}$, could you define it in terms of the position of the right triangle's legs from the central angle? Does it hold true for both $\triangle ABC$ and $\triangle AED$?
- **j.** How will your definition in part **i.** help you define tangent when the right triangle is not associated with a circle?
- **k.** Label the diagram using the words tangent, radius, sine, and cosine.

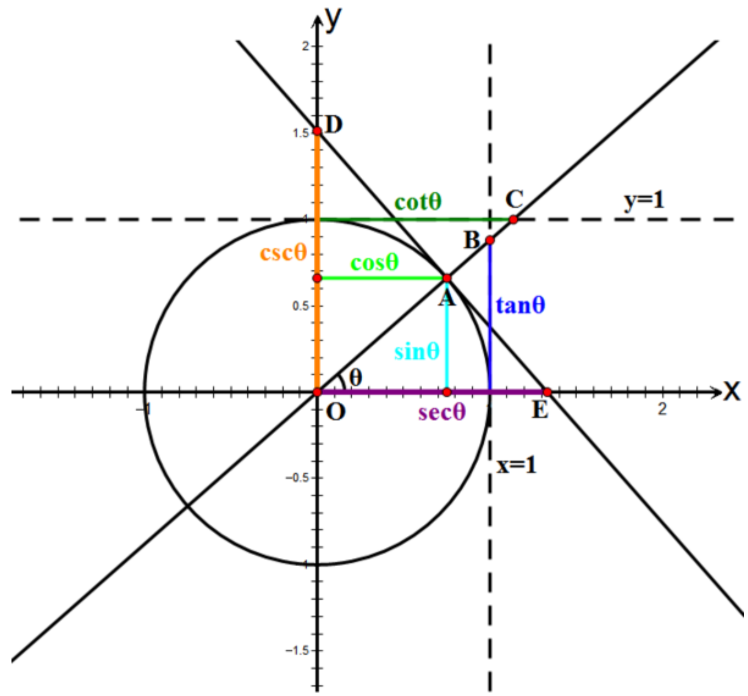
Discussion: Students should recognize that $\triangle ABC$ and $\triangle AED$ are similar because of AA-Triangle Similarity Criteria, so the side lengths are proportional. Students should write the proportion $\frac{tangent}{radius} = \frac{\overline{CB}}{\overline{AB}} = \frac{\overline{DE}}{\overline{AE}}$. Since $\overline{DE}$ is the half-chord sine, and $\overline{AE}$ is cosine, $\frac{tangent}{radius} = \frac{sine}{cosine} = \frac{sin\theta}{cos\theta}$, and since the radius is 1, $tangent = \frac{sin\theta}{cos\theta}$. The tangent-to-radius ratio can also be defined as $\frac{opposite}{adjacent}$, which can be useful when a right triangle is not connected to the circle. This triangle can be extended by adding lines for cosecant, secant, and cotangent for more advanced students (See diagram on the left of the next page).

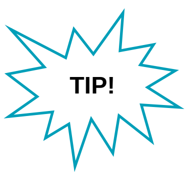


INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.SRT.6-8)

On the left shows a diagram that represents sine, cosine, and tangent with the addition of secant, cosecant, and cotangent following from the definition of sine as the half chord. The diagram on the right is an alternative diagram.







Historically, tangents were not part of trigonometry. Whereas sines and cosines focused on trigonometry and astronomy, tangents dealt with shadows. Triangle trigonometry began by determining the length of a shadow cast by a vertical stick, called a shadow stick or gnomon, given the angle of the sun because the line length was tangent, the function was known as tangent. An example of this is a sundial.

EXAMPLE

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.SRT.6-8)

Draw or create at least two right triangles that meet the criteria for each of the following. Label their sides. Approximations are acceptable. You may use a calculator, and it may be helpful to use dynamic geometric software to create the triangles.

- **a.** sin 30°
- **b.** tan 45°
- **c.** sin 20°
- **d.** cos 40°
- **e.** sin 70°
- **f.** tan 60°

Discussion: The idea behind this example is to emphasize the ratio nature of sine, cosine, and tangent. Parts **a.** and **b.** are fairly easy as students can create any triangle whose side lengths have a ratio of $\frac{1}{2}$ for the opposite to hypotenuse (half-chord to radius) ratio in part **a.,** and whose side lengths have a ratio of 1 for the opposite to adjacent (sine to cosine) in part **b.** Parts **c.-f.** are more difficult. For part **c**. students can use their calculator to find the sine of 20 which is approximately 0.34. (It may be helpful to have students round to the nearest hundredth for this activity). Then they could draw a triangle with a leg of 17 and a hypotenuse of 50 or a leg of 0.34 and a hypotenuse of 1 or a leg of 8.5 and a hypotenuse of 25. *Note: This is also laying foundational understanding for advanced math courses.* Another approach could be to have the students build right triangles with a fixed acute angle using geometric software and have them manipulate the side lengths. If dynamic geometric software is used, make sure students see the equivalent ratios of the two sides. If the program allows it, students may want to measure their side lengths in thousandths or hundred thousandths to see the ratios.

GENERAL INFORMATION ABOUT TRIGONOMETRIC RATIOS

If one of the acute angles in a right triangle is known, then all right triangles with that acute angle are similar to one another. Because these triangles are similar to each other, their within-figure ratios are equal within this set of triangles. Students may discover that there are six possible ratios for comparing pairs of sides in these triangles. The emphasis in this course is sine, cosine, and tangent. Cosecant, secant, and cotangent will be explored in Math 3 but can be used as an extension during this course.

EXAMPLE

How many trigonometric ratios can there be in a right triangle? Explain.

Discussion: Students need to get to the idea that three sides can only be paired three different ways. If students differentiate the trigonometric ratios from their reciprocals, then they may find six different ratios. This can be tied into the fundamental counting principle of probability where $3 \cdot 2 = 6$.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.SRT.6-8)

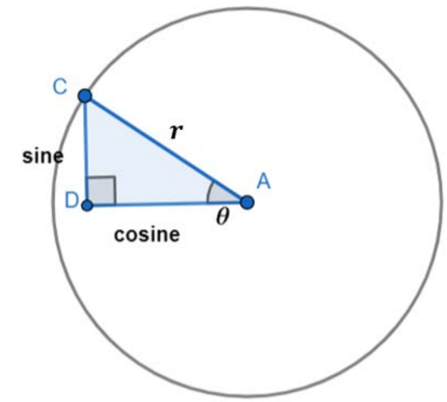
Have students explore using dynamic geometric software whether sine, cosine, and tangent have maximum values.

Observe that, as the size of the acute angle increases, sines and tangents increase while cosines decrease. Sines and cosines also have maximum values, whereas tangents do not.

EXAMPLE

Using geometric software, create a right triangle (with a hypotenuse of 1) inside a circle where the radius is the hypotenuse. Label the sine and cosine.

Figure 1



Part 1

Drag the intersection point of sine and the radius (point C in Figure 1) around the circle to change the size of θ.

- **a.** As θ increases, what happens to sine?
- **b.** As θ decreases, what happens to sine?
- **c.** Does sine have a maximum and/or minimum value? Explain?
- **d.** As θ increases, what happens to cosine?
- **e.** As θ decreases, what happens to cosine?
- **f.** Does cosine have a maximum and/or minimum value? Explain?
- **g.** How does sine behave in relation to cosine? Explain.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.SRT.6-8)

Part 2

Use geometric software, create a right triangle as shown. Drag the intersection point of the hypotenuse and tangent (point C in Figure 2) along the tangent line to change the size of θ.

- **a.** As θ increases, what happens to tangent?
- **b.** As θ decreases, what happens to tangent?
- **c.** Does tangent have a maximum and/or minimum value? Explain.

Figure 2

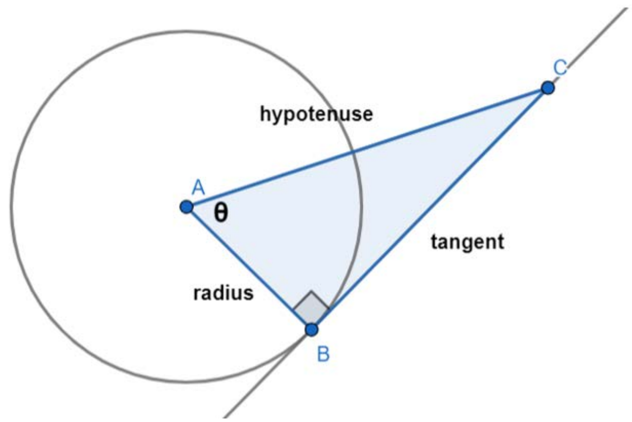
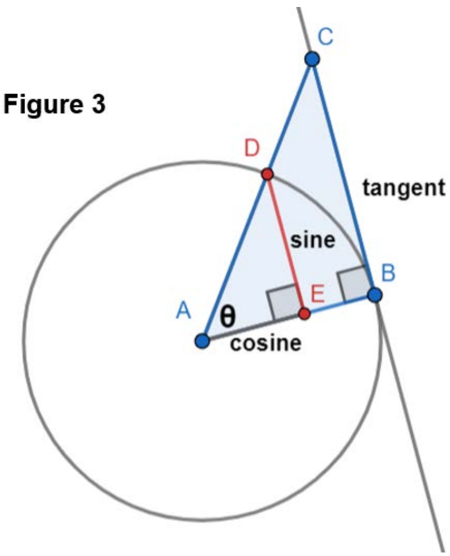


Figure 3



Part 3

Using geometric software, create two right triangles as shown. Drag the intersection point of the hypotenuse and tangent (point C in Figure 3) along the tangent line to change the size of θ.

- **a.** What happens to sine as tangent increases? Explain.
- **b.** What happens to sine as tangent decreases? Explain.
- **c.** What happens to cosine as the tangent increases? Explain.
- **d.** What happens to cosine as the tangent decreases? Explain.
- **e.** How do sine, cosine, and tangent relate to each other?

TIP!

Some students do not realize that opposite and adjacent sides need to be identified with respect to a particular acute angle in a right triangle.

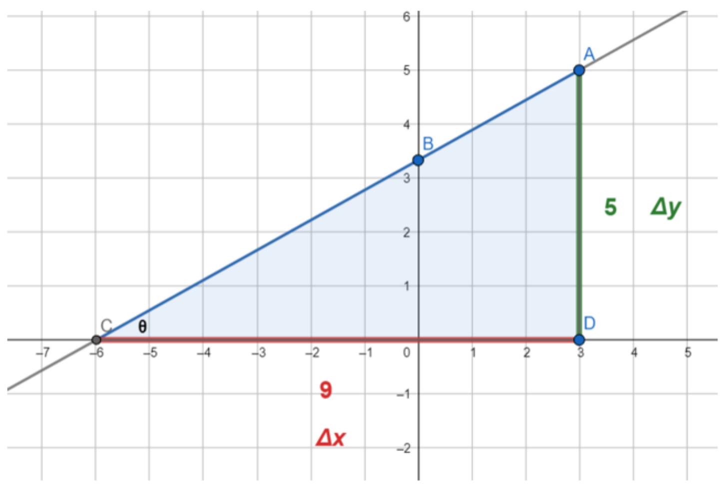
Some students incorrectly believe that the trigonometric ratios defined in this cluster apply to all triangles, but they are only defined for acute angles in right triangles.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.SRT.6-8)

Give opportunities for students to make connections between slope of a line and the tangent of the angle between the line and the horizontal axis.

EXAMPLE

- **a.** Draw a line on a coordinate plane.
- **b.** Find the slope of the line.
- **c.** Create a right triangle using your line as the hypotenuse and the horizontal axis and a vertical line as the two legs.
- **d.** Find the sine, cosine, and tangent of the acute angle created by the horizontal axis and the hypotenuse.
- **e.** What conclusions can you make? Justify your thinking.
- **f.** Compare your triangle to the triangles of your classmates. Does your conclusion generalize for any other examples? Explain.



$$\sin\theta = \frac{5}{\sqrt{106}} = \frac{5\sqrt{106}}{106}$$

$$\cos\theta = \frac{9}{\sqrt{106}} = \frac{9\sqrt{106}}{106}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{(5\sqrt{106})/106}{(9\sqrt{106})/106} = \frac{5}{9}$$

or

$$\tan\theta = \frac{o}{a} = \frac{5}{9}$$

$$m = \frac{\Delta y}{\Delta x} = \frac{5}{9}$$

Discussion: Students should make the connection that the tangent ratio $\left(\frac{sin\theta}{cos\theta}\right)$ is the same as the slope of the line, because the sine is the same as the vertical line of the slope triangle (Δy) and the cosine is the same as the horizontal line of the slope triangle (Δx).

SOLVING PROBLEMS USING RIGHT TRIANGLES

Special Right Triangles

In Math 2, students should be able to determine the trigonometric values of special right triangles; fluency is not the priority at this level. In Math 3, however, students will develop fluency of special right triangles such as 30°-60°-90° or 45°-45°-90°using the unit circle.

Have students apply the concept of similar triangles and the Pythagorean Theorem to find properties of special triangles such as 30°-60°-90° and 45°-45°-90°. It may be beneficial to expose more advanced students to rationalizing the denominator of simple radical expression (with a single term in the denominator).

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.SRT.6-8)

EXAMPLE

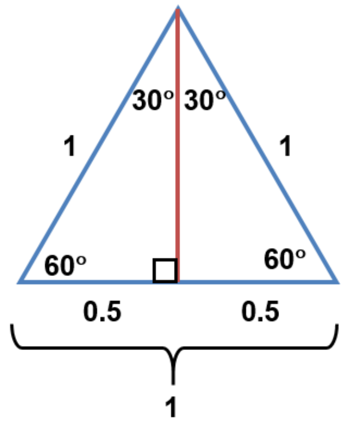
Part 1

a. Create an equilateral triangle inside a circle where the radius is a side of the triangle whose length is 1. Label the angle measures and side lengths.

b. Draw an altitude from one of the vertices to create two right triangles. Label the angle measures and side lengths of the right triangle that are apparent.

c. Find the length of the altitude. Explain what you did.

d. Use your knowledge of similar triangles to fill in the chart showing the relationships between sides of a 30°-60°-90° triangle. In the chart, ℓ refers to the length of the shorter leg, L refers to the length of the longer leg, and H refers to the length of the hypotenuse. Leave numbers in radical form.



ℓ	L	H
		1
1		
	1	
		H
ℓ		
	L	

e. Choose how you prefer to define the relationship of the sides of a 30°-60°-90° triangle (ℓ, L, or H). Draw a triangle to illustrate the relationship using your chosen preference. Then write the relationship in words using the words hypotenuse, shorter leg, and longer leg.

f. Draw and label the side lengths of the three similar 30°-60°-90° triangles using your preferred relationship.

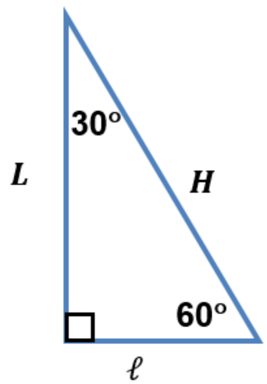
g. Solve the following problems using your preferred relationship for 30°-60°-90° triangles.
- If the length of the hypotenuse is 10 find the lengths of the other two legs.
- If the length of the short leg is 4 find the length of the hypotenuse and the long leg.
- If the length of the long leg is 9 find the length of the short leg and the hypotenuse.

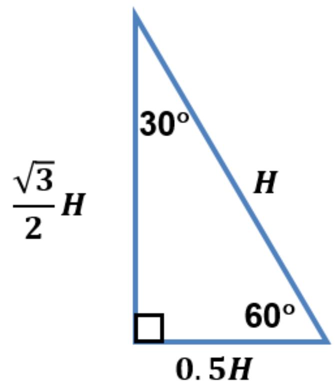
Discussion: In part **a.**, you may wish to remove the triangle from the circle and just have the students draw an equilateral triangle with side lengths of 1. If the triangle is presented in a circle, it reinforces the relationships between circles and triangles and builds informal foundations of the unit circle. Students may define the relationships in terms of ℓ, L, or H. There are different advantages to the students' choice. Typically, most Geometry textbooks define the relationship in terms of the short leg ℓ since the numbers are easier, especially if an equilateral triangle with a side of 2 is used to illustrate the relationship. However, the advantage of using H to define the relationship is that it will carry over to understanding of the unit circle in later courses. It is important for a teacher to describe the relationships using the words hypotenuse, shorter leg, and longer leg. For example, the length of the longer leg is $\sqrt{3}$ times that of the shorter leg and the length of the hypotenuse is twice the length of the shorter leg

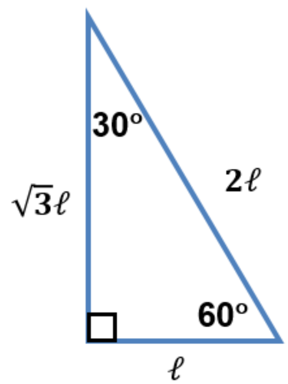
INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.SRT.6-8)

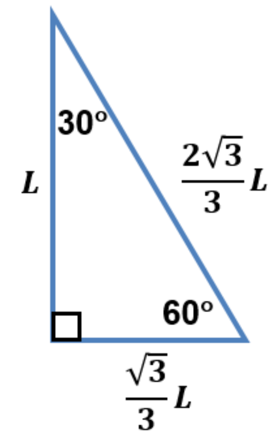
or the length of the shorter leg is half the length of the hypotenuse and the length of the longer leg is $\frac{\sqrt{3}}{2}$ times the length of the hypotenuse. (You may need to draw attention to the fact that $\frac{\sqrt{3}}{2}$ is less than one

ℓ	L	H
0.5	$\frac{\sqrt{3}}{2}$	1
1	$\sqrt{3}$	2
$\frac{1}{\sqrt{3}}$ or $\frac{\sqrt{3}}{3}$	1	$\frac{2}{\sqrt{3}}$ or $\frac{2\sqrt{3}}{3}$
$0.5H$	$\frac{\sqrt{3}}{2}H$	H
ℓ	$\sqrt{3}\ell$	2ℓ
$\frac{1}{\sqrt{3}}L$ or $\frac{\sqrt{3}}{3}L$	L	$\frac{2}{\sqrt{3}}L$ or $\frac{2\sqrt{3}}{3}L$









Students who memorize the side length relationships of 30°-60°-90° triangles as $1:2:\sqrt{3}$ oftentimes incorrectly interpret $\sqrt{3}$ as the hypotenuse. It may be best to avoid this memory trick or draw attention to the fact that the value of $\sqrt{3}$ is less than 2, and therefore cannot be the hypotenuse which is always the longest side of the triangle.

Part 2

Using your preferred relationship for the side lengths of 30°-60°-90° triangles find the following. Leave values in radical form:

- **a.** $\sin 30°$
- **b.** $\cos 30°$
- **c.** $\tan 30°$
- **d.** $\sin 60°$
- **e.** $\cos 60°$
- **f.** $\tan 60°$

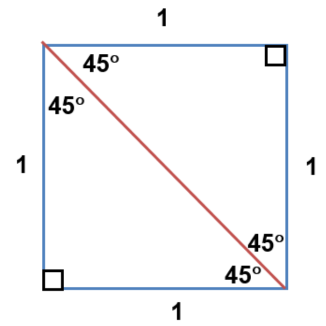
Discussion: Connect the 30°-60°-90° triangle relationships to the ratios needed for the unit circle in Math 3. This will not only lay some foundational understanding but also reinforce the finding of trigonometric ratios.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.SRT.6-8)

EXAMPLE

Part 1

a. Create a square with side lengths of 1. Draw a diagonal in the square creating two right triangles. Label the angle measures and the side of the triangle.
b. What kind of triangle is created? Explain why.
c. Find the missing side length in the triangle. Explain how you found it.
d. Use your knowledge of similar triangles to fill in the chart showing the relationships between sides of a 45°-45°-90° triangle. In the chart, ℓ refers to length of the legs, and H refers to the length of the hypotenuse. Leave numbers in radical form.
e. Choose how you prefer to define the relationship of the sides of a 45°-45°-90° (ℓ or H). Draw a triangle to illustrate the relationship using your chosen preference. Then write the relationship in words using the words hypotenuse and leg.
f. Draw and label the side length of three similar 45°-45°-90° triangles using your preferred relationship.
h. Solve the following problems using your preferred relationship for 45°-45°-90° triangles.
- If the length of the hypotenuse is 10 find the lengths of the other two legs.
- If the length of a leg is 4 find the length of the hypotenuse.



ℓ	H
	1
1	
	H
ℓ	

Discussion: Students may define the relationships in terms of ℓ or H. It is important for a teacher to describe the relationships using words such as hypotenuse and leg. (In this case since it is an isosceles triangle there is not a shorter or longer leg.) For example, the length of the hypotenuse is $\sqrt{2}$ times the length of the leg and the length of the leg is $\frac{\sqrt{2}}{2}$ times the length of the hypotenuse.

Part 2 (Extension)

Using your preferred relationship for the side lengths of the 45°-45°-90° triangles find the following. Leave values in radical form.

a. $\sin 45°$
b. $\cos 45°$
c. $\tan 45°$

Discussion: Connect the 45°-45°-90° triangle relationships to the ratios needed for the unit circle in Math 3. This will not only lay some foundational understanding but also reinforce the finding of trigonometric ratios.

Pythagorean Triples

Pythagorean triples (and their families) should be explored in connection with similar triangles to show that for example, the sine of an acute angle in a 3:4:5 triangle is equal to the sine of the corresponding angle in a 6:8:10 triangle.

Connecting the Pythagorean Theorem to Trigonometry

EXAMPLE

Find $\tan\theta$, given $\sin\theta = \frac{1}{3}$.

Discussion: Students could draw a right triangle labeling the opposite side of θ as 1and the hypotenuse as 3. Then they could use the Pythagorean Theorem to solve for the missing side length:

$$1^2 + b^2 = 3^2$$
$$1 + b^2 = 9$$
$$b^2 = 8$$
$$b = \sqrt{8} \approx 2.828$$

Thus $\tan\theta = \frac{1}{\sqrt{8}}$ or $\frac{\sqrt{8}}{8}$.

Application Problems

Right triangle trigonometry is one of the most applicable areas of mathematics. Give students the opportunity to solve many problems where they can apply trigonometric ratios in a real-world context. Use cooperative learning in small groups for discovery activities and outdoor measurement projects. Have students work on applied problems and projects, such as using clinometers and trigonometric ratios to measure the height of the school building or a flag pole. See the Instructional Resources/Tools sections for application problems involving right triangles.

When solving a right triangle, some students incorrectly use the right angle as a reference angle.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.SRT.6-8)

Instructional Tools/Resources

These tools and resources may be helpful for instruction. Before use, they should be evaluated at the local level to determine their appropriateness for instruction.

Manipulatives/Technology

- Cutouts of right triangles
- Rulers
- Protractors
- Compasses
- Scientific calculators
- Dynamic geometric software (Geometer's Sketchpad®, Desmos®, Cabri®, or GeoGebra®)
- Trig Trainer® instructional aids
- Clinometers (can be made by the students)
- Websites for the history of mathematics:
    - o Trigonometric Functions
    - o Trigonometric Course
    - o Beyond SOH-CAH-TOA: An Example of How History Helps Us to Understand Trigonometry by Eureka Math.

Defining Trigonometric Ratios

- Defining Trigonometric Ratios by Illustrative Mathematics is a task where students use the notion of similarity to define the sine and cosine of an acute angle.
- Tangent of Acute Angles by Illustrative Mathematics is a task where students focus on studying values of tan x for special angles and conjecturing from these values how the function tan x varies.
- Trigonometry by A B Cron is a GeoGebra unit that helps students understand trigonometric functions.
- Sine and Cosine Explained Visually by Victor Powell is an interactive website that explains sine and cosine visually.
- Roadblocks to Success in Trigonometry is a worksheet that deals with students struggle of seeing opposites and connecting trigonometric ratios with similar figures.
- Trig Ratios Part 4 Sine by LearnWithJeff is a YouTube video that explores the meaning behind SOHCAHTOA and connects sine to a half-chord.
- How to Learn Trigonometry Intuitively from Better Explained is a website with an accompanying video on how to approach trigonometry intuitively.
- Slope and Tangent by Texas Instruments is an TInspire activity where students explore the relationship between the slope of a line and the tangent of the angle between the line and the horizontal axis.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.SRT.6-8)

The Complementary Relationship Between Sine and Cosine

- Sine and Cosine of Complementary Angles by Illustrative Mathematics is a task where students provide a geometric explanation for the relationship between sine and cosine.
- Trigonometric Function Values by Illustrative Mathematics is a task where students explore the relationship between sine and cosine for special benchmark angles.

Solving Problems Involving Right Triangles

- Neglecting the Curvature of the Earth by Illustrative Mathematics is a task where students apply tangents and circles of right triangles to a modeling situation.
- Ask the Pilot by Illustrative Mathematics is a task where students apply tangents and circles of right triangles to a modeling situation.
- Access Ramp by Achieve the Core is a CTE task where students design an access ramp using ADA requirements.
- Miniature Golf by Achieve the Core is a CTE task where students design a mini golf course.
- Range of Motion by Achieve the Core is a CTE task where students explore a man's range of motion after a bicycle accident.
- Launch Altitude Tracker by NASA has students use a simple altitude track to indirectly measure the altitude of rockets they construct.
- Problem 19: Beyond the Blue Horizon from NASA's Space Math III is an activity where students determine the height of a transmission antenna to insure proper reception.
- Solving Problems with Circles and Triangles from Mathematics Assessment Project is a lesson where students have to solve problem involving a triangle inscribed in a circle inscribed in a triangle.
- Calculating Volumes of Compound Objects by Mathematics Assessment Project has student use right triangles and their properties to solve volume problems.
- Solving Problems with Circles and Triangles by Mathematics Assessment Project is a lesson where students solve problems by determining the lengths of the sides in right triangles.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.SRT.6-8)

Curriculum and Lessons from Other Sources

- EngageNY, Geometry, Module 2, Topic E, Lesson 25: Incredibly Useful Ratios, Lesson 26: The Definition of Sine, Cosine, and Tangent, Lesson 27: Since and Cosine of Complementary Angles and Special Angles, Lesson 28: Solving Problems Using Sine and Cosine, Lesson 29: Applying Tangents, Lesson 30: Trigonometry and the Pythagorean Theorem are lessons that pertain to this cluster*. Note: Trigonometric Identities, Inverse Trigonometric ratios, and using trigonometric ratios and the Pythagorean to solve problems when an acute angle is not given has been moved to Math 3 in Ohio.*
- Mathematics Vision Project, Geometry, Module 4: Similarity and Right Triangle Trigonometry have tasks that pertain to this cluster.
- Georgia Standards of Excellence Curriculum Framework, Unit 3: Right Triangle Trigonometry has many tasks that pertain to this cluster.
- Illustrative Mathematics, Geometry, Unit 4: Right Triangle Trigonometry, Lesson 1: Angles and Steepness, Lesson 2: Half a Square, Lesson 3: Half a Triangle, Lesson 4: Ratios in Right Triangles, Lesson 5: Working with Ratios in Right Triangles, Lesson 6: Working with Trigonometric Ratios, and Lesson 7: Applying Ratios in Right Triangles, Lesson 8: Sine and Cosine in the Same Right Triangle are lessons that pertain to this cluster.

General Resources

- Arizona 7-12 Progression on Geometry is an informational resource for teachers. This cluster is addressed on page 17.
- Arizona High School Progression on Modeling is an informational resource for teachers. This cluster is addressed on page 17.
- High School Coherence Map by UnboundED is a map of the relationships among high school mathematics standards. Please note: This resource is aligned to the Common Core not Ohio's Learning Standards. Although the tool is useful, it needs to be checked with respect to Ohio's Learning Standards.
- van Hiele Model of Geometric Thinking is a pdf created by ODE that summarizes the van Hiele levels.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.SRT.6-8)

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STANDARDS	MODEL CURRICULUM (G.C.1)
Geometry **CIRCLES** **Understand and apply theorems about circles.** **G.C.1** Prove that all circles are similar using transformational arguments.	**Expectations for Learning** In middle school, students have worked with measurements of circles such as circumference and area. In Math 1, students solve problems using the relationships among the arcs and angles created by radii, chords, and tangents. They construct inscribed and circumscribed circles of a triangle. In this course, they extend their understanding of similarity to circles. The student understanding of this cluster aligns begins at van Hiele Level 1 (Analysis) and moves to Level 2 (Informal Deduction/Abstraction). **ESSENTIAL UNDERSTANDINGS** • All circles are similar as follows: translate one circle so that its center maps onto the center of the other, and then dilate about the common center by the ratio of the radii. **MATHEMATICAL THINKING** • Use accurate mathematical vocabulary. • Make connections between concepts, terms, and properties. • Recognize, apply, and justify mathematical concepts, terms, and their properties. • Solve mathematical and real-world problems accurately. • Determine reasonableness of results. • Consider mathematical units involved in a problem. • Make sound decisions about using tools. **INSTRUCTIONAL FOCUS** • Use transformational arguments to prove that all circles are similar. **Content Elaborations** **OHIO'S HIGH SCHOOL CRITICAL AREAS OF FOCUS** • Math 2, Number 5, page 9 **CONNECTIONS ACROSS STANDARDS** • Understand similarity in terms of similarity transformations (G.SRT.2). • Find arc length and areas of sectors of circles (G.C.5).

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.C.1)

Instructional Strategies

Note: The model curriculum is not a collection of lessons nor a full curriculum; it does not suggest pace, sequence, or amount of time spent on topics. It provides information about a topic related to the standards including ideas for examples, strategies for teaching, common misconceptions, and possible connections between topics. In addition, the emphasis of the model curriculum is on instruction not on assessment; therefore, the examples in the Instructional Strategies section are written for instructional purposes not necessarily as assessment items. Examples may need to be rewritten to accommodate the needs of each individual classroom.

It is important to reinforce the precise definition of a circle from G.CO.1 as it is the foundation for the rest of the learning in this cluster.

VAN HIELE CONNECTION

In Math 2 students are expected to move from Level 1(Analysis) to Level 2 (Informal Deduction/Abstraction). In van Hiele Level 1 students may list all the properties of a shape, but not see the relationships between properties. Whereas van Hiele Level 2 can be characterized by the student doing some or all of the following:

- noticing relationships within and between properties and figures;
- forming meaningful definitions;
- justifying reasoning using informal arguments;
- drawing logical maps and diagrams; and/or
- not yet understanding formal deduction.

This cluster when combined with G.SRT.4-5 cluster heading "Prove and apply theorems both formally and informally involving similarity using a variety of methods." will also move into Level 3 (Deduction) where students start to construct proofs and understand the necessity of proofs.

See the van Hiele pdf on ODE's website for more information about van Hiele levels.

SIMILARITY OF CIRCLES

Two figures are similar if there is a sequence of rigid motions and dilations that maps one figure onto the other. Another way to look at it is that a figure and its dilated image are similar. In previous clusters, students looked at similarity of polygons. Now they extend that understanding to circles.

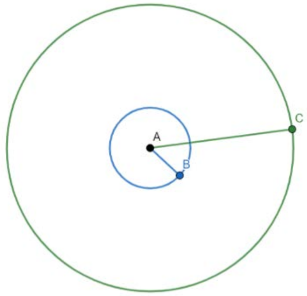
Given any two circles in a plane, guide students to discover the transformations that map one circle onto the other. It may be helpful to have students start with circles that have the same center using the center as the center of dilation. From there challenge students to prove that all circles are similar despite their location on the plane. Students should come to the conclusion that transformations must include a dilation (possibly with a scale factor of 1) but could also include a translation since a translation allows centers of any two circles to coincide. This experience can help students use transformation to understand all circles are similar.

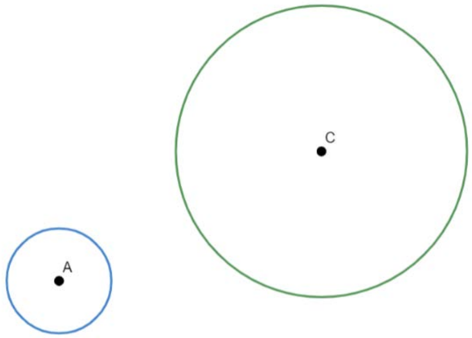
INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.C.1)

EXAMPLE

Part 1

Map the circle with radius AB onto the circle with radius AC. Explain your sequence of transformations.





Part 2

Map Circle A onto Circle C. Explain your sequence of transformations.

Part 3

Create any two circles in the plane. Name one Circle R and the second Circle J. Map Circle R onto Circle J. Explain your sequence of transformations.

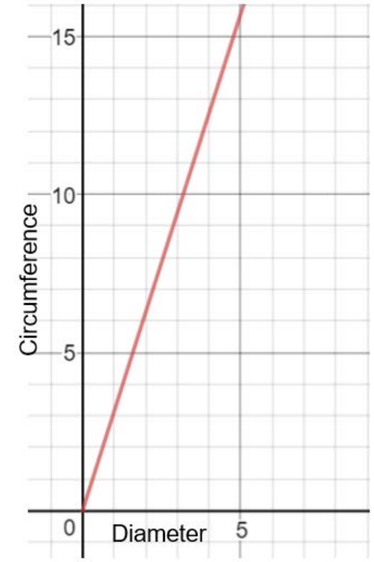
Part 4

- **a.** Can any two circles be mapped onto each other? Explain.
- **b.** What transformation(s) must be included in order to map two noncongruent circles onto each other?
- **c.** Are all circles similar? Explain why or why not.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.C.1)

EXAMPLE

Marcus was cleaning out his old notebooks and found a graph he made from Pi Day in 7th grade. In the activity, he measured the circumference and diameter of various circular objects to find pi.



a. How is pi represented on the graph?
b. Write the equation of the line.
c. Using his graph as a guide and your knowledge of circles, list the circumference and diameter of possible circular objects he might have measured. Fill out your information in a table.
d. In his high school Math 2 class yesterday, Marcus used rigid motions to prove that all circles are similar. How does what Marcus learned about pi in 7th grade relate to similarity of circles?

Discussion: Students should realize that pi is represented by the slope or $\frac{circumference}{diameter}$. The equation of the line is $C = \pi d$. In 7th grade students learned that a line that goes through the origin represents a proportion and the slope of that line is the constant of proportionality. In this case π is the constant of proportionality which can also be called the scale factor. Because the scale factor is constant, the ratio of $\frac{circumference}{diameter}$ stays the same, so the relationship between the circumference and diameter of a circle is proportional, thus all circles are similar.

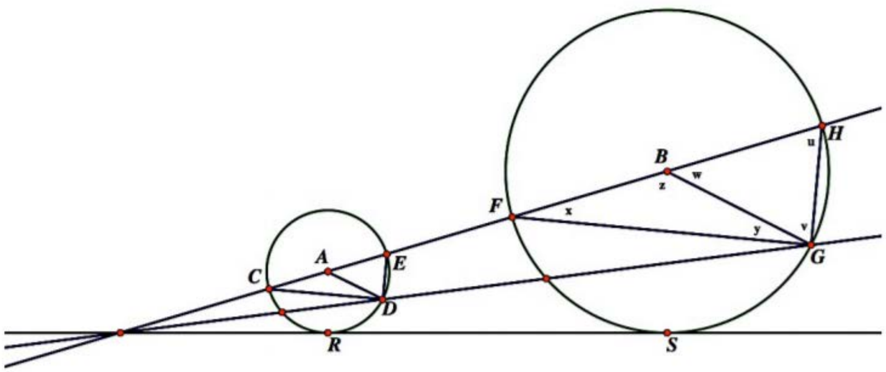


Image taken from Mathematics Vision Project, Geometry, Module 5 [Circles: A Geometric Perspective.]

EXAMPLE

Circle A is dilated by a scale factor of 2.5 about Point O to produce Circle B and $m\angle z = 131°$ and $\overline{AC} = 4.2$. Find the measure of the following:

a. $\angle x$
b. $\angle ADC$
c. $\angle w$
d. $\angle v$
e. $\angle AED$
f. $\overline{AD}$
g. $\overline{FG}$
h. $\overline{AE}$
i. $\overline{ED}$

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.C.1)

Instructional Tools/Resources

These tools and resources may be helpful for instruction. Before use, they should be evaluated at the local level to determine their appropriateness for instruction.

Manipulatives/Technology

- Tracing paper (patty paper)
- Transparencies
- Graph paper
- Ruler
- Protractor
- Compass
- Mira
- Computer dynamic geometric software (such as Geometer's Sketchpad®, Desmos, Cabri®, or GeoGebra®).
- Graphing calculators and other handheld technology such as TI-Nspire™.
- GeoGebra Geometry is the geometry application of GeoGebra.

Similar Circles

- Similar Circles by Illustrative Mathematics is a task that has students prove that circles are similar using a coordinate plane. There is an attached GeoGebra file.

Curriculum and Lessons from Other Sources

- EngageNY, Geometry, Module 5, Topic B, Lesson 7: The Measure of the Arc, Lesson 8: Arcs and Chords are lessons that pertain to this cluster.
- Mathematics Vision Project, Geometry, Module 5: A Geometric Perspective has a lesson, 5.2, that pertain to this cluster.
- Georgia Standards of Excellence Curriculum Frameworks, Geometry, Unit 4: Circles and Volume has many lessons that pertain to this cluster.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.C.1)

General Resources

- Arizona 7-12 Progression on Geometry is an informational document for teachers. This cluster is addressed on page 17.
- High School Coherence Map by UnboundED is a map of the relationships among high school mathematics standards. Please note: This resource is aligned to the Common Core not Ohio's Learning Standards. Although the tool is useful, it needs to be checked with respect to Ohio's Learning Standards.
- van Hiele Model of Geometric Thinking is a pdf created by ODE that summarizes the van Hiele levels.
- Prove that All Circles Are Similar 1 and Prove that All Circles Are Similar 2 are threads on Bill McCallum's Mathematical Musings blog that discuss similarity of circles.

References

- Common Core Standards Writing Team. (2016, March 24). *Progressions for the Common Core State Standards in Mathematics (draft). Grades 7–8, HS, Geometry*. Tucson, AZ: Institute for Mathematics and Education, University of Arizona.

STANDARDS	MODEL CURRICULUM (G.C.5)
Geometry **CIRCLES** **Find arc lengths and areas of sectors of circles.** **G.C.5** Find arc lengths and areas of sectors of circles. **a.** Apply similarity to relate the length of an arc intercepted by a central angle to the radius. Use the relationship to solve problems. **b.** Derive the formula for the area of a sector, and use it to solve problems.	**Expectations for Learning** In middle school, students are limited to working with measurements of circles such as circumference and area. This cluster spans Math 2 and Math 3. In Math 2, students are using part-to-whole proportional reasoning to find arc lengths and sector areas, in which the arc or central angle is measured in degrees. Math 3, students derive and use formulas relating degree and radian measure. The student understanding of this cluster aligns with van Hiele Level 2 (Informal Deduction/Abstraction). *Note: Since in Math 2 students focus on quadratics with leading coefficients of 1 with occasional uses of other simple coefficients, geometry standards should only apply to equations where the squared terms have a coefficient of 1 or occasionally other simple leading coefficients.* **ESSENTIAL UNDERSTANDINGS** • A central angle that turns through n one-degree angles is said to have an angle measure of n degrees. • The measure of an arc is equal to the measure of the corresponding central angle and is expressed in degrees, while the length of an arc is expressed in units of linear measure. • The arc length is a part of the circumference of a circle. • The ratio of the central angle to 360 degrees is equal to the ratio of the length of the arc to the circumference of the circle. • The sector area is a part of the area of a circle. • The ratio of the central angle to 360 degrees is equal to the ratio of the area of the sector to the area of the circle. • Because all circles are similar, if the radius of the circle is scaled by k, the corresponding arc length is multiplied by k and the sector area is multiplied by k^2. *Continuation on next page*

STANDARDS	MODEL CURRICULUM (G.C.5)
	Expectations for Learning, continued **MATHEMATICAL THINKING** • Consider mathematical units involved in a problem. • Make connections between concepts and terms. • Generalize concepts based on patterns. • Use proportional reasoning (part to whole). • Draw a picture to make sense of a problem. • Solve real-world and mathematical problems accurately. • Plan a solution pathway. • Attend to the meaning of quantities. **INSTRUCTIONAL FOCUS** • Develop understanding of the formulas for arc length and area of a sector through derivation. • Solve problems using arc lengths and areas of sectors of circles. **Content Elaborations** **OHIO'S HIGH SCHOOL CRITICAL AREAS OF FOCUS** • Math 2, Number 5, page 9 **CONNECTIONS ACROSS STANDARDS** • Understand and apply theorems about circles (G.C.1). • Explain volume formulas and use them to solve problems (G.GMD.1). • Understand similarity in terms of similarity transformations (G.SRT.2). • Create equations that describe numbers or relationships (A.CED.1, 2, 4). • Understand the relationships between lengths, areas, and volumes (G.GMD.5-6).

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.C.5)

Instructional Strategies

Note: The model curriculum is not a collection of lessons nor a full curriculum; it does not suggest pace, sequence, or amount of time spent on topics. It provides information about a topic related to the standards including ideas for examples, strategies for teaching, common misconceptions, and possible connections between topics. In addition, the emphasis of the model curriculum is on instruction not on assessment; therefore, the examples in the Instructional Strategies section are written for instructional purposes not necessarily as assessment items. Examples may need to be rewritten to accommodate the needs of each individual classroom

ARC LENGTHS

Students need to understand that angle measure and angular arc measure are each an amount of rotation and are measured in the same units. The measure of the central angle of the entire circle as well as the angular arc measure of the entire circle are equal to 360°. One way to show this is to relate a central angle of 90° to the arc measure of a quarter of a circle.

Connect back to the 4th grade standard 4.MD.5, where angles are defined as an amount of turning. The angular measure of an arc as a measure can be thought of in a similar way. For example, consider that doing a 180 is a half turn, 90 is a quarter turn, etc.

EXAMPLE

Part 1

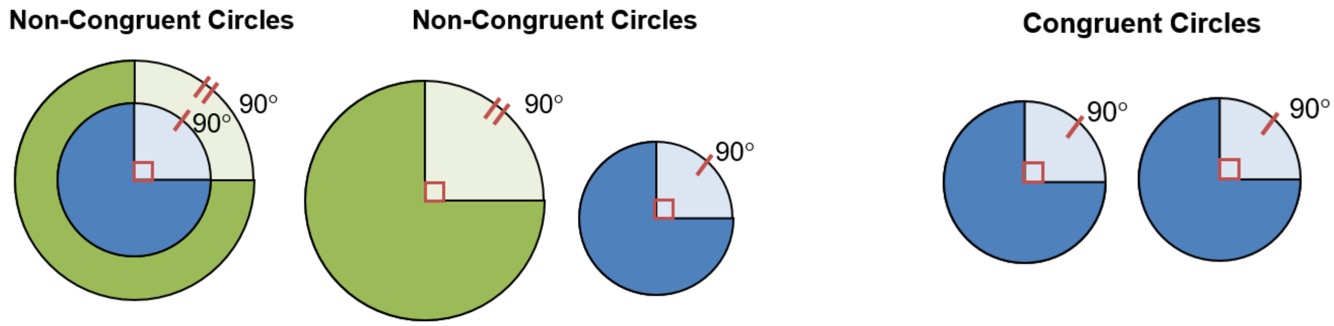
One method to model this is to have one student stand at the center of a circle and another student stand on the circumference. "Connect" the two students with a piece of string to represent the radius. As the student at the center turns through the central angle and does not move his position from the center, the student on the circle will rotate the same amount but will travel along the circle the distance equaling to the length of the arc.

Part 2

Now draw two concentric circles on the sidewalk and have three students hold the string. One student should continue to anchor the radius (string) at the center. The second student should stand on the outside concentric circle and hold the other end of the string. The third student should hold the string at the place where it intersects the inside concentric circle. Mark the starting point. As the student standing at the center turns through the central angle, both students will rotate the same amount, but will travel different lengths. Mark the end of the turn, and have students measure the arc length.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.C.5)

Discuss what it means for arcs to be congruent. Students need to understand that congruent central angles in non-congruent circles have non-congruent corresponding intercepted arcs. Therefore, congruent arcs will have equal angular measures and equal arc lengths and either be on the same or congruent circles.



Many students incorrectly think that arc measure (the amount of rotation about the center) and arc length (a distance along the arc) are the same. Show students two concentric circles that have the same angular measure but different arc lengths.

Some students may incorrectly think that the arc of a larger size circle will have the same size and shape (length and curvature) of a smaller circle. Have students map one circle onto the other (without using dilation), so they can see the overlap is not perfect because the curvatures are different.

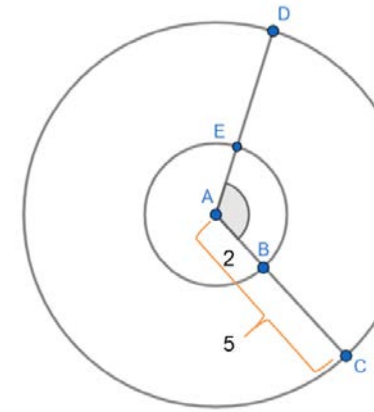
Discuss how arc length is related to circumference. Begin by calculating lengths of arcs that are simple fractional parts of a circle (e.g., $\frac{1}{6}$), and do this for circles of various radii so that students discover a proportional relationship. Students should discover that the ratio of the arc length to the circumference is proportional to the measure of the central angle to 360° or $\frac{arc\ length}{circumference\ (2\pi r)} = \frac{central\ angle\ \theta}{360°}$. The focus should be more on understanding the concepts of arc length rather than memorizing formulas.

In G.C.1, students discovered that all circles are similar. Therefore, the dilation of a circle by a scale factor of *k* results in a dilation of any fixed arc along the circle by the same scale factor, *k*, and the length of the dilated arc equals the product of the length of the arc and factor *k*. (This also connects to G.GMD.5 and 6 that discusses the proportional relationships between lengths and areas.)

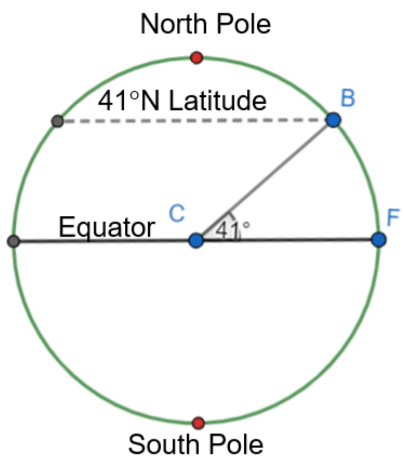
INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.C.5)

EXAMPLE

If the length of $\widehat{EB}$ is $\frac{4\pi}{3}$, how long is $\widehat{DC}$?



Discussion: Students should realize that since all circles are similar, the scale factor between two concentric circles shown is $\frac{AC}{AB}$ and the image of $\widehat{EB}$ is $\widehat{DC}$ whose length is $|\widehat{DC}| = \frac{AC}{AB}|\widehat{EB}|$ where $|\widehat{DC}|$ is the arc length of $\widehat{DC}$ and $|\widehat{EB}|$ is the arc length of $\widehat{EB}$. Therefore $|\widehat{DC}| = \frac{5}{2}\left(\frac{4\pi}{3}\right)$ or $\frac{10\pi}{3}$.

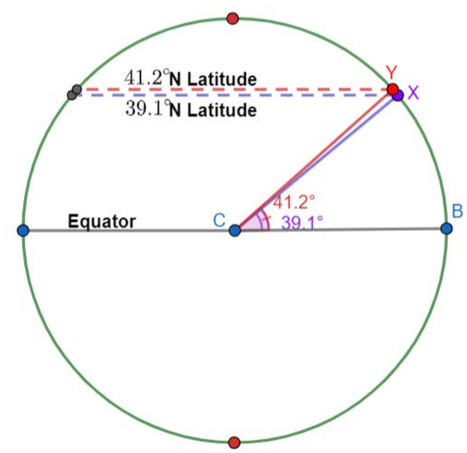


EXAMPLE

On the surface of Earth, the latitude of Point B is equal to the measurement of the angle formed by Point B, the center of Earth, C, and a second Point F along the same line of longitude intersecting the equator. This relationship is depicted at the cross sectional at the left. The radius of the Earth is approximately 3,960 miles.

- **a.** The latitude of Cincinnati, Ohio is 39.1°N. Approximately, how far away is Cincinnati from the equator?
- **b.** Defiance, Ohio is due north of Cincinnati, Ohio. The latitude of Defiance, Ohio is 41.2°N. Approximately how far apart are the two cities?

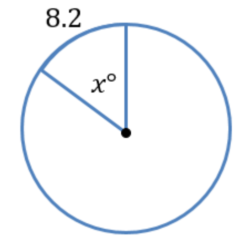
Discussion: In Part **a.,** to find the length of $\widehat{XB}$, students can use a proportion, $\frac{39.1°}{360°} = \frac{a}{2 \cdot 3960\pi}$, to find that the length of XB is 2,702.4 miles. In Part **b.,** to find the distance between the two cities, students need to find the arc length between the two altitudes. To find degree measure of the central angle, subtract the two given angles $41.2° - 39.1°$ to get $2.1°$. Then use a proportion to find the arc length: $\frac{2.1°}{360°} = \frac{a}{2 \cdot 3960\pi}$. After solving the proportion students should get approximately 145.1 miles between the two cities.



INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.C.5)

EXAMPLE

If the arc length intercepted by the central angle, x, is 8.2, what would be the length of the intercepted arc if you double x?



Discussion: Students should see the relationship between the central angle and the arc length. If the central angle doubles, the arc length will also double.

EXAMPLE

For a circle with each specified radius, use similarity to list the length of an arc intercepted by each of the following central angles.

	central angles							
radius	**30°**	**60°**	**90°**	**120°**	**150°**	**180°**	**270°**	**360°**
6	π							
12								
18								
24								
30								

a. Fill out the chart.
b. What patterns do you notice?
c. What conclusions can you make about arc length?

AREAS OF SECTORS OF CIRCLES

Discuss the relationship between the central angle and the area of a sector. Begin by calculating the areas of the sectors that are simple fractional parts of a circle (e.g. $\frac{1}{6}$) and do this for circles of various radii so that students discover a proportional relationship. Students should discover that the ratio of the area of the sector to the area of a circle is equal to the measure of the central angle to 360°, or $\frac{\text{area of a sector}}{\text{area of a circle } (\pi r^2)} = \frac{\text{central angle } \theta}{360°}$. The focus should be more on understanding the concepts of sector area rather than memorizing formulas.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.C.5)

Although, circle graphs are not mentioned in the standards, they are an application of sector area of a circle.

EXAMPLE

Take a survey of your classmates using categorical data with at least four categories. Create a circle graph illustrating your data. Then find the area of each sector. Make sure to show your calculations.

Discussion: After students finish the activity discuss with students why circle graphs are not always the best way to display data. Save the Pies for Dessert by Stephen Few of Perceptual Edge is an article that could be used for discussion.

TIP!

Use coffee filters or paper plates to have students create sectors of circles.

EXAMPLE

Choose a local pizza company.

- **a.** How big is a slice of medium round pizza? How big is a slice of a large round pizza?
- **b.** Compare the cost per slice of a large round pizza versus a cost per slice of a medium round pizza.
- **c.** Which pizza is the better deal? Explain.

Discussion: This question is intentionally left open. Students can discuss the meaning of the word *big*. Does it mean longer, wider, more weight, more area, or heavier? Eventually settle on the idea of area of a sector. NPR has an article titled 74,476 Reasons You Should Always Get the Bigger Pizza and published on February 26, 2014 that could supplement this activity. Each pizza may have a different number of slices, styles of crusts, or thickness depending on crust style (factoring in volume). The suggestion is to explore how the best deal can be determined through examining the features of pizzas.

In G.C.1, students discovered the fact that all circles are similar, therefore the dilation of a circle by a scale factor of k results in area of a sector multiplied by a factor of k^2. (This also connects to G.GMD.5 and 6 that discusses the proportional relationships between lengths and areas.)

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.C.5)

EXAMPLE

For a circle with each specified radius, use similarity to list the area of a sector of a circle intercepted by each of the following central angles.

	central angles							
radius	**30°**	**60°**	**90°**	**120°**	**150°**	**180°**	**270°**	**360°**
6	3π							
12								
18								
24								
30								

a. Fill out the chart.
b. How does the area change when the central angle changes by a factor of *k*? Explain.
c. How does the area change when the radius changes by a factor or *k*? Explain.

TIP!

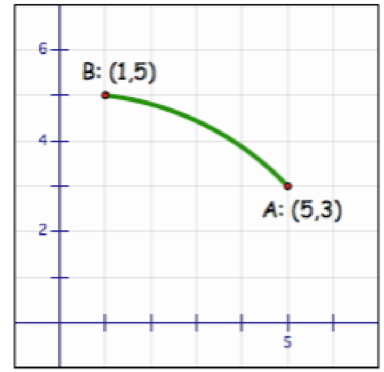
Sectors and segments are often used interchangeably in everyday conversation. Care should be taken to distinguish these two geometric concepts.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.C.5)

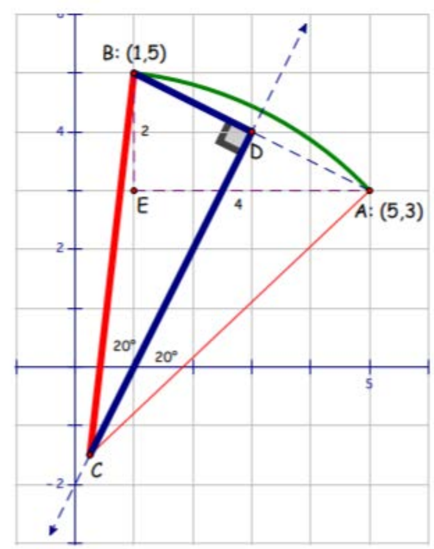
EXAMPLE

Arc to Area

The arc on the right has a measure of 40°, and its endpoints are at (1,5) and (5, 3). Find the area of the circle that contains the arc.

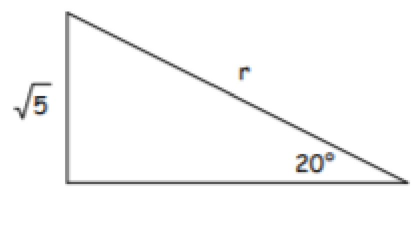


Discussion: This is an example of a problem of the week. There are several methods students can use to solve this problem. One method is to draw a chord connecting the two end points of the arc, and then draw CD that goes through the center of the circle and is a perpendicular bisector of a chord, which would cut the central angle of 40° in half. Then use the Pythagorean Theorem to find the length of $\overline{AB}$ (see purple dash lines) to get $2\sqrt{5}$ for the length of $\overline{AB}$. Since $\overline{BD}$ was formed by a perpendicular bisector, the length of $\overline{BD}$ is half the length of $\overline{AB}$ or $\sqrt{5}$. Triangle BDC is also a right triangle, so sine of half the central angle can be used to find the length of the radius $\overline{BC}$. Thus $\overline{BC} = \frac{\sqrt{5}}{\sin 20}$. So students can use $A = \pi r^2$ or $A = \pi\left(\frac{\sqrt{5}}{\sin 20}\right)^2$ which is approximately 134.28 square units. A more detailed explanation of this problem can be found on NCTM Problems of the Week Arc to Area.



Instructional Tools/Resources

These tools and resources may be helpful for instruction. Before use, they should be evaluated at the local level to determine their appropriateness for instruction.



Manipulatives/Technology

- Ruler
- Compass
- Protractor
- String
- Coffee Filters
- Paper Plates
- Computer dynamic geometric software (such as Geometer's Sketchpad®, Desmos, Cabri®, or GeoGebra®).
- Graphing calculators and other handheld technology such as TI-Nspire™.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.C.5)

Arc Length and Area of Sectors

- My Favorite Slice by CPalms is a lesson that uses pizzas as application to calculate the area of a sector.
- Measures of Arcs and Sectors by Rafranz D. is a lesson plan that connects arc length and circumference and a sector as a fractional part of a circle.
- Arc Lengths (Degrees) by Toh Wee Teck is a GeoGebra applet where students can explore arc length.
- Area of Sectors (Degrees) by Toh Wee Teck is a GeoGebra applet where students can explore area of sectors.
- Area Length and Sector Area by James Dunseith is a lesson where students apply similarity to find arc length and sector areas of circles with different radii.
- Save the Pies for Dessert by Stephen Few of Perceptual Edge is an article about why Pie Charts are not the best representative of graphical displays.
- Calculating Arcs and Areas of Sectors of Circle by Mathematics Assessment Project is a lesson where students compute perimeters, areas, and arc lengths of sectors and find the relationships between arc lengths and areas of sectors after scaling.

Curriculum and Lessons from Other Sources

- EngageNY, Geometry, Module 5, Topic B, Lesson 7: The Angle Measure of an Arc, Lesson 8: Arcs and Chords, Lesson 9: Arc Length and Areas of Sectors, and Lesson 10: Unknown Length and Area Problems are lessons that pertain to this cluster.
- Mathematics Vision Project, Geometry, Module 5: A Geometric Perspective has many lessons that pertain to this cluster.
- Georgia Standards of Excellence Curriculum Frameworks, Geometry, Unit 4: Circles and Volume has many lessons that pertain to this cluster.
- Illustrative Mathematics, Geometry, Unit 7, Lesson 8: Arcs and Sectors, Lesson 9: Part to Whole and Lesson 10: Angles, Arcs, and Radii are lessons that pertain to this cluster.

General Resources

- Arizona 7-12 Progression on Geometry is an informational resource for teachers.
- High School Coherence Map by UnboundED is a map of the relationships among high school mathematics standards. Please note: This resource is aligned to the Common Core not Ohio's Learning Standards. Although the tool is useful, it needs to be checked with respect to Ohio's Learning Standards.
- van Hiele Model of Geometric Thinking is a pdf created by ODE that summarizes the van Hiele levels.

References

- Cooper, L & Dennis, E. (August 2016). Estimating Earth's circumference with an app. *Mathematics Teaching in the Middle School, 22, (1*), 47-50.

STANDARDS	MODEL CURRICULUM (G.GPE.1)
Geometry **EXPRESSIONS GEOMETRIC PROPERTIES WITH EQUATIONS** **Translate between the geometric description and the equation for a conic section.** **G.GPE.1** Derive the equation of a circle of given center and radius using the Pythagorean Theorem; complete the square to find the center and radius of a circle given by an equation.	**Expectations for Learning** In middle school, students use the Pythagorean Theorem to find distances between points within the coordinate system. In the high school, students complete the square to solve quadratic equations. In this cluster, students derive the equation of a circle using the Pythagorean Theorem. They also complete the square to find the center and radius of a circle. The student understanding of this cluster begins at van Hiele Level 2 (Informal Deduction/Abstractions) and moves to Level 3 (Deduction). **ESSENTIAL UNDERSTANDINGS** • The equation of a circle relates a fixed center, a fixed radius, and a set of variable points, which are the points on the circle. • Just as the Distance Formula is an application of the Pythagorean Theorem so is the equation of a circle. **MATHEMATICAL THINKING** • Use precise mathematical language. • Discern and use a pattern or structure. • Make connections between concepts, terms, and properties within the grade level and with previous grade levels. • Justify relationships in geometric figures. • Use technology strategically to deepen understanding. • Solve routine and straightforward problems accurately. **INSTRUCTIONAL FOCUS** • Use the Pythagorean Theorem to derive the equation of a circle. • Given the equation of a circle that is not in standard form, find the center and radius of the circle by completing the square. *Continued on next page*

STANDARDS	MODEL CURRICULUM (G.GPE.1)
	Content Elaborations **OHIO'S HIGH SCHOOL CRITICAL AREAS OF FOCUS** • Math 2, Number 5, page 9 **CONNECTIONS ACROSS STANDARDS** • Solve equations and inequalities in one variable (A.REI.4). • Prove that all circles are similar (G.C.1). • Prove theorems about triangles (G.SRT.4). • Write equations in two variables and use them to solve problems (A.CED.2). • Solve quadratic equations in one variable (A.REI.4). • Solve systems of linear and quadratic equations (A.REI.7).

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.GPE.1)

Instructional Strategies

Note: The model curriculum is not a collection of lessons nor a full curriculum; it does not suggest pace, sequence, or amount of time spent on topics. It provides information about a topic related to the standards including ideas for examples, strategies for teaching, common misconceptions, and possible connections between topics. In addition, the emphasis of the model curriculum is on instruction not on assessment; therefore, the examples in the Instructional Strategies section are written for instructional purposes not necessarily as assessment items. Examples may need to be rewritten to accommodate the needs of each individual classroom.

Standards for Mathematical Practice
This cluster focuses on but is not limited to the following practices:
MP.2 *Reason abstractly and quantitatively.*
MP.4 *Model with mathematics.*
MP.5 *Use appropriate tools strategically.*
MP.6 *Attend to precision.*
MP.7 *Look for and make use of structure.*

VAN HIELE CONNECTION

In Math 2 students are expected to move from Level 2 (Informal Deduction/Abstraction) toward Level 3 in shapes and transformations.

Shapes

Level 3 (Deduction). Van Hiele Level 2 can be characterized by the student doing some or all of the following with respect to shapes:

- noticing relationships within and between properties and figures;
- forming meaningful definitions;
- justifying reasoning using informal arguments;
- drawing logical maps and diagrams; and/or
- not yet understanding formal deduction.

Van Hiele Level 3 can be characterized by the student doing some or all of the following with respect to shapes:

- constructing proofs and understanding the necessity of proofs;
- understanding the role and relationships between of axioms, theorems, definitions, and postulates;
- differentiating between necessary and sufficient conditions; and/or
- making logical conclusions.

Transformations

Van Hiele Level 2 can be characterized by the student doing some or all of the following with respect to transformations:

- analyzing relationships within and between transformations;
- algebraically expressing transformations: reflections about the *x*- and *y*-axes, and rotations of 45°, 90°, and 180° about the origin;
- understanding and justifying how to use perpendicular bisectors of corresponding points to locate the turn center of a rotation;
- understanding how the composition of two reflections can be a translation or a rotation; and/or
- following a rationale if shown proof.

Van Hiele Level 3 can be characterized by the student doing some or all of the following with respect to transformations:

- Understanding and creating formal proofs using transformations.

RELATING THE DISTANCE FORMULA AND/OR THE PYTHAGOREAN THEOREM TO THE EQUATION OF A CIRCLE

The Instructional Strategies for G.GPE.4-7 address how to use the Pythagorean Theorem to derive the Distance Formula. It is up to a district to decide whether or not deriving the Distance Formula is best placed in G.GPE.7 or G.GPE.1. Regardless of where the Distance Formula is introduced, in this standard, students should use the Distance Formula to derive the equation of a circle. Review the definition of a circle as a set of points whose distance from a fixed point is constant.

EXAMPLE

Part 1.

- **a.** If a circle has a radius of 3, how many circles can you make on a coordinate grid?
- **b.** What information do you need to write the equation of a specific circle on the coordinate plane?

Part 2

- **a.** If a circle has a center of $(4, -1)$, how many circles can you make on a coordinate grid?
- **b.** What information do you need to write the equation of a specific circle on a coordinate plane?

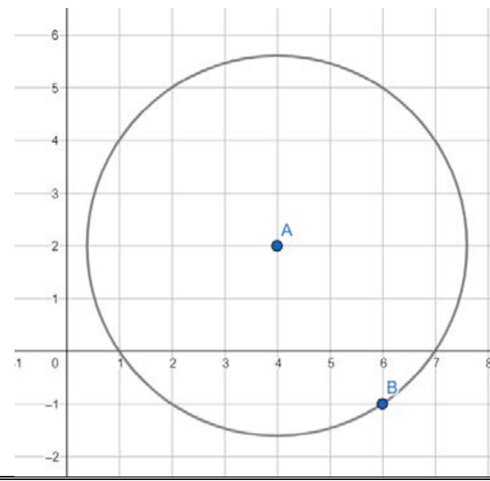
Discussion: Students should come to the conclusion that in order to define a circle both a center and radius needs to be stated.

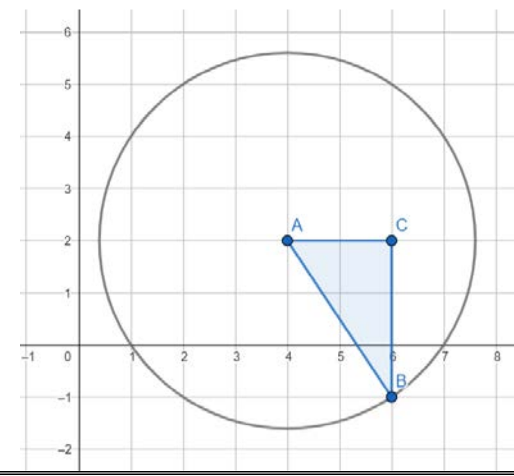
EXAMPLE

Part 1

- **a.** How could you find the length of a radius of a circle in a coordinate plane?
- **b.** Draw a circle on the coordinate plane and find the length of the radius of the circle.

Discussion: In our example on the right a student drew a circle with center $(4, 2)$ and a point on the circle $(6, -1)$. Students should connect the Pythagorean Theorem or the Distance Formula to the length of the radius. The length of $\overline{AB}$ is $\sqrt{(6-4)^2+(-1-2)^2}=r$ to get $\sqrt{2^2+(-3)^2}=\sqrt{13}$, so the radius is $\sqrt{13}$.

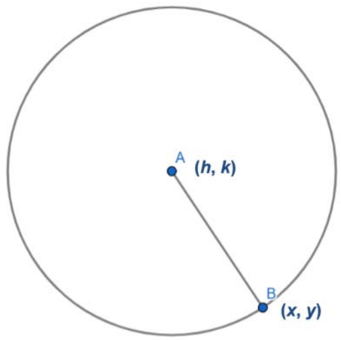




INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.GPE.1)

Part 2

Using the method you used in Part 1, create a formula for finding the radius of a circle with the center at (h, k) and with any arbitrary point on a circle is (x, y).



Discussion: Students should recognize that they can just substitute (*x*, *y*) and (*h*, *k*) for (x_1, y_1) and (x_2, y_2) into the Distance Formula, $D = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ respectively. So, the equation of the circle is $\sqrt{(x-h)^2 + (y-k)^2} = r$ or $(x-h)^2 + (y-k)^2 = r^2$.

Part 3

- **a.** Using the formula you derived in Part 2, write the equation of a circle with the center (4, 2) and a radius of $\sqrt{13}$.
- **b.** How is that similar to what you did in Part 1?
- **c.** Write an equivalent equation for the equation you wrote in part **a.**

Discussion: Draw attention the different forms of the equation of a circle such as center-radius (standard) form and general form.

TIP! The Distance Formula involves subscripted variables whose differences are squared and then added. The notation and multiplicity of steps can be a serious stumbling block for some students. Therefore, it is important that students conceptually understand the Distance Formula and connect it to the Pythagorean Theorem.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.GPE.1)

EXAMPLE

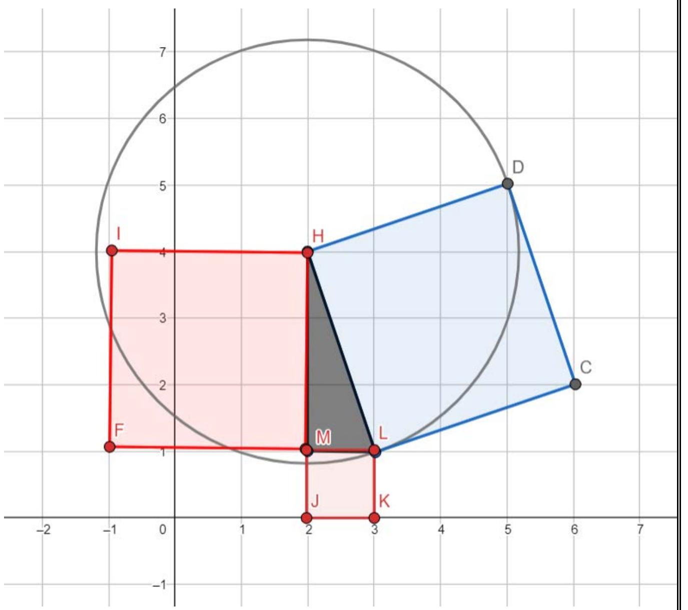
Use the diagram on the right to answer the questions:

a. What's the relationship among the red squares and the blue square?

b. The equation of the above circle is $(x-2)^2+(y-4)^2=10$.

- **i.** Which part of the diagram represents $(x-2)^2$?
- **ii.** Which part of the diagram represents $(y-4)^2$?
- **iii.** Which part of the diagram represents 10?
- **iv.** Which part of the diagram represents x?
- **v.** Which part of the diagram represents y?
- **vi.** Which part of the diagram represents $(2,4)$?
- **vii.** Which part of the diagram represents 2?
- **viii.** Which part of the diagram represents 4?
- **ix.** What relationship does point M have to point L?
- **x.** What relationship does point M have to point H?
- **xi.** Which part of the diagram represents $(x-2)$?
- **xii.** Which part of the diagram represents $(y-4)$?

c. Using geometric software create a circle and radius. Create a right triangle where the radius is the hypotenuse. Then create squares on each side of the triangles. Drag the endpoint around and see what happens.

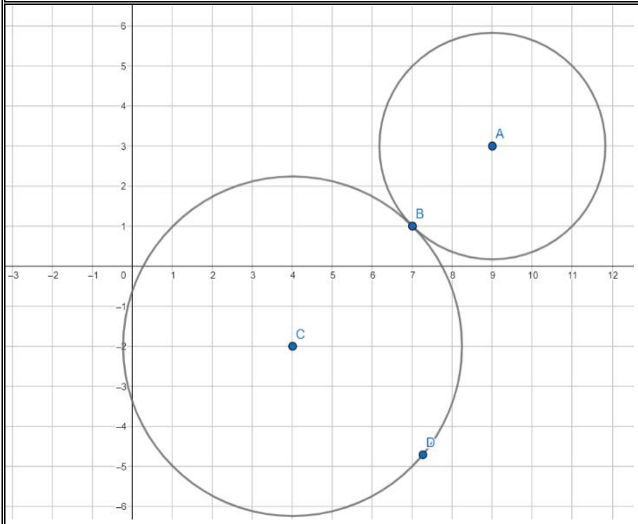


Discussion: Students should note that the sum of the areas of the red squares equals the area of the blue squares.

- **i.** $(x-2)^2$ is the area of the red square $MLKJ$.
- **ii.** $(y-4)^2$ is the area of the red square $HMFI$.
- **iii.** 13 is the area of the blue square $HDCL$.
- **iv.** x is the x -coordinate of point L (or technically any point on the circle).
- **v.** y is the y -coordinate of point L (or technically any point on the circle).
- **vi.** (2, 4) is the center of the circle.
- **vii.** 2 is the x-coordinate of the center of the circle and the distance point H is from the y-axis.
- **viii.** 4 is the y-coordinate of the center of the circle and distance point H is from the origin.
- **ix.** Point M has the same y-coordinate as L
- **x.** Point M has the same x-coordinate as H.
- **xi.** $(x-2)$ is the length of $\overline{LM}$
- **xii.** $(y-4)$ is the length of $\overline{HM}$.

Note: The bullet points above are not meant to be problems completed on a worksheet, but rather points of discussion that support the visual for the relationship between the Pythagorean Theorem and the equation of a circle.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.GPE.1)



EXAMPLE

What is the distance between the centers of the circles in the diagram?

Discussion: Draw attention to the fact that points $A, B,$ and C are collinear, and point B is between points A and C. Therefore, students could use the Distance Formula to find the distances between point A and point C. Some students may want to find the length of radius AB and radius BC and add them together. Discuss why either method works. Discuss how they would find the distance if the points are not collinear.

EXAMPLE

Write an equation of a circle where $(-8, 3)$ and $(-2, 1)$ are endpoints of the diameter of the circle.

Discussion: First, students can find the center by using the midpoint formula or some other method. Then, they can write the equation of the circle using the center and radius.

EXAMPLE

Find the radius of a circle centered at $(-2, 6)$ where one of the points on the circle is $(1, 3)$.

Discussion: Students can utilize the equation of the circle, the Pythagorean Theorem, or the Distance Formula to find the radius.

EXAMPLE

Write an equation of a circle that is tangent to $y = -1, y = -7$ and $x = 1$.

Discussion: There are two possible circles that satisfy the given conditions. Students may want to use a coordinate grid to help them visualize the process. One circle would be centered at $(4, -4)$ to the right of $x = 1$ and the other circle would be centered at $(-2, -4)$ to the left of $x = 1$. Both circles will have a radius of 3 that is half the distance from $y = -1$ and $y = -7$.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.GPE.1)

EXAMPLE

Part 1

What could be the coordinates of a point on a circle with a center at the origin and a radius of $\sqrt{26}$?

Part 2

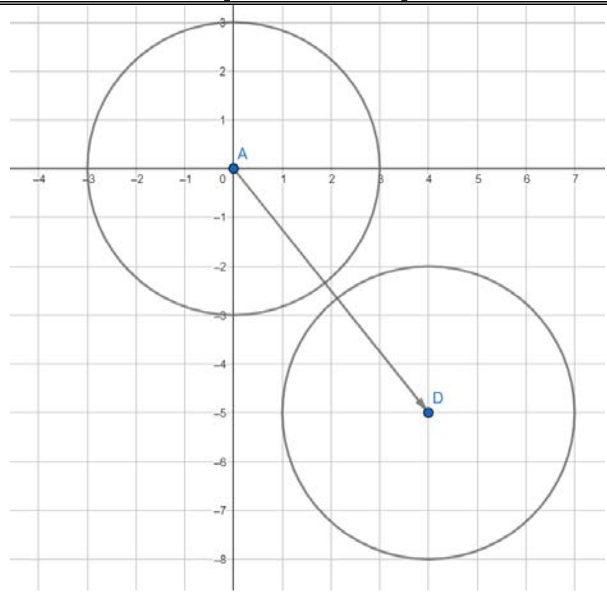
What could be coordinates of a point on a circle with a center of $(-4, 2)$ and a radius of $\sqrt{26}$?

Discussion: This is an example of an open-ended problem that could be solved a variety of ways with multiple solutions. One way that students can approach this is by writing an equation of the circle: $(x+4)^2 + (y-2)^2 = 26$. Students could visualize the Pythagorean Theorem and realize that 26 is the sum of two perfect squares. The perfect squares less than 26 are 1, 4, 9, 16, and 25. Only 25 and 1 equal 26. Therefore, $(x+4)^2$or $(y-2)^2$ has to equal 25 and the other squared binomial has to equal 1. If students chose to set $(x+4)^2 = 25$ and $(y-2)^2 = 1$, then $x = 1, -9$ and $y = 1, 3$. Therefore coordinates of a point on the circle could be $(1, 1), (1, 3), (-9, 1)$ or $(-9, 3)$. If students choose to set $(x+4)^2 = 1$ and $(y-2)^2 = 25$, then $x = -3, -5$ and $y = 7, -3$. Therefore the coordinates of the point on the circle could be $(-3, 7), (-3, -3), (-5, 7), (-5, -3)$.
Note: Students may choose to find non-integer coordinates.

The formula for an equation of a circle can also be connected to translations of a circle using coordinate notation. A translation can be represented by $(x, y) \rightarrow (x+2,\ y-3)$ to describe a figure that translates two units to the right in the horizontal direction and three units down in the vertical direction.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.GPE.1)

EXAMPLE



a. Draw a circle on the coordinate plane with a radius of your choosing and centered at the origin.
b. Write the equation of the circle you drew for part **a.**
c. Translate your circle 4 units to the right and 5 units down.
d. Describe your translation using coordinate notation.
e. Write the equation of a circle for the image you drew in part **c.**
f. Compare the coordinate notation of your translated image in part **d.** to the equation of the circle in part **e.** What do you notice?
g. Perform another translation of the original circle centered at the origin to anywhere in the coordinate plane.
h. Describe your translation in part **g.** using coordinate notation.
i. Write the equation of the translated circle from part **g.**
j. Compare the coordinate notation of your translated image in part **h.** to the equation of the circle in part **i.** What do you notice?
k. Explain how the equation of the circle is really just a way of describing a translated image of a circle centered at the origin. Look for connections between the equations of the two circles and the translations of the two circles?

Discussion:

a. A student may choose a radius of 3.
b. The equation of the circle for part **a.** would be $x^2 + y^2 = 9$.
d. Coordinate notation describing the translation in part **c.** would be $(x, y) \rightarrow (x + 4, y - 5)$.
e. The equation of the translated circle would be $(x - 4)^2 + (y + 5)^2 = 9$.
f. Students should recognize that there is a 4 and 5 present both in the coordinate notation of the transformation and the equation of the circle, but their corresponding signs are different.
g. By translating it a second time, they should see that the same pattern emerges.
h.-k. Since students choose their own translations for part **g.**, they could discuss with their classmates that this is a consistent finding. The big question then becomes "Why are the signs different?" For example, if the students used our example, they could make the observation that although the plane moves 4 units to the right (+4) and 5 units down (–5) to get the translated image back to the center, it would have to move 4 units to the left (–4) and 5 units up (+5).

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.GPE.1)

Students oftentimes incorrectly think that the signs in front of *h* and *k* in the equation of the circle correspond to the signs of the coordinate of the center of the circle. Allow students to discover that they actually indicate how far the image is translated to a congruent circle centered at the origin. Therefore, the signs are opposite of the actual center of the circle.

EXAMPLE

The local hardware store sells pop-up sprinkler heads that have an adjustable 6 ft. to 15 ft. radius for $2.96 each. If your backyard measures 60 ft. by 100 ft., how many sprinklers should you place on your lawn to get maximum coverage and by spending the least amount of money? Where would you place the sprinklers and why?

- Illustrate the lawn in the first quadrant of the coordinate plane with one of the corners at the origin.
- Illustrate the placement of the sprinklers by drawing circles on your coordinate plane with the center and radius marked. Where would you place the sprinklers and why?
- Write an equation of the circle to represent each setting.
- How much will it cost?
- Compare your work to your classmates and discuss your findings. Justify your reasoning.

Discussion: This is a modeling problem, so there is more than one correct answer. Not only do students have to figure out where to place the sprinklers and decide on the radius of each sprinkler, they also have to determine if it is better to overlap sprinklers, so the lawn is completely watered or if it is better to leave small gaps and save money by not wasting water or buying extra sprinklers. Students may discuss if it is appropriate to water over the boundaries of their backyard which may hit a neighbor's lawn, a building, a garden, or a sidewalk. Teachers and students can also utilize Google maps to zoom in and out of farmland, so they can see this concept implemented at a larger scale.

FIND THE CENTER AND RADIUS OF A CIRCLE BY COMPLETING THE SQUARE

Starting with any quadratic equation in two variables (*x* and *y*) in which the coefficients of the quadratic terms are equal, complete the squares in both *x* and *y* and obtain the equation of a circle in standard form. Students in Math 2 use completing the square to solve quadratic equations and write quadratic equations in vertex form. They may need to review the algebraic method of completing the square and connect it to a geometric model. Algebra tiles are helpful for students to visualize this problem. Connect with standards A.SSE.3, A.REI.4, and F.IF.8.

TIP!

The method of completing the square is a multi-step process that takes time to assimilate. A geometric demonstration of completing the square can be helpful in promoting conceptual understanding.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.GPE.1)

EXAMPLE

a. How does the equation of the circle relate to the Pythagorean Theorem?

b. The areas of all three squares, the center, and the radius can be seen in center-radius form, $(x-h)^2+(y-k)^2=r^2$. If the equation, $x^2+y^2-2x+6y=3$, is in general form for the equation of the circle, can you see the expression of the area of the squares in this form? Can you see the coordinate of the center or the radius of the circle in this form?

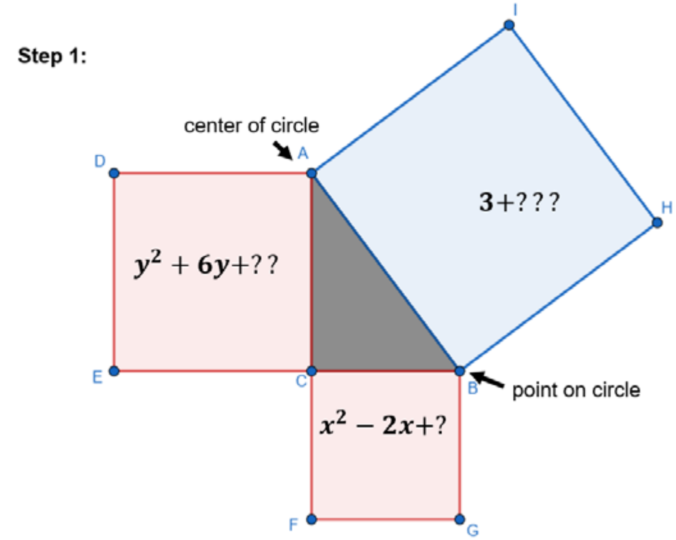
c. Show the area of the squares with algebra tiles? Are the squares complete? If not, how can you complete them? Discuss as a class or within your group.

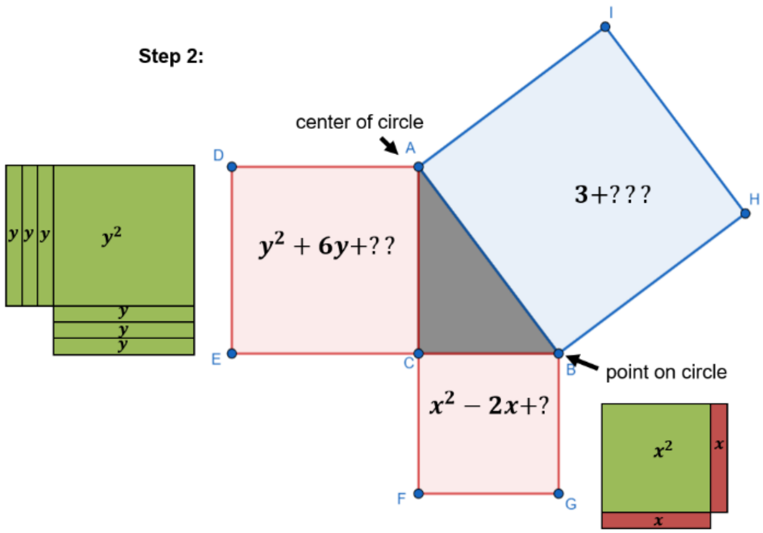
d. Find the length of each side of the triangle using the squares.

e. Write your equation in center-radius form.

f. Identify the center and radius of the circle.

g. Draw the circle on the coordinate plane.





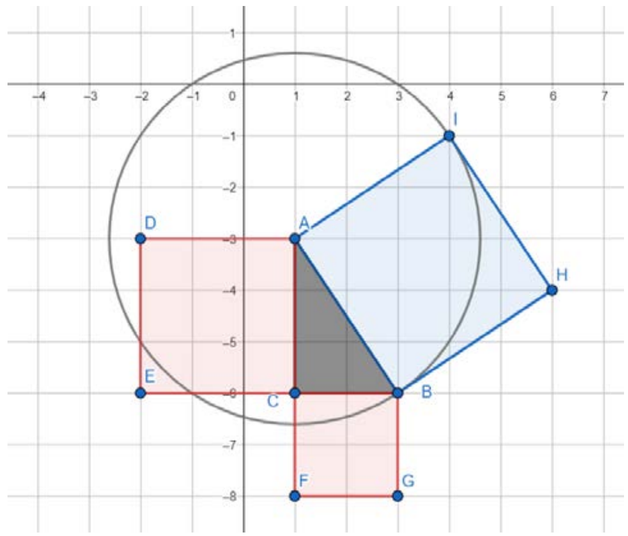
Discussion:

The purpose of this example is to use completing the square to convert an equation of a circle in general form to the center-radius form.

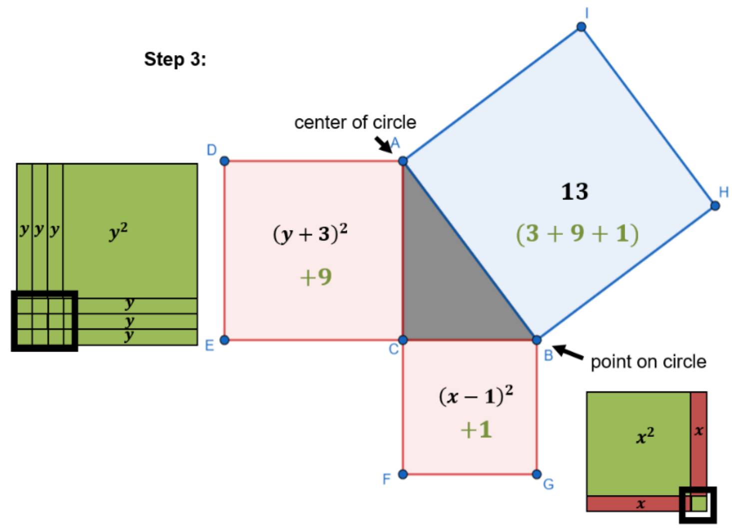
- In part **a.** students should be able to state that a right triangle can be formed where the hypotenuse is the radius. Thus, $(x-h)^2$ would represent the area of the square formed on one leg, and $(y-k)^2$ would represent the area of the square formed on the other leg, and r^2 would represent the area of the square formed on the hypotenuse. In group discussion, reinforce the fact that three squares need to be formed by each side of the triangle so that they satisfy the Pythagorean Theorem.
- In part **b.** students should realize that the general form does not lend itself to seeing the areas of the squares or the center or the radius.
- In part **c.** students should come up with the idea of completing the square. If they are struggling to get there, connect the concept to the visual proof of the Pythagorean Theorem where there is a square on each of the sides of the right triangle. Explain to students that the center of the circle and a point on the circle are endpoints of the hypotenuse. If students are having trouble visualizing it, create a circle with geometric software where the hypotenuse of a right triangle is the radius, and then rotate the triangle around the center of the circle.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.GPE.1)

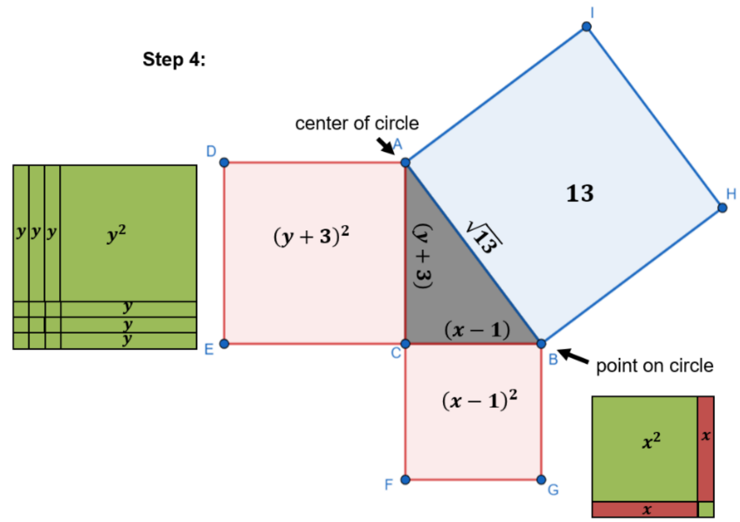
- In part **d.** ask students which terms would go in which square? The x-terms should be placed in the square below the horizontal line of the triangle, and the y-terms should be placed in the square next to the vertical line of the triangle, and the constant term should be placed in the square of the hypotenuse (See Step 1). Use Algebra tiles to illustrate completing the square (See Step 2). Ask students how many unit squares are needed to complete each of the two smaller squares. Students should see that 9 more unit squares are needed for the red square that is represented by $y^2 + 6y + ?$ and 1 unit square is needed for red square that is represented by $x^2 - 2x + ?$. Since the sum of the areas of both red squares along the legs has to equal the area of the blue square along the hypotenuse, the extra area added to the red squares has to be added to the blue square (Step 3). Then students can see that the area of the blue square is $3 + 9 + 1 = 13$. They can also see that $x^2 - 2x + 1$ and $y^2 + 6y + 9$ are perfect squares, where side lengths are $(x - 1)$ and $(y + 3)$ respectively, so the area of the squares can be represented by $(x + 1)^2, (y + 3)^2$, , and 13 (Step 4).
- In part **e.**, students can write the equation of the circle in center-radius form as , $(x + 1)^2 + (y + 3)^2 = 13$.
- In part **f.**, student should identify the center as $(1, 3)$ and the radius as $\sqrt{13}$.



Step 3:



Step 4:



INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.GPE.1)

Students should also understand that for this course radii must be greater 0. In later courses students will explore circles with negative radii which are imaginary.

EXAMPLE

a. How many possible solutions are there of a system involving an equation of a line and a circle. Explain.
b. How can you tell how many solutions a system involving an equation of a circle and a line would have?
c. Find the intersection point of $(x+3)^2+(y-2)^2=10$ and $y=3x+1$ algebraically. Then check your work using a graphing utility.

Discussion: The purpose of this activity is for students to determine the ways a circle and a line can intersect. They should be able to find the point(s) of intersection between a circle and a line.

Ask students how to graph a circle using a graphing utility. Have students rearrange the equations by connecting to A.REI.6. If students are using a graphing utility other than Desmos, lead them to discover that it will take two different equations $y_1=\sqrt{r-(x-h)^2}-k$ and $y_2=-\sqrt{r-(x-h)^2}-k$ to create a circle graph.

TIP!

Be precise with language of equations versus language of functions because not every quadratic equation represents a function, e.g., a circle is not a function.

Instructional Tools/Resources

These tools and resources may be helpful for instruction. Before use, they should be evaluated at the local level to determine their appropriateness for instruction.

Manipulatives/Technology

- Dynamic geometric software (Geometer's Sketchpad®, Cabri®, Desmos, or GeoGebra®)
- Graph paper

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.GPE.1)

Equation of Circles

- Explaining the Equation of a Circle by Illustrative Mathematics is a task that has students derive the formula for an arbitrary circle in a plane.
- Equations of Circles is a GeoGebra applet where students explain how the equation of a circle affects its graph.
- Slopes and Circles by Illustrative Mathematics is a task that has students figure out algebraically that point X is on the circle of diameter AB whenever $\angle AXB$ is a right angle.
- Earthquake is a Desmos activity where the goal is to find out whose houses are within five miles from the epicenter. This activity could be extended by having the student write the equation of a circle.
- Back to School: Deriving Circles with Desmos by Fishing4Tech is a blog that describes an activity using equation of circles and the context of earthquakes. Untitled Graph is a Desmos applet that accompanies this activity where students can write the equation of a circle for the first, second, and third waves of an earthquake.
- Intro: Equation of Circles by Lauren Olson is a Desmos activity that has students write equations of circles in standard and general form to match various descriptions and constraints.
- Sorting Equations of Circles 1 and Sorting Equation of Circles 2 are lessons from the Mathematics Assessment Project where students use the Pythagorean Theorem to derive the equation of a circle, translate between the geometric features of circles and their equations, and sketch a circle from its equation.

Applications using the Equation of a Circle

- All Wet by NCTM is a Problem of the Week where students explore a sprinkler's radius. It may have to be slightly adapted to stress the focus of this cluster. *NCTM now requires a membership to view their lessons.*
- How Can We Water All of That Grass by Robert Kaplisky is a modeling lesson involving sprinklers. It may have to be slightly adapted to stress the focus of this cluster.
- Landscape Irrigation: Circles upon Circles by Conceptual Learning Circles is a modeling lesson involving sprinklers. The extension has students write equations for the circles drawn.

Translation of Circles

- Translation of Circles is a GeoGebra activity that relates the translation of circles to the equation of a circle.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.GPE.1)

Algebra Tiles

- Algebra Tiles Applet by NCTM Illuminations is a link to a virtual algebra tiles applet. *NCTM now requires a membership to view their lessons.*
- Virtual Algebra Tiles is an applet from Michigan Virtual University that allows students to use algebra tiles. This applet allows for positive and negative representation of the tiles.
- CPM Tiles is an applet from the CPM Educational Program that allows students to use algebra tiles. The benefit of this applet is that students can change the dimensionality of x and y. However, it is limited by not allowing for a negative representation of the tiles.
- Algebra tile templates on the SMART Exchange has a variety of useful models that teachers can use if they have access to a SMART Board.

Curriculum and Lessons from Other Sources

- EngageNY, Geometry, Module 5, Topic D, Lesson 17: Writing an Equation for a Circle, Lesson 18: Recognizing Equations of Circles, Lesson 19: Equations for Tangent Lines to Circles are lessons that pertain to this cluster.
- Georgia Standards of Excellence, Geometry, Unit 5: Geometry and Algebraic Connections has many tasks that pertain to this cluster.
- Mathematics Vision Project, Geometry, Module 6: Connecting Algebra and Geometry has several tasks related to this cluster.
- Illustrative Mathematics, Geometry, Unit 6, Lesson 4: Distances and Circles, and Lesson 6: Completing the Square are lessons that pertain to this cluster.

General Resources

- Arizona 7-12 Progression on Geometry is an informational resource for teachers. This cluster is addressed on pages 17-18.
- High School Coherence Map by UnboundED is a map of the relationships among high school mathematics standards. Please note: This resource is aligned to the Common Core not Ohio's Learning Standards. Although the tool is useful, it needs to be checked with respect to Ohio's Learning Standards.
- van Hiele Model of Geometric Thinking is a pdf created by ODE that summarizes the van Hiele levels.

References

- Common Core Standards Writing Team. (2016, March 24). *Progressions for the Common Core State Standards in Mathematics (draft). Grades 7–8, HS, Geometry*. Tucson, AZ: Institute for Mathematics and Education, University of Arizona.

STANDARDS	MODEL CURRICULUM (G.GPE.4, 6)
Geometry **EXPRESSIONS GEOMETRIC PROPERTIES WITH EQUATIONS** **Use coordinates to prove simple geometric theorems algebraically and to verify specific geometric statements.** **G.GPE.4** Use coordinates to prove simple geometric theorems algebraically and to verify geometric relationships algebraically, including properties of special triangles, quadrilaterals, and circles. *For example, determine if a figure defined by four given points in the coordinate plane is a rectangle; determine if a specific point lies on a given circle.* (G, M2) **G.GPE.6** Find the point on a directed line segment between two given points that partitions the segment in a given ratio.	**Expectations for Learning** In middle school, students find the distance between two points in a coordinate system; work with linear functions; solve linear equations; and apply the Pythagorean Theorem in the coordinate system. In addition, they used square root symbols to represent solutions to equations, and they evaluate square roots of rational numbers. In Math 1, students use the coordinate system to justify slope criteria for parallel and perpendicular lines and compute perimeters and areas of geometric figures. In this course, these strategies are used for proof of geometric relationships with respect to properties of figures and partitioning line segments. The student understanding of this cluster begins at van Hiele Level 2 (Informal Deduction/Abstractions) and moves to Level 3 (Deduction). *Note: In Math 2, revisit G.CO.10-11 with respect to coordinate and transformational proofs. Since G.GPE.4 is no longer placed in Math 1, students may not have had exposure to these types of proofs for that content.* **ESSENTIAL UNDERSTANDINGS** • Coordinate proof is a method that uses algebraic techniques to prove geometric theorems and properties. • Properties of geometric figures, especially special quadrilaterals, can be proven on a coordinate plane using lengths of segments, slopes of lines, and equations of lines. • Coordinate proof can be used to prove that figures are congruent or similar. • Partitioning a line segment into a given ratio is an application of similar triangles. **MATHEMATICAL THINKING** • Use accurate mathematical vocabulary to represent geometric relationships. • Make connections between terms and formulas. • Recognize, apply, and justify mathematical concepts, terms, and their properties. • Compute using strategies or models. • Use formal reasoning with symbolic representation. • Determine reasonableness of results. • Solve multi-step problems accurately. • Discern and use a pattern or structure. *Continued on next page*

STANDARDS	MODEL CURRICULUM (G.GPE.4, 6)
	Expectations for Learning, continued **INSTRUCTIONAL FOCUS** • Know and use the Distance Formula. • Use coordinates to prove simple geometric theorems algebraically. • Verify geometric relationships algebraically. • Partition a line segment given a ratio. **Content Elaborations** **OHIO'S HIGH SCHOOL CRITICAL AREAS OF FOCUS** • Math 2 Number 4, page 8 • Math 2 Number 5, page 9 **CONNECTIONS ACROSS STANDARDS** • Understand and apply theorems about circles (G.C.1, 5). • Prove theorems involving similarity (G.SRT.4-5). • Understand the relationships between lengths, areas, and volumes in similar figures (G.GMD.6). • Create equations that describe numbers or relationships (A.CED.4). • Represent and solve equations graphically (A.REI.4). • Apply geometric concepts in modeling situations (G.MG.3).

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.GPE.4, 6)

Instructional Strategies

Note: The model curriculum is not a collection of lessons nor a full curriculum; it does not suggest pace, sequence, or amount of time spent on topics. It provides information about a topic related to the standards including ideas for examples, strategies for teaching, common misconceptions, and possible connections between topics. In addition, the emphasis of the model curriculum is on instruction not on assessment; therefore, the examples in the Instructional Strategies section are written for instructional purposes not necessarily as assessment items. Examples may need to be rewritten to accommodate the needs of each individual classroom.

Standards for Mathematical Practice
This cluster focuses on but is not limited to the following practices:
MP.2 *Reason abstractly and quantitatively.*
MP.4 *Model with mathematics.*
MP.5 *Use appropriate tools strategically.*
MP.6 *Attend to precision.*
MP.7 *Look for and make use of structure.*

This cluster connects geometric to algebraic concepts by using the coordinate plane. The emphasis is on justification. The main skill that should be gleaned from these standards is for students to be able to explain reasons for their thinking and why/how something is true, much like in the ELA writing standards where *evidence* must be included for any claim. Students continue to use precise language and relevant vocabulary to justify steps in their work and construct viable arguments that defends their method of solution. See clusters G.SRT.4-5 for expectations surrounding proof.

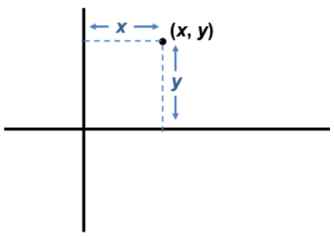
Formerly, students found the distance between two points using the Pythagorean Theorem. Now, using the Pythagorean Theorem, students should understand how to derive the Distance Formula. They should be able to explain how the Pythagorean Theorem, the slope formula, and the Distance Formula are connected.

TIP!

The Euclidean Distance Formula is the shortest possible path between two points in a plane. There are other distance formulas such as geodesic distance which is the shortest path along a curved surface such as the Earth. In this document, the Distance Formula will reference the Euclidean Distance Formula, but it might be helpful to have a discussion so that students are aware that other types of distance formulas exist.

TIP!

Show the dispalcements of points within a quadrant rather than on the *x*- and *y*- axes, so students can view the the points' distance *from* the axes in addition to its distances *along* the axes. Viewing an ordered pair as displacement of a point from the axes is more useful in proving theorems than viewing the ordered pair as horizontal and vertical distances along the axes.



Ohio | Department of Education

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.GPE.4, 6)

VAN HIELE CONNECTION

In Math 2 students are expected to move from Level 2 (Informal Deduction/Abstraction) toward Level 3 (Deduction). Van Hiele Level 2 can be characterized by the student doing some or all of the following:

- analyzing relationships within and between transformations;
- algebraically expressing transformations: reflections about the *x*- and *y*-axes, and rotations of 45°, 90°, and 180° about the origin; *Note: In high school, students should be given an opportunity to experience rotations around any point (not just the origin) and onto any angle, not just multiples of 90. Also, reflections are not limited to the x-and y-axes but could be about any line.*
- understanding and justifying how to use perpendicular bisectors of corresponding points to locate the turn center of a rotation;
- understanding how the composition of two reflections can be a translation or a rotation; and/or
- following a rationale if shown proof.

Van Hiele Level 3 can be characterized by the student doing some or all of the following:

- understanding and creating formal proofs using transformations.

See the van Hiele pdf on ODE's website for more information about van Hiele levels.

TIP! It is important when connecting algebra concepts with geometry concepts that students do not lose sight of the underlying geometric meanings and reduce the geometric concepts to procedures.

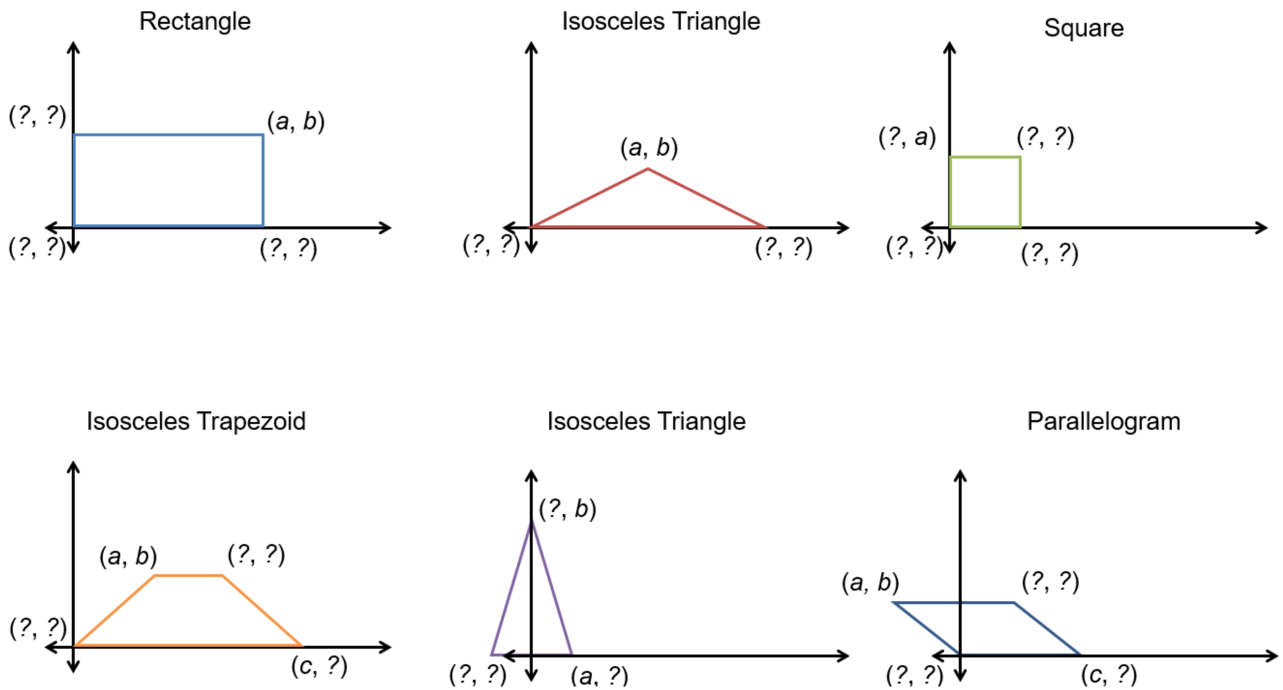
GEOMETRIC THEOREMS WITH COORDINATES

TIP! When approaching a coordinate proof with polygons, encourage students to choose a convenient placement of the axes on the coordinate place such as the origin or by placing a vertex on one of the axes or by using lines of symmetry. This will make calculation easier. It may be helpful to have students give the missing coordinates without introducing any new variables.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.GPE.4, 6)

EXAMPLE

Find the missing coordinates. Only use the variables given in the diagram.



Taken from Houghton Mifflin Geometry.

The part of geometry where students use algebraic proofs to prove geometric theorems is called analytic geometry. Students utilize the coordinate plane to prove geometric theorems algebraically. This includes the properties of special triangles among others. Students need to show all algebraic steps with justification that prove simple geometric theorems.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.GPE.4, 6)

EXAMPLE

Part 1

- In your group, create a rectangle in the coordinate plane. Create three more rectangles. What are the coordinates of each vertex of your rectangle?
- In your group, create a quadrilateral that is not a rectangle in the coordinate plane. Create three more quadrilaterals that are not rectangles. What are the coordinates of each vertex of your quadrilateral?
- Discuss what you notice about the coordinate points that are vertices of rectangles compared to the coordinate points that are vertices of non-rectangles.
- Can you come up with any criteria to determine if four coordinate points make a rectangle?

Discussion: After students experiment in the coordinate plane with various groups of four points that make rectangles and those that do not, they may falsely come to the conclusion that two pairs of points share the same x-coordinate and that two pairs of vertices of a quadrilateral share the same y-value such as (1, 2), (1, 4), (6, 4), (6, 2). This sets them up for the thinking required in Part 2.

Part 2

- Do the coordinates (5, 6), (7, 4), (9, 10), and (11, 8) make a rectangle? Explain, justifying your answer using the coordinate plane.
- Do the criteria that you wrote to create rectangles in Part 1 hold true? If not, revise your criteria to determine if four points create a rectangle.
- Given four points in a coordinate plane, find a complete set of criteria to determine if these points create a rectangle.

Discussion: At this point some students may come up with the idea that the opposite sides have to have the same slope and that opposite sides are congruent. However, Part 3 pushes them to discover that a parallelogram that is not a rectangle also has opposite congruent sides with equal slopes.

Part 3

- Do (2, –1), (3, 2), (5, 0), and (4, –3) create a rectangle? Justify your answer with the coordinate plane.
- Does the criteria that you created for rectangles in Part 2 hold true? If not, revise your criteria to determine if four points create a rectangle.

Discussion: Part 3 pushes students to think that they must have some kind of criteria to create right angles in a rectangle. Therefore, not only do the opposite sides have to be parallel and thus have equal slopes, but the adjacent sides must be perpendicular, so they must have slopes that are opposite reciprocals.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.GPE.4, 6)

Use slopes and the Distance Formula to solve problems about figures in the coordinate plane such as the following:

- Given three noncollinear points, decide if they are vertices of an isosceles, equilateral, or right triangle?
- Given four points, decide if they are vertices of a parallelogram, a rectangle, a rhombus, or a square?
- Given the equation of a circle and a point, does the point lie outside, inside, or on the circle?
- Given the equation of a circle and a point on it, find an equation of the line tangent to the circle at that point.

Students could explore the medians of a triangle intersecting at a point. This could be done using dynamic geometric software or patty paper. See the Instructional Tools/Resources for different applets and resources. However, students could also use coordinate geometry to prove that the medians of a triangle are concurrent. Students could also explore the orthocenter, the circumcenter, and the incenter of a triangle using geometric software and use coordinate geometry to prove their conjectures.

EXAMPLE

Part 1

Use geometric software to discover the relationship between the medians of a triangle.

Discussion: This exploration connects with G.CO.10. Students should discover that all three medians intersect at a single point inside the triangle, so they are concurrent. The concurrent point formed by the medians is called the centroid.

Part 2

Give students Doritos (or cardboard cutouts of triangles or both), and have them balance them on their finger. Have them mark the balancing point as best as they can. Ask them what they notice about the balancing point.

Discussion: Students should notice that the balancing point is the centroid of the triangle—the point of concurrency formed by the three medians.

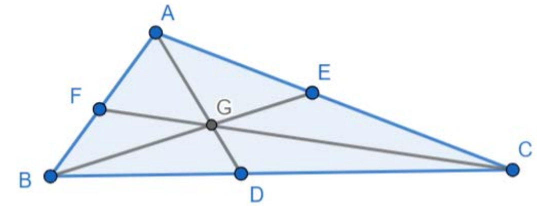
Example continued on next page

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.GPE.4, 6)

Part 3

- Find the coordinates of the centroid of $\triangle TED$ with vertices $T(-3,3), E(5,\ 4)$ and $D(1,-1)$ in two different ways.
- How can you find the centroid of any triangle using vertices $A(x_1,y_1), B(x_2,y_2), C(x_3,y_3)$? Prove that your method is correct.

Discussion: The centroid of triangle TED is (1, 2). Although there are many ways to find the centroid (all of which are acceptable routes for students to pursue), the idea of balancing a Dorito should be fresh in their minds. After discussion, students may come to the idea that the balance point would be the mean of the *x*-coordinates, $\frac{x_1+x_2+x_3}{3}$, and the mean of the *y*-coordinates, $\frac{x_1+x_2+x_3}{3}$. Have them prove their ideas using geometric software. Another way to find the coordinates of the centroid is by creating a system of two equations describing the lines containing two of the medians. The centroid, or point of intersection of the median, is the solution of the system. An alternative way is to apply Ceva's Theorem which states that if D, E, and F are points on sides AB, BC, and CA, respectively, then lines AD, BE, and CF are concurrent if and only if $\frac{BD}{DC}\cdot\frac{CE}{EA}\cdot\frac{AF}{FB}=1$. (See the diagram at the right) Ceva's Theorem may be unfamiliar to many students, but proving it could be used as an extension.



Part 4

a. In $\triangle TED$ from Part 3, find the distance from each vertex to the centroid, and the distance of each midpoint to the centroid. What do you notice about the relationship between the distances?

b. Use geometric software to establish the relationship between the centroid and any median on the triangle

Discussion: Students should discover that the centroid divides each median into a ratio of 2:1, which means that the centroid will always be $\frac{2}{3}$ of the distance from the vertex o the midpoint of the opposite side. In this example, part **a.** and part **b.** could be switched depending what you want to emphasize. Having students find the distances first allows for reinforcement of the Distance Formula, and then use the geometric software to generalize and informally derive the theorem that the centroid divides each median into a ratio of 2:1. Using geometric software first, allows students to generalize and informally derive the theorem that the centroid divides each median into a ratio of 2:1, and then reinforce concepts of partitioning line segments.

Other simple coordinate proofs may include theorems such as the diagonals of a parallelogram bisect each other, the diagonals of a rectangle are congruent, the diagonals of a square are perpendicular, and a point on the perpendicular bisector of a segments it equidistant from the endpoints of the segment.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.GPE.4, 6)

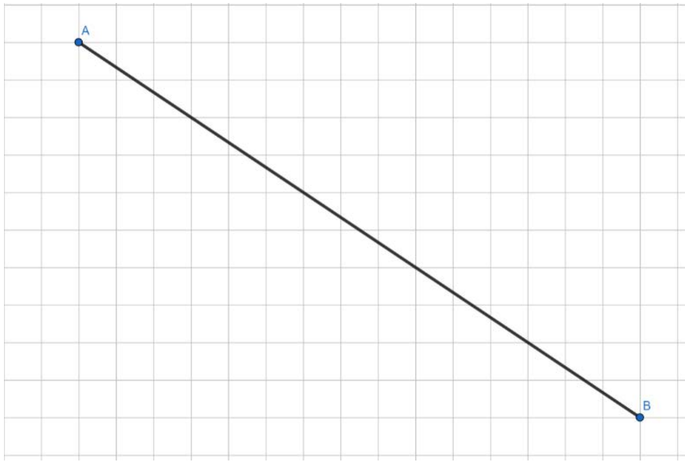
Students often incorrectly think that bisectors divide triangles in half or act as lines of symmetry. Misconceptions around bisectors can be challenging to correct. Students may have a difficult time drawing valid conclusions about angle bisectors, line segment bisectors, and perpendicular bisectors. When presented with a bisector ask students, "What is being bisected?" or "What type of bisector is this?". Then reinforce the definition for the appropriate type of bisector. Formally teaching about and distinguishing between bisectors before teaching students to prove may be more beneficial. Draw attention to the fact that a perpendicular bisector is a special type of line segment bisector.

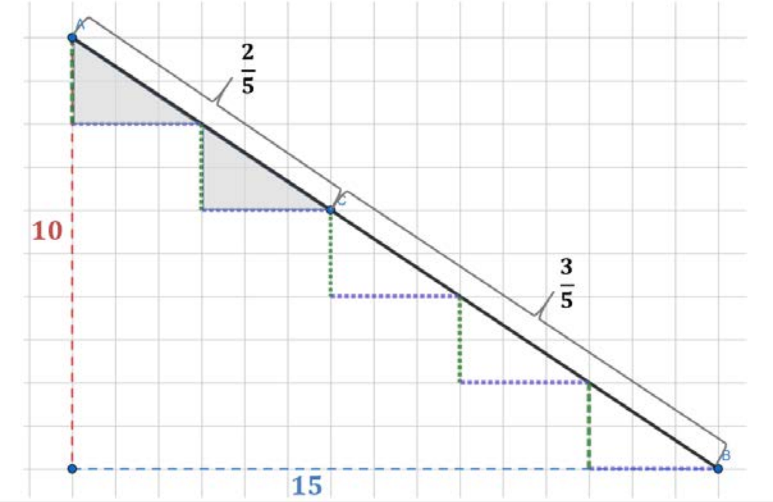
A POINT ON A DIRECTED LINE SEGMENT BETWEEN TWO GIVEN POINTS

Students will find the point that partitions a line segment into a given ratio.

EXAMPLE

a. On the line segment $\overline{AB}$, Plot point C so that it is $\frac{2}{5}$ of the distance from point A to point B. Show that point C divides the segment into a ratio of 2:3.
b. Find the ratio of $AC: AB$.
c. Find the ratio of $BC: AB$.
d. Find the ratio of $BC: AC$.
e. Find the ratio of $AC: BC$.





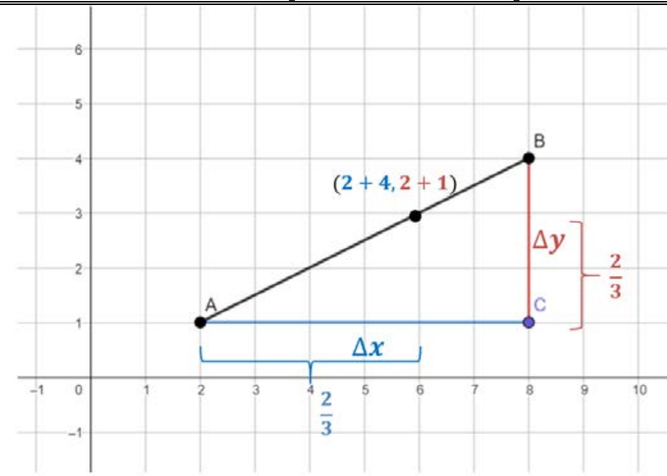
Discussion: Note that the vertical distance (rise) from point A to point B is 10, and the horizontal distance (run) from point A to point B is 15. In order to partition the segment into 5 equivalent parts, these distances can be divided by 5 (since there are 5 sections or partitions) to show the smaller increment of down 2 and right 3. Since each increment is $\frac{1}{5}$ of the line, two increments would be $\frac{2}{5}$ of the line, so point C would be at the end of the second increment. That would leave 3 increments between points C and B, so $\overline{AB}$ would be divided into a ratio of 2:3. Some students may notice that since $\frac{2}{5}$ of 15 is 6 and $\frac{2}{5}$ of 10 is 4, using this information, they would just have to add 6 units to the horizontal direction and 4 units to the vertical direction.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.GPE.4, 6)

EXAMPLE

Part 1

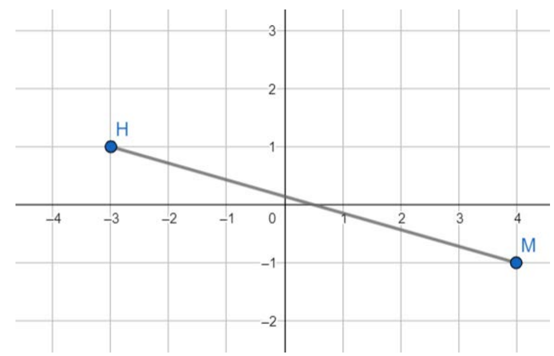
Find the point D which is $\frac{2}{3}$ of the distance along of $\overline{AB}$ from $A(2, 1)$ to $B(8, 4)$.



Discussion: One way to approach this problem is to think about the line segment in terms of the slope triangle and find $\frac{2}{3}$ of the horizontal distance (Δx), and $\frac{2}{3}$ of the distance along the vertical distance (Δy). The horizontal distance is $8 - 2 = 6$ and the vertical distance is $4 - 1 = 3$. Since $\frac{2}{3}$ of 6 is 4, then 4 units need to be added to $x_1 = 2$. Since $\frac{2}{3}$ of 3 is 2, then 2 units need to be added to $y_1 = 1$. Because $2 + 4 = 6$ and $1 + 2 = 3$, point D which is $\frac{2}{3}$ of the distance along of $\overline{AB}$ is $(6, 3)$.

Part 2

Find the point A that is $\frac{2}{3}$ of the distance along of $\overline{HM}$ from $H(-3, 1)$ to $M(4, -1)$.



Discussion: Students are encouraged to make a diagram. This problem is a little more complex than the problem in Part 1, because the length of the line segment does not divide evenly and because the position of point A is down and to the right from point H along the line segment. Therefore, students have to rely more on generalizing the algebraic calculations than counting the slope from the graph. The horizontal distance of $\overline{HM}$ is $|4 - (-3)| = 7$ and the vertical distance is $-1 - 1 = 2$. Since point A is to the right from point H horizontally, $\frac{2}{3}$ of 7 is $\frac{14}{3}$, and $\frac{14}{3}$ needs to be added to H's x-coordinate -3 to get $\frac{5}{3}$. Likewise, since point A is lower than point H vertically, $\frac{2}{3}$ of 2 is $\frac{4}{3}$, and $\frac{4}{3}$ needs to be subtracted from H's y-coordinate 1 to get $-\frac{1}{3}$. Therefore, point A is $\left(\frac{5}{3}, -\frac{1}{3}\right)$.

Midpoint Formula

Given two points, use the Distance Formula to find the coordinates of the midpoint. Students should realize that given two points, they can derive the Midpoint Formula by finding the average of the coordinates. Generalize this for two arbitrary points to derive the Midpoint Formula; beginning on a number line may be helpful.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.GPE.4, 6)

TIP!

For an extension, students can derive the Distance Formula and the Midpoint Formula for three-dimensions. They may also find the formula for the length of a diagonal of a box.

EXAMPLE

If the midpoint P of $\overline{RT}$ is (3, 1), and point R is (−2, 5), what are the coordinates of point T?

Discussion:

Students should realize that because the horizontal distance between points R and P is 5 units, the same horizontal distance of 5 units to the right is from point P to point T. So, the x-coordinate of point T is $3 + 5 = 8$. The vertical distance between points R and P is 4 units. The same distance of 4 units down is from point P to point T. So, the y-coordinate of point T is $1 - 4 = -3$. Therefore, the coordinates of point T are $(8, -3)$.

Instructional Tools/Resources

These tools and resources may be helpful for instruction. Before use, they should be evaluated at the local level to determine their appropriateness for instruction.

Manipulatives/Technology

- Graph paper
- Patty paper
- Scientific or graphing calculators
- Dynamic geometric software (Geometer's Sketchpad®, Cabri®, Desmos®, or GeoGebra®)

Geometric Theorems

- Medians Centroid Theorem (Proof without Words) by Tim Brzezinski is a GeoGebra applet that illustrates the theorem.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.GPE.4, 6)

Points of Concurrency

- Proofs that the Median of a Triangle are Concurrent by Michael McCallum from the University of Georgia is an explanation of the proof. It includes a link to a Geometer's Sketchpad demonstration.
- Triangles: Points of Concurrency by Tim Brzezinski is a GeoGebra webpage that has many applets about the points of concurrency in triangles.
- Points of Concurrency in Triangles by knwilson is a GeoGebra applet where students can see the orthocenter, circumcenter, incenter, and centroid.
- Points of Concurrency in a Triangle by Ed Bernal is a GeoGebra applet where students can drag the vertices of triangles and observe the point of concurrency.
- Finding Centroid by Julia Finneyfrock is a Desmos lesson where students find the centroid.

Partitioning Line Segments

- Finding Triangle Coordinates by Illustrative Mathematics is a task where students use similar triangles in order to study the coordinates of points which divide a line segment into a given ratio.
- Scaling a Triangle in the Coordinate Plane by Illustrative Mathematics is a task where students apply dilations to a triangle in the coordinate plane.

Midpoint Formula

- A Midpoint Miracle by Illustrative Mathematics is a task where students have the opportunity to discover that joining the midpoints of a quadrilateral will form a parallelogram.

Curriculum and Lessons from Other Sources

- EngageNY, Geometry, Module 4, Topic A, Lesson 1: Searching a Region in the Plane, Lesson 2: Finding Systems of Inequalities That Describe Triangular and Rectangular Regions, Lesson 3: Lines that Pass Through Regions, Lesson 4: Designing a Search Robot to Find a Beacon are lessons that pertain to this cluster.
- EngageNY, Geometry, Module 4, Topic B, Lesson 5: Criterion for Perpendicularity, Lesson 6: Segments that Meet at Right Angles, Lesson 7: Equations for Lines Using Normal Segments, Lesson 8: Parallel and Perpendicular Lines are lessons that pertain to this cluster.
- EngageNY, Geometry, Module 4, Topic C, Lesson 9: Perimeter and Area of Triangles in the Cartesian Plane, Lesson 10: Perimeter and Area of Polygonal Regions in the Cartesian Plane, Lesson 11: Perimeters and Areas of Polygonal Regions Defined by Systems of Inequalities are lessons that pertain to this cluster.
- EngageNY, Geometry, Module 4, Topic D, Lesson 12: Dividing Segments Proportionately, Lesson 13: Analytic Proofs of Theorems Previously Proved by Synthetic Means, Lesson 14: Motion Along a Line—Search Robots Again (extension), Lesson 15: The Distance from a Point to a Line are lessons that pertain to this cluster.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.GPE.4, 6)

Curriculum and Lessons from Other Sources, continued

- Mathematics Vision Project, Geometry, Module 6: Connecting Algebra and Geometry has a task that pertains to this cluster.
- Georgia Standards of Excellence Curriculum Frameworks, Geometry, Unit 5: Geometric and Algebraic Connections has many tasks that align to this cluster.
- Illustrative Mathematics, Geometry, Unit 6, Lesson 7: Distances and Parabolas, Lesson 10: Parallel Lines in a Plane, Lesson 11: Perpendicular Lines in the Plane, Lesson 12: It's All on the Line, Lesson 14: Coordinate Proof, Lesson 15: Weighted Averages, Lesson 16: Weighted Averages in a Triangle, Lesson 17: Lines in Triangles are lessons that pertain to this cluster.

General Resources

- Arizona 7-12 Progression on Geometry is an informational document for teachers. This cluster is addressed on pages 17-18.
- Arizona High School Progression on Modeling is an informational document for teachers.
- High School Coherence Map by UnboundED is a map of the relationships among high school mathematics standards. Please note: This resource is aligned to the Common Core not Ohio's Learning Standards. Although the tool is useful, it needs to be checked with respect to Ohio's Learning Standards.
- van Hiele Model of Geometric Thinking is a pdf created by ODE that summarizes the van Hiele levels.
- G-GPE.4 is a blog thread on Bill McCallum's Mathematical Musings blog is a discussion about the example listed in the standard G.GPE.4.
- G-GPE.6 is a blog thread on Bill McCallum's Mathematical Musings blog is a discussion about the rationale for including the specific skill listed in the standard.
- G.GPE.5 is a blog thread on Bill McCallum's Mathematical Musings blog is a discussion about the depth of understanding in this standard.
- Points of Concurrency is a blog thread on Bill McCallum's Mathematical Musings blog is a discussion about points of concurrency.
- Providing the Slope Criteria is a blog thread on Bill McCallum's Mathematical Musings blog is a discussion about circular reasoning in G.GPE.5 and G.CO.4.

References

- Common Core Standards Writing Team. (2016, March 24). *Progressions for the Common Core State Standards in Mathematics (draft). Grades 7–8, HS, Geometry*. Tucson, AZ: Institute for Mathematics and Education, University of Arizona.
- Common Core Standards Writing Team. (2013, July 4). P*rogressions for the Common Core State Standards in Mathematics (draft). High School, Modeling.* Tucson, AZ: Institute for Mathematics and Education, University of Arizona.
- Jurgensen, R., Brown, R., & Jurgensen, J. (1985). *Geometry: Teacher's Edition.* Houghton Mifflin Geometry: Boston, MA.
- Matsuura, R. & Sword, S. (February 2015). Illuminating coordinate geometry with algebraic symmetry. *Mathematics Teacher*, *108, (6),* 470-473.

STANDARDS	MODEL CURRICULUM (G.GMD.1, 3)
Geometry **GEOMETRIC MEASUREMENT AND DIMENSION** **Explain volume formulas, and use them to solve problems.** **G.GMD.1** Give an informal argument for the formulas for the circumference of a circle, area of a circle, and volume of a cylinder, pyramid, and cone. *Use dissection arguments, Cavalieri's principle, and informal limit arguments.* **G.GMD.3** Use volume formulas for cylinders, pyramids, cones, and spheres to solve problems. ★	**Expectations for Learning** In middle school, students use established circumference, area, and volume formulas for two- and three-dimensional figures. Instead of using area and volume formulas rotely, students in this cluster give informal justifications for these formulas and use them to solve problems. The student understanding of this cluster begins at van Hiele Level 1 (Analysis) and moves to Level 2 (Informal Deduction/Abstraction). **ESSENTIAL UNDERSTANDINGS** • A three-dimensional solid can be viewed as a stack of layers. • If all of the layers of a three-dimensional solid have the same area, then the volume is the area of the base times the height. • The volume remains unchanged when layers parallel to the base in a three-dimensional solid are shifted. • A cone's volume is $\frac{1}{3}$ of the volume of a cylinder if their base areas are equal and their heights are congruent. • A pyramid's volume is $\frac{1}{3}$ of the volume of a prism if their base areas are equal and their heights are congruent. • Volume, like area, is additive, so to find the volume of a composite figure, cut the figure into pieces of known volume, and add or subtract as appropriate. • The cross sections of a cylinder are circles of equal area. • The cross sections of a prism are congruent to the base, so therefore the areas are equal. **MATHEMATICAL THINKING** • Draw a picture or create a model to make sense of a problem. • Make and modify a model to represent mathematical thinking. • Attend to meaning of quantities. • Consider mathematical units involved in a problem. • Solve real-world and mathematical problems accurately. • Determine reasonableness of results. • Use informal reasoning. Continued on next page.

STANDARDS	MODEL CURRICULUM (G.GMD.1, 3)
	Expectations for Learning, continued **INSTRUCTIONAL FOCUS** • Explain and justify the formulas for the circumference of a circle, area of a circle, and volume of a cylinder, pyramid, and cone. • Apply volume formulas in real-world and mathematical problems. • (+) Informally apply Cavalieri's principle for the formulas for the volume of a sphere and other solid figures. **Content Elaborations** **OHIO'S HIGH SCHOOL CRITICAL AREAS OF FOCUS** • Math 2, Number 6, page 10 **CONNECTIONS ACROSS STANDARDS** • Understand and apply theorems about circles (G.C.5). • Understand the relationships between lengths, areas, and volumes (G.GMD.5-6). • Apply geometric concepts in modeling situations (G.MG.1-3). • Create equations that describe numbers or relationships (A.CED.2, 4).

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.GMD.1, 3)

Instructional Strategies

Note: The model curriculum is not a collection of lessons nor a full curriculum; it does not suggest pace, sequence, or amount of time spent on topics. It provides information about a topic related to the standards including ideas for examples, strategies for teaching, common misconceptions, and possible connections between topics. In addition, the emphasis of the model curriculum is on instruction not on assessment; therefore, the examples in the Instructional Strategies section are written for instructional purposes not necessarily as assessment items. Examples may need to be rewritten to accommodate the needs of each individual classroom.

Standards for Mathematical Practice
This cluster focuses on but is not limited to the following practices:
MP.3 *Construct viable arguments and critique the reasoning of others.*
MP.4 *Model with mathematics.*
MP.5 *Use appropriate tools strategically.*
MP.7 *Look for and make use of structure.*
MP.8 *Look for and express regularity in repeated reasoning.*

In Grade 8, students were required to know and use the formulas for volumes of cylinders, cones, and spheres. In this cluster those formulas are derived by a combination of concrete demonstrations and formal reasoning. Students who eventually go on to study calculus will formally derive formulas for the volume of a pyramid, cone, sphere and other solids using definite integrals.

Connect G.GMD.4 with this cluster as it is closely related.

It might be helpful to tie in concepts of area related probability concepts such as unions, intersections, subsets, and complements in S.CP.1. For example, the area of the union of two regions is the sum of the areas minus the area of the intersection. Also, the area of the difference of two regions, where one is contained in the other or subset, is the difference of the areas.

VAN HIELE CONNECTION

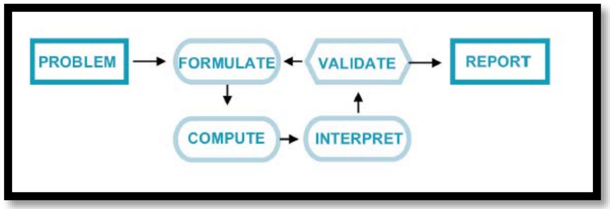
In Math 2 students are expected to move from Level 1(Analysis) to Level 2 (Informal Deduction/Abstraction). Van Hiele Level 2 can be characterized by the student doing some or all of the following:

- fully understanding the procedures and formula for determining measurement of simple, familiar shapes;
- generalizing unit-measure enumeration to include fractional units;
- understanding and making unit conversions; and/or
- generalizing measurement to non-squares for area and non-cubes for volume.

See the van Hiele pdf on ODE's website for more information about van Hiele levels.

MODELING

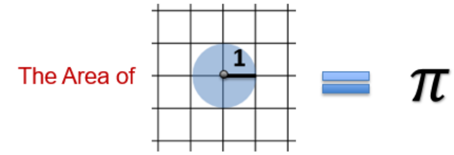
G.GMD.3 is a modeling standard. See page 12 for more information about modeling.



Formulas are mathematical models of relationships among quantities. Some formulas describe laws of nature such as gravity, others are measurements of one quantity in terms of another. This ties in with A.CED.4.

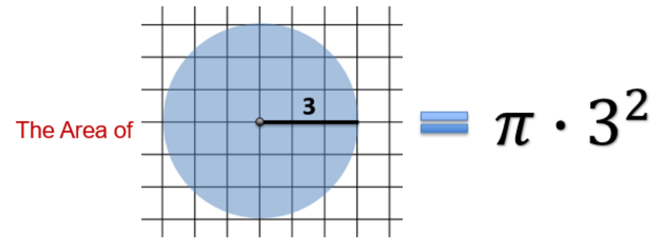
INFORMAL ARGUMENTS FOR FORMULAS

Students should draw a picture, write a paragraph, demonstrate, or describe orally the rationale for the development of area, circumference, and volume formulas for circles, cylinders, pyramids, cones, and spheres. Give the opportunities for students to explore multiple ways to derive area and volume formulas.



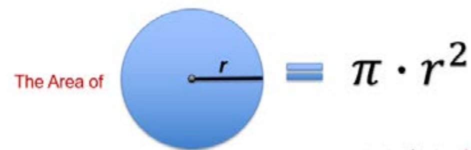
Circles

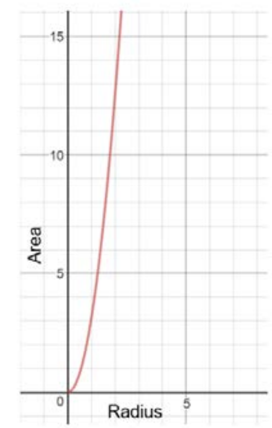
Building on the concepts of scale factor (G.GMD.6) and on the fact that all circles are similar (G.C.1), students should know that when a figure is scaled by a factor of x, the area changes by a factor of x^2. Define the constant π as the area of the region inside a circle when the radius is 1 unit.



EXAMPLE

- π can be defined as the area of the region inside a circle when the radius is 1 unit. Draw an example of π using this definition.
- Enlarge a circle by a scale factor of your choosing and draw a picture representing your circle. How does the radius of the circle change? How does the area of the circle change?
- Enlarge a circle by a scale factor of r and draw a picture representing your new circle. How does the radius of the circle change? How does the area of the circle change?
- How do your pictures represent the formula for area of a circle?
- Why is it beneficial to define π in this way?



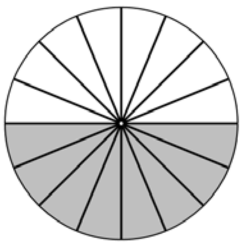


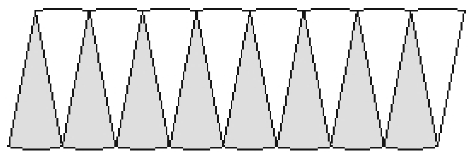
Discussion: Enlarging a circle by a scale factor of r means that the radius changes by a scale factor of r, and the area changes by a scale factor of r^2; hence the formula for the area of a circle is $A = \pi r^2$. Draw students' attention to the fact that although there is a relationship between the area and radius, it is not directly proportional; the graph of $A = \pi r^2$ is not a straight line, but rather it is squarely proportional because the graph is not a straight line.

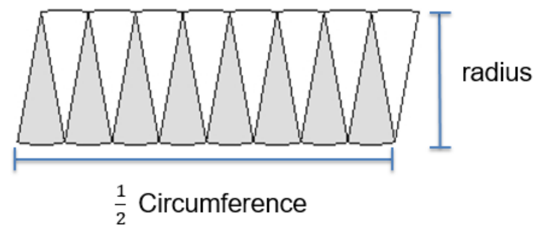
INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.GMD.1, 3)

TIP! Refer to the area of the circle as the area of the region inside the circle, since the circle itself is just the curved outline of the figure.

Another visual for understanding the area of a circle can be modeled by cutting up a circular disc (for example a paper plate) into pieces along diameters and reshaping the pieces into a parallelogram. Cut a cardboard circular disk into 8 congruent sectors and rearrange the pieces to form a shape that looks like a parallelogram with two scalloped edges. Repeat the process with 16 sectors and note how the edges of the parallelogram look "straighter." Discuss what would happen in the case where the number of sectors becomes infinitely large.







Ask students to identify what represents the circumference and the radius in their new parallelogram-looking shape. Identifying the radius gives cause for conversations. Students should come to the conclusion that the parallelogram's height is the radius and its length is $\frac{1}{2}$ of the circumference. Therefore, another formula for area is $A = \frac{1}{2}Cr$.

Since students already found in the previous example that the area of circle is $A = \pi r^2$ or $A = \pi rr$, then by substitution $\frac{1}{2}Cr = \pi rr$. After dividing each side by r, the formula simplifies to $\frac{1}{2}C = \pi r$. They can then solve for C by multiplying each side by 2 to get $C = 2\pi r$. Since $2r = d$, $C = 2\pi r$ is equivalent to $C = d\pi$.

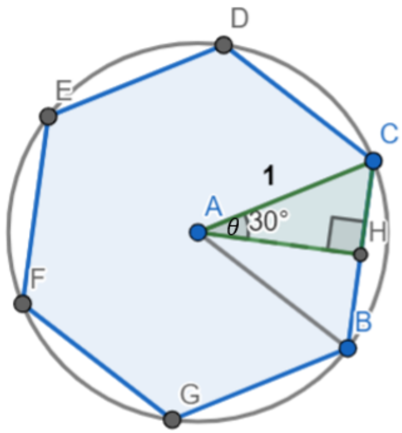
INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.GMD.1, 3)

Students can use informal limit arguments to find the area of a circle such as finding the area of polygons inscribed in circles as the number of sides increases.

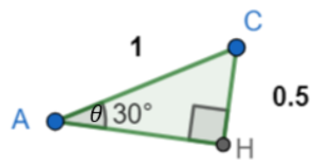
EXAMPLE

Part 1

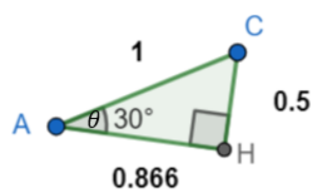
- Find the area of a hexagon inscribed in a circle with a radius of 1.
- How does the area of the inscribed hexagon compare to the area of the circle? Explain.



$$\theta = \frac{360}{6} \div 2 = 30°$$



$$\text{half–chord} = \sin 30° = \frac{x}{1} = 0.5$$



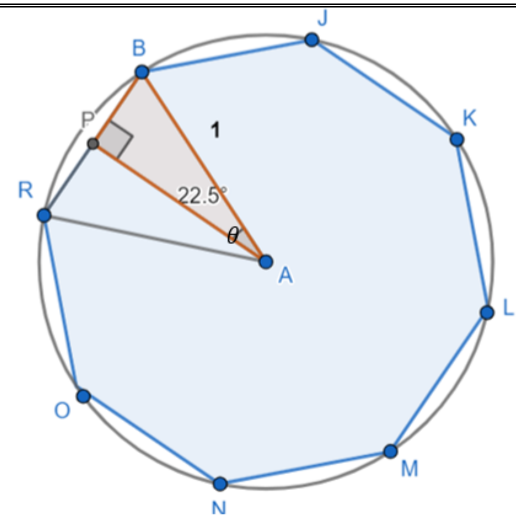
$$\text{altitude} = \cos 30° = \frac{x}{1} \approx 0.866$$

$$A_{hexagon} \approx 12\left(\frac{1}{2}(0.5)(0.866)\right) = 2.598$$

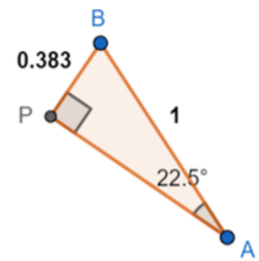
INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.GMD.1, 3)

Part 2

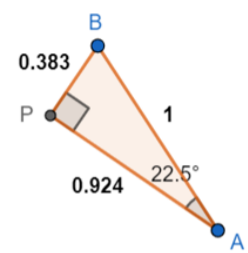
- Find the area of an octagon inscribed in a circle with a radius of 1.
- How does the area of the inscribed hexagon in Part 1 compare to the area of the octagon? Explain.
- Which area is closer to the area of the circle: the hexagon in part 1 or the octagon in part 2? Explain.
- What type of regular polygons would have an area that would be the closest to the area of the circle? Explain.



$$\theta = \frac{360}{8} \div 2 = 22.5°$$



$$\text{half–chord} = \sin 22.5° = \frac{x}{1} \approx 0.383$$



$$\text{altitude} = \cos 22.5° = \frac{x}{1} \approx 0.924$$

$$A_{octagon} \approx 16\left(\frac{1}{2}(0.383)(0.924)\right) = 2.831$$

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.GMD.1, 3)

Part 3

- Use an Excel spreadsheet to show the area of different regular polygons inscribed in a circle with a radius of 1 as the number of sides increases.
- How can you use this information to determine the area of a circle? Explain.

	A	B	C	D	E	F
1	Number of Sides	Angle Measure	Sine Side	Altitude or Cosine	Area	
2	6	30	0.5	0.866025	2.598076	
3	7	25.71429	0.433884	0.900969	2.73641	
4	8	22.5	0.382683	0.92388	2.828427	
5	9	20	0.34202	0.939693	2.892544	
6	10	18	0.309017	0.951057	2.938926	
7	11	16.36364	0.281733	0.959493	2.973524	
8	12	15	0.258819	0.965926	3	
9	13	13.84615	0.239316	0.970942	3.020701	
10	14	12.85714	0.222521	0.974928	3.037186	
11	15	12	0.207912	0.978148	3.050525	
12	16	11.25	0.19509	0.980785	3.061467	
13	17	10.58824	0.18375	0.982973	3.070554	
14	18	10	0.173648	0.984808	3.078181	

Discussion: The goal of this activity is for students to see that as they increase the number of sides of a regular polygon inscribed in a circle, the area of the polygon approaches that of a circle. The reason a unit circle is used is that in a unit circle, the area equals π (which is how π was defined in an earlier example). It also gives students a chance to apply what they have learned in trigonometry using a unit circle. One way to fill out the Excel Spreadsheet is as follows:

- Type the number "6" in cell A2. Type the formula "=A2+1" into cell A3, and drag down the formula for Column A.
- To get the angle measure of central angle, in B2 type the formula "=360/(2*A2)" or another equivalent formula. Then drag the formula through the rest of the column.
- To get sine which is the half-chord in a circle with the radius of 1, type "=SIN(RADIANS(B2))" in cell C2. Then drag the formula through the rest of the column.
- Students can either use the Pythagorean Theorem to find the altitude (or apothem) or realize that the cosine is the altitude in a right triangle with a radius of 1. If students recognize this, they can type =COS(RADIANS(B2)) into cell D2 and then drag the formula through the rest of the column.
- To find the area of the triangle, students can type "=C2*D2*A2" into cell E2, and then drag the formula through the rest of the columns. *Note: Some students may want to have more columns showing more steps, so the formula may not be quite as condensed. Individual student methods should be encouraged.*

In middle school students should have discovered that the circumference is a little more than three times the diameter of the circle. Now they should build on that to explore the formula for circumference.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.GMD.1, 3)

EXAMPLE

Part 1

- Find the perimeter of a hexagon inscribed in a circle with a radius of 1.
- How does the perimeter of the inscribed hexagon compare to the circumference of the circle? Explain.

Part 2

- Find the perimeter of an octagon inscribed in a circle with a radius of 1.
- How does the perimeter of the inscribed hexagon in Part 1 compare to the perimeter of the octagon? Explain.
- Which perimeter is closer to the circumference of the circle: the hexagon in Part 1 or the octagon in Part 2? Explain.
- What type of regular polygons would have a perimeter that would be the closest to the circumference of the circle? Explain.

Part 3

- Using the spreadsheet from the previous example show what happens to the perimeter of different regular polygons inscribed in a circle with a radius of 1 as the number of sides increases.
- How can you use this information to determine the circumference of a circle? Explain.

Discussion: Students should realize that in a circle with a radius of 1, the circumference is 2π. Therefore, as the number of sides in a regular polygon inscribed in a circle increases, the perimeter of the regular polygon approaches the circumference or 2π which is approximately 6.283185307. One way to fill out the Excel Spreadsheet is as follows:

- Since each side is a chord, the perimeter of a regular polygon is twice the half-chord (sine) times the number sides. In cell F2, type "=2*C2*A2" or another equivalent formula. *Note: Some students may want to have more columns showing more steps shown, so the formula may not be quite as condensed. Individual student methods should be encouraged.*

	A	B	C	D	E	F	G
1	Number of Sides	Angle Measure	Sine Side	Altitude or Cosine	Area	Perimeter	
2	6	30	0.5	0.866025	2.598076	6	
3	7	25.71429	0.433884	0.900969	2.73641	6.074372348	
4	8	22.5	0.382683	0.92388	2.828427	6.122934918	
5	9	20	0.34202	0.939693	2.892544	6.15636258	
6	10	18	0.309017	0.951057	2.938926	6.180339887	
7	11	16.36364	0.281733	0.959493	2.973524	6.198116251	
8	12	15	0.258819	0.965926	3	6.211657082	
9	13	13.84615	0.239316	0.970942	3.020701	6.222207271	
10	14	12.85714	0.222521	0.974928	3.037186	6.230586151	
11	15	12	0.207912	0.978148	3.050525	6.237350725	
12	16	11.25	0.19509	0.980785	3.061467	6.242890305	
13	17	10.58824	0.18375	0.982973	3.070554	6.247483606	
14	18	10	0.173648	0.984808	3.078181	6.251334396	
15	19	9.473684	0.164595	0.986361	3.084645	6.254594431	
16	20	9	0.156434	0.987688	3.09017	6.257378602	
17	21	8.571429	0.149042	0.988831	3.094929	6.259775179	
18	22	8.181818	0.142315	0.989821	3.099058	6.261852884	
19	23	7.826087	0.136167	0.990686	3.102663	6.263665858	
20	24	7.5	0.130526	0.991445	3.105829	6.265257227	
21	25	7.2	0.125333	0.992115	3.108624	6.266661678	
22	26	6.923077	0.120537	0.992709	3.111104	6.267907373	
23	27	6.666667	0.116093	0.993238	3.113314	6.269017363	
24	28	6.428571	0.111964	0.993712	3.115293	6.270010662	

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.GMD.1, 3)

EXAMPLE

Wind a piece of string or rope upon itself to form a circular disk (like a circular rug) or use an onion with concentric rings. Cut the figure along its radius. Stack the pieces to form a triangular shape with base *C* ($2\pi r$) and altitude *r*. Again, discuss what would happen if the string or onion ring became thinner and thinner so that the number of pieces in the stack became infinitely large. Then calculate the area of the triangle to derive the formula $A = \pi r^2$.

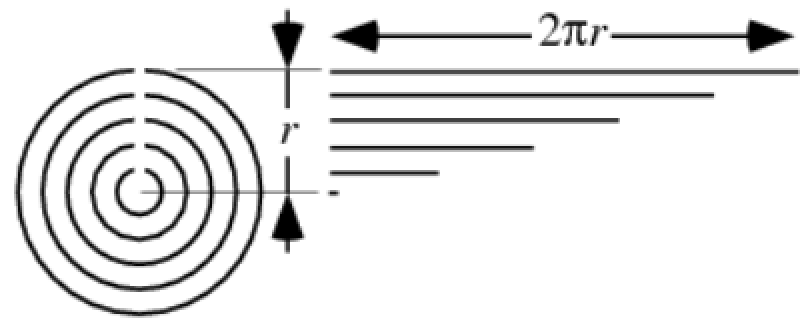


Image from Calculus Proof for the Area of a Circle from Math Stack Exchange.

Pi

Have students investigate the history of pi. There are several major questions to be answered:

- When and why was the symbol π chosen to represent this number?
- How did the "formula" for the area of a circle evolve?
- How is it possible to compute more than a billion digits of the number π?
- What is meant by saying that π is a transcendental number?

Students need to understand that π is an irrational number that can be placed on a number line. Discuss the difference between usefulness and precision. Students need to understand that 3.14 or $\frac{22}{7}$ are approximations of π, and although the approximations are useful, they are not the most precise. Throughout history different fractional values other than $\frac{22}{7}$ were used for π. Many times the fraction chosen to represent π was the most useful, not necessarily the most precise. That is why many people use 3.14 (or $\frac{277}{150}$) instead of $\frac{22}{7}$ even though $\frac{22}{7}$ is technically more precise. Even the π button on the calculator is just an approximation of π as the value is infinitely long. The level of precision needed depends on the application that is being used. For example, a construction worker may need more precision in building a large structure such as a skyscraper versus a small structure such as a small patio. A physicist might need more precision than a carpenter, since there needs at least 30 digits of π to approximate the size of the universe. Context must be taken into account when determining the level of precision that is needed.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.GMD.1, 3)

Many students want to incorrectly think of infinity as a number. Avoid this by talking about a quantity that becomes larger and larger with no upper bound.

Students may incorrectly believe that π is an exact number such as 3.14 or $\frac{22}{7}$ rather than understanding that 3.14 and $\frac{22}{7}$ are just approximations of π. Students can do activities with measuring circles or by placing π on number lines to break this misconception.

Cylinders

There are varying definitions of cylinders. Some mathematicians emphasize the distinction between solids and surfaces which can be distinguished in three-dimensional figures. A surface is the boundary of a three-dimensional figure. The region enclosed by the surface together with the surface itself is a solid. The differences between a brick and an empty box illustrate this concept. These mathematicians define prisms and cylinders (or rather circular cylinders) as subsets of general cylinders (cylindrical surface). A cylindrical solid can also be viewed as the set of points between a region (base) and its translated image in space, including the region and its image. Using this definition, a cylinder and a prism are special cases of a cylindrical solid. On the other hand, circular cylinder and prism are also defined as the surfaces of a cylindric solid whose base is a circle and polygon respectively. A cylindrical solid may be formed using any shape as a base not just circles and polygons. *Note: This definition of a cylinder and prism mentions surfaces; however, the common use of the term refers to solids. It is like defining polygons and other closed figures such as circles by their boundaries but not their interior points.* See Cylinder by Wolfram MathWorld and Cylinder by Wikipedia.

Others define a cylinder as any solid whose cross sections are perpendicular to some axis running through the solid are all the same. Again, in this viewpoint prisms and circular cylinders (what many commonly call cylinders) are subsets of cylinders.

Another viewpoint is to define a cylinder and a prism as separate solids defined by the shape of their bases: A cylinder is a solid with two congruent circular bases and a prism is a solid with two congruent polygon bases. In this instance a circular cylinder can be thought of as a prism where the base has an infinite number of sides. It is up to each district/teacher to decide how to define three-dimensional figures for their students.

Regardless, there is a connection between prisms and cylinders, so it is appropriate to teach prisms and figures with two congruent irregular parallel bases even though these types of figures are not explicitly called out in the high school standards. These types of irregular figures connect to the modeling standards. See standard G.MG.1.

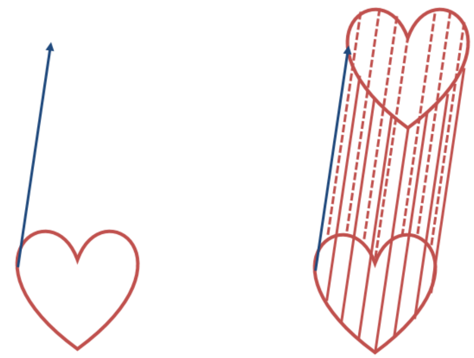
As one figure can by defined as a subset of the other, therefore it may make more sense to generalize the formulas as $V = Bh$, where B is the area of the Base. This formula will be true for all cylindrical solids instead of the figure specific formulas: $V_{Cylinder} = \pi r^2 h$ and $V_{prism} = lwh$. Also, the volume can be viewed as the product of the area of the preimage region and the distance between the planes of the bases formed by the preimage and image. It may be helpful to connect with G.CO.14 and have students create a hierarchy of three-dimensional figures to show the nuances between the definitions.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.GMD.1, 3)

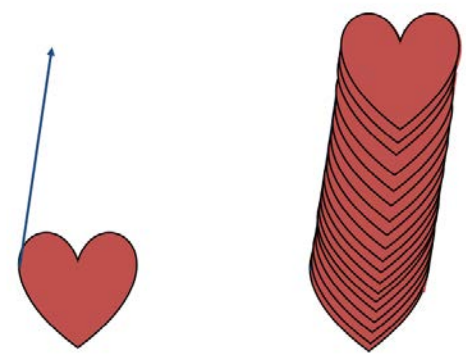
Cylinders can be explored in terms of translations.

EXAMPLE

- Create a surface by translating the figure along a directed line segment (vector).



- Create a solid by translating a region along a directed line segment (vector).



- Compare and contrast the two resulting figures in part **a.** and part **b.**
- How could you find the volume of the two figures?
- How does the volume of the two figures compare?

Discussion: The benefit of this type of problem is that it builds upon transformational geometry that has been emphasized in this course. It also has the ability to lay a foundation for students who will be pursuing advanced mathematics courses. Students should compare cases and come to the realization that the first figure is just a surface. It is "empty" like a jar, can, or empty box. The second figure is a solid made up of disks like a stack of tissues or a stack of coins or a mold of Jell-O. Part **b.** can be connected to G.GMD.4 with respect to the cross section that is parallel to the base. The volume of the figures can be found using $V = Bh$ which is the same formula that can be used for cylinders and prisms, so to calculate the volumes, the area of the base (preimage) and the height would be needed. Regardless of whether the figure is a solid or a surface (empty solid), the volume is the same. This problem can be extended to a discussion of surface areas. Surface area could be illustrated by using a piece of notebook paper to show the lateral area in combination with the two bases. *Note: Although not needed for this problem, students may want to find the area of the heart in order to calculate the volume. In order to find the area of the heart, they would need to use dissection arguments.*

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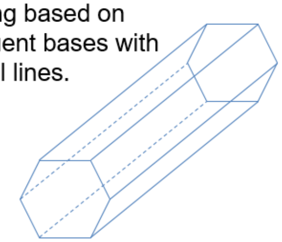
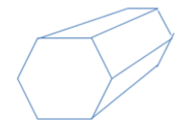
INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.GMD.1, 3)

TIP! It may be helpful to have students draw examples of prisms and cylinders based on definitions and then based on perspective. Have them explain any differences. For example, the corresponding edges on the bases of the perspective drawing would not be parallel. Instead they would be drawn as if they intersected at the horizon. The figure would change depending on the viewer's position.

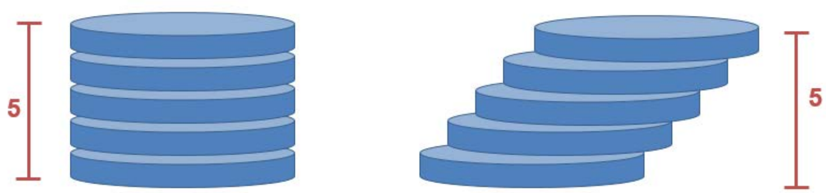
Drawing based on congruent bases with parallel lines.



Perspective drawing with non-parallel lines



Understanding the volume of cylinders builds upon the concepts of G.GMD.4: Identify the shapes of two-dimensional cross-sections of three-dimensional objects. As students identify the shapes of the cross-sections that are parallel to the base, they can make the connection that figures are just composed of layers where the height could represent the number of layers. In prisms and cylinders the layers are figures congruent to the base and in pyramid figures the layers are figures that are similar to the base.



Students should also explore Cavalieri's principle. Introduce Cavalieri's principle using a concrete model, such as a deck of cards. Use Cavalieri's principle with cross sections of cylinders, pyramids, and cones to justify their volume formulas. It can also be shown by stacking coins to make a right cylinder, and then shifting the coins to make an oblique cylinder illustrating that the volume remains constant.

Although the standard only calls for the volume of cylinders, it may be beneficial to also have students find the surface area of cylinders as finding the surface area of cylinders is missing from the standards.

Invite an architect or an engineer to visit the class to demonstrate some uses of cylinders, pyramids, cones or spheres in their work.

Pyramids and Cones (Cases of Conic Solids)

Just as some mathematicians consider prisms and circular cylinders special cases of generalized cylinders, some mathematicians consider circular cones and pyramids special cases of general cones. One definition of a conic solid is the set of points between the vertex and all the points of its base (which can be any shape) together with the vertex and the base. Using this definition, a cone is the surface of a conic solid whose base is a circle, and a pyramid is the surface of a conic solid whose base is a polygon.

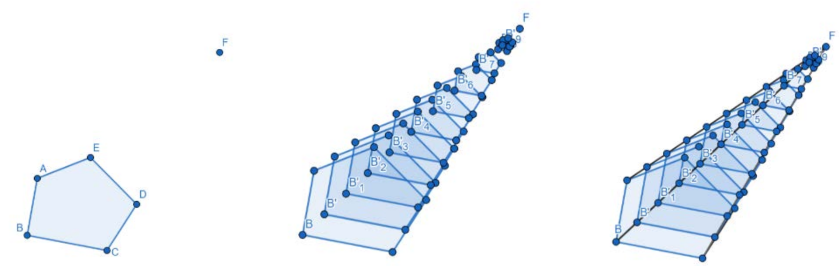
A conic solid can also be defined in terms of dilations. A conic solid is the set of points between a region and its dilated image in space, including the region and its image. In this course the scale factor of the dilated image is between 0 and 1.

Another definition of a conic solid is the set of all line segments from the apex to the base. Some mathematicians make a distinction between a cone and a pyramid by defining the figures in terms of their base. Again, in this instance a cone can be thought of a pyramid whose base has infinitely many sides. It is up to a district to determine how to define figures; however, using more inclusive definitions of generalized cones allows students to generalize the formula $V_{cone} = \frac{1}{3}Bh$, where B is the area of the Base, to a variety of figures. For more information regarding defining cones, see Cone by Wikipedia and Generalized Cone by Wolfram Mathworld.

Just as a cylinder can be related to cross sections congruent to the base that is created by translations, a cone can be related to the cross sections similar to the base, that is created by dilations with a scale factor *k* (where $1 > k > 0$) about a fixed center of dilation.

EXAMPLE

- Using dynamic geometric software create a shape and a center of dilation outside the figure.
- Dilate the original figure around the dilation point with a scale factor between 0 and 1.
- Choose a different scale factor between 0 and 1 and dilate the original figure again.
- Repeat the process in part **c.** an additional 12 times.
- Connect any vertices in your figure. What do you notice?
- Justify any observations you had in part **e.**

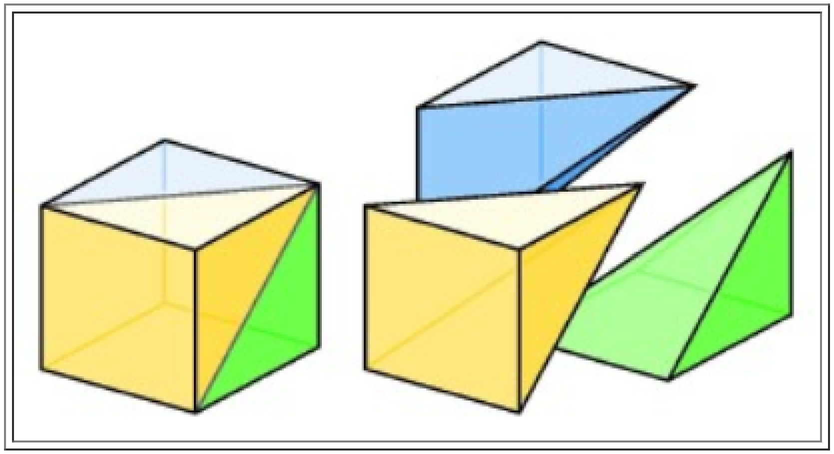


Discussion: Students should come to the conclusion that a conic solid or pyramid was formed (depending on the terminology used in the classroom). Each cross section is a dilation of the original 2D shape by a scale that is less than 1 that pushes the plane towards the center of dilation. This is because each of the cross sections are similar figures to the original figure (which becomes the base) and dilations less than 1 pull the plane towards the center of dilation. Discuss why a dilation maps a plane to a parallel plane. An interesting extension for students who will be pursuing advanced math courses would be to ask students "What would the cone would look like if the scale factor was not limited to positive numbers?"

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.GMD.1, 3)

For pyramids and cones, the presence of the factor $\frac{1}{3}$ in the formulas for volumes need some explanation. Using a set of plastic shapes, pour liquid or sand from one shape into another to informally demonstrate the relationship between the capacity (volume) of a cone and the capacity (volume) of a cylinder with the same base and height and a pyramid and prism with the same base and height. Another way to help students understand presence of the factor $\frac{1}{3}$ for pyramids is by using Geoblocks®. The set includes three pyramids with equal bases and altitudes that will stack to form a cube. An algebraic approach involves the formula for the sum of squares ($1^2 + 2^2 + \ldots + n^2$).

Although pouring water from pyramids into cubes (and cones into cylinders) is a good informal argument to illustrate that the volume of a pyramid is $\frac{1}{3}$ the volume of the prism with the same base and height, a more formal argument is disecting a cube into congruent pyramids. After the presence of the coefficient $\frac{1}{3}$ has been justified for the formula of the volume of the pyramid ($A = \frac{1}{3}Bh$), it can be argued that it must also apply to the formula of the volume of the cone by considering a cone to be a pyramid that has a base with infinitely many sides. There are applets and lesson plans with pyramid net templates in the Instructional Resources/Tools section.



Three congruent pyramids meet along a diagonal of a cube.

Image taken from Volume Patterns for Pyramids by Brown University
http://www.math.brown.edu/~banchoff/Beyond3d/chapter2/section02.html

TIP! The inclusion of the coefficient $\frac{1}{3}$ in the formulas for the volume of a pyramid or cone and $\frac{4}{3}$ in the formula for the volume of a sphere remains a mystery for many students. In high school, students should attain a conceptual understanding of where these coefficients come from. Concrete demonstrations, such as pouring water from one shape into another should be followed by more formal reasoning.

Using arguments based on informal limits can also help students understand the formula for a pyramid or cone.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.GMD.1, 3)

EXAMPLE

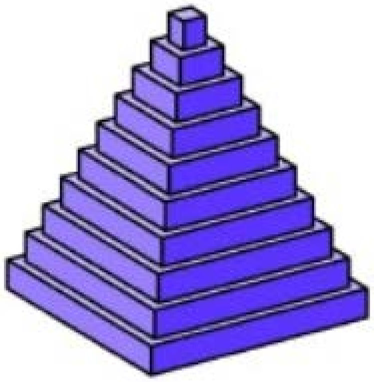
Note: This example is conducive to using a spreadsheet.

Part 1

- Draw or create a cube that is 10 cm x 10 cm x 10 cm.
- Find the volume of your cube.

Part 2

- Draw or create a pyramid with base that is 10 cm by 10 cm units and a height of 10 cm.
- Then draw or create a pyramid with base that is 10 cm by 10 cm consisting of 10 layers where each layer is 1 cm in height.
- What do you know about each of the layers?
- Find the volume of your figure.
- How does the drawing of your figure in part **b.** compare to your pyramid in part **a.**? Is there any way to get a more accurate pyramid?
- Draw or create a pyramid with base that is 10 cm by 10 cm consisting of 20 layers where each layer is 0.5 cm in height.
- Does the volume of your pyramid in part **f.** match the value of your pyramid in part **a.**? How could you get it closer?
- As a class decide how many layers you want in your structure to get it as close as you can to your figure in part **a**. You may want to use a spreadsheet to determine your calculations.
- What happens if you keep making thinner and thinner layers?
- How does your volume compare to the original cube in Part 1?
- What do you think the formula for a pyramid should be?



Discussion: Students should realize that each successive layer is a square with shorter sides. Since all squares are similar, they can calculate the volume by adding the layers together: $100 + 81 + 64 + 49 + 36 + 25 + 16 + 9 + 4 + 1 = 385\ cm^3$. Students should realize that although the volume of the figure in part **b.** is close to that of their pyramid in part **a.**, it is not exact because there are "gaps" between the two figures. Cavalieri's principle can also be applied to cones and pyramids. Through discussion, student should be able to come up with the idea that the more layers there are, the more closely the volume of the pyramid will align to the pyramid in part **a.** A pyramid with 20 square layers would have the volume of $(10^2 + 9.5^2 + 9^2 + 8.5^2 + 8^2 + 7.5^2 + 7^2 + 6.5^2 + 6^2 + 5.5^2 + 5^2 + 4.5^2 + 4^2 + 3.5^2 + 3^2 + 2.5^2 + 2^2 + 1.5^2 + 1^2 + 0.5^2 +)0.5 = 358.75\ cm^3$. As a class decide how precise you want to be. For example, suppose your students want 100 layers that are 0.1 cm high. In a spreadsheet label column A in cell A1 with the title "Base Length" and type "10" into cell A2. In cell A3 type "=A2 – 0.1." then drag the formula for 100 or so rows. Then label column B in cell B1 with the title "Area of the Base". Type "=A2^2" in cell B2 and drag down the formula. Label cell A103 as "Sum of Area Square", and write the formula "=Sum(B2:B102)" in B103. Then multiply that by the height of the layers which is 0.1 or use the formula "=B103*0.1" in another cell which would result in 338.35 cm^3. If students are more ambitious, you could repeat the same procedure in the spreadsheet using 1,000 layers to get 333.8335 cm^3. Students should realize that as the number of layers increases and the difference

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.GMD.1, 3)

between the side lengths of each of the subsequent layers get smaller and smaller the volume approaches $333.\overline{3}$ cm^3, which is $\frac{1}{3}$ the volume of the original cube in part 1.

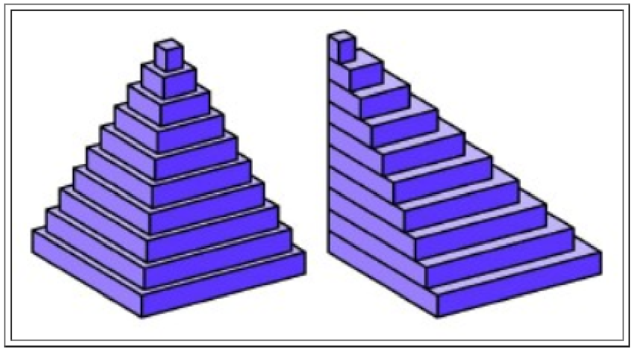
Part 3.

- Draw or create a cylinder with a radius of 10 cm and a height of 10 cm.
- Find the volume of your cylinder.

Part 4

- Draw or create a cone with a radius of 10 cm and a height of 10 cm.
- Draw or create a cone with layers like you did for the pyramid in Part 2. Find its volume.
- Use your spreadsheet to find the volume of your cone.
- How does the volume of the cone compare to the volume of the cylinder in Part 3?

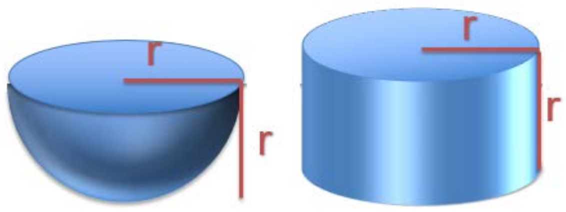
Note: Part 4 is similar to the process in part 2 with a circular base instead of a square base. The spreadsheet formulas need to be adjusted accordingly.



The same thin slabs approximate the volume of both a centered square-based pyramid and a pyramid with the same base and its top over one corner.

Image taken from [Volume Patterns for Pyramids] by Brown University

Students can generalize the formula for a pyramid to a cone using dissection arguments, transformations of layers or informal limit arguments.



Spheres

Consider using the following argument for the volume of a sphere. It may help to have students visualize a hemisphere "inside" a cylinder with the same height and "base." The radius of the circular base of the cylinder is also the radius of the sphere and the hemisphere. The area of the "base" of the cylinder and the area of the section created by the division of the sphere into a hemisphere is πr^2. The height of the cylinder is also r, so the volume of the cylinder is πr^3. Students can see that the volume of the hemisphere is less than the volume of the cylinder and more than half the volume of the cylinder. Illustrating this with concrete materials such as rice or water will help students see the relative difference in the volumes. At this point, students can reasonably accept that the volume of the hemisphere of radius r is $\frac{2}{3}\pi r^3$ and therefore volume of a sphere with radius r is twice that or $\frac{4}{3}\pi r^3$. There are several websites with explanations for students who wish to pursue the reasons in more detail, including the YouTube video Visualizing the Volume of a Sphere Formula: Deriving the Algebraic Formula with Animations by Kyle Pearce.

USING FORMULAS TO SOLVE PROBLEMS

This section involves geometric modeling. The formulas for volumes of cylinders, pyramids, cones and spheres can be applied to a wide variety of problems such as finding the capacity of a pipeline; comparing the amount of food in cans of various shapes; comparing capacities of cylindrical, conical and spherical storage tanks; using pyramids and cones in architecture, etc. Use a combination of concrete models and formal reasoning to develop conceptual understanding of the volume formulas. See the Instructional Resources/Tools section for examples of problems.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.GMD.1, 3)

Instructional Tools/Resources

These tools and resources may be helpful for instruction. Before use, they should be evaluated at the local level to determine their appropriateness for instruction

Manipulatives/Technology

- Rope or string
- Concrete models of circles cut into sectors and cylinders, pyramids, cones and spheres cut into slices.
- Rope or string
- Geoblocks® or comparable models of solid shapes
- Volume relationship set of plastic shapes
- Web site on Archimedes and the volume of a sphere,
- Dynamic geometric software (Geometer's Sketchpad®, Desmos®, Cabri®, or GeoGebra®)
- Video: *The Story of Pi* from Project MATHEMATICS!

Circumference of a Circle

- Circumference of a Circle by Illustrative Mathematics is a task where students find the circumference of a circle highlighting two different approaches: similar triangles and similarity of circles.
- Circumference = ? (Animation) by Tim Brzezinski is a GeoGebra applet illustrates the meaning of pi in terms of circumference divided by diameter.

Area of a Circle

- Area of a Circle by Illustrative Mathematics is a task where students find the area of a circle using Archimedes argument.
- Circle Area (by Peeling), Circle Area (by Peeling!), Area of a Circle (by Peeling), Area of a Circle-without Words (Animation 38) by Tim Brzezinski are GeoGebra applets that illustrate the formula for the are a circle by peeling.
- Area of a Circle by minoomath is a GeoGebra applet that shows how to find the area of a circle using a dissection argument.

Pi

- Improving Approximation for Pi with GeoGebra by tchoi99 is a GeoGebra Applet that allows students to test different values for polygon sides and compare the approximations of pi between Archimedes' and Snell-Huygens' methods.

Cavalieri's Principle

- Oblique and Right Pyramid-Cavalieri by Ted Coe is a GeoGebra applet that illustrates Cavalieri's principle.
- Cavalieri's Principle by Andreas Lindner is a GeoGebra applet that illustrates Cavalieri's principle.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.GMD.1, 3)

Cylinders and Prisms

- Volume Formulas for Cylinders and Prisms by Illustrative Mathematics is a task where students establish formulas for cylinders and prisms based on dissections.
- Volume of a Special Pyramid by Illustrative Mathematics is a task where students calculate the volume of a specific pyramid with a square base imbedded in a cube.
- Unwrapping a Cylinder by Tim Brzezinski is a GeoGebra applet that illustrates the surface area of cylinders.
- Unwrapping a Cylinder: Revamped! by Tim Brzezinski is a GeoGebra applet that illustrates the surface area of cylinders with an augmented reality addition.
- Right Triangular Prism! and Adjustable Triangular Prism by Tim Brzezinski is a GeoGebra applet that allows students to manipulate prisms in order to help them make sense of 2-D drawings of 3-D figures.
- Three Figures That Form a Cube by Wolfram Demonstrations Project is an applet that shows how a pyramid is $\frac{1}{3}$ of a cube.
- You Pour, I Choose, by Dan Meyer is a 3-act task that explores the volume of two different cylinders.
- Water Tank by Dan Meyer is a 3-act task that explores filling up a water tank.

Pyramids and Cones

- Trisecting the Cube into 3 Pyramids, Volume of Pyramids by Anthony OR 柯志明 is a GeoGebra applet that shows dissecting a cube into 3 pyramids.
- Square Pyramid: Underlying Anatomy by Tim Brzezinski is a GeoGebra applet that illustrates the difference between the height and the lateral height.
- Net of a Cone and Curved Surface Area of Cones (Combined Version) by Anthony OR 柯志明 is a GeoGebra applet that illustrates the surface area of a cone.
- Dissecting the Cube by Philip Busse and Beth McNabb is a lesson plan where students cut-out nets of square pyramids, make the pyramids, and then use them to make cubes thereby allowing them to compare the volumes of a cube and a pyramid.
- TI Online User Group October 2014 has 5 TI-Nspire geometry files. One of which is Pyramids and Cones which explores the volume formulas in connection with a cube.
- Arguments for Volume Formulas for Pyramids and Cones by Anthony Carruthers is a Better Lesson plan where students give informal arguments for volume.
- Prisms and Pyramids: How Many Pyramids Does It Take to Fill the Prism? by Kyle Pearce from Tap Into Teen Minds is a 3-act task where students explore the relationship between a pyramid and cube.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.GMD.1, 3)

Spheres

- Volume of Spheres and Volume of Spheres with Proofs by Anthony OR 柯志明 is a GeoGebra applet that shows the formula for the volume of a sphere.
- A Sphere in a Cylinder by Ted Coe is a GeoGebra applet that shows how to make sense of the formula for the volume of a sphere.
- TI Online User Group October 2014 has 5 TI-Nspire geometry files. One of which is Volume of a Sphere which uses Cavalier's Principal to find the volume of a sphere.
- Cones and Spheres: How Many Cones Does It Take to Fill the Sphere? by Kyle Pearce from Tap Into Teen Minds is a 3-act task where students explore the relationship between a cone and sphere.

Using Formulas to Solve Problems

- Greenhouse Management by Achieve the Core is a CTE task where students need to help their manager produce a crop of Easter Lilies. In one of the questions, students have to find the volume of a cylindrical pot.
- World's Largest Coffee Cup by Dan Meyer is a 3-act task where students need to apply their knowledge of the volume formula for a cylinder. Tap Into Teen Minds: Hot Coffee has accompanying resources for the task.
- Big Nickel by Andrew Stadel from Tap Into Teen Minds is a 3-act task where students need to find the volume of a giant nickel.
- Mix, Then Spray by Kyle Pearce from Tap Into Teen Minds is a 3-act task where students need to find the volume of spray bottle by using composite figures.
- Mustard Mayhem: How Many Bottles of Mustard Was Used? by Kyle Pearce from Tap Into Teen Minds is a 3-act task where students need to use the volume in a bottle of mustard to compare it to the amount of mustard in condiment cups.
- Water Tank: How Long Will It Take to Fill Up the Water Tank? by Dan Meyer is a 3-act task where students need to find how long it will take to fill a water tank.
- You Pour, I Choose: Which Glass Contains More Soda? by Dan Meyer is a 3-act task where students need to figure out which glass has more pop.
- Centerpiece by Illustrative Mathematics is a task where students have to apply the formula for the volume of a cylinder to model a real-life scenario.
- Doctor's Appointment by Illustrative Mathematics is a task where students have to analyze a real-life scenario involving a cone.
- Volume Estimation by Illustrative Mathematics is a modeling task where students have to apply volume formulas to a figure that is neither quite a cylinder nor a cone.
- The Great Egyptian Pyramids by Illustrative Mathematics is a task where students have to find the volume of a pyramid or using the volume to find a pyramid's base and height.
- Cubed Cans by NCTM Illuminations is a lesson where students try to find out how many cans fit in a prism. *Note: NCTM now requires a membership to view their lessons.*
- Sand Castles from Mathematics Vision Project and its accompanying teacher notes in partnership with the Utah State Office of Education has students explore the area and volume of pyramids.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.GMD.1, 3)

Curriculum and Lessons from Other Sources

- EngageNY, Geometry, Module 3, Topic A, Lesson 1: What is Area?, Lesson 2: Properties of Area, Lesson 4: Proving the Area of the Disk are lessons that pertain to this cluster.
- EngageNY, Geometry, Module 3, Topic B, Lesson 5: Three-Dimensional Space, Lesson 6: General Prisms and Cylinders and Their Cross-Sections, , Lesson 8: Definition and Properties of Volume, , Lesson 10: The Volume of Prisms and Cylinders and Cavalieri's Principle, Lesson 11: The Volume Formula of a Pyramid and Cone, Lesson 13: How Do 3D Printers Work? are lessons that pertain to this cluster.
- Georgia Standards of Excellence Curriculum Frameworks, Geometry, Unit 4: Circles and Volumes has many tasks that pertain to this cluster.
- Mathematics Vision Project, Geometry, Module 7: Modeling with Geometry has several tasks that pertain to this cluster.
- Illustrative Mathematics, Geometry, Unit 4, Lesson 11: Approximating Pi is a lesson that pertains to this cluster.
- Illustrative Mathematics, Geometry, Unit 5: Solid Geometry, Lesson 9: Cylinder Volumes, Lesson 10: Cross Sections and Volume, Lesson 11: Prisms Practice, Lesson 12: Prisms and Pyramids, Lesson 13: Building a Volume Formula for a Pyramid, Lesson 14: Working with Pyramids, Lesson 15: Putting all the Solids Together, Lesson 16: Surface Area and Volume, Lesson 18: Volume and Graphing are lessons that pertain to this cluster.
- Illustrative Mathematics, Geometry, Unit 7, Lesson 10: Angles, Arcs and Radii is a lesson that pertains to this cluster.

General Resources

- Arizona 7-12 Progression on Geometry is an instructional research for teachers. This cluster is addressed on page 19.
- Arizona High School Progression on Modeling is an informational resource for teachers. This cluster is addressed on page 17.
- High School Coherence Map by UnboundED is a map of the relationships among high school mathematics standards. Please note: This resource is aligned to the Common Core not Ohio's Learning Standards. Although the tool is useful, it needs to be checked with respect to Ohio's Learning Standards.
- van Hiele Model of Geometric Thinking is a pdf created by ODE that summarized the van Hiele levels.
- "Know the Formula" by Bill McCallum (one of the authors of the Common Core) is a thread from his blog Mathematical Musings. It gives some clarification surrounding the intent of knowing the formulas and some of the other issues surrounding the Geometry standards. Note that his blog refers to the Common Core, and not Ohio's Learning standards. During the standards revision, Ohio may have addressed some of these concerns with different outcomes than he suggests.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.GMD.1, 3)

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STANDARDS	MODEL CURRICULUM (G.GMD.4)
Geometry **GEOMETRIC MEASUREMENT AND DIMENSION** **Visualize relationships between two-dimensional and three-dimensional objects.** **G.GMD.4** Identify the shapes of two-dimensional cross-sections of three-dimensional objects, and identify three-dimensional objects generated by rotations of two-dimensional objects.	**Expectations for Learning** In middle school, students identify cross-sections as a result of slicing right rectangular prisms and pyramids. In this cluster, which supports the previous cluster, students extend the identification of cross-sections to include other three-dimensional solids. Students will also identify three-dimensional objects created when a two-dimensional object is rotated about a line. The student understanding of this cluster begins at van Hiele Level 1 (Analysis) and moves to Level 2 (Informal Deduction/Abstraction). **ESSENTIAL UNDERSTANDINGS** • Two-dimensional figures can be used to understand three-dimensional solids. • A three-dimensional figure can be created by rotating a two-dimensional figure about a line. **MATHEMATICAL THINKING** • Draw a picture or create a model to make sense of a problem. • Use technology strategically to deepen understanding. • Make connections between concepts, terms, and properties. **INSTRUCTIONAL FOCUS** • Identify two-dimensional cross-sections of three-dimensional objects. • Identify three-dimensional objects formed by rotations of two-dimensional objects. **Content Elaborations** **OHIO'S HIGH SCHOOL CRITICAL AREAS OF FOCUS** • Math 2, Number 6, page 10 **CONNECTIONS ACROSS STANDARDS** • Explain volume formulas and use them to solve problems (G.GMD.1, (+) 2, 3). • Understand the relationships between lengths, areas, and volumes (G.GMD.5-6). • Apply geometric concepts in modeling situations (G.MG.1-3).

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.GMD.4)

Instructional Strategies

Note: The model curriculum is not a collection of lessons nor a full curriculum; it does not suggest pace, sequence, or amount of time spent on topics. It provides information about a topic related to the standards including ideas for examples, strategies for teaching, common misconceptions, and possible connections between topics. In addition, the emphasis of the model curriculum is on instruction not on assessment; therefore, the examples in the Instructional Strategies section are written for instructional purposes not necessarily as assessment items. Examples may need to be rewritten to accommodate the needs of each individual classroom.

Standards for Mathematical Practice
This cluster focuses on but is not limited to the following practices:
MP.1 *Make sense of problems and persevere in solving them.*
MP.4 *Model with mathematics.*
MP.5 *Use appropriate tools strategically.*

This cluster should be connected to standards G.GMD.1, 3. Slices of rectangular prisms and pyramids were explored in Grade 7. In high school, the concept is extended to a wider class of solids. Students who eventually take calculus will learn how to compute volumes of solids by methods involving cross-sections.

VAN HIELE CONNECTION

In Math 2 students are expected to move from Level 1(Analysis) to Level 2 (Informal Deduction/Abstraction). In van Hiele Level 1 can be characterized by the student doing some or all of the following:

- showing a greater degree of attention to properties of shapes and solids;
- building 3D figures from 2D images and 2D drawings from 3D figures;
- viewing a figure from front, back, left, and right positions of solids;
- visualizing cross-sections when slicing solids;
- comparing solids based on properties;
- using observation as a basis for explanations; and/or
- understanding that a movement is made, then after observing the result, the next movement is selected, etc.

Whereas van Hiele Level 2 can be characterized by the student doing some or all of the following:

- using mathematical analysis of solids before performing any movements;
- giving informal justifications based on isolated properties; and/or
- analyzing beginning and final positions and making a plan to transform figures using a sequence of movements.

(Note: The significant difference between Level 1 and Level 2 is using logical reasoning. Level 1 activities can be turned into Level 2 by logical reasoning).

See the van Hiele pdf on ODE's website for more information about van Hiele levels.

TWO-DIMENSIONAL CROSS SECTIONS OF THREE-DIMENSIONAL OBJECTS

Two-dimensional cross sections of three-dimensional objects have many useful applications. Biologists uses cross sections to study cells, and geologists use cross sections to illustrate layers of the earth. Using cross sections can help students understand the concept of volume and helps build spatial reasoning.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.GMD.4)

Slice various solids to illustrate their cross sections. This could be done using clay, play-dough, hard-boiled eggs, fruit such as oranges, or even a Jell-O mold. Rubber bands may also be stretched around a solid to show a cross section. There are many useful applets listed in the Instructional Tool/Strategies section. It may be helpful to start with the cross sections in G.GMD.1,3 and continue in G.GMD.4. Ask students to find possible cross-sections of a cube. A square cross-section is the most obvious. Finding cross-sections in the shape of triangles, parallelograms, rectangles and hexagons may be more challenging.

Post-It notes or tag board can be used to illustrate cross sections as well.

TIP! Students may believe the only cross section for a three-dimensional figure is parallel or perpendicular to the base, when in fact there are many other ones as well. *Note: some sources such as EngageNY make a distinction between a slice and a cross section. Ohio does not make such a distinction.*

TIP! Students may have difficulty distinguishing shadows/projections from slices/cross sections.

TIP! Use 3-D Printers to illustrate cross sections to reinforce concepts of volume.

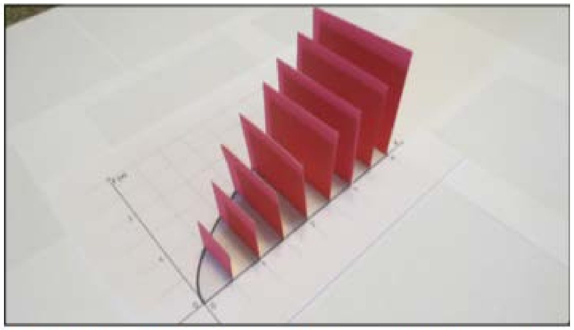

Fig. 2 Pieces of tag board taped perpendicular to a region can be used to model volume by cross sections.

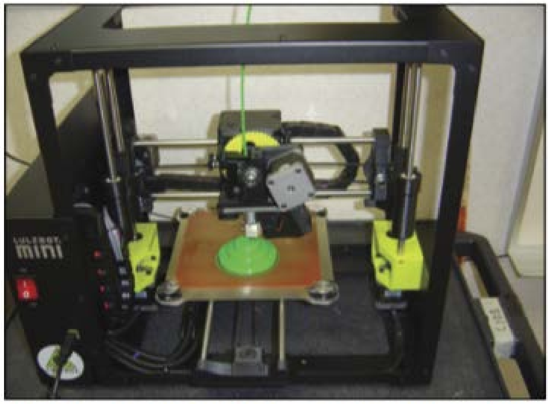

Fig. 3 We used a LulzBot Mini™ 3D printer to print models for volumes of revolution and cross section.

Images taken from Popelka, S. & Langlois, J. (March 2018). Getting out of Flatland.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.GMD.4)

Have students explore the sequence of cross-sections parallel to the base of 3D-objects. The thickness of the cross-sections may vary but as cross-sections become thinner, the lateral faces of the three-dimensional objects get smoother and the figure looks more and more like the original object.

TIP!

The book *Flatland: A Romance of Many Dimensions*, by Edwin Abbot can be used to illustrate the idea of cross sections and dimensions.

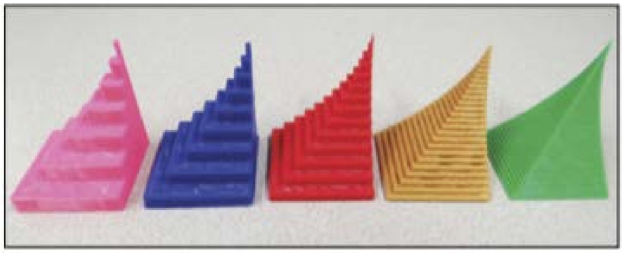

Fig. 5 These models for volume by cross sections show the progression from 7 squares on the left to a "smooth" surface on the right.

THREE-DIMENSIONAL OBJECTS GENERATED BY ROTATIONS OF TWO-DIMENSIONAL OBJECTS

Manufacturers can create objects such as axels, funnels, pills, bottles, and pistons by rotating an object around an axis of revolution. Graphic designers can also use these methods to generate images. The simplest of these types of solids is a cylinder where a rectangle is rotated about an axis coinciding to one side of a rectangle.

Students may have difficulty grasping that a rotation of a two-dimensional object creates a circular base for the three-dimensional object, therefore it is vital that students have concrete experiences rotating two-dimensional objects. Use a two-dimensional shape and spin it to show its three-dimensional representation. An example of this is taping a shape to a pencil and using a drill to spin it. Another method is to cut a half-inch slit in the end of a drinking straw and insert a cardboard cutout shape. Rotate the straw and observe the three-dimensional solid of revolution generated by the two-dimensional cutout. Party decorations like a "honeycomb paper bell" that starts as a two-dimensional shape and fans to be a three-dimensional shape are good visual representations of this effect.

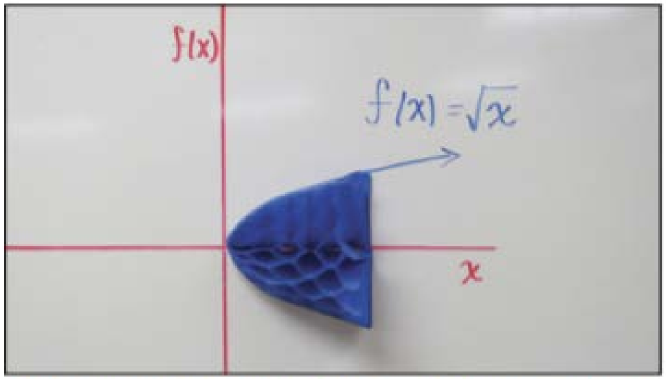

Fig. 1 Honeycomb cut-outs can be used to model a surface of revolution.

Images taken from Popelka, S. & Langlois, J. (March 2018). Getting out of Flatland.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.GMD.4)

TIP! Encourage students to make drawings of solids with highlighted cross sections or drawings of solids generated by rotation two-dimensional shapes.

MODELING CONNECTION

People use cross sections in real-life to model different situations which enables them to answer questions or solve problems. One example is telling the age of a tree by the rings in its stump (or, in mathematics terms, cross-sections of the tree). Another example is when the Thai soccer team was stuck in a cave in Thailand (See diagram at the right). They used cross sections to show the size of the different areas of the cave. Cross sections are also used in science and social studies to illustrate the world around us. Use illustrations of cross sections from other content areas such as eyes, volcanoes, and cells to illustrate their usefulness.

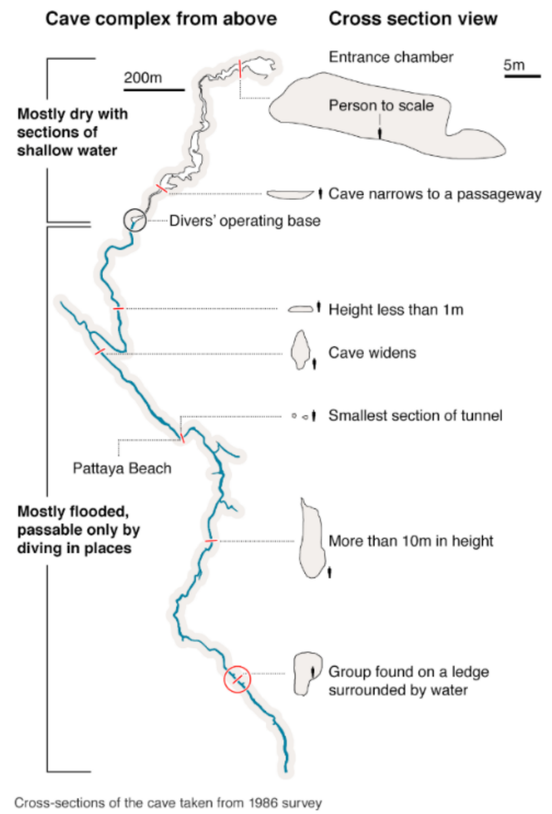

Image taken from BBC News, [Cave Rescue: Four More Boys Rescued by Thai Navy Divers]() on July 9, 2018.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.GMD.4)

Instructional Tools/Resources

These tools and resources may be helpful for instruction. Before use, they should be evaluated at the local level to determine their appropriateness for instruction.

Manipulatives/Technology

- Concrete models of solids such as cubes, pyramids, cylinders, and spheres. Include some models that can be sliced, such as those made from Styrofoam.
- Rubber bands
- Cardboard cutouts of 2-D figures (e.g. rectangles, triangles, circles)
- Drinking straws
- Geometric software
- *Flatland: A Romance of Many Dimensions*, by Edwin Abbott
- Web sites, that illustrate geometric models. Some examples are The Geometry Junkyard and Wolfram Mathworld

Two-dimensional Cross Sections of Three-dimensional Objects

- Sections of Rectangular Prisms (Cuboids) by Anthony OR 柯志明 is a GeoGebra applet that shows the cross sections of a rectangular prism.
- Sections of Triangular Prisms by Anthony OR 柯志明 is a GeoGebra applet that shows the cross sections of a triangular prism.
- Sections of Cylinders by Anthony OR 柯志明 is a GeoGebra applet that shows the cross sections of a cylinder.
- Sections of Rectangular Pyramids by Anthony OR 柯志明 is a GeoGebra applet that shows the cross sections of a rectangular pyramid.
- Sections of Triangular Pyramids by Anthony OR 柯志明 is a GeoGebra applet that shows the cross sections of a triangular pyramid.
- Sections of Cones by Anthony OR 柯志明 is a GeoGebra applet that shows the cross sections of a cones.
- Sections of Spheres by Anthony OR 柯志明 is a GeoGebra applet that shows the cross sections of a spheres.
- Exploring Sections of Cubes by Anthony OR 柯志明 is a GeoGebra applet that shows the cross sections of a cubes.
- Cube Dissection Problem by Anthony OR 柯志明 is a GeoGebra applet that has students figure out how many pieces would be obtained by dissecting a cube with planes through all the diagonals of the six faces.
- The Flatland Game by Julian Fleron and Volker Ecke in the NSF funded book, *Discovering the Art of Geometry*, is game where students determine the identity of a solid object from a series of parallel cross sections taken at regular intervals.
- Slicing a Cube by NCTM Illuminations is a task that has students answer questions about a cross section of a cube. *Note: NCTM now requires a membership to view their lessons.*
- Ants Marching by NCTM Illuminations is a task that has students answer questions about a cross section of a cube. *Note: NCTM now requires a membership to view their lessons.*
- Cross Section Flyer by Shodor is an applet that shows cross sections and can be manipulated for various cross sections of the same three-dimensional figure.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.GMD.4)

Two-dimensional Cross Sections of Three-dimensional Objects, continued

- Can You Cut It? Slicing Three-Dimensional Figures is a CPalms lesson where students sketch, model, and describe cross-sections formed by a plan passing through a three-dimensional figure.
- Sections of a Cube by AssocTeachers math is a YouTube video that shows cutting through a cube in a number of different ways and examining the cross section of each.
- Math Shorts Episode 8—Slicing Three Dimensional Figures by PlanetNutshell is a YouTube video about slicing three-dimensional figures.
- Tennis Balls in a Can by Illustrative Mathematics is a task where students explore cross sections in the context of tennis balls in a can.
- Problem 6: Plasticine Geometry from the University of Waterloo is a worksheet that has students predict shapes of cross sections and then find an actual cross section using fishing line and plasticine.

Three-dimensional Objects Generated by Rotations of Two-Dimensional Objects

- Rotate Triangle by Sobarrera is a GeoGebra applet that shows a cone formed by rotating a triangle.
- Rotate Rectangle by Sobarrera is a GeoGebra applet that shows a cylinder formed by rotating a rectangle.
- Rotate Circle by Sobarrera is a GeoGebra applet that shows a sphere formed by rotating a cylinder.
- Rotate Triangle by Sobarrera is a GeoGebra applet that shows the figure formed by rotating a trapezoid.
- Investigation 6.4.4 Creating Solids of Rotation Using GeoGebra is a lesson where students rotate figures using GeoGebra.

Curriculum and Lessons from Other Sources

- EngageNY, Geometry, Module 3, Topic B, Lesson 5: Three-Dimensional Space, Lesson 6: General Prisms and Cylinders and Their Cross-Sections, Lesson 7: General Pyramids and Cones and Their Cross-Sections and Lesson 13: How Do 3D Printers Work? are lessons that pertain to this cluster. are lessons that pertain to this cluster.
- Georgia Standards of Excellence Curriculum Frameworks, Geometry, Unit 4: Circles and Volumes has many tasks that pertain to this cluster.
- Mathematics Vision Project, Geometry, Module 7: Modeling with Geometry has several tasks that pertain to this cluster.
- Illustrative Mathematics, Geometry, Unit 5: Solid Geometry, Lesson 1: Solids of Rotation, Lesson 2: Slicing Solids, Lesson 3: Creating Cross Sections by Dilating are lessons that pertain to this cluster.

General Resources

- Arizona 7-12 Progression on Geometry is an informational resource for teachers.
- High School Coherence Map by UnboundED is a map of the relationships among high school mathematics standards. Please note: This resource is aligned to the Common Core not Ohio's Learning Standards. Although the tool is useful, it needs to be checked with respect to Ohio's Learning Standards.
- van Hiele Model of Geometric Thinking is a pdf created by ODE that summarizes the van Hiele levels.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.GMD.4)

References

- Cochran, J., Cochran, X., Laney, K., & Dean, M. (May 2016). Expanding geometry understanding with 3D printing. *Mathematics Teaching in the Middle School*, *21, (9),* 534-542.
- Popelka, S. & Langlois, J. (March 2018). Getting out of Flatland. *Mathematics Teacher*, *111, (5),* 352-400.

<table>
<tr><th>STANDARDS</th><th>MODEL CURRICULUM (G.GMD.5-6)</th></tr>
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Geometry

GEOMETRIC MEASUREMENT AND DIMENSION

Understand the relationships between lengths, areas, and volumes.

G.GMD.5 Understand how and when changes to the measures of a figure (lengths or angles) result in similar and non-similar figures.

G.GMD.6 When figures are similar, understand and apply the fact that when a figure is scaled by a factor of k, the effect on lengths, areas, and volumes is that they are multiplied by k, k^2, and k^3, respectively.

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Expectations for Learning

In middle school, students solve problems involving two-dimensional similar figures and calculate the volumes of three-dimensional figures. In this cluster, students extend their knowledge of similarity to explore and understand how changes to length or angle measure in one figure will result in similar or non-similar figures. Students will also understand the effect that a scale factor has on the length, area, and volume of similar figures and use this relationship to solve problems.

The student understanding of this cluster begins at van Hiele Level 1 (Analysis) and moves to Level 2 (Informal Deduction/Abstraction).

ESSENTIAL UNDERSTANDINGS

- Changes to the lengths and/or angle measures of a figure result in similar and non-similar figures.
- When changes to a figure result in similar figures with a scale factor of k, the lengths of the resulting figures are a multiple of k.
- When changes to a figure result in similar figures with a scale factor of k, the areas of the resulting figures are a multiple of k^2.
- When changes to a figure result in similar figures with a scale factor of k, the volume of the resulting figures are a multiple of k^3.

MATHEMATICAL THINKING

- Use precise mathematical language.
- Draw a picture or create a model to make sense of a problem.
- Determine reasonableness of results.
- Solve multi-step problems accurately.
- Plan a solution pathway.
- Solve mathematical and real-world problems accurately.
- Consider mathematical units involved in a problem.
- Attend to the meaning of quantities.
- Recognize, apply, and justify mathematical concepts, terms, and their properties.
- Generalize concepts based on patterns.

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STANDARDS	MODEL CURRICULUM (G.GMD.5-6)
	Expectations for Learning, continued **INSTRUCTIONAL FOCUS** • Classify objects as similar or non-similar when the lengths or angles of figures are changed. • Explain the types of changes to a figure that result in similar and non-similar figures. • Use geometry and algebra to explain how length, area, and volume are affected when scaling is applied. • Solve problems involving length, area, and volume of figures under scaling. **Content Elaborations** **OHIO'S HIGH SCHOOL CRITICAL AREAS OF FOCUS** • Math 2, Number 6, page 10 **CONNECTIONS ACROSS STANDARDS** • Explain volume formulas and use them to solve problems (G.GMD.1, (+) 2, 3). • Understand similarity in terms of similarity transformations (G.SRT.1-2). • Apply geometric concepts in modeling situations (G.MG.2-3). • Create equations that describe numbers or relationships (A.CED.2, 4). • Solve quadratic equations in one variable (A.REI.4). • Build a function that models a relationship between two quantities (F.BF.1a).

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.GMD.5-6)

Instructional Strategies

Note: The model curriculum is not a collection of lessons nor a full curriculum; it does not suggest pace, sequence, or amount of time spent on topics. It provides information about a topic related to the standards including ideas for examples, strategies for teaching, common misconceptions, and possible connections between topics. In addition, the emphasis of the model curriculum is on instruction not on assessment; therefore, the examples in the Instructional Strategies section are written for instructional purposes not necessarily as assessment items. Examples may need to be rewritten to accommodate the needs of each individual classroom.

Standards for Mathematical Practice
This cluster focuses on but is not limited to the following practices:
***MP.2** Reason abstractly and quantitatively.*
***MP.4** Model with mathematics.*
***MP.5** Use appropriate tools strategically.*
***MP.7** Look for and make use of structure.*

VAN HIELE CONNECTION

In Math 2 students are expected to move from Level 1(Analysis) to Level 2 (Informal Deduction/Abstraction) and even touches on Level 3 (Deduction). Van Hiele Level 2 can be characterized by the student doing some or all of the following:

- fully understanding the procedures and formula for determining measurement of simple, familiar shapes;
- generalizing unit-measure enumeration to include fractional units;
- understanding and making unit conversions; and/or
- generalizing measurement to non-squares for area and non-cubes for volume.

Van Hiele Level 3 can be characterized by the student doing some or all of the following:

- comparing shapes by property-preserving transformations/decompositions;
- using geometric properties and variables to understand and solve problems involving formulas for non-familiar shapes; and/or
- comparing length, area, and volume by using geometric properties or transformations.

See the van Hiele pdf on ODE's website for more information about van Hiele levels.

SCALING

Students need to understand that scaling produces similar figures, but for figures to remain similar, the scale needs to be applied to each dimension. Give students time to build things by scaling. Rep-tiles, pattern-blocks, Legos, and cubes may help. A rep-tile is a shape that can be dissected into copies of the same shape that is used in tessellations. (More resources on rep-tiles can be found in the Instructional Tools/Resources section.) The goal is to help students achieve an intuition about scaling so that they think "I doubled the side lengths, so I get four figures that are the same as the original," etc. Students should explore non-similar solids as well. For example, multiply the length and width of a prism but not the height results in non-similar solids.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.GMD.5-6)

Many students do not see a difference between the unit of measures in the three different dimensions, and so they struggle with the correct use of unit of measure for the dimension they should be working in. For example, they may confuse square inches with cubic inches. Teachers could use multiple visual examples to help physically demonstrate lengths are one-dimensional, areas are two-dimensional, and volumes are three-dimensional and tie this into appropriate unit for the measurement.

EXAMPLE

Part 1

a. Create similar triangles using rep-tiles.
b. Create similar parallelograms using rep-tiles.

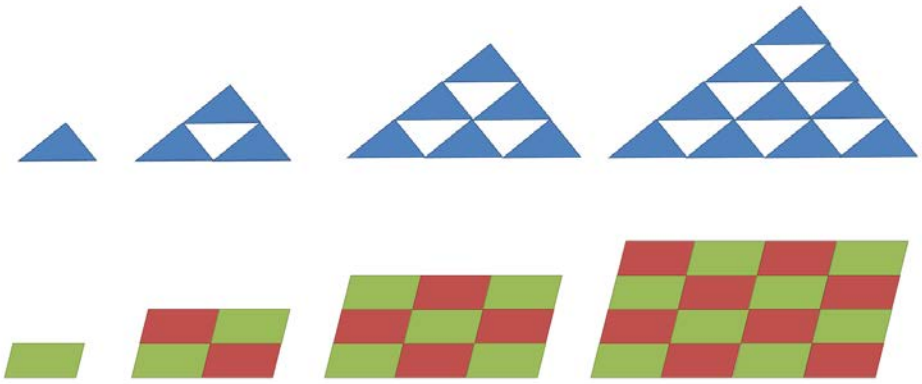
Discussion: The purpose of this activity is that students can understand that to properly scale a figure, the figure must scale by the same factor in all dimensions. Once students are able to visualize this, they will be ready for questions about perimeter, area, and volume.

Part 2:

Create a figure made out of tiles and change the side lengths of the figure as stated below:

a. How many tiles did you get when you double the side lengths of the figure? Explain.
b. How many tiles did you get when you triple the side lengths of the figure? Explain.
c. How many tiles did you get when you quadruple the side lengths of the figure? Explain.
d. How many tiles would you get if you changed the side length by a factor of k? Explain.
e. How does the perimeter in two similar figures relate?
f. How does the area in two similar figures relate?

Discussion: If you double the side lengths of a triangle (as shown in the first diagram below), then four of the original triangles can fit in the new figure. If you triple the side lengths of a triangle, then 9 triangles can fit in, etc. This may be clearer for parallelograms (as shown in the second diagram), but it is harder to see with triangles because the triangles require rotation. These diagrams are built using figures called "rep-tiles." In addition to the geometric (visual), you can also use an algebraic representation. Think about the area formulas for parallelograms and triangles. If you dilate the parallelogram, you multiply the base by k and the height by k, then the perimeter is multiplied by k and the area is multiplied by k^2. Students should come to the conclusion that the number of tiles changes by a factor of k^2.



INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.GMD.5-6)

Part 3

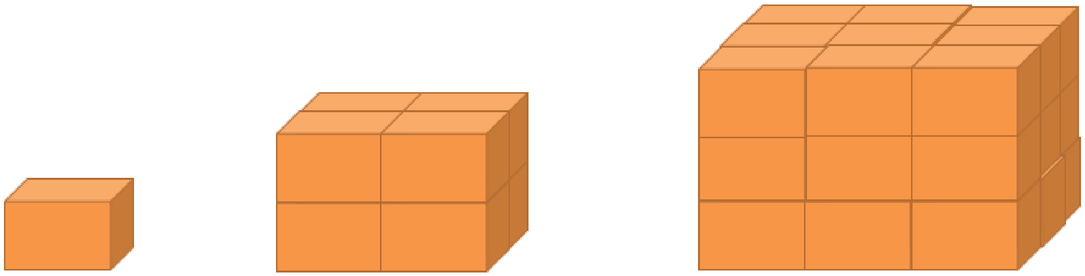
- **a.** Create similar figures to a 2 by 2 Lego brick.
- **b.** Create similar figures to a 2 by 3 Lego brick.
- **c.** Create similar figures to a 2 by 4 Lego brick.

Discussion: The purpose of this activity is that students can understand that to properly scale a three-dimensional figure, the figure must scale by the same factor in all three dimensions. Many students will incorrectly only want scale in two directions.

Part 4

- **a.** How many Legos do you get when you double the side lengths of the figure? Explain.
- **b.** How many Legos do you get when you triple the side lengths of the figure? Explain.
- **c.** How many Legos do you get when you quadruple the side lengths of the figure? Explain.
- **d.** How many Legos would you get if you changed the side length by a factor of k? Explain.
- **e.** How does the volume in two similar figures relate? Explain.
- **f.** How does the surface area of the two figures relate? Explain.

Discussion: If you double the side lengths of a prism (as shown in the first diagram below), then eight Legos fit in the new figure. If you triple the side lengths of a prism, then 27 Legos can fit in, etc. The idea in Parts 1 and 2 can be extended to 3D figures by multiplying the length, width, and height of a prism by the same scale factor. Explain what happens to the surface area (multiplied by k^2) and volume (multiplied by k^3). Students should come to the conclusion that the number of Legos changes by a factor of k^3 and can be connected to volume.



INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.GMD.5-6)

The misconception of units extends into the use of proportions. Students assume that if the ratio of the lengths is *a*:*b* then the ratio of the areas or volumes is also *a*:*b*, which is incorrect. They may also incorrectly think that objects that are dilated with a scale factor *k*, then $\frac{a}{b} = \frac{a^2}{b^2} = \frac{a^3}{b^3}$. That is why it is important for students to do hands-on activities to show that if an object grows multiplicatively in all directions or it grows the same amount in the other direction or dimension. For example, if a rectangle with side lengths a and b doubles by a scale factor of *k*, the proportions describing the side length would be $\frac{a}{b} = \frac{2a}{2b}$, but the area would be changed by a factor of k^2. Teachers should stress that proportions only work if both ratios are in the same dimension. For example, $\frac{3cm}{4cm} = \frac{6cm}{8cm}$ or $\frac{1cm^2}{4cm^2} = \frac{12cm^2}{48cm^2}$, but $\frac{4cm}{8cm} \neq \frac{12cm^2}{24cm^2}$.

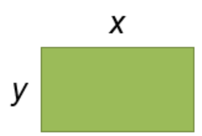
COMPARING SIMILAR AND NON-SIMILAR FIGURES

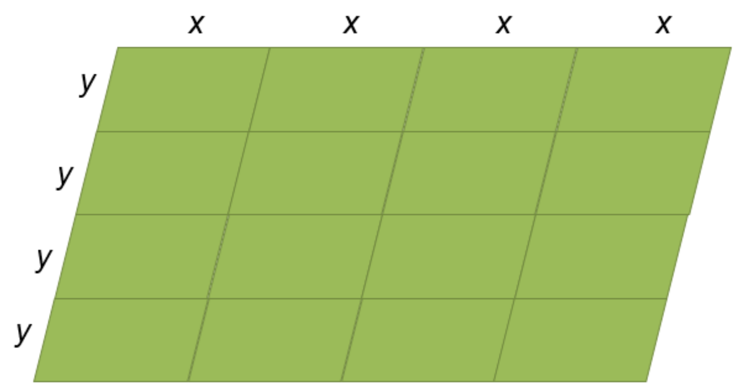
Students should explore what changes would create similar figures compared to non-similar figures. This connects to dilations in G.SRT.1-3. Since in similar figures corresponding angles are congruent and sides are proportional, changes that do not keep angles congruent and sides proportional result in non-similar figures. An example of this would be an additive change versus a multiplicative change. It is also important for students to discover that in order to keep similarity, the multiplicative change must be applied to all dimensions. Students should discover that this concept applies to three-dimensional figures as well as two-dimensional figures. It is important for students to know that although congruent corresponding angles prove triangle similarity, this does not hold true for other polygons. For example, adding the same amount to the sides of a rectangle results in non-similar rectangles, yet the corresponding angles are still congruent. See Model Curriculum 7.G.1-3 for scaffolding ideas about similar figures and see Model Curriculum 6.RP.1-3 and 7.RP.1-3 for more information about multiplicative versus additive relationships.

EXAMPLE

Marissa said the two figures on the right are similar because their sides are proportional. Is she correct? Explain.

Discussion: Although there are four copies of the shape in each direction students make think the figures are similar. They may also say they are not similar because "I can see they aren't." Push students to more precisely state that the corresponding angles are not congruent, so the figures are not congruent.



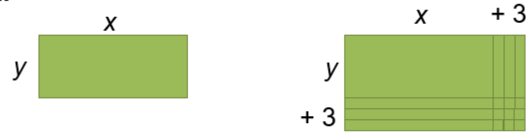


INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.GMD.5-6)

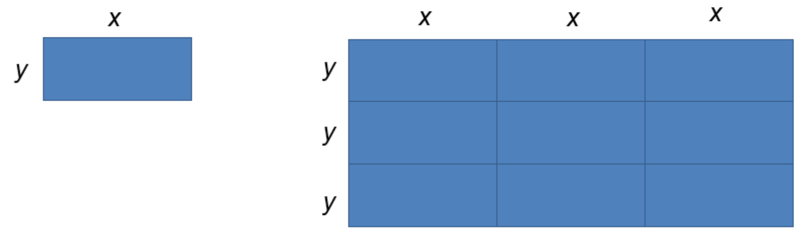
EXAMPLE

Which of the following situations shows a set of similar figures? Explain why each set either represents or does not represent similar figures.

Set A:

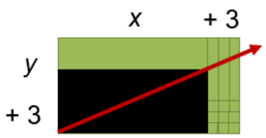


Set B:

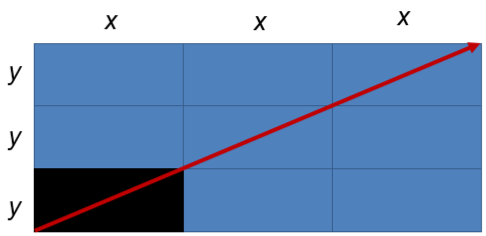


Set C:

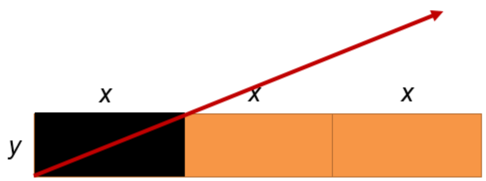
Set A:



Set B:



Set C:



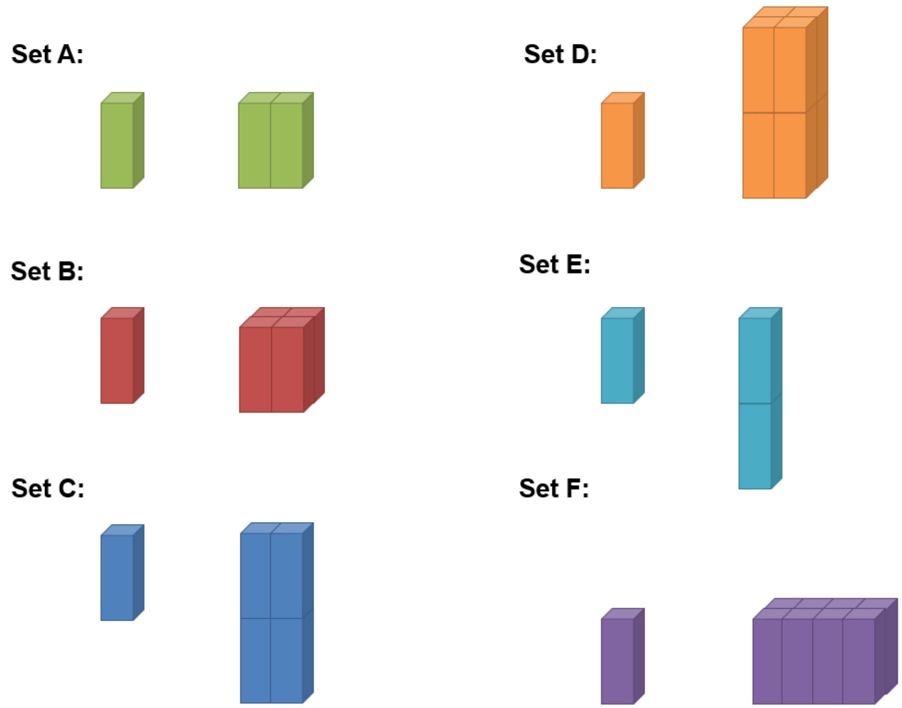
Discussion:

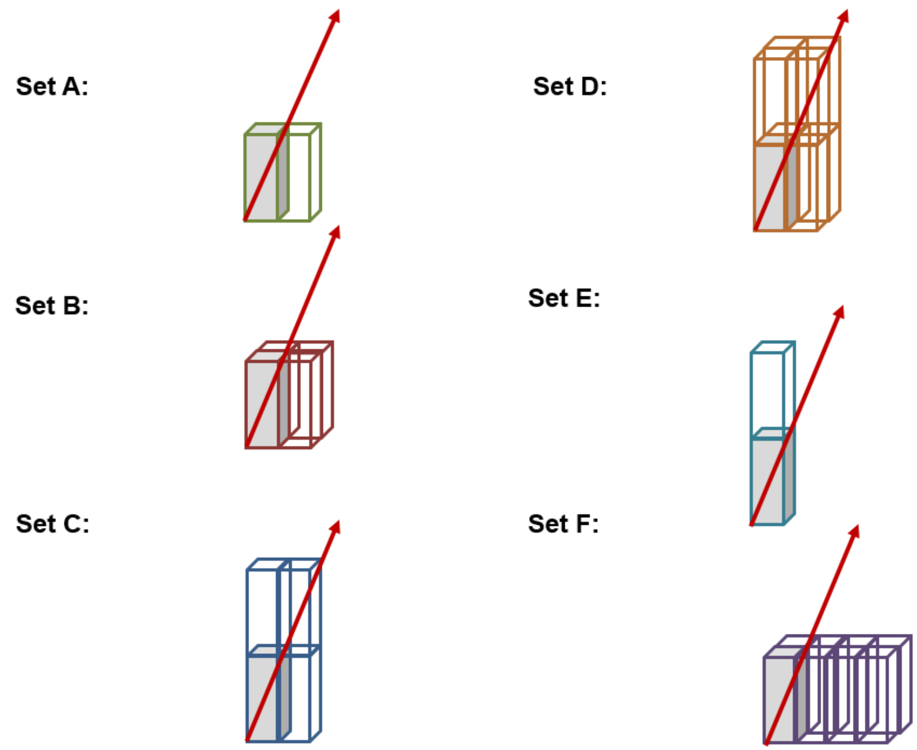
- Students should come to the realization that Set A does not show similar figures because it shows an additive relationship, and therefore the sides are not proportional. Set B shows a set of similar figures because both sides are proportional and angle measures are congruent. Set C does not show a set of similar figures because it the figure only changes in one-direction.
- Draw a connection between similar figures and dilations using the nesting method. Nested figures share a common vertex and vertex angle. Students can see that when nesting the figures atop of one another, only Set B has a line of dilation that goes through both corresponding vertices.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.GMD.5-6)

EXAMPLE

Which of the following situations (on the left) show a set of similar figures? Explain why each set either represents or does not represent similar figures.



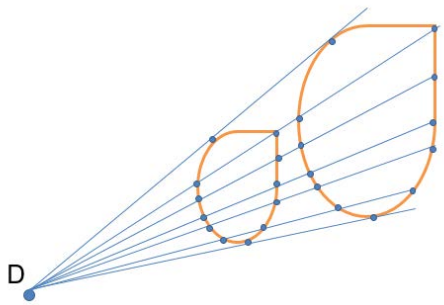


Discussion:

- Students may give a variety of explanations about which sets show similar figures. However, Set D is the only set that represents similar figures because the dimensions increase by a factor of 2 in each direction and the angles remain congruent.
- Students can also use the nesting method to help them in this situation as well. A line that connects the left-front vertex and the back-right vertex of the original prism should also go through the left-front vertex and the back-right vertex of the resulting prism. This is shown in the diagram by Set D. Have students explain why this is so by relating it to concepts of dilations.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.GMD.5-6)

This idea of using rays generating from the same point of dilation that pass through corresponding points also generalized for irregular figures.



Change the orientation of similar figures and have students explain why they are still similar. This prevents them from creating a misconception that similar figures must have the same orientation.

Fundamental Theorem of Similarity

There are several versions of the Fundamental Theorem of Similarity. It is up to each district to choose how they want to name the Fundamental Theorem of Similarity. The important thing is that students understand the concepts underlying both versions. Here are two examples:

The University of Chicago Mathematics Project: Geometry textbook states:
If $G \sim G'$ and k is the ratio of similitude, then

a. $Perimeter\ (G') = k \cdot Perimeter\ (G)$ or $\frac{Perimeter\ (G')}{Perimeter\ (G)} = k$;

b. $Area\ (G') = k^2 \cdot Area\ (G)$ or $\frac{Area\ (G')}{Area\ (G)} = k^2$; and

c. $Volume\ (G') = k^3 \cdot Volume\ (G)$ or $\frac{Volume\ (G')}{Volume\ (G)} = k^3$

H. Wu states (whose thoughts EngageNY is based upon): "Given a dilation with center O and a scale factor r, then for any two points P and Q in the plane so that $O, P,$ and Q are not collinear, the lines PQ and $P'Q'$ are parallel, where $P' = Dilation(P)$ and $Q' = Dilation(Q)$, and furthermore, $|P'Q'| = r|PQ|$."

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.GMD.5-6)

TIP! The object's unit of measure can help students remember the scale factors associated with each measurement. Since area is measured in square units, k is squared. Since volume is measured in cubic units, k is cubed.

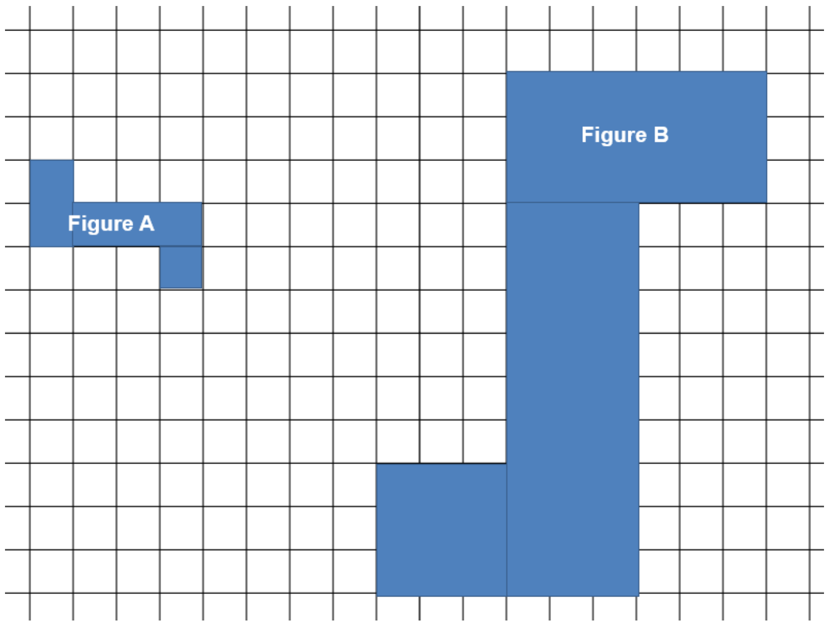
EXAMPLE

Figure A and B are similar. Figure B has been created by rotating and dilating Figure A by a scale factor of 3.

a. If the area of Figure A is 6 cm^2, what is the area of Figure B? How can you find out without counting the squares of Figure B?

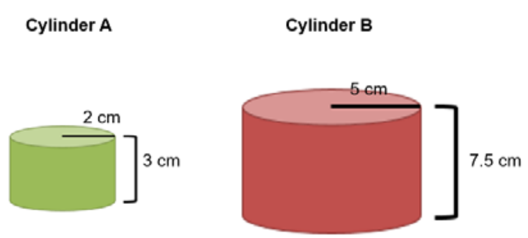
b. Check to see if you are correct.

Discussion: Students should make the connection that area changes by a scale factor of k^2, so the new area would be $6 \cdot 3^2$ or 54 cm^2.



EXAMPLE

Cylinder B is the result of similarity transformation of Cylinder A.



a. What is the ratio of the radii for the two figures?

b. What is the ratio of the surface area for the two figures?

c. What is the ratio of the volume for the two figures?

Discussion: The ratio of the radii for the two cylinders is $\frac{5}{2}$. Although students could calculate the surface areas and volumes of the cylinders to find the ratios, some students may realize that the area changes by a factor of k^2 under a similarity transformation and volume changes by a factor of k^3, so the ratio of the surface area is $\frac{25}{4}$ and the ratio of the volume is $\frac{125}{8}$.

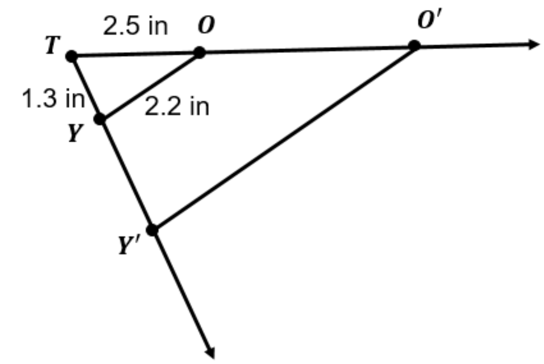
INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.GMD.5-6)

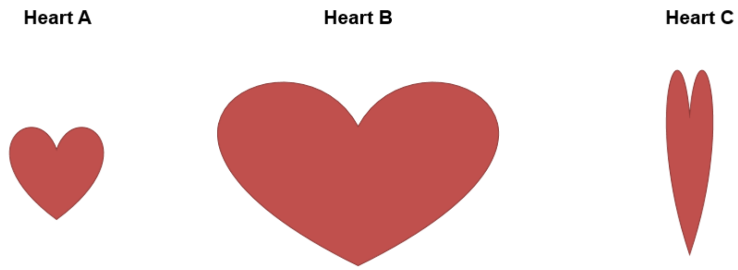
EXAMPLE

Note: Figure not drawn to scale

a. A dilation was performed on $\overline{OY}$ from the center of dilation T with a scale factor of 2.9. What is the perimeter of $\triangle TO'Y'$?

b. What is the length of $\overline{YY'}$?





EXAMPLE

a. If Heart A has an area of 16 in^2, and Heart B's height was doubled, and its width was tripled, what would the area of Heart B be?

b. If Heart A has an area of 16 in^2, and Heart C's height was doubled, and its width was halved, what would the area of Heart C be?

c. Are any of the hearts similar? Explain.

EXAMPLE

What would Lebron James look like if he was average height? Scale 6'8" LeBron down to 5'10."

Discussion: Lebron would be reduced by a scale factor of $\frac{7}{8}$ (6'8'' is 80 in and 5'10'' is 70 in). Students can explore what happens to his waist-size (original waist size multiplied by $\frac{7}{8}$ because waist size is a length), amount of fabric in jersey (original amount of fabric is multiplied by $\frac{7}{8}$ squared because amount of fabric is an area), etc.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.GMD.5-6)

EXAMPLE

a. Marcus wants to carpet a room that is 25 yd^2. Draw a scale drawing of the room in inches.
b. Fill in the blank: 1 yd^2 = ___ft^2 = ___in^2
c. Julia has a pool that has a capacity of 375 m^3. Draw a scale model in mm.
d. Fill in the blank: 1 m^3 = ___ cm^3 = ___ mm^3

EXAMPLE

Create a scale model of something in real-life. Include the original dimensions, your scale, and the new dimensions.

Instructional Tools/Resources

These tools and resources may be helpful for instruction. Before use, they should be evaluated at the local level to determine their appropriateness for instruction.

Manipulatives/Technology

- Tracing paper (patty paper)
- Transparencies
- Graph paper
- Ruler
- Protractor
- Compass
- Legos
- Pattern blocks or other shapes that can be tiled
- Computer dynamic geometric software (such as Geometer's Sketchpad®, Desmos, Cabri®, or GeoGebra®).
- Graphing calculators and other handheld technology such as TI-Nspire™.

Rep-Tiles

- Rep-Tiles by Mela Hardin is an activity where students discover and explore algebraic expressions through a special kind of tiling of the plane.
- Rep-Tiles by NCTM Illuminations is a lesson plan using rep-tiles. *Note: NCTM now requires a membership to view their lessons.*
- Rep-Tiles, Or How Mathematicians Start to Puzzle and Open Up Questions by Maxwell's Demon Vain Attempts to Construct Order is a blog that shows examples of rep-tiles.
- Rep-Tile by Wolfram MathWorld gives examples of rep-tiles.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.GMD.5-6)

Fundamental Theorem of Similarity

- Explore Fundamental Theorem of Similarity by HJ is a GeoGebra applet where students can explore FTS.
- Fundamental Theorem of Directly Similar Figures by Steve Phelps is a GeoGebra applet where students can explore FTS.

Scaling

- Geometry: Nested Similar Triangles by Texas Instruments is a lesson where students explore nested triangles to determine similarity.
- Scaling Away by NCTM Illuminations is a lesson where students compute the surface area and volume of a scale model. *Note: NCTM now requires a membership to view their lessons.*
- Scaling the City: Ground Truthing the Size of SimCity Objects by NCTM Illuminations is a lesson where students compare computer dimensions of objects to dimensions in real-life. *Note: NCTM now requires a membership to view their lessons.*
- Demonstrate the Effects of Scaling on Volume by Terry Lee Lindenmuth is a GeoGebra applet that allows students to scale a prism.
- Volume and Surface Areas of Similar 3D Figures by Anthony OR 柯志明 is a GeoGebra applet that allows students to scale a prism.
- 6.3 Investigating Connections between Measurements and Scale Factors of Similar Figures by NCTM is a lesson where students scale shapes on their applet and can see the corresponding graph of the scale factors. *Note: NCTM now requires a membership to view their lessons.*
- Blue Squares and Beyond by NCTM is a lesson where students construct and interpret figures using scale factors.

Curriculum and Lessons from Other Sources

- EngageNY, Grade 8, Module 3, Topic A, Lesson 4: Fundamental Theorem of Similarity and Lesson 5: First Consequences of FTS are lessons that pertain to this cluster.
- EngageNY, Geometry, Module 2, Topic A, Lesson 5: Scale Factors is a lesson that pertains to this cluster.
- EngageNY, Geometry, Module 3, Topic A, Lesson 3: The Scaling Principle for Areas is a lesson that pertains to this cluster.
- EngageNY, Geometry, Module 3, Topic B, Lesson 9: Scaling Principle for Volumes is a lesson that pertains to this cluster.
- Illustrative Mathematics, Geometry, Unit 5: Solid Geometry, Lesson 4: Scaling and Area, Lesson 5: Scaling and Unscaling, Lesson 6: Scaling Solids, Lesson 7: The Root of the Problem, and Lesson 8: Speaking of Scaling are lessons that pertain to this cluster.

General Resources

- Arizona 7-12 Progression on Geometry is an informational resource for teachers. This cluster is addressed on page 19.
- High School Coherence Map by UnboundED is a map of the relationships among high school mathematics standards. Please note: This resource is aligned to the Common Core not Ohio's Learning Standards. Although the tool is useful, it needs to be checked with respect to Ohio's Learning Standards.
- van Hiele Model of Geometric Thinking is a pdf created by ODE that summarizes the van Hiele levels.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.GMD.5-6)

References

- Coxford, A., Usiskin, Z., & Hirschorn, D. (1991). *The University of Chicago Mathematics Project: Geometry, Teacher's Edition.* Scott Foresman and Company, Glenview: IL.
- Ronau, R., Meyer, D. Crites, T., & Doughtery, B. (2015*). Putting essential understanding of geometry into practice*. The National Council of Teachers of Mathematics: Reston, VA.
- Snapp, B. & Findell, B. (2016). *Parallels in Geometry*. Creative Commons. Retrieved from https://people.math.osu.edu/findell.2/ParallelsInGeometry2016.pdf
- Wu, H. (2013). Teaching geometry in grade 8 and high school according to common core standards. Retrieved from https://math.berkeley.edu/~wu/CCSS-Geometry_1.pdf

STANDARDS	MODEL CURRICULUM (G.MG.1-3)
Geometry **MODELING WITH GEOMETRY** **Apply geometric concepts in modeling situations.** **G.MG.1** Use geometric shapes, their measures, and their properties to describe objects, e.g., modeling a tree trunk or a human torso as a cylinder. ★ **G.MG.2** Apply concepts of density based on area and volume in modeling situations, e.g., persons per square mile, BTUs per cubic foot. ★ **G.MG.3** Apply geometric methods to solve design problems, e.g., designing an object or structure to satisfy physical constraints or minimize cost; working with typographic grid systems based on ratios. ★	**Expectations for Learning** In middle school, students work with nets, area, and volume; use appropriate tools to represent situations; and solve real-life and mathematical problems. In this cluster, students make sense of the world around them by using geometric models and their properties to solve more sophisticated problems. The student understanding of this cluster begins at van Hiele Level 1 (Analysis) and moves to Level 2 (Informal Deduction/Abstraction). **ESSENTIAL UNDERSTANDINGS** • Composite figures can be analyzed by approximating them with traditional two- and three-dimensional figures. • Many real-life scenarios are related to length, area, and volume. **MATHEMATICAL THINKING** • Use accurate mathematical vocabulary to represent geometric relationships. • Make connections between terms and properties. • Recognize, apply, and justify mathematical concepts, terms, and their properties. • Use formal reasoning with symbolic representation. • Determine reasonableness of results. • Use proportional reasoning. • Plan a solution pathway. • Connect mathematical relationships to real-world encounters. • Draw a picture or create a model to represent a problem. **INSTRUCTIONAL FOCUS** • Use geometric shapes, their measures, and their properties to describe objects. • Identify useful quantities for modeling situations. • Apply concepts of density based on area and volume. • Solve design problems geometrically. *Continued on next page*

STANDARDS	MODEL CURRICULUM (G.MG.1-3)
	Content Elaborations **OHIO'S HIGH SCHOOL CRITICAL AREAS OF FOCUS** • Math 2, Number 6, page 10 **CONNECTIONS ACROSS STANDARDS** • Solve problems involving right triangles (G.SRT.8). • Use volume formulas to solve problems (G.GMD.3-4, 6). • Model a relationship given a verbal description (F.IF.4). • Show key features of a function (F.IF.7). • Write a function that describes a relationship between two quantities (F.BF.1). • Interpret parts of an expression (A.SSE.1). • Write expressions in equivalent forms (A.SSE.3). • Create equations to describe relationships (A.CED.1-2, 4).

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.MG.1-3)

Instructional Strategies

Note: The model curriculum is not a collection of lessons nor a full curriculum; it does not suggest pace, sequence, or amount of time spent on topics. It provides information about a topic related to the standards including ideas for examples, strategies for teaching, common misconceptions, and possible connections between topics. In addition, the emphasis of the model curriculum is on instruction not on assessment; therefore, the examples in the Instructional Strategies section are written for instructional purposes not necessarily as assessment items. Examples may need to be rewritten to accommodate the needs of each individual classroom.

Standards for Mathematical Practice

This cluster focuses on but is not limited to the following practices:

***MP.1** Make sense of problems and persevere in solving them.*
***MP.2** Reason abstractly and quantitatively.*
***MP.4** Model with mathematics.*
***MP.5** Use appropriate tools strategically.*
***MP.7** Look for and make use of structure.*

VAN HIELE CONNECTION

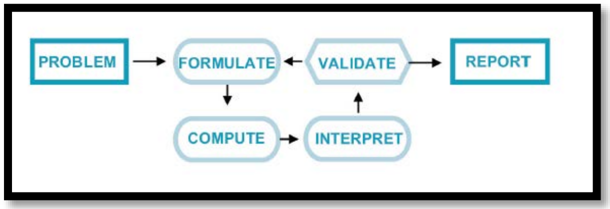
In Math 2 students are expected to move from Level 1(Analysis) to Level 2 (Informal Deduction/Abstraction) and towards Level 3(Deduction).

Level 2 can be characterized by the student doing some or all of the following:

- fully understanding the procedures and formula for determining measurement of simple, familiar shapes;
- generalizing unit-measure enumeration to include fractional units;
- understanding and making unit conversions; and/or
- generalizing measurement to non-squares for area and non-cubes for volume.

Level 3 can be characterized by the student doing some or all of the following:

- comparing shapes by property-preserving transformations/decompositions;
- using geometric properties and variables to understand and solve problems involving formulas for non-familiar shapes; and/or
- comparing length, area, and volume by using geometric properties or transformations.



MODELING

This cluster is dependent on the modeling standards. See page 12 for more information about modeling.

In the standards, modeling means using mathematics or statistics to describe (i.e., model) a real-world situation to deduce additional information about the situation by mathematical or statistical computation and analysis. Modeling in high school uses problems that are not precisely formulated and may not necessarily have a "correct" answer. When making models, students need to figure out not only what to include in the model, but also what to exclude. Students also should be able to analyze and communicate the limitations of the model that they choose.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.MG.1-3)

TIP!

The Consortium for Mathematics and Its Applications (COMAP) offers a High School Mathematical contest in Modeling which offers students the opportunity to compete in a team setting using mathematics to present solutions to real-world modeling problems. See their website for more information.

Genuine mathematical modeling typically involves more than one conceptual category. For example, modeling a herd of wild animals may involve geometry, measurement, proportional reasoning, estimation, probability and statistics, functions, and algebra. It would be somewhat misleading to try to teach a unit with the title of "modeling with geometry." Instead, these standards can be woven into several other content clusters.

The challenge for teaching modeling is finding problems that are interesting and relevant to high school students and, at the same time, solvable with the mathematical tools at the students' disposal.

TIP!

Modeling activities are a good way to show connections among various branches of STEM fields. The sciences (e.g., phyiscs) use mathematics to model real-world phenomena.

DESCRIBING OBJECTS WITH GEOMETRIC MODELS

Provide students with opportunities to understand when real-life scenarios are related to length, area, and volume. For example, heating is related to volume, crop coverage is related to area, and traffic accidents are related to miles driven using probability concepts. Once they build this understanding, then they can use mathematics such as geometric shapes, their measures, and their properties to describe objects or situations. See the Instructional Resources/Tools section for examples.

Technology such as graphing calculators and spreadsheets can help students work with large amounts of data to make models of the real world.

Connecting Geometric Modeling with Probability.

Geometric modeling concepts can be integrated into probability situations. One example is the game of darts. The Instructional Resources/Tools sections has several links connecting probability, darts, and modeling at various levels.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.MG.1-3)

EXAMPLE

Part 1

- Diameter of playing circle: $13\frac{1}{4}$ inches
- Width of inner and outer rings: $\frac{1}{4}$ inches
- Diameter including the inner ring: $6\frac{1}{8}$inches
- Diameter of outer (double) bullseye: $1\frac{1}{4}$ inches
- Diameter of inner (triple) bullseye: $\frac{1}{2}$ inch

If Randy has no skill at darts and all his throws are random but hit the dartboard, find the theoretical probability of hitting the following targets:

a. inner bullseye
b. outer bullseye
c. area between the inner ring and outer bullseye
d. area between inner ring and outer ring
e. inner ring in any section
f. outer ring in any section
g. area of a section between the outer bullseye and inner ring
h. area of a section between the inner ring and outer ring

Discussion: This activity can be extended by having students create their own dartboard.

Information from this example taken from
https://www.quia.com/files/quia/users/kimminr/GameDevelopment/Dartboard-Geometry.pdf_3

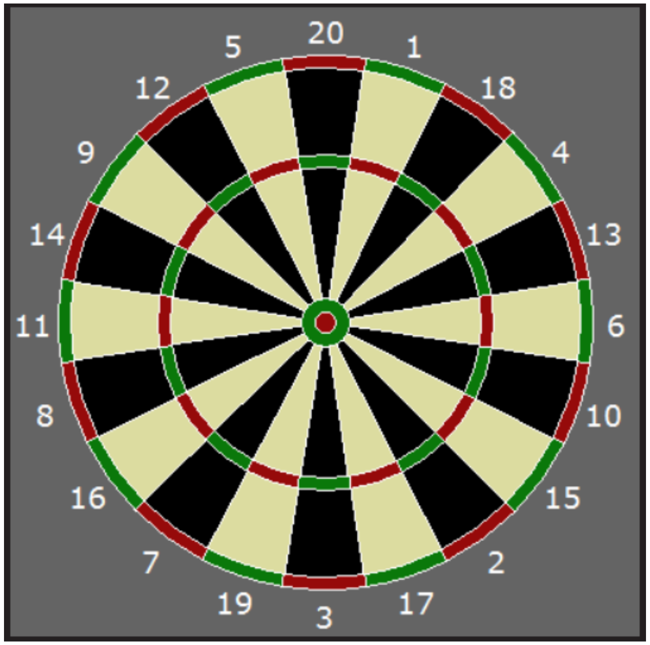


Image taken from dartboards.
http://datagenetics.com/blog/january12012/index.html

Part 2

Darts can be scored with point values ranging from 1 to 20 depending on the sector with the corresponding number. If a dart lands in the outer ring, it is double the sector value, and if it lands in the inner ring it is triple the sector value. The inner bullseye is 50 points and the outer bullseye is 25 points.

a. Find the probability of the following assuming that the throwing is random but lands on the board:
- three inner bullseyes in a row
- the triple 20 three times in a row
- an inner bullseye, an inner ring of 17, and the black sector of 10

b. Where should someone aim when they play darts? Explain.

Discussion: There is no simple answer to part **b.**, for it depends on the skill of the dart player. This will lead to an interesting classroom discussion. Is it better to aim towards the center or towards the outer circle and miss the board entirely? A Geek Plays Darts, a blog by DataGenetics has some interesting ideas for extension.

CONCEPTS OF DENSITY

Students should understand and be able to apply concepts of density based on area and volume in modeling situations.

EXAMPLE

1. The article states that 3.7 million people may have attended the papal Mass in an area of 497,000 square meters. By this count, what was the crowd density? Does this number match that stated in the clip?
2. Assume that the crowd area was a rectangular region with the beach along one side. The full article states that the beach is 4 km long. How wide was the crowd area?
3. The firm Datafohla and other researchers say that a good rule is 2 to 3 people per m^2 at a packed event. Given the size of the crowd area, find an interval to estimate the number of people at the event.
4. The research director for Datafohla estimated that there were between 1.2 million and 1.5 million people at the event. According to these estimates, what would be a reasonable range of the density of the crowd?

> **Counting the Crowd**
>
> The Vatican said an historic 3.7 million people were at the Sunday event, an eye-popping number that would have made it the second largest papal Mass ever. . . . The problem was, the count released by Vatican and Brazilian officials was a guesstimate that statisticians say grossly inflated the crowd figures. The research director of Datafolha, one of Brazil's top polling and statistic firms, said that based on the size of the crowd area and reasonable density estimates, he would put Sunday's turnout at between 1.2 million and 1.5 million people. . . .
>
> McPhail [an emeritus professor of sociology at the University of Illinois who has studied crowd counts for four decades] first chuckled when he heard the Vatican's crowd estimate in an area of Copacabana beach and adjoining streets that encompassed about 497,000 square meters (594,400 square yards).
>
> By the Vatican's count, the crowd density throughout the entire area would have been 7.4 people per square meter, which wouldn't allow for movement of any kind, let alone the jumping, arm waving, singing and dancing seen at the papal events. . . .

Source: "Numbers Don't Add Up for Papal Mass Crowd Count," Associated Press, Aug. 3, 2013, http://newsinfo.inquirer.net/457893/numbers-dont-add-up-for-papal-mass-crowd-count

5. An example from the Programme for International Student Assessment (PISA) reads:
   At a rock concert, a rectangular field 100 m × 50 m was reserved for fans to stand. The concert was sold out. Approximately how many fans were in attendance? Given your answers to the previous questions, determine which of the answers is the most reasonable.
   - **a.** 2,000
   - **b.** 5,000
   - **c.** 20,000
   - **d.** 50,000
   - **e.** 100,000
6. If the general rule is 2 to 3 people per m^2 in a packed event, what is a better estimate of the number of people at the concert in question 5?
7. Determine a good rule for determining the density of people at a packed event.

Example taken from Ebert, D., Morriss, J, Broude, E., & Lesser, R. (October 2015). Counting the crowd; taxing the garden. *Mathematics Teacher,* 109, (3), 168-171.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.MG.1-3)

EXAMPLE

How many people would be in the world if the world had the same population density as—

- **a.** Cleveland
- **b.** Columbus
- **c.** Lima
- **d.** Chillicothe
- **e.** Rio Grande
- **f.** Ohio
- **g.** your city

Discussion: Students will have to look up and/or calculate population density of these cities to answer the question.

TIP! Use masking tape to mask out square meters on the floor and have students stand in the squares to give students a concrete idea of population density.

APPLY GEOMETRIC METHODS TO SOLVE PROBLEMS

Students need to apply geometric methods to solve design problems. This is a good opportunity to make connections to careers and even possibly utilize the career tech standards. Applications could include the following:

- Amount/cost of flooring needed for a house or room
- Amount/cost of paint needed for a room or rooms
- Amount/cost of concrete needed to pour concrete for a patio
- Amount/cost of fencing needed to fence in a yard
- Amount/cost of heating or cooling a large central in BTUs/cubic feet.

Initially modeling problems may need to be scaffolded, but true modeling occurs once the scaffolding is removed. As students gain more exposure to modeling problems the scaffolding should be reduced. The GAIMME Report, Appendix D has some ideas on how to assess modeling in the classroom.

Three-act tasks can also be used in the classroom to promote modeling. See the Instruction Tools/Resources section for ideas.

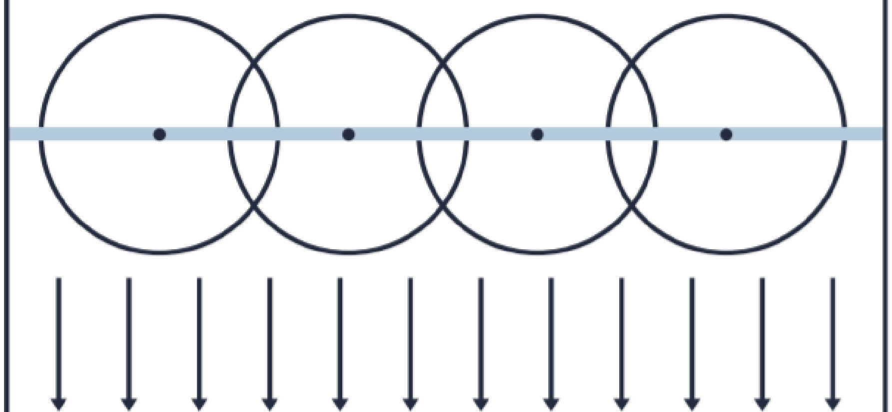
Students may incorrectly believe the process of solving may have only one pathway or that there is always one solution to a problem. Give students messy problems to confront this misconception.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.MG.1-3)

Students may incorrectly believe that answers must always be whole numbers, yet in real-life contexts decimal or fractional solutions occur all the time. It is important to give students problems that result in non-whole number answers.

EXAMPLE

A linear irrigation system consists of a long water pipe set on wheels that keep it above the level of the plants. Nozzles are placed along the pipe, and each nozzle sprays water in a circular region. The entire system moves slowly down the field at a constant speed, watering the plants beneath as it moves. You have 300 feet of pipe and 6 nozzles available. The nozzles deliver a relatively uniform spray to a circular region 50 feet in radius. How far apart should the nozzles be placed to produce the most uniform distribution of water on a rectangular field 300 feet wide?

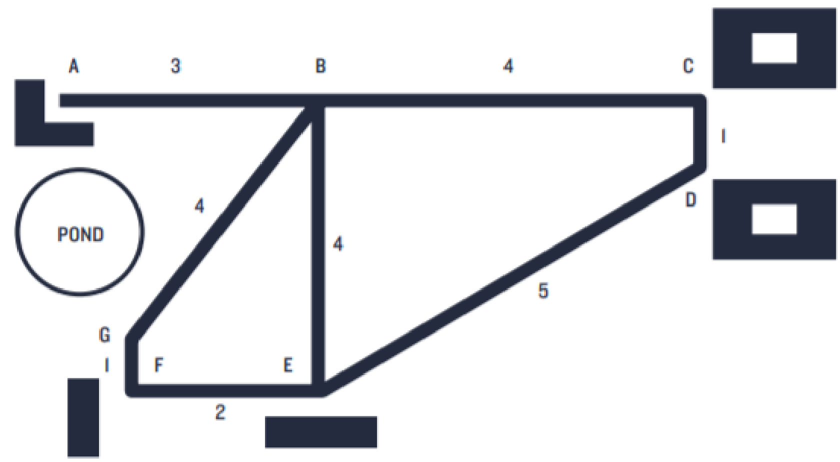


PROBLEM 3.4: THE IRRIGATION PROBLEM

Example taken from the GAIMME report.

EXAMPLE

A map of a portion of a college campus is shown below. The map shows the walking paths and dormitories in this section of campus and the approximate distances (in 100 feet) between locations (the distance between the dorm at D and the dorm at E is 500 feet). Your roommate has convinced you to open a hot dog cart during lunch hours on weekends at one of the intersections along the walkways. You would like the location to be as convenient as possible for the students. Where on campus should you set up your stand?



PROBLEM 3.7: THE HOT DOG CART[10]

Example taken from the GAIMME report.

Discussion: These problems are discussed on pages 74-75 and page 78-79 of the GAIMME report.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.MG.1-3)

Instructional Tools/Resources

These tools and resources may be helpful for instruction. Before use, they should be evaluated at the local level to determine their appropriateness for instruction.

Manipulatives/Technology

- Graphing calculators
- Dynamic geometric software (Geometer's Sketchpad®, Desmos®, Cabri®, or GeoGebra®)
- Rulers
- Protractors
- Compasses

Three-Act Tasks

- Captain's Wheel by When Math Happens is a 3-act task where students answer questions about spinning a wheel.
- Lava Field by When Math Happens is a 3-act task where students answer questions about how long it will take lava to cover a field.
- Closest to the Pin by When Math Happens is a 3-act task where students answer questions about which golf ball is closest to the hole.
- Equidistant Arena by When Math Happens is a 3-act task where students answer questions about where to place an arena.
- Will the Rims Fit? by When Math Happens is a 3-act task where students answer questions about fitting rims onto a car.
- Will the Court Fit? by When Math Happens is a 3-act task where students answer questions about whether a basketball court would be able to fit on a stage.
- Pancakes by When Math Happens is a 3-act task where students answer questions about how many pancakes can be made.
- How Many Houses? by When Math Happens is a 3-act task where students answer questions about how many houses are in a development.
- Air Mattress by When Math Happens is a 3-act task where students answer questions about how long it will take to fill an air mattress.
- Meatballs by Dan Meyer is a 3-act task where students answer questions about whether adding meatballs to a pot of spaghetti will make it overflow. Here is an explanation on how to teach the task.
- Car Caravan by Dan Meyer is a 3-act task where students answer questions about how many toy cars are in a circle. Here is some commentary on the tasks.
- World's Largest Hot Coffee by Dan Meyer is a 3-act task where students answer questions about how many gallons of coffee are in the largest coffee cup.
- Apple Mothership by Dan Meyer is a 3-act task where students answer questions about how many square feet each employee gets.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.MG.1-3)

Describing Objects with Geometric Models

- Toilet Roll by Illustrative Mathematics is a task where students use modeling to deduce algebraic relationships between variables stemming from geometric constraints.
- The Lighthouse Problem by Illustrative Mathematics is a task where students model phenomena on the surface of the Earth.
- Tennis Balls In a Can by Illustrative Mathematics is a task where students explore tennis balls in a can.
- Hexagonal Pattern of Beehives by Illustrative Mathematics is a task where students explore the design of beehives.
- Tilt of Earth's Axis and the Four Seasons by Illustrative Mathematics is a task where students relate their weather experiences with a simple geometric model to explain why the seasons occur.
- Solar Eclipse by Illustrative Mathematics is a task where students apply their knowledge of similar triangles to a solar eclipse.
- Modeling Motion: Rolling Cups by Mathematics Assessment Project where students produce a model to illustrate rolling a cup.
- Simpson Sunblocker by YouCubed has students explore geometric proportionality in the context of Mr. Burns placing a circular disk to block the sun over Springfield.

Concepts of Density

- How Many Leaves on a Tree? and How Many Leaves on a Tree? (Version 2) by Illustrative Mathematics is a task where students have to make a reasonable estimate for something that is too large to count.
- How Many Cells Are in the Human Body? by Illustrative Mathematics is a task where students have to apply concepts of mass, volume, and density in a real-world context.
- How Thick Is a Soda Can? Variation 1 and How Thick is a Soda Can? Variation 2 by Illustrative Mathematics is a task where students apply concepts of density to find the thickness of a soda can.
- Archimedes and the King's Crown by Illustrative Mathematics is a task where students combine the ideas of ratio and proportion with the context of density of matter.
- Indiana Jones and the Golden Statue by Illustrative Mathematics is a task where students are introduced to the subtle use of density and units related to density.
- A Ton of Snow by Illustrative Mathematics is a task where students examine a mathematical statement about the mass of snow.
- Density Word Problems by Khan Academy are word problems related to concepts of density.
- Where is Everybody? by Jessica Woolard is an NCTM Illuminations lesson where students explore the population densities in Canada and the United States. *NCTM now requires a membership to view their lessons.*
- Why is California So Important?, How Could that Happen?, and A Swath of Red by Kimberly Morrow-Leong is an NCTM Illuminations unit about the electoral college and population density. *NCTM now requires a membership to view their lessons.*
- Density and Specific Gravity—Practice Problems from The Math You Need, When You Need It is a website with math problems that make a distinction between density and weight.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.MG.1-3)

Apply Geometric Methods to Solve Problems

- Access Ramps by Achieve the Core is a CTE task where students design an access ramp.
- Fences by Achieve the Core is a CTE task where students add a fence to a pool.
- Framing a House by Achieve the Core is a CTE task where students frame a house.
- Storage Sheds by Achieve the Core is a CTE task where students build storage sheds.
- Stairway by Achieve the Core is a CTE task where students design a stairway for a custom home.
- Miniature Golf by Achieve the Core is a CTE task where students design three new holes of a golf course.
- Grain Storage by Achieve the Core is a CTE task where students have to design a new storage facility for grain.
- Ice Cream Cone by Illustrative Mathematics is a task where students develop a formula for surface area and estimate the maximum number of wrappers that could be cut from a rectangular piece of paper.
- Curriculum Burst 102: A Dart Probability by James Tanton, MAA Mathematician in Residence, is a lesson about the probability of a dart hitting a dart board.
- Dart Board Geometry by Quia is lesson on geometric modeling and probability using a dartboard.
- Another Dartboard by NZMaths connect geometric modeling and probability to the area under a parabola and above the x-axis.
- A Geek Plays Darts by DataGenetics is a blog that connects geometric modeling and probability and extends to more advanced topics.
- Darlene's Dart Board by NCTM is a problem of the week pertaining to a dart board. *NCTM now requires a membership to view their lessons.*
- Optimal Shooting Angle by DataGenetics is a blog that discusses the optimal shooting angle of a soccer ball.

Curriculum and Lessons from Other Sources

- EngageNY, Geometry, Module 2, Topic A, Lesson 1: Scale Drawings is a lesson that pertains to this cluster.
- EngageNY, Geometry, Module 2, Topic C, Lesson 19: Families of Parallel Lines and the Circumference of the Earth and Lesson 20: How Far Away Is the Moon? are lessons that pertain to this cluster.
- EngageNY, Geometry, Module 3, Topic B, Lesson 13: How Do 3D Printers Work? is a lesson that pertains to this cluster.
- Georgia Standards of Excellend and Curriculum Frameworks, Geometry, Unit 5: Geometric and Algebraic Connections has many tasks that pertain to this cluster.
- Mathematics Vision Project, Geometry, Module 7: Modeling with Geometry has several tasks that pertain to this cluster.
- Illustrative Mathematics, Geometery, Unit 3, Lesson 2: Scale of the Solar System is a lesson that pertains to this cluster.
- Illustrative Mathematics, Geometery, Unit 5: Solid Geometery, Lesson 17: Volume and Density and Lesson 18: Volume and Graphing are lessons that pertain to this cluster.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (G.MG.1-3)

General Resources

- Arizona 7-12 Progression on Geometry is an informational resource for teachers. This cluster is addressed on page 17.
- Arizona High School Progression on Modeling is an informational resource for teachers. This cluster is addressed on page 17.
- High School Coherence Map by UnboundED is a map of the relationships among high school mathematics standards. Please note: This resource is aligned to the Common Core not Ohio's Learning Standards. Although the tool is useful, it needs to be checked with respect to Ohio's Learning Standards.
- van Hiele Model of Geometric Thinking is a pdf created by ODE that summarizes the van Hiele levels.

References

- Common Core Standards Writing Team. (2019). *Progressions for the Common Core State Standards for Mathematics (draft February 8, 2019).* Tucson, AZ: Institute for Mathematics and Education, University of Arizona.
- Ebert, D., Morriss, J, Broude, E., & Lesser, R. (October 2015). Counting the crowd; taxing the garden. *Mathematics Teacher,* 109, (3),
- *GAIMME: Guidelines for Assessment and Instruction in Mathematical Modeling Education,* Sol Garfunkel and Michelle Montgomery, editors, COMAP and SIAM, Philadelphia, 2016. View the entire report, available freely online, at https://siam.org/Publications/Reports/Detail/Guidelines-for-Assessment-and-Instruction-in-Mathematical-Modeling-Education
- *Sourcebook of Applications of School Mathematics,* compiled by a Joint Committee of the Mathematical Association of America and the National Council of Teachers of Mathematics (1980).
- *Mathematics: Modeling our World*, Course 1 and Course 2, by the Consortium for Mathematics and its Applications (COMAP).
- *Geometry & its Applications* (GeoMAP). An exciting National Science Foundation project to introduce new discoveries and real-world applications of geometry to high school students. Produced by COMAP.
- Williams, N. & Bruels, C. (February 2011). Target geometry and probability using a dartboard. *Mathematics Teaching in Middle School*, *16, (6),* 375-378.

STANDARDS

Statistics and Probability

CONDITIONAL PROBABILITY AND THE RULES OF PROBABILITY

Understand independence and conditional probability, and use them to interpret data.

S.CP.1 Describe events as subsets of a sample space (the set of outcomes) using characteristics (or categories) of the outcomes, or as unions, intersections, or complements of other events ("or," "and," "not").★

S.CP.2 Understand that two events A and B are independent if and only if the probability of A and B occurring together is the product of their probabilities, and use this characterization to determine if they are independent.★

S.CP.3 Understand the conditional probability of A given B as $^{P(A \text{ and } B)}/_{P(B)}$, and interpret independence of A and B as saying that the conditional probability of A given B is the same as the probability of A, and the conditional probability of B given A is the same as the probability of B.★

Continued on next page

MODEL CURRICULUM (S.CP.1-5)

Expectations for Learning

In middle school, students develop basic probability skills including probability as relative frequencies; probabilities of compound events; the development a uniform/non-uniform probability model; and the use of tree diagrams. Also, students are introduced to two-way frequency tables in middle school. However, students' only prior exposure to the concept of independence was in S.ID.5 (Math 1). This cluster focuses on the concept of independence between two categorical variables. It also focuses on the understanding of independence rather than symbolic notation and formulas. Fluency with independence is expected by the end of Math 2.

ESSENTIAL UNDERSTANDINGS

- Approximations for the true probability of an event can be found by looking at the long-run relative frequency.
- The sample space of a probability experiment can be modeled with a Venn diagram.
- The union of an event and its complement represent the entire sample space.
- The intersection of an event and its complement represent the empty set.
- Conditional probability is the probability of event A occurring given that event B has occurred. It is denoted by $A|B$ and is read "A given B."
- Two events occurring in succession are said to be independent if the outcome of one event has no effect on the outcome of the other, e.g., a coin tossed twice. Otherwise, the events are dependent, e.g., two cards are drawn in succession from a standard deck of cards.
- The intersection of two sets A and B is the set of elements that are common to both set A and set B. It is denoted by $A \cap B$ and is read "A intersection B" as well as "A and B."
- The union of two sets A and B is the set of elements, which are in A or in B or in both. It is denoted by $A \cup B$ and is read "A union B" as well as "A or B."
- If A and B are events that have no outcomes in common $(A \cap B \neq 0)$, they are said to be mutually exclusive. Mutually exclusive events cannot occur together.

Continued on next page

STANDARDS

S.CP.4 Construct and interpret two-way frequency tables of data when two categories are associated with each object being classified. Use the two-way table as a sample space to decide if events are independent and to approximate conditional probabilities. *For example, collect data from a random sample of students in your school on their favorite subject among math, science, and English. Estimate the probability that a randomly selected student from your school will favor science given that the student is in tenth grade. Do the same for other subjects and compare the results.* ★

S.CP.5 Recognize and explain the concepts of conditional probability and independence in everyday language and everyday situations. *For example, compare the chance of having lung cancer if you are a smoker with the chance of being a smoker if you have lung cancer.* ★

MODEL CURRICULUM (S.CP.1-5)

Expectations for Learning, continued

MATHEMATICAL THINKING

- Use appropriate vocabulary.
- Attend to precision.

INSTRUCTIONAL FOCUS

- Recognize and explain for two successive events, whether the outcome of the first event affects the outcome of the second event.
- Recognize and justify conceptually whether two events are independent.
- Make connections between conditional probability and independence. Recognize sample space subsets in everyday contexts.
- Identify an event and its complement.
- Identify which components of the sample space represent the union and intersection of two events.
- Explain what a conditional probability means within a context.
- Distinguish between a conditional probability (A given B) and the probability of an intersection (A and B).
- Use a two-way frequency table to determine the following:
    - conditional probabilities;
    - probabilities of the sample space subsets;
    - event independence by comparing joint probabilities (P(A and B)) and the product of the separate probabilities (P(A) × P(B)); and
    - event independence by comparing the conditional probability (P(A given B)) and the probability P(A).

Content Elaborations

OHIO’S HIGH SCHOOL CRITICAL AREAS OF FOCUS

- Math 2, Number 1, page 3

CONNECTIONS ACROSS STANDARDS

- This will lead into the cluster (S.CP.6-9) which includes the calculations of conditional probabilities, and the use of probability formulas and set notation with probability.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (S.CP.1-5)

Instructional Strategies

Note: The model curriculum is not a collection of lessons nor a full curriculum; it does not suggest pace, sequence, or amount of time spent on topics. It provides information about a topic related to the standards including ideas for examples, strategies for teaching, common misconceptions, and possible connections between topics. In addition, the emphasis of the model curriculum is on instruction not on assessment; therefore, the examples in the Instructional Strategies section are written for instructional purposes not necessarily as assessment items. Examples may need to be rewritten to accommodate the needs of each individual classroom.

Standards for Mathematical Practice

This cluster focuses on but is not limited to the following practices:

MP.4 *Model with mathematics.*

MP.6 *Attend to precision.*

MP.7 *Look for and make use of structure.*

GAISE MODEL

The GAISE model is a framework for teaching students to be statistically literate in a world that is driven by data. The GAISE model has four steps.

Step 1: Formulate Questions
Step 2: Collect Data
Step 3: Analyze Data
Step 4: Interpret Data

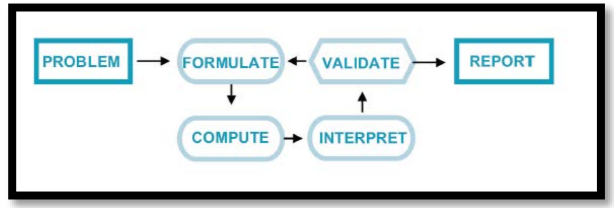
This cluster provides an opportunity for continued use of the GAISE model which is foundational for statistics taught in courses from middle school and beyond Math 3. Students in Math 2 should be formulating their own questions of interest including starting to form questions that make a generalization about a population. They should be collecting data and begin discussing random selection and random assignment. This includes describing potential sources of error in data collection and analyzing if a sample is representative of the population.

EXAMPLE

Students can conduct a survey designed to explore possible independence or association between two events. Keep in mind that the data collected should ideally be qualitative as probability analysis is usually more appropriate for qualitative data than other methods of analysis. Quantitative data is better suited for other high school courses as it relates to other clusters (S.ID and S.IC). After students have collected their data from the surveys, they can organize it into a two-way frequency table. Then they can consider the resultant probabilities to evaluate whether their events are independent or associated. The project may also culminate in a presentation and/or paper summarizing their process and work.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (S.CP.1-5)

MODELING



The Standard for Mathematical Practice (SMP.4), *modeling* is important for working with all forms of probability. In fact, all the standards in Statistics and Probability conceptual category are modeling standards. Instruction should stress the usefulness and applicability to real-world scenarios and using data driven probabilities in context. Students should use the probability tools addressed in this cluster to model real-life information. See page 12 for more information about modeling.

Students should be able to interpret and explain concepts of probability, including those expressed in percentages, in real-life situations.

TIP! Many games that use playing cards or number cubes can be used to explore sample space. Students in middle school may have explored why 7 is a lucky number based on viewing the sample space of rolling two number cubes. At this level, it may be fun to use familiar games such as Yahtzee, Farkle, or even Monopoly to explore more advanced concepts of probability.

CHARACTERISTICS OF OUTCOMES

The "A and B" and "A or B" language is easy for students to misinterpret. With additional contextual language they can make sense of it, but will take a lot of practice to master the vocabulary of "or," "and," "not" with the mathematical notation of union (∪) and intersection (∩). Districts may choose which notation to use for a complement. Commonly used notations for the complement of event A include A^C, A', or $\bar{A}$.

In Algebra students learned about compound inequalities using the words "and" and "or." Make the connection between the solutions of compound inequalities and unions and/or intersections of sets. Begin by developing the concept that an "and" statement is an overlap (intersection) and that an "or" statement is a "union" by using familiar categorical contexts, emphasizing the words "union" and "intersection." After students become familiar with the contexts of "and" and "or" move to real-world examples modeled by compound inequalities that are solved and then graphed on a number line.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (S.CP.1-5)

EXAMPLE

Part 1

Have two students list their favorite fruits.

Beth:

Jennifer:

a. What are the favorite fruits of Beth *and* Jennifer?
b. What are the favorite fruits of Beth *or* Jennifer?

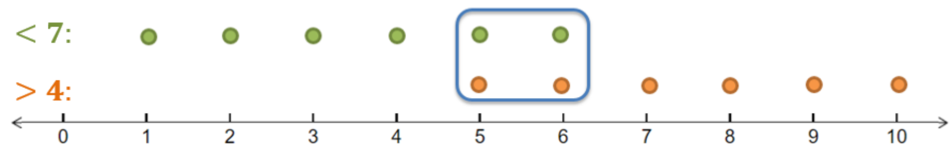
Discussion: Draw attention to the fact that the word "and" refers to the overlap (intersection) of the two sets of fruit. Therefore, the solution to part **a.** is bananas and oranges since both girls list those fruits as one of their favorites. The solution in part **b.** is each fruit listed either by Beth *or* Jennifer which would be strawberries, oranges, grapes, apples, bananas, cherries, pears, lemons, which would be a union of both sets of fruit. Notice that repeated fruit that is common to both sets is only listed once, not twice.

Part 2

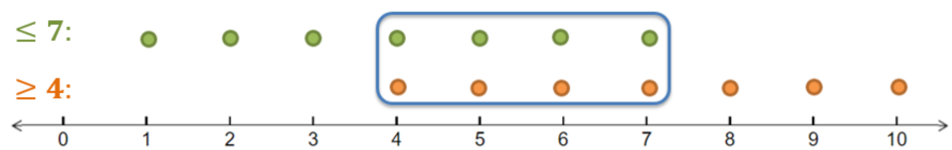
a. Use a number line to show the whole numbers between 1-10 that are bigger than 4 *and* smaller than 7?
b. Use a number line to show the whole numbers between 1-10 that are bigger than or equal to 4 *and* smaller than or equal to 7?
c. Use a number line to show the whole numbers between 1-10 that are bigger than 4 *or* smaller than 7?

Discussion: Students should build off the idea behind "and" and "or" in the previous fruit context. Draw attention to the fact that "and" refers to the overlap of the solutions between the two sets and "or" refers to each number listed in either set. So the solution to part **a.** is 5 and 6. The solution to part **b.** is 4, 5, 6, and 7. The solution to part **c.** is 1, 2, 3, 4, 5, 6, 7, 8, 9, and 10. Notice that the numbers 5 and 6 are listed only listed once, not twice.

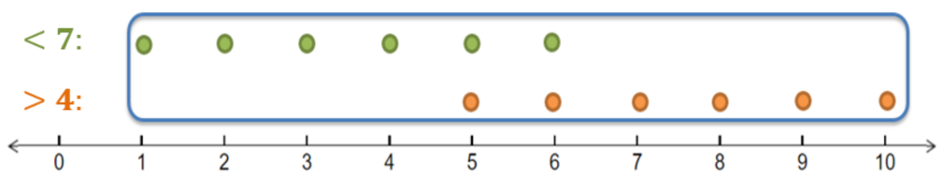
Bigger than 4 ***and*** smaller than 7



Bigger than or equal to 4 ***and*** smaller than or equal to 7



Bigger than 4 ***or*** smaller than 7



INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (S.CP.1-5)

Part 3

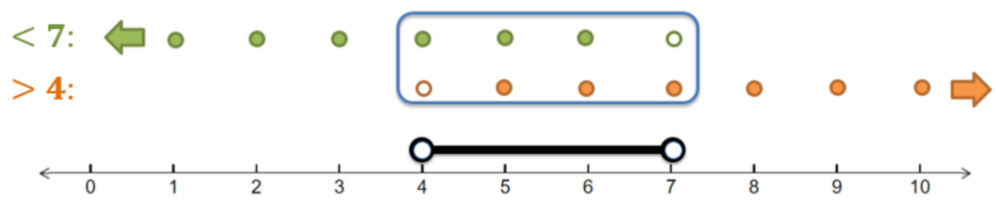
a. Use a number line to show all numbers satisfying the conditions $x > 4$ *and* $x < 7$.

b. Use a number line to show all numbers satisfying the conditions $x \geq 4$ *and* $x \leq 7$.

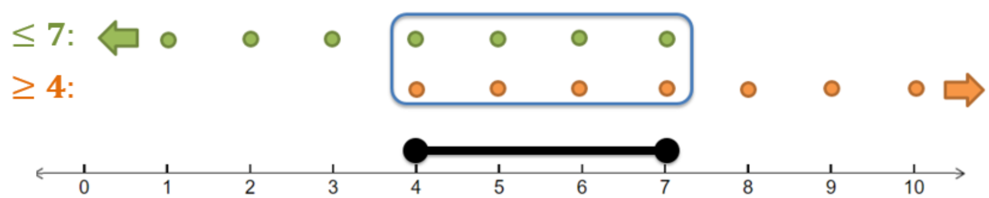
c. Use a number line to show all numbers satisfying the conditions $x > 4$ *or* $x < 7$.

Discussion: Connect the inequalities to the previous two examples. Draw attention to the fact that "and" still mean an overlap or intersection of the two statements. The solution to part **a.** is $4 < x < 7$, and the solution to part **b.** is $4 \leq x \leq 7$. The solution to part **c.** is all real numbers since the graph covers the entire number line. *Note: The solutions can be written in other formats, e.g., set builder notation, interval notation, or inequalities.*

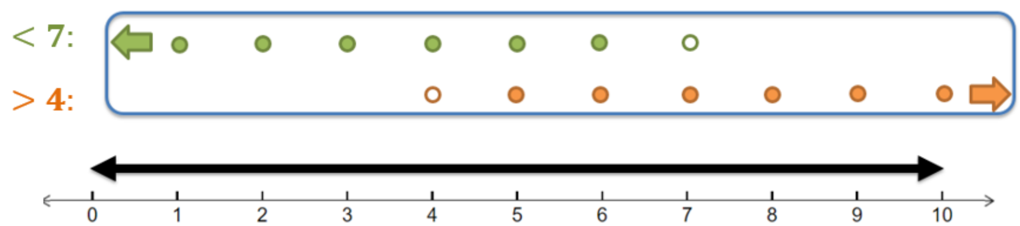
Greater than 4 ***and*** less than 7



Greater than or equal 4 ***and*** less than or equal to 7



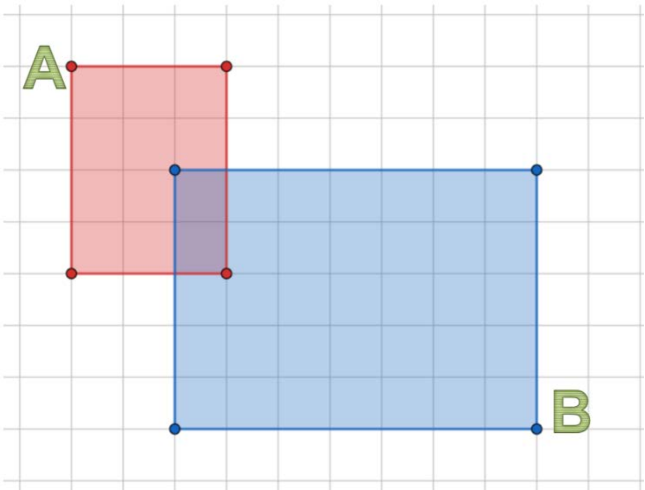
Greater than 4 ***or*** *less* than 7



INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (S.CP.1-5)

To help students understand probability and Venn diagrams, it may be helpful to connect the idea of intersections and unions to area models. This will set them up for the Addition Rule in standard S.CP.6.

EXAMPLE



Part 1

- **a.** What is the area of region A?
- **b.** What is the area of region B?
- **c.** What is the area of region A *and* B?
- **d.** If A and B are regions in the plane, then $A \cap B$ denotes the intersection of the two regions, that is, all points that lie in both A and B. What is $A \cap B$?

Discussion: Students have been doing composite shapes since at least middle school and may have developed inaccurate language. However, now is the time to push for more precision in language. Some students will say that the area is the area of both rectangles minus the overlapping regions. Others may say it is only the overlapping region. This is a good discussion to have. Bring them back to the previous example regarding the fruit. The region of A and B is the overlapping piece because it includes both. Therefore, the area of A and B or the intersection of A and B (or $A \cap B$) is 2 units.

Part 2

- **a.** What is the area of region A *or* region B?
- **b.** Do you count the overlapping squares? Explain.
- **c.** If A and B are regions in the plane, then $A \cup B$ denotes the union of the two regions (i.e., all points that lie in A or in B), including points that lie in both A and B. What is $A \cup B$?

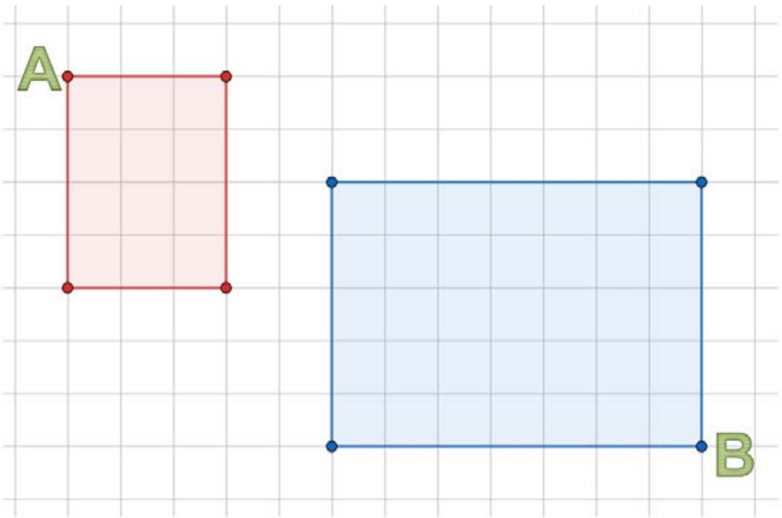
Discussion: Again, bring students back to the concept of the fruit. Draw attention to the fact that when students were finding the area of composite shapes there were actually finding the area of the union of the two shapes. Students should realize that just as they never counted the overlapping section twice in composite shapes, neither should they count twice the overlapping region in union. Note that often times the area of region A *or* B is also called the union of A *or* B and notated $A \cup B$. The area of $A \cup B$ is 45 square units.

TIP! Even though in the past students may have used the word "and" to refer to both shapes, they really should have been using the word "or," since from a mathematical standpoint the use of "and" is incorrect.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (S.CP.1-5)

EXAMPLE

a. What is the area of region A?
b. What is the area of region B?
c. What is the area of region A *and* B?
d. What is the area of region A *or* B?



Discussion: In this example the events are disjoint, so their intersection in the empty set and their union is the sum of the areas of the two rectangles.

TIP! To help students represent the probability symbols, the sign for union, ∪, looks like a capital U. The sign for intersection ∩ looks like an A for "And."

The Standard for Mathematical Practice (SMP.7), *making use of structure* is important for working with all forms of probability. Specifically, students may use the structure of Venn diagrams to build two-way frequency tables and vice-versa.

Venn diagrams provide a way to display a sample space visually.

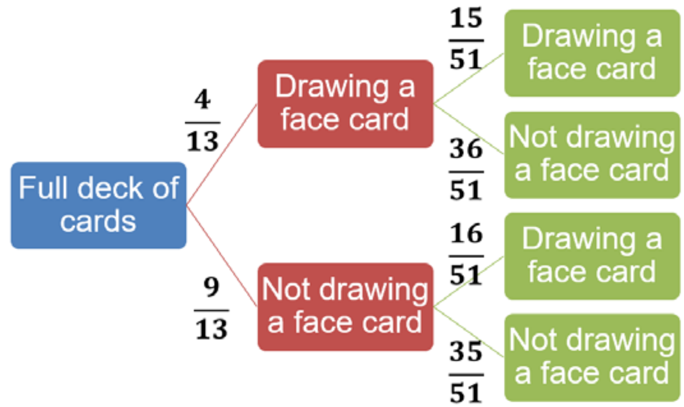
A good activity for working with sample space is through shading different sections of a Venn diagram to represent the various subsets of the sample space. Another way is to identify the subset's sample space represented by a shaded diagram. The Math Vision Project link in the *resources* section below provides excellent activities of this type.

Two-way frequency tables can also be used to identify all probabilities addressed in this cluster including marginal, union, intersection, conditional, and complementary probabilities. They allow students to use table data to evaluate independence of the events represented. It is also recommended that students practice transitioning back and forth between Venn diagrams and two-way frequency tables as well as other data displays of the teachers' choosing.

Although students are familiar with tree diagrams from middle school, the standards in this cluster do not mention the use of tree diagrams which is the traditional way to treat conditional probabilities. Instead, probabilities of conditional events are to be found using a two-way table wherever possible. However, tree diagrams may be a helpful tool for some problems, but students may have difficulty realizing that the second set of branches are conditional probabilities.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (S.CP.1-5)

Students may incorrectly believe that multiplying across branches of a tree diagram has nothing to do with conditional probability, when in fact a tree diagram is set up to illustrate conditional probabilities.



The probability of drawing two face cards successively without replacement is $\frac{4}{13} \cdot \frac{15}{51} = \frac{20}{221}$.

TWO-WAY TABLES

There are two-types of frequency tables: frequency tables and relative frequency tables. In Grade 8 students worked mostly with frequency tables. In Math 1, they were exposed to and expected to work with relative frequency tables.

Frequency Table

	More than 60 min	60 min or less	TOTAL
Phone	61	48	109
No Phone	11	40	51
TOTAL	72	88	160

Relative Frequency Table

	More than 60 min	60 min or less	TOTAL
Phone	$^{61}/_{160} \approx$ 0.38 = 38%	$^{48}/_{160} \approx$ 0.30 = 30%	$^{109}/_{160} \approx$ 0.68 = 68%
No Phone	$^{11}/_{160} \approx$ 0.07 = 7%	$^{40}/_{160} \approx$ 0.25 = 25%	$^{51}/_{160} \approx$ 0.32 = 32%
TOTAL	$^{72}/_{160} \approx$ 0.45 = 45%	$^{88}/_{160} \approx$ 0.55 = 55%	$^{160}/_{160} =$ 1.00 = 100%

TIP!

Calculating relative frequencies compared to frequencies can be difficult to understand. See https://opinionator.blogs.nytimes.com/2010/04/25/chances-are/. It may be helpful for students to transform a relative frequency table into a frequency table with the total as 1,000. Although, not exact, the numbers will be close enough for this level of mathematics.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (S.CP.1-5)

Marginal Frequency

Row totals and column totals constitute the marginal frequencies. These are located in the margins of the table and can also be calculated by adding across columns or rows.

	More than 60 min	60 min or less	TOTAL
Phone	61	48	109
No Phone	11	40	51
TOTAL	72	88	160

	More than 60 min	60 min or less	TOTAL
Phone	61	48	109
No Phone	11	40	51
TOTAL	72	88	160

Joint Frequency

Joint frequency is where the two variables "join" such as keeping the phone in the bedroom and taking longer than 60 minutes to fall asleep. These can be found in the body of the table. Joint frequencies can be connected to the term "intersection."

Using a two-way table begins with calculation of marginal probabilities. Conditional probabilities and determination of independent events follow. A Bayes' Problem presented in a tree-diagram context (where a question asks for a particular prior event having happened in the first set of branches when the given information is about what specifically happened in the second set of branches) becomes straightforward when using a two-way table, while also avoiding a lot of confusing tree-diagram notation.

RELATIONSHIPS BETWEEN SETS AND PROBABILITY

Intersection

EXAMPLE

A survey was given asking about the social network people prefer. The results are summarized in the table.

	Snapchat	Facebook	Total
Ages 18-24	75	15	90
Ages 25+	12	92	104
Total	87	107	194

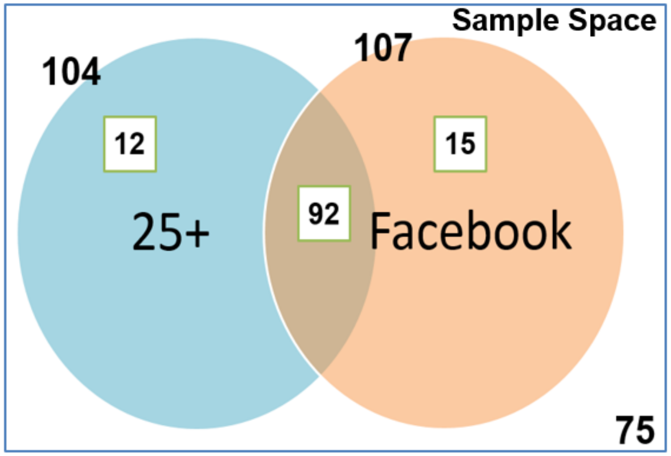
Example continued on next page

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (S.CP.1-5)

Once the data are organized into a table, the table can be used to find the probability that someone prefers Facebook AND is 25+ which can be written P(FB ∩ 25) which would be $\frac{92}{194}$ or approximately 47.42%.

	Snapchat	Facebook	Total
Ages 18-24	75	15	90
Ages 25+	12	92	104
Total	87	107	194

Note: Using two-way frequency tables should serve as a more formalized extension of the informal usage in cluster S.ID.5 from Math 1.



The data can also be represented by a Venn diagram, where the intersection of 92 is the shaded region. The total number of people in the survey could be found by adding up the frequencies in each region: the blue region representing those who were 25+ and did not prefer Facebook (12), the peach region representing those who were not 25+ but did prefer Facebook (15), the intersection of the regions representing the participants who were 25+ and prefer Facebook (92), and those not included in the Venn diagram (75) to get a total of 194.

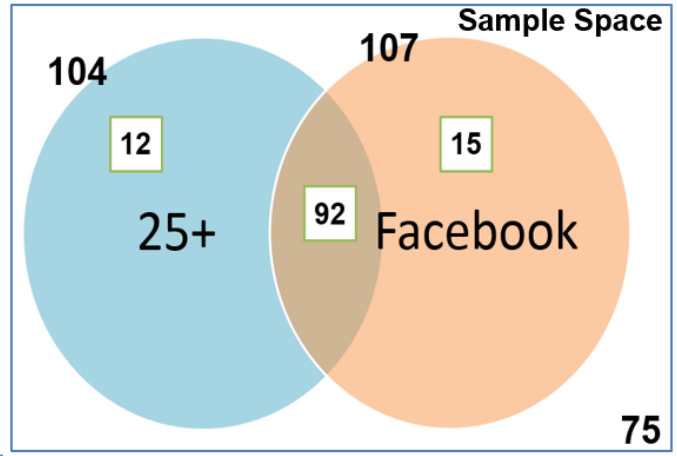
Since intersections involve the word "and" students often incorrectly think the probability of two events happening together is greater than the probability of each individual event. However, it is actually smaller since it is a more specific (restrictive) event.

Union

The Venn diagram can also be used to find the probability if someone prefers Facebook OR is 25+ which can be written P(FB ∪ 25), by adding up all the numbers in the parts of the circles and dividing by the total number of people surveyed: $\frac{(12+92+15)}{194}$. (See previous example for complete data.)

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (S.CP.1-5)

Students should be able to make the connection between the Venn diagram and the table.



	Snapchat	Facebook	Total
Ages 18-24	75	15	90
Ages 25+	12	92	104
Total	87	107	194

For more information on how to use the table to find the probability in a union, see cluster S.CP.6-9.

Since unions involve the word "or," students often incorrectly think that they have to choose between two events or that they will lose an event, so they incorrectly think the probability is smaller than either of the events individually. However, it is minimally at least equal to the probability of the larger probability of the two events.

EXAMPLE

If the probability of liking soccer is 43% and the probability of being a freshman is 26%.

- Is the probability of liking soccer *or* being a freshman? Choose the most accurate statement.
  - **a.** 43%
  - **b.** 26%
  - **c.** At least 43%
  - **d.** At least 26%
  - **e.** At most 43%
  - **f.** At most 26%
- Is the probability of liking soccer *and* being a freshman? Choose the most accurate statement.
  - **a.** 43%
  - **b.** 26%
  - **c.** At least 43%
  - **d.** At least 26%
  - **e.** At most 43%
  - **f.** At most 26

Discussion: The probability of liking soccer *or* being a freshman is at least 43%, whereas The probability of liking soccer *and* being a freshman is at most 26%.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (S.CP.1-5)

Complement

Some problems can be solved much faster using the complement of the event. The complement of the event is the set of all outcomes that are not the event. The $P(event) + P(complement) = 1$. This formula can be rearranged to $P(event) = 1 - P(complement)$ or $P(complement) = \quad 1 - P(event)$ depending on the situation. Commonly used notations for the complement of event A include A^C, A', or $\bar{A}$.

EXAMPLE

- **a.** What is the probability of rolling a 4 when rolling a number cube?
- **b.** What is the probability of not rolling a 4 when rolling a number cube?
- **c.** Would rolling a 4 be better, worse, or the same chance if you had two rolls? Explain.
- **d.** What is the probability of rolling at least one 4 when rolling a number cube twice?
- **e.** You wanted to roll a 4. You rolled the number cube but rolled a 5. What is the chance that you will roll a 4 if you roll again?

Discussion:

- Part **a.** and part **b.** are $P(4) = \frac{1}{6}$ and $P(not\ 4) = \frac{5}{6}$.
- Part **c**. lends itself to discussion that should set the stage for more in-depth analysis in Part **d.**
- Part **d.** gets tricky as it would be easier to solve the problem using complement: $P(not\ 4, not\ 4) = \frac{5}{6} \cdot \frac{5}{6} = \frac{25}{36} \approx 69.4\%$, so the probability of rolling a 4 when rolling the number cube twice is $P(rolling\ at\ least\ one\ 4) = 1 - \frac{25}{36} = \frac{11}{36} \approx 30.6\%$, so you are more likely to roll a 4 if you have two chances to roll.
- Since probability only deals with what is not yet known and the outcomes of rolling a number cube is independent, the first roll does not affect the second roll. Therefore, the first failure in part **e**. is irrelevant, and $P(4) = \frac{1}{6}$.

Disjoint/Mutually Exclusive

When two events are *mutually exclusive,* also known as *disjoint*, the intersection of the events is the empty set. For example, possible student grades in a course are mutually exclusive because a student cannot have both an 'A' and a 'B' in a course at the same time.

Students may incorrectly think that independent and mutually exclusive events are the same thing. In reality, *independence* is a probability concept, which means that the occurrence of one event does not affect the occurrence of another. In contrast, *mutually exclusive* is an event concept, which means that the two events cannot occur at the same time.

CONDITIONAL PROBABILITY

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (S.CP.1-5)

In the Standard for Mathematical Practice *precision* (SMP.6), it is important for working with conditional probability. Attention to the definition of an event along with the writing and use of probability function notation are important requisites for communication of that precision. For example, given event $A(female)$ and event $B(survivor)$, what does $P(A|B)$ mean?

	More than 60 min	60 min or less	TOTAL
Phone	61	48	109
No Phone	11	40	51
TOTAL	72	88	160

Conditional probabilities are determined by focusing on a specific row or column of the table. For example, if a person sleeps with their phone, what is the probability it will take him or her more than 60 minutes to fall asleep? ($^{61}/_{109}$ ≈ 56%.).

Notice the example above is different than "What is the probability that a person that takes more than 60 minutes to fall asleep sleeps with their phone?" ($^{61}/_{72}$ ≈ 85%). *Note: The use of a vertical line for the conditional "given" is not intuitive for students and they often confuse the events $B|A$ and $A|B$.*

TIP!

For conditional probabilities, students should recognize that in the formula $P(A|B) = \frac{P(A \cap B)}{P(B)}$ the numerator represents the value belonging to both events and the denominator is the total of the "given" event. For example, if event A is more than 60 minutes to fall asleep and event B is sleeping with a phone, then $P(More\ than\ 60\ min|Phone) = \frac{P(A \cap B)}{P(B)} = \frac{61}{109}$. Emphasize the distinction between this probability as compared to the probability of the intersection of two events—sleeping with their phone and taking more than 60 minutes to fall asleep—which would be $\frac{61}{160}$; students often confuse the two.

Exploring situations where the outcomes of tests can be false-positive is a good real-world example of conditional probabilities. See the Instructional Tools/Resources for ideas. A good example could be discussing the usefulness of medical testing despite the possibility of false positives using probability to defend a stance.

TIP!

Students often find identifying a conditional difficult when the problem is expressed in words in which the word "given" is omitted. For example, find the probability that a female is a survivor. It may be helpful to have students rewrite conditional probabilities using different verbal expressions to show flexibility in language.

Conditional probability is addressed in S.CP.3, S.CP.5, and S.CP.6.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (S.CP.1-5)

- Standard S.CP.3 defines and calculates conditional probability using mathematical symbolism. The probability, $A\ given\ B$, are stated with respect to the original (whole) sample space.
- Standard S.CP.5 conceptualizes and explains in words the conditional probability of S.CP.3.
- Standard S.CP.6 differs from S.CP.3 in that it forces the student to consider that event B is the sample space (which is reduced compared to the whole sample space) and that students need to use the part of B that belongs to A.

EXAMPLE

Roll two fair number cubes (one number cube is red, and the other is green). Let event A represent the sum of the rolled numbers on the number cube equals 8 or more, and event B is both number cubes rolled numbers are prime numbers.

- **a.** What are the possible outcomes of event A?
- **b.** What are the possible outcomes of event B?
- **c.** What is the probability of event A?
- **d.** What is the probability of event B?
- **e.** What is $P(A\ given\ B)$?

Discussion:

- The possible outcomes of event A are $\{(2,6)(3,5),(3,6),(4,4),(4,5)(4,6),(5,3),(5,4),(5,5)(5,6),(6,2),(6,3),(6,4),(6,5),(6,6)\}$.
- The possible outcomes of event B are $\{(2,2),(2,3),(3,2),(2,5),(5,2),(3,3),(3,5),(5,3),(5,5)\}$.
- The probability of event A is $\frac{15}{36}$ and the probability of event B is $\frac{9}{36}$.
- S.CP.3 emphasizes that the $P(A\ given\ B)$ is $P(A\ and\ B)$ divided by $P(B)$. Since $(A\ and\ B) = [(3,5),(5,3),(5,5)]$, then $P(A\ and\ B)$ is $\frac{3}{36}$ because there are 36 parings for rolling two number cubes. Therefore, $P(A\ given\ B)$ can be calculated by $\frac{3}{36} \div \frac{9}{36} = \frac{3}{9}$ or $\frac{1}{3}$. Notice the probabilities are with respect to the original sample space of 36 outcomes. S.CP.6 emphasizes the reduced sample space of event $B = [(2,2),(2,3),(3,2),(2,5),(5,2),(3,3),(3,5),(5,3),(5,5)]$. Each outcome has the probability $\frac{1}{9}$. Then $P(A\ given\ B)$ is the number of outcomes of A that belong to the 9 outcomes of B, namely 3 of them, $(3,5),(5,3),(5,5)$ that is $\frac{3}{9}$ or $\frac{1}{3}$.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (S.CP.1-5)

INDEPENDENCE

Students in Grade 8 learn how to find association between two quantitative variables using the Quadrant Count Ratio (QCR). Then in Math 1, they used a comparable method called the Agreement-Disagreement-Ratio (ADR) which can be employed for categorical data in a 2 by 2 table for two Yes-No variables. It is calculated by taking the sum of the agreements minus the sum of the disagreements divided by the total or

ADR $= \dfrac{(a+d)-(b+c)}{T}$.

Agreement

Disagreement

Question 2

Question 1

	Yes	No	Row Total
Yes	a	b	$r_1 = a + b$
No	c	d	$r_2 = c + d$
Column Total	$c_1 = a + c$	$c_2 = b + d$	$T = a + b + c + d$

Tests for Independence

- $P(A \cap B) = P(A) \cdot P(B)$
- $P(A|B) = P(A)$
- $P(B|A) = P(B)$
- $P(B|A) = P(B|NotA)$

Now, students are building on the concept of association to understand independence. Compared to association, independent events have a very precise definition written in terms of probabilities of occurrence in the framework of having defined a sample space. Association can be thought of as dependence. *If two events are independent, then their variables are not associated. If two variables are dependent on each other, then there is some association (although it may be weak).* There are several ways to test for independence.

Using Conditional Probabilities

However, it is far more intuitive to introduce the independence of two events in terms of conditional probability (stated in Standard S.CP.3), especially where calculations can be performed in two-way tables: $P(A|B) = P(A)$ or $P(B|A) = P(B)$ In other words, if knowing that B has occurred does not affect A occurring, then the events are independent and vice versa. Referring to the table in the example on page 318, comparing $P(L)$ to $P(L|D)$ or comparing $P(D)$ to $P(D|L)$ will evaluate whether selecting diet is independent from selecting lemonade.

Step 1: $P(A|B) = \dfrac{P(A \cap B)}{P(B)}$

Step 2: $P(A|B) \cdot P(B) = P(A \cap B)$

Step 3: If events A and B are independent, then $P(A|B) = P(A)$.

Step 4: Therefore, $P(A) \cdot P(B) = P(A \cap B)$.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (S.CP.1-5)

EXAMPLE

Is the probability that a person who sleeps with their phone (Event B) and who takes more than 60 minutes to fall asleep (Event A) independent?

	More than 60 min	60 min or less	TOTAL
Phone	61	48	109
No Phone	11	40	51
TOTAL	72	88	160

Discussion:

Method 1: One way to show that events A and B are independent is to compare $P(A|B)$ with $P(A)$. The probability of a randomly selected person who takes more than 60 minutes to fall asleep given the person sleeps with his or her phone is $^{61}/_{109} \approx 56\%$, and the probability that a person sleeps more than 60 minutes is $^{72}/_{160} \approx 45\%$. Since $56\% \neq 45\%$ the events are not independent (or in this case dependent). Students can also compare $P(B|A)$ with $P(B)$.

Method 2: Another way to show that events A and B are independent is to compare $P(A|B)$ with $P(A|NotB)$. The probability of a randomly selected person who takes more than 60 minutes to fall asleep given the person sleeps with his or her phone is $^{61}/_{109} \approx 56\%$. The probability of a randomly selected person who takes more than 60 minutes to fall asleep given the person does NOT sleep with his or her phone is $^{11}/_{51} \approx 22\%$. Since $56\% \neq 22\%$ the events are not independent (or in this case dependent). Students can also compare $P(B|A)$ with $P(B|NotA)$.

There are many good problems that can appeal to students' sensitivities of fairness and justice in society. Students can formulate their questions that concern how certain characteristics of their own identity groups are viewed by society and understand how conditional probability is often misunderstood by society as whole.

Using Product of the Probabilities

The independence of two events is defined in Standard S.CP.2 using the notion of intersection: $P(A \cap B) = P(A) \cdot P(B)$. Two events, A and B, are independent if the fact that A occurs does not affect the probability of B occurring. Students can derive this formula by rearranging the rule for conditional probability which is more intuitive.

TIP!

Students may want to know why they multiply when finding the probability of two events. It may be clearer to think about probability as the fraction of the time that something will happen. If event A happens $\frac{1}{2}$ of the time, and event B happens $\frac{1}{3}$ of the time, and events A and B are independent, then event B will happen $\frac{1}{3}$ of the times that event A happens. To find $\frac{1}{3}$ of $\frac{1}{2}$, multiply the probabilities. The probability that events A and B both happen is $\frac{1}{6}$. Note also that adding two probabilities will give a larger number than either of them; but the probability that two events both happen cannot be greater than either of the individual events, so it would make no sense to add probabilities in this situation. From Dr. Peterson, http://mathforum.org/library/drmath/view/74065.html

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (S.CP.1-5)

Using the Knowledge of the Complement

Students can also test for independence comparing $P(A|B) = P(A|notB)$. In other words, if one event does not change whether we know that the other event has occurred or not occurred then $P(A|B) = P(A|notB)$.

EXAMPLE

One hundred people are surveyed regarding their favorite soft drink from the choices {Diet Cola, Diet Lemonade, Regular Cola, Regular Lemonade}. Results are shown in the table.

	Cola (C)	Lemonade (L)	Total
Diet (D)	20	13	33
Regular (R)	24	43	67
Total	44	56	100

- Test for independence of the probability of preferring to drink Cola compared to preferring to drink Diet:
    - using the product of their probabilities;
    - using conditional probabilities; and
    - comparing the probability of an event to the probability of its complement.
- If the events are not independent, figure out if there is a strong or weak association based on past methods such as the Agreement-Disagreement Ratio (ADR).

Discussion:

a. The probability that people prefer Cola is $\frac{44}{100}$ or 44% or 0.44. The probability that people prefer Diet is $\frac{33}{100}$ or 33% or 0.33. Therefore if preferring Cola is independent of preferring Diet, 44% of the 33% of the Diet drinkers should have preferred Diet Cola or (0.33)(0.44) should equal 0.20, (see upper left cell in the table) but instead it equals 0.1452. Since 0.1452 does not equal 0.20, the events are not independent.

b. If the Event C is independent from the Event D then the probability of preferring Cola should be equivalent to the probability of preferring Cola given that the drink is Diet. Since P(Cola) = 44% and the P(Cola given Diet) is $\frac{20}{33}$ or 60.6% and 44% ≠ 60.6% the events are not independent.

c. If an event is independent, the probability of drinking Cola given that the drink is diet should be equivalent to the probability of drinking Cola given that the drink is Regular. If the probabilities of the two events are equal, this would prove that they are not associated. Since P(Cola given Diet) is $\frac{20}{33}$ or about 60.6% and the P(Cola given Regular) is $\frac{24}{67}$ or about 35.8%, the events are dependent.

d. After students explore the three methods of independence, they should realize that the events are not independent, so they have an association. They could use the ADR or Pearson's Coefficient for quantitative variables to test for strength of the association. In this case there is an association, but it is a weak association. Discuss when it would be appropriate to test for independence versus when it would be more appropriate to figure out the strength of the association.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (S.CP.1-5)

Note: On one hand S.CP.4 looks at the same questions as would be asked in 8.SP.4 and S.ID.5 but from a probability point of view. The difference between relative frequency and probability is very subtle as we define probability as long-run relative frequency. So whereas S.ID.5 and 8.SP.4 are answering questions within the given data set using relative frequencies, S.CP.4 is randomly sampling a subject from the data set and asking for the chance that the chosen subject satisfies some classification of interest. The probability is assigned by calculating the respective relative frequency. On the other hand, S.CP.4 muddies the waters by asking about whether or not two events defined in the two-way table are independent. S.ID.5 and 8.SP.4 talk about association, a more vague concept than independence. Independent events have a very precise definition written in terms of probabilities of occurrence in the framework of having defined a sample space. The definition of independent events using the conditional probability point of view plays a prominent part here. Therefore, S.ID.5 is basically a repeat of 8.SP.4. Note that S.CP.4 asks similar questions to those of S.ID.5 and 8.SP.4; however, in S.CP.4 the questions are now written in terms of a) random sampling of subjects, b) considering the table as a sample space with values of the variables considered as simple events of the sample space, and c) asking if the events are independent events (a more formal representation of questions regarding association asked in S.ID.5 and 8.SP.4).

Instructional Tools/Resources

These tools and resources may be helpful for instruction. Before use, they should be evaluated at the local level to determine their appropriateness for instruction.

Manipulatives/Technology

- Surveys
- Census Data
- StatTrek is a good website to clarify the vocabulary in this cluster.
- "Chances Are" by Steven Strogatz is an interesting article about probability.
- Understanding Uncertainties: Visualizing Probabilities by Mike Pearson and Ian Short from +Plus Magazine in an article that illustrates probabilities using pictures.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (S.CP.1-5)

Events and Sample Space

- Return to Fred's Fun Factory (with 50 cents) by Illustrative Mathematics is a task where students address standards regarding sample space, independence, probability distributions, and permutations/combinations. Some key components of this task challenge common misconceptions surrounding probability.
- Describing Events by Illustrative Mathematics is a task where students review the definition of sample space and events.
- "Go" in the news—Man Versus Machine by YummyMath is a lesson that has students figure out how many plays are possible in the game *Go* and apply the concept of the Fundamental Counting Principle.
- Too Early in the Day for So Many Choices by YummyMath is a lesson that has students figure out how many possible combinations of hot chocolate Dunkin' Donuts has.
- Doritos Roulette: Hot or Not? How Many Chips are Hot? by Tap Into Teen Minds is a 3-act task that explores how many hot Doritos are in a bag.
- Darius Washington—Free Throws for the Win: Will Darius Washington Score Enough Free Throws to Force Overtime or a Win? by Tap Into Teen Minds is a 3-act task that explores probability.
- Chance Experiments by Desmos is an introduction to probability using a spinner game.
- Probability: Union and Intersection by Jennifer Vadnais is a Desmos activity where students explore unions and intersections.

Independence

- Cards and Independence by Illustrative Mathematics is a task where students explore the concept of independence of events.
- The Titanic 2 by Illustrative Mathematics is a task where students explore the concepts of independence. This is the 2nd task in a series of three.
- Rain and Lightning by Illustrative Mathematics is a task where students explore the concept of independence of events and conditional probability.
- Lucky Envelopes by Illustrative Mathematics is a task where students explore the concept of independence of events.
- Finding Probabilities of Compound Events is a task where students explore the concept of independence of events.
- Breakfast Before School by Illustrative Mathematics is a task where students recognize and explain independence in everyday situations.
- The Egg Roulette Game by Amanda Walker from Statistics Education Web (STEW) is a probability lesson that follows the GAISE model analyzing Jimmy Fallon playing the Egg Roulette with celebrities. The first part of this lesson has to do with the cluster. The second part connects to statistical concepts. There is a note on how to scaffold down the activity as well.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (S.CP.1-5)

Conditional Probability

- The Titanic 1 by Illustrative Mathematics is a task where students explore the concepts of probability as a fraction of outcomes and using two-way data tables with emphasis on understanding conditional probability. This is the first task in a series of three.
- False Positives by Illustrative Mathematics is a task where students explore a common fallacy where two conditional probabilities are confused.
- Representing Conditional Probabilities 1 and Representing Conditional Probabilities 2 by Mathematics Assessment Project is task where students demonstrate their understanding of conditional probabilities, represent events as a subset of a sample space and communicate their reasoning.
- Representing Probabilities: Medical Testing by Mathematics Assessment Projects is a lesson where students understand and calculate conditional probability based on the real-life situation of medical testing.
- False Positives by Achieve the Core is a CTE Task that integrates the concept of probability into medical testing surrounding false positives.
- The False-Positive Paradox as a Class Activity/Discussion Point by Like Teaching—Assume *m* is positive is hands-on activity that can introduce the idea of false positive.
- Conditional Probability and Crime by Cris Wellington and Anne Quinn published in NCTM's Mathematics Teacher, Volume 109, (2) in September 2015. *NCTM now requires a membership to view their lessons.*

Two-Way Tables

- The Titanic 3 by Illustrative Mathematics is a task where students make conclusions based on data using a two-way table. This is the 3rd task in a series of three.
- How Do You Get to School? by Illustrative Mathematics is a task where students use a two-way table to calculate a probability and a conditional probability.
- Two-Way Tables and Probabilities by Illustrative Mathematics is a task where students use a two-way table to calculate a probability and a conditional probability.
- A Sweet Task by Elizabeth Fiedler, Maryann Huey, Brandon Jenkins, and Sharon Flinspach from Statistical Education Website (STEW) is a lesson based on the GAISE model where students gain an understanding of how to create a two-way frequency table and calculate probabilities based on colors of candy.
- Probability and 2-Way Tables-TPS4e Chapter 5 by Bob Lochel is a Desmos activity on two-way tables and probability.
- Performance Assessment Task: Winning Spinners by Inside Mathematics is a task where students need to demonstrate an understanding of the concept of constructing and interpreting two-way tables as a sample space.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (S.CP.1-5)

Curriculum and Lessons from Other Sources

- Georgia Standards of Excellence Curriculum Frameworks, Geometry, Unit 6: Applications of Probability has many tasks that align to this cluster.
- UC San Diego's Computer Science and Engineering page has a variety of probability problems that can be adapted for the course by using two-way tables, Venn diagrams, and tree diagrams.
- EngageNY, Algebra 2, Module 4, Topic A, Lesson 1: Chance Experiments, Sample Spaces, and Events, Lesson 2: Calculating Probabilities of Events Using Two-Way Tables, Lesson 3: Calculating Conditional Probabilities and Evaluating Independence Using Two-Way Tables, Lesson 4: Calculating Conditional Probabilities and Evaluating Independence Using Two-Way Tables, Lesson 5: Events and Venn Diagrams are many tasks that align to this cluster.
- Illustrative Mathematics, Geometry, Unit 8: Conditional Probability, Lesson 1: Up to Chance, Lesson 2: Playing with Probability, Lesson 3: Sample Spaces, Lesson 4: Tables of Relative Frequencies, Lesson 5: Combing Events, Lesson 7: Related Events, Lesson 8: Conditional Probabilities, Lesson 9: Using Tables for Conditional Probabilities, Lesson 10: Using Probabilities to Determine if Events are Independent, Lesson 11:Probabilities in Games are lessons that pertain to this cluster.
- The Mathematics Vision Project, Secondary Math Two, Module 9: Probability has many tasks that align to this cluster.
- The University of Florida has an Open Learning Textbook on Biostatistics that has good explanations about probability.
- Probability through Data: Interpreting Results from Frequency Tables by P. Hopfensperger, H. Kranendonk, R. Scheaffer is a module from Dale Seymour Publications.
- Probability Models by P. Hopfensperger, H. Kranendonk, R. Scheaffer is a pdf of a module by Dale Seymour Publications.

General Resources

- Arizona's High School Progression on Statistics and Probability is an informational document for teachers. This cluster is addressed on pages 13-15.
- Arizona's High School Progression on Modeling is an informational document for teachers. Statistics and Probability is discussed on page 10.
- High School Coherence Map by UnboundED is a map of the relationships among high school mathematics standards. Please note: This resource is aligned to the Common Core not Ohio's Learning Standards. Although the tool is useful, it needs to be checked with respect to Ohio's Learning Standards.
- Statistics Teacher is an online journal published by the American Statistical Association (ASA) and NCTM. It includes articles, lesson plans, and professional development opportunities.
- Significance is a magazine that demonstrate the practical use of statistics and shows how statistics benefits society.
- Chance is a magazine about statistics for anyone who is interested in using data to advance science, education, and society.
- Levels of Conceptual Understanding in Statistics (LOCUS) is an NSF funded project that has assessment questions around statistical understanding.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (S.CP.1-5)

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STANDARDS	MODEL CURRICULUM (S.CP.6-9)
Statistics and Probability **CONDITIONAL PROBABILITY AND THE RULES OF PROBABILITY** **Use the rules of probability to compute probabilities of compound events in a uniform probability model.** **S.CP.6** Find the conditional probability of A given B as the fraction of B's outcomes that also belong to A, and interpret the answer in terms of the model.★ **S.CP.7** Apply the Addition Rule, P(A or B) = P(A) + P(B) − P(A and B), and interpret the answer in terms of the model.★ (+) **S.CP.8** Apply the general Multiplication Rule in a uniform probability model[G], P(A and B) = P(A)·P(B|A) = P(B)·P(A|B), and interpret the answer in terms of the model.★ (+) **S.CP.9** Use permutations and combinations to compute probabilities of compound events and solve problems.★	**Expectations for Learning** In middle school, students develop basic probability skills including probability as relative frequencies; probabilities of compound events; development of a uniform/non-uniform probability model; and the use of tree diagrams. Also, they are introduced to two-way frequency tables in middle school. Now in Math 2 this cluster formalizes the concepts of conditional probability and independence in S.CP.1-5. The focus of this cluster is developing the Addition Rule and the (+) Multiplication Rule in everyday contexts. Although permutations and combinations are part of Math 2 for students who pursue advanced mathematics, these concepts would also be appropriate in a fourth-year course. Exploration of the Fundamental Counting Principle and factorials (!) may also be addressed in a fourth-year course. **ESSENTIAL UNDERSTANDINGS** • Compound probabilities model real-world scenarios and must be interpreted within a context. • The conditional probability of A *given* B is the fraction of B's outcomes that also belong to A. This can be expressed by $P(A\|B) = P(A \cap B)/P(B)$. • The addition rule is $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$ and can also be expressed as $P(A \cup B) = P(A) + P(B) - P(A \cap B)$. • (+) The Multiplication Rule is $P(A \text{ and } B) = P(A) * P(B\|A) = P(B) * P(A\|B)$. • (+) Permutations and combinations are strategies for counting the outcomes of a sample space. **MATHEMATICAL THINKING** • Use precise mathematical language. • Look for and make use of structure. • Compute accurately and efficiently. *Continued on next page*

STANDARDS	MODEL CURRICULUM (S.CP.6-9)
	Expectations for Learning, continued **INSTRUCTIONAL FOCUS** • Recognize and justify mathematically whether two events are independent. • Generalize probability rules using patterns. • Compute probabilities accurately and efficiently. • (+) Recognize and apply counting methods to compute probabilities. • Compute the conditional probability of A given B. • Interpret and explain conditional probability within a context. • Apply the Addition Rule. • Interpret and explain the Addition Rule within a context. • (+) Apply the Multiplication Rule. • (+) Interpret and explain the Multiplication Rule within a context. • (+) Know and explain the difference between a permutation and a combination. • (+) Calculate probabilities using permutations and combinations. • (+) Interpret and explain probabilities using permutations and combinations within a context. **Content Elaborations** **OHIO'S HIGH SCHOOL CRITICAL AREAS OF FOCUS** • Math 2, Number 1, page 3 **CONNECTIONS ACROSS STANDARDS** • Understand independence and conditional probability (S.CP.1-5). • Interpret and use the structure of expressions (A.SSE.1-3).

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (S.CP.6-9)

Instructional Strategies

Note: The model curriculum is not a collection of lessons nor a full curriculum; it does not suggest pace, sequence, or amount of time spent on topics. It provides information about a topic related to the standards including ideas for examples, strategies for teaching, common misconceptions, and possible connections between topics. In addition, the emphasis of the model curriculum is on instruction not on assessment; therefore, the examples in the Instructional Strategies section are written for instructional purposes not necessarily as assessment items. Examples may need to be rewritten to accommodate the needs of each individual classroom.

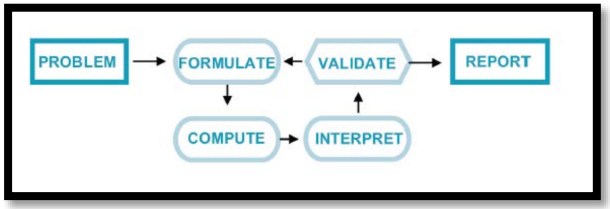
Standards for Mathematical Practice
This cluster focuses on but is not limited to the following practices:
***MP.2** Reason abstractly and quantitatively.*
***MP.4** Model with mathematics.*
***MP.6** Attend to precision.*

The Standard for Mathematical Practice (SMP.6), *attending to precision* is a vital component in this cluster. Precision is important from reading and understanding the probability question to then selecting the proper rules of probability to use in answering the question.

The Standard for Mathematical Practice (SMP.2), *quantitative and abstract reasoning* is applicable to this cluster because of the need to represent the data symbolically and manipulating those symbols to make sense of the Addition Rule (S.CP.7) and the Multiplication Rule (+S.CP.8).

MODELING

The Standard for Mathematical Practice (SMP.4), *modeling* is critical to all probability standards because emphasis must be placed on the real-world context and applicability of what students are learning. In this cluster, students explore the rules that model probabilities.



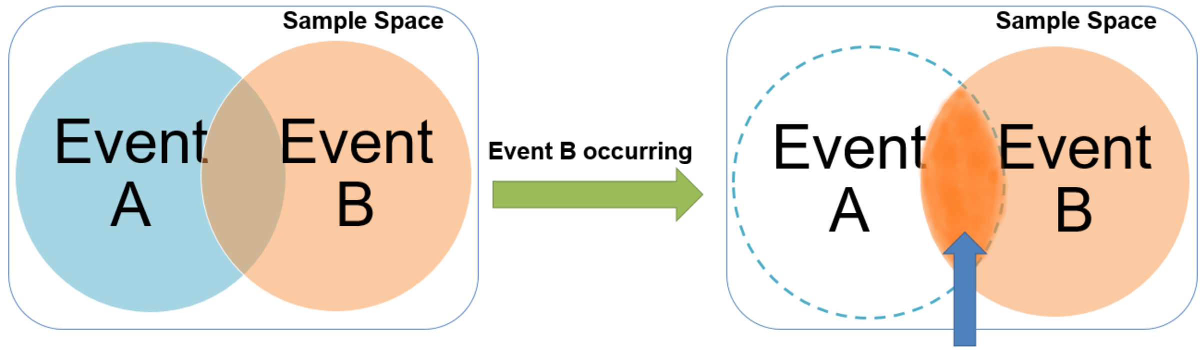
CONDITIONAL PROBABILITY

Standard S.CP.6 ties into S.CP.3 and S.CP.5 in the previous cluster and should be taught together. Whereas S.CP.3 emphasizes computation from the original sample space, S.CP.6 emphasizes computation from the reduced sample space. The standard S.CP.6 emphasizes conditional probability of A given B as the fraction of B's outcomes that belong to A. It can be illustrated using a Venn diagram or table. See cluster S.CP.1-5 for more information differentiating between the standards regarding conditional probability.

Students may incorrectly believe that the probability of A and B is always the product of the two events individually, not realizing that one of the probabilities may be conditional. Emphasize to students that they need to read the problem carefully and differentiate whether the events are independent or dependent. From there they can figure out the probability. Using Venn diagrams and two-way tables may help students gain conceptual understanding.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (S.CP.6-9)

In situations involving unions and intersections, the sample space includes both Event A and Event B. However, in situations involving conditional probability the sample space changes once Event B occurs; the only part of Event A that is left to be considered is the part that overlaps with Event B.



The sample space changes to only include the sample space of Event B. The only part of Event A's outcomes that are left, is those represented by the overlap with Event B.

Viewing conditional probability from this perspective is more intuitive than the formula $P(A|B) = \frac{P(A \text{ and } B)}{P(B)}$. Students can use their understanding of sample space in this cluster to derive the rule in S.CP.2: If and only if $P(A \text{ and } B) = P(A) \cdot P(B)$, then the events are independent. Notice that standard S.CP.6 calls for the interpretation of the answer in terms of the model, so when students are asked to find the conditional probability, they should be given a contextual situation.

	Event B	Event Not B	Total
Event A	Event A and Event B	Event A and Not B	Outcomes of Event A
Event Not A	Event B and Event Not A	Event Not A and Event Not B	Outcomes of Event Not A
Total	Outcomes of Event B	Outcomes of Event Not B	Total Sample Space

Event B occurring

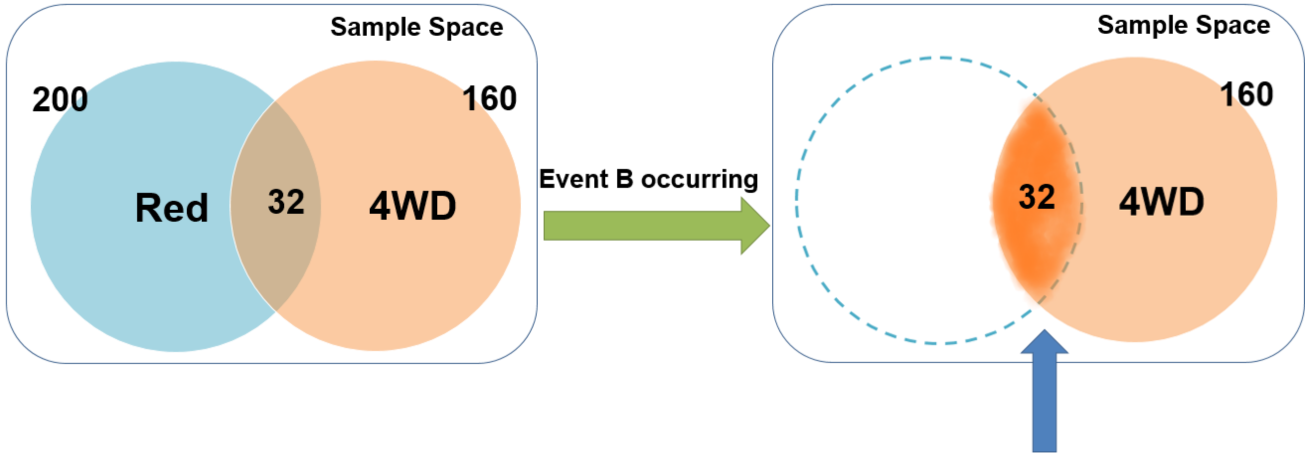
	Event B		
Event A	Event A and Event B		
Event Not A	Event B and Event Not A		
Total	Reduced Sample Space determined by Event B		

Sample Space changes to only include Event B. The only part of Event A that is left is the space that overlaps Event B.
Note: Event B can be in a row or column.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (S.CP.6-9)

EXAMPLE

Twenty percent of the cars on the used car lot were red. Sixteen percent of the cars have four-wheel drive. 3.2% of cars are red and have four-wheel drive. If the probability of car being red and a car having four-wheel drive are independent, and Chloe randomly chooses a car with 4WD what is the probability that it is red?



The sample space changes to only include the cars that are 4WD. The part of the reduced sample space that is red is the overlap of 32. All of those 32 cars also have 4WD.

Discussion: The problem is stated in relative frequencies. Many students and adults struggle with calculating probabilities using relative frequencies. Although, students may choose to do this problem with relative frequencies (which would be slightly more precise). For this course they may wish to convert the relative frequencies to frequencies. When converting relative frequencies to frequencies the grand total could be in any multiple of 10 such as 100, 1,000, 10,000 etc., but for greater precision the grand total needs to be at least 1,000 for this course. Since the events are independent, the first step is to determine the probability of choosing a car that is red and is four-wheel drive. To find that students can multiply 0.16 and 0.20 using relative frequencies or $\frac{160}{1,000}$ and $\frac{200}{1,000}$ if converting to frequencies. They should get either 0.032 or 32 depending on their method. Using a Venn diagram or a two-way table, students can then see that the sample space changes to 0.16 or 160 (depending on whether relative frequencies or frequencies are used). From there they can calculate the conditional probability of $\frac{0.032}{0.16}$ or $\frac{32}{160} = 0.20\ or\ 20\%$.

	Red	Not Red	Total
4WD	32	128	160
Not 4WD	168	672	840
Total	200	800	1,000

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (S.CP.6-9)

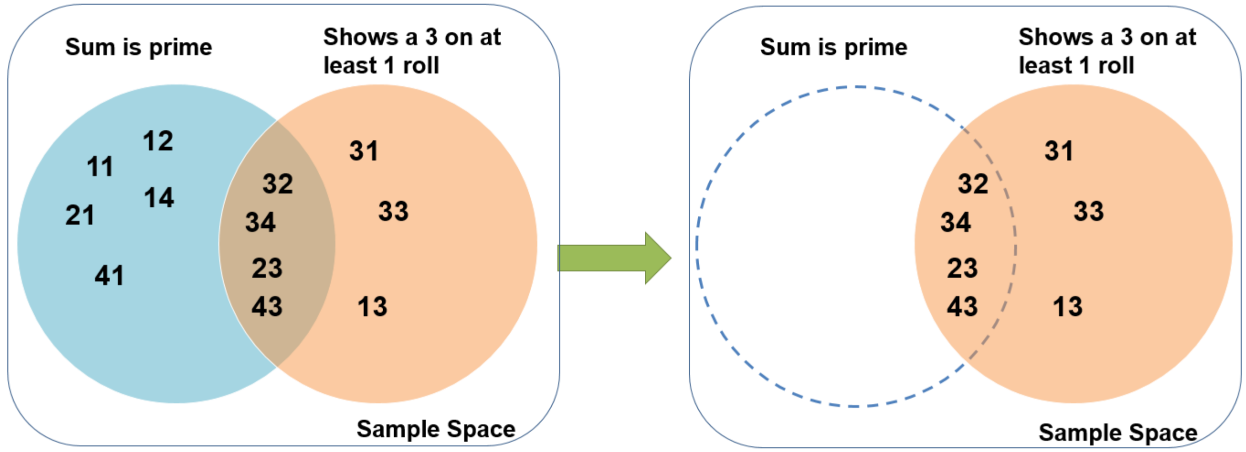
Identifying whether a probability is conditional when the word "given" is omitted can be difficult for students. For example, the wording "of students that have 'A's who are freshmen" is a way of stating "students who have an 'A' given they are freshmen."

Students can play games with number cubes to help reinforce conditional probability and explain sample space.

EXAMPLE

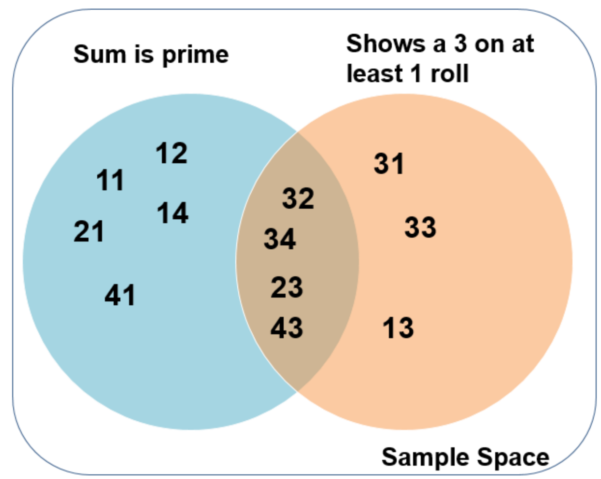
a. If a balanced tetrahedron with faces 1, 2, 3, 4 is rolled twice, what is the probability that the sum of the rolled numbers is prime (A) of those that show a 3 on at least one roll (B)?

b. What is the probability that the sum of two rolls of a fair tetrahedron is prime (A) or at least one of the rolls is a 3 (B)?



Discussion:

a. Deciding whether the situation translates into $P(A\ and\ B)$, $P(A\ or\ B)$, or $P(A|B)$ may be problematic for students. The question can be rephrased as finding the probability that the sum is prime (A) given at least one roll shows 3 (B). One way to calculate the probability is to count the elements of B by listing them if possible. In this example, there are 7 paired outcomes {(3,1), (3,2), (3,3), (3,4), (1,3), (2,3), (4,3)} in Event B. Of those 7 there are 4 whose sum is prime {(3,2), (3,4), (2,3), (4,3)} which is the intersection of events $A\ and\ B$. Hence in the long run, 4 out of 7 times of rolling a fair tetrahedron twice, the sum of the two rolls will be a prime number under the condition that at least one of its rolls shows the digit 3. Showing the outcomes in a Venn Diagram may be helpful.



b. The wording in part **b.** is more obvious because of the presence of "or" in the sentence. It translates into $P(A\ or\ B)$ which is denoted as $P(A \cup B)$ in set notation. The sample space of a tetrahedron is 16, since each tetrahedra has 4 sides and $4 \cdot 4 = 16$. Again, it is often useful to appeal to a Venn Diagram in which A consists of the pairs as $A = \{(1,1), (1,2), (1,4), (2,1), (2,3), (3,2), (3,4), (4,1), (4,3)\}$ and $B = \{(1,3), (2,3), (3,1), (3,2), (3,3), (3,4), (4,3)\}$. Finding $P(A\ or\ B)$, by adding probability $P(A)$ and $P(B)$ is incorrect because the set of outcomes for $P(A)$ and $P(B)$ would include a duplicate of the two events, namely $23, 32, 34$ and 43. So $P(A\ or\ B)$ is $\frac{9}{16} + \frac{7}{16} - \frac{4}{16} = \frac{12}{16}$ or 75%, so 75% of the time, the result of rolling a fair tetrahedron twice will result in the sum being prime, or at least one of the rolls showing a 3, or perhaps both will occur.

ADDITION RULE

The Addition Rule is a formal way to find the union of two events: $P(A \cup B) = P(A) + P(B) - P(A \cap B)$. Students can derive both the Addition Rule and the Multiplication Rule as an extension. Permutations and combinations also provide opportunities for enrichment.

Though the use of strictly numeric problems can be used in this cluster as a platform for practicing with the rules, it is strongly recommended that students also be exposed to a healthy amount of contextual problems.

EXAMPLE

One hundred people are surveyed regarding their favorite soft drink from the choices {Diet Cola, Diet Lemonade, Regular Cola, Regular Lemonade}. Results are shown in the table. What is the probability that a person likes Cola or that they like Diet drinks?

	Cola (C)	Lemonade (L)	Total
Diet (D)	20	13	33
Regular (R)	24	43	67
Total	44	56	100

Discussion: These data were also used in S.CP.1-5. This example is asking for the union of $(A \cup B)$. Students can approach this task in a couple of ways. One way is to use the joint frequencies in the body of the table $\frac{20+13+24}{100}$ to get $\frac{57}{100}$. Another way is to use marginal frequencies, $\frac{33+44}{100}$ and then subtract the overlap of the joint frequency $\frac{20}{100}$ to get $\frac{57}{100}$. Although, the first way may be more intuitive when using the table, the second way leads to the deriving of the Addition Rule: $P(A\ or\ B) = P(A) + P(B) - P(A\ and\ B)$. The Addition Rule may be more efficient when the probabilities are not in tabular form.

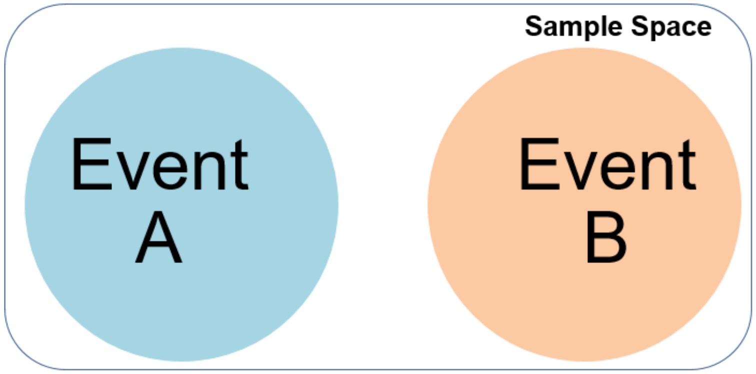
	Cola (C)	Lemonade (L)	Total
Diet (D)	20	13	33
Regular (R)	24	43	67
Total	44	56	100

	Cola (C)	Lemonade (L)	Total
Diet (D)	20	13	33
Regular (R)	24	43	67
Total	44	56	100

Students may incorrectly believe that the probability of A or B is always the sum of the two events individually not taking into account the overlap. Use Venn diagrams and two-way tables to confront this misconception.

Disjoint (Mutually Exclusive) Events

Disjoint events are events that cannot happen at the same time. For example, a person cannot be in New York City and in Hawaii at the same time. They are mutually exclusive, so their intersection is 0. Since there is no overlap, when a situation calls for the union of two events, one simply adds the probabilities of the two events together: $P(A \cup B) = P(A) + P(B)$, since $P(A\ and\ B) = 0$. If one of the two events must happen, then the events are complementary. For example, when rolling a number cube, one can either roll a "6" or "not a 6." In this case, rolling a "6" is an event and rolling "not 6" is its complement. The sum of their probabilities equals one.



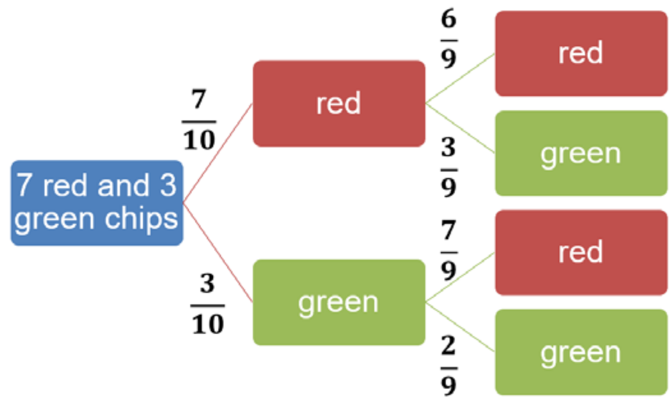
Sometimes students incorrectly think that disjoint events are independent, but they are not independent unless one event is impossible. Disjoint events never occur at the same time; their intersection is impossible. The occurrence of one event prohibits the occurrence of another. For example, being a teenager and being a senior citizen cannot happen at the same time because no one can be both a teenager and senior citizen. In contrast, independent events are unrelated. For example, being a teenager and being born in March are independent events.

MULTIPLICATION RULE (+)

It should be noted that the Multiplication Rule in Standard S.CP.8 is designated as a plus (+) standard. Whereas the Specific Multiplication Rule in S.CP.2 $P(A\ and\ B) = P(A) \cdot P(B)$ only works for independent events, the General Multiplication Rule $P(A\ and\ B) = P(A) \cdot P(B|A)$ in S.CP.8 works for either independent or dependent events. In reality, $P(A\ and\ B) = P(A) \cdot P(B)$ is just a subcase of $P(A\ and\ B) = P(A) \cdot P(B|A)$ when A and B are independent, since when A and B are independent $P(A|B) = P(A)$ and $P(B|A) = P(B)$. Note: Even though S.CP.8 is a (+) standard, it is connected to the Addition Rule in S.CP.7 and the *Specific Multiplication Rule* in S.CP.2.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (S.CP.6-9)

The Multiplication Rule $P(A\ and\ B) = P(A) \cdot P(B|A)$ is useful when the sample space changes. It is best introduced in a two-stage setting in which A denotes the outcome of the first stage, and B, the second. For example, suppose a jar contains 7 red and 3 green chips. If one draws two chips without replacement from the jar, the probability of getting a red followed by a green is $P(red\ on\ first, green\ on\ second) = P(red\ on\ first) \cdot P(green\ on\ second\ given\ a\ red\ on\ first) = \frac{7}{10} \cdot \frac{3}{9} = \frac{21}{90}$ or $\frac{7}{30}$. Demonstrated on a tree diagram indicates that the conditional probabilities are on the second set of branches. It may be helpful to students to point out that sum of all the probabilities of $P(A\ and\ B)$ for each branch equal 1.



The probability of drawing a red on the first draw and a green on the second draw without replacement is $\frac{7}{10} \cdot \frac{3}{9} = \frac{7}{30}$.

- $1st\ branch: \frac{7}{10} \cdot \frac{6}{9} = \frac{42}{90}$
- $2nd\ branch: \frac{7}{10} \cdot \frac{3}{9} = \frac{21}{90}$
- $3rd\ branch: \frac{3}{10} \cdot \frac{7}{9} = \frac{21}{90}$
- $4th\ branch: \frac{3}{10} \cdot \frac{2}{9} = \frac{6}{90}$
- $Sum\ of\ branches: \frac{42}{90} + \frac{21}{90} + \frac{21}{90} + \frac{6}{90} = 1$

EXAMPLE

In a standard playing deck, what is the probability that you will lay down two kings in row.

Discussion: The probability of getting the first king is $\frac{4}{52}$; however, once you lay down a king, there are only 3 kings left out of 51 cards, so the probability of the second king is $\frac{3}{51}$. Therefore the $P(king\ and\ king) = \frac{4}{52} \cdot \frac{3}{51}$ which equals $\frac{12}{2652}$ or $\frac{1}{221}$.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (S.CP.6-9)

EXAMPLE

One hundred people are surveyed regarding their favorite soft drink from the choices {Diet Cola, Diet Lemonade, Regular Cola, Regular Lemonade}. Results are shown in the table.

	Cola (C)	Lemonade (L)	Total
Diet (D)	20	13	33
Regular (R)	24	43	67
Total	44	56	100

- **a.** Is the probability that a person likes regular Cola and Diet drinks independent?
- **b.** What is the probability that if a person likes Cola and then he or she likes Diet drinks?
- **c.** Does the rule $P(C \text{ and } D) = P(C) \cdot P(D)$ work in this circumstance? Explain.
- **d.** Explain why the General Multiplication Rule $P(C \text{ and } D) = P(C) \cdot P(D|C)$ works to find $P(C \text{ and } D)$.
- **e.** Does the other version of the General Multiplication Rule $P(C \text{ and } D) = P(D) \cdot P(C|D)$ also work to find $P(C \text{ and } D)$? Explain.

Discussion:

- Students can choose to use any of the four rules for independence to determine that the events are dependent.
- The probability of $P(D|C) = \frac{20}{44}$ or approximately 45%, whereas $P(D) = \frac{33}{100} = 33\%$. Since $33\% \neq 45\%$, the events are not independent.
- In this example, looking at the upper left cell of the table students can easily see that the intersection of Cola and Diet Drinkers is $\frac{20}{100}$ or 20%. No, the Multiplication Rule $P(C \text{ and } D) = P(C) \cdot P(D)$, does not hold true because $\frac{44}{100} \cdot \frac{33}{100} = \frac{363}{2500}$ does not equal $\frac{20}{100}$. Therefore students should realize that the events are dependent.
- Using the General Multiplication Rule students can see that $P(C \text{ and } D) = \frac{44}{100} \cdot \frac{20}{44} = \frac{20}{100}$ which is the same value that the upper left cell in the table indicates as the intersection. One reason that students may state is that the rule works because the numerator of one of the fractions will equal the denominator of the other. Push them to explain if it will always work. They may generalize this observation by showing that $P(C) \cdot P(D|C) = P(D)$. For example, $P(C) \cdot P(D|C) = \frac{\text{Space of Event } C}{\text{Total Space}} \cdot \frac{\text{Space of Event } D}{\text{Space of Event } C} = \frac{\text{Space of Event } D}{\text{Total Space}}$.
- The same holds true for the other version of the formula $P(C \text{ and } D) = \frac{33}{100} \cdot \frac{20}{33} = \frac{20}{100}$.

PERMUTATIONS AND COMBINATIONS (+)

When addressing (+)S.CP.9, be aware that students have likely had no prior exposure to many of the prerequisites. For example, if listing of outcomes is not possible, the counting techniques such as Fundamental Counting Principle, permutations, or combinations may be required. Students should understand the difference between permutations (a number of different ordered arrangements of a fixed set of elements) and combinations (a number of unordered fixed sets of elements taken from the given set), and then, use permutations and combinations to find probabilities of compound events.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (S.CP.6-9)

Instructional Tools/Resources

These tools and resources may be helpful for instruction. Before use, they should be evaluated at the local level to determine their appropriateness for instruction.

Manipulatives/Technology

- Surveys
- Census Data
- StatTrek is a good website to clarify the vocabulary in this cluster.
- "Chance Are" by Steven Strogatz is an interesting article about probability.
- Understanding Uncertainties: Visualizing Probabilities by Mike Pearson and Ian Short from +Plus Magazine in an article that illustrates probabilities using pictures.
- Roll Dice Online is an applet for rolling dice.
- Dice and Spinners is an applet for dice and spinners. The number of sides on the dice can be changed.
- Virtual Dice by Curriculumbits.com is an applet for rolling dice.

Conditional Probability

- The Egg Roulette Game by Statistics Education Web (STEW) is a lesson that follows the GAISE model. It uses a probability game and computer simulation to explore the law of large numbers, conditional events, sampling distributions, and the central limits theorems using Jimmy Fallon's bit from the Late Night Show.
- Stick or Switch? by NCTM Illuminations is a lesson that explores the probability of compound events using methods such as tree diagrams and area models. *NCTM now requires a membership to view their lessons.*
- Three Shots: Should You Ever Foul at the Buzzer? By Mathalicious is a lesson where students calculate the conditional probability of a win or loss for the defensive team, given that they foul or do not foul.
- Conditional Probability and Probability of Simultaneous Events by Shodor is an activity where students explore conditional probability.
- The Dog Ate My Homework! by NRICH Math models the interpretation of statistics for testing involving false positives.
- Who is Cheating? by NRICH Math models the interpretation of statistics for testing. It can be used to establish the difference between $P(A\ given\ B)$ and $P(B\ given\ A)$.
- Conditional Probability is Important for All Students! by NRICH Math is an article that explains the importance of conditional probability.
- The Titanic 1, The Titanic 2, and The Titanic 3 by Illustrative Mathematics is a series of tasks developing conditional probability.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (S.CP.6-9)

Conditional Probability, continued

- How Do You Get to School? by Illustrative Mathematics uses a two-way table to calculate conditional probability.
- POM: Friends You Can Count On by Inside Mathematics is a series of open-ended tasks. Tasks D and E align to this grade level and cluster. Task D has students calculate the probabilities for simple and compound events. Task E has to do with false positives.
- POM: Got Your Number by Inside Mathematics is a series of open-ended tasks. Tasks D and E align to this grade level and cluster. Tasks D and E have students think about probabilities to determine a detailed strategy for playing the game.
- POM: Party Time by Inside Mathematics is a series of open-ended tasks. Task D aligns to this grade level and cluster. Tasks D has students reason about a fair game.

Addition Rule

- Odd or Even? The Addition and Complement Principles of Probability is a lesson that follows the GAISE model and analyzing the game *Odd or Even*?
- Coffee and Mom's Diner by Illustrative Mathematics is a task where students use the addition rule to compute a probability.
- Rain and Lightning by Illustrative Mathematics is a task where students explore the Addition Rule as part of the task.
- The Addition Rule by Illustrative Mathematics is a task where students to develop the addition rule for calculating the probability of the union of two events

Multiplication Rule

- Games for Teaching Probability #3: Conditional Probability and the Multiplication Rule is a blog by Board Game Geek that explains modifying the game *No Thanks* to teach the Multiplication Rule. The game No Thanks is explained on this thread.
- P5: Playing Craps by Bowling Green State University is a lesson that integrated the Multiplication Rule in the context of craps.
- Multiplication Rule Probability: Definition, Examples is an article by StatisticsHowTo that describes the difference between the general and specific multiplication rule.
- Independence and Dependence by NRICH Math is an article that discusses, independence, dependence, and sampling with and without replacement including the Multiplication Rule.
- POM: Diminishing Returns by Inside Mathematics is a series of open-ended tasks. Task E challenges students to work with repeating decimals and probabilities to calculate the probability of being born male in an urban North American location.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (S.CP.6-9)

Permutations and Combinations

- POM: Cubism by Inside Mathematics is a series of open-ended tasks. Task E has student analyze permutations of a Rubik's Cube.
- POM: Rod Trains by Inside Mathematics is a series of open-ended tasks. Tasks D and E explore combinations of trains.
- Random Walk III and Random Walk IV by Illustrative Mathematics is part of a progression of tasks, starting with Random Walk and Random Walk II which stressed the function aspect of this situation, transitioning to the probability and statistics side.
- Alex, Mel, and Chelsea Play a Game by Illustrative Mathematics is a task where students combine the concept of independent events with computational tools for counting combinations, requiring fluent understanding of probability in a series of independent events.
- Return to Fred's Fun Factory (with 50 cents) by Illustrative Mathematics is a task where students address concepts regarding sample space, probability distributions, and permutations/combinations.

Curriculum and Lessons from Other Sources

- Georgia Standards of Excellence Curriculum Frameworks, Geometry, Unit 6: Applications of Probability has many tasks that align to this cluster.
- UC San Diego's Computer Science and Engineering page has a variety of probability problems that can be adapted for the course by using two-way tables, Venn diagrams, and tree diagrams.
- EngageNY, Algebra 2, Module 4, Topic A, Lesson 2: Calculating Probabilities of Events Using Two-Way Tables, Lesson 3: Calculating Conditional Probabilities and Evaluating Independence Using Two-Way Tables, Lesson 4: Calculating Conditional Probabilities and Evaluating Independence Using Two-Way Tables, Lesson 5: Events and Venn Diagrams, Lesson 6: Probability Rules, and Lesson 7: Probability Rules are many tasks that align to this cluster.
- EngageNY, Precalculus and Advanced Topics, Module 5, Topic A, Lesson 1: The General Multiplication Rule, Lesson 2: Counting Rules—The Fundamental Counting Principle and Permutations, Lesson 3: Counting Rules—Combinations, Lesson 4: Using Permutations and Combinations to Compute Probabilities are tasks that align to this cluster.
- The Mathematics Vision Project, Secondary Math Two, Module 9: Probability has many tasks that align to this cluster.
- Illustrative Mathematics, Geometry, Unit 8, Lesson 6: The Addition Rule,
- The University of Florida has an Open Learning Textbook on Biostatistics that has good explanations about probability.
- Probability through Data: Interpreting Results from Frequency Tables by P. Hopfensperger, H. Kranendonk, R. Scheaffer is a module from Dale Seymour Publications.
- Probability Models by P. Hopfensperger, H. Kranendonk, R. Scheaffer is a pdf of a module by Dale Seymour Publications.

INSTRUCTIONAL SUPPORTS FOR THE MODEL CURRICULUM (S.CP.6-9)

General Resources

- The Mathematics of Games of Pure Chance and Games of Pure Strategy by Sam Smith is a pdf explaining the probability of game theory.
- Arizona's High School Progression on Statistics and Probability is an informational document for teachers. This cluster is addressed on pages 15-17.
- Arizona's High School Progression on Modeling is an informational document for teachers. Statistics and Probability is discussed on page 10.
- High School Coherence Map by UnboundED is a map of the relationships among high school mathematics standards. Please note: This resource is aligned to the Common Core not Ohio's Learning Standards. Although the tool is useful, it needs to be checked with respect to Ohio's Learning Standards.
- Statistics Teacher is an online journal published by the American Statistical Association (ASA) and NCTM. It includes articles, lesson plans, and professional development opportunities.
- Significance is a magazine that demonstrate the practical use of statistics and shows how statistics benefits society.
- Chance is a magazine about statistics for anyone who is interested in using data to advance science, education, and society.
- Levels of Conceptual Understanding in Statistics (LOCUS) is an NSF funded project that has assessment questions around statistical understanding.

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